Reflections on the matter of 3D $\mathcal{N}=1$ vacua and local $Spin(7)$ compactifications

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We use Higgs bundles to study the 3D $\mathcal{N}=1$ vacua obtained from M-theory compactified on a local $Spin(7)$ space given as a four-manifold M_4 of ADE singularities with further generic enhancements in the singularity type along one-dimensional subspaces. There can be strong quantum corrections to the massless degrees of freedom in the low energy effective field theory, but topologically robust quantities such as "parity" anomalies are still calculable. We show how geometric reflections of the compactification space descend to 3D reflections and discrete symmetries. The parity anomalies of the effective field theory descend from topological data of the compactification. The geometric perspective also allows us to track various perturbative and nonperturbative corrections to the 3D effective field theory. We also provide some explicit constructions of well-known 3D theories, including those which arise as edge modes of 4D topological insulators, and 3D $\mathcal{N}=1$ analogs of grand unified theories. An additional result of our analysis is that we are able to track the spectrum of extended objects and their transformations under higher-form symmetries.

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I. INTRODUCTION

Geometric engineering provides a promising way to recast difficult questions in quantum field theory (QFT) in terms of the geometry of extra dimensions in string theory. One of the underlying themes in much of this progress has been the use of holomorphic structures and its close connection with supersymmetric QFT in flat space.

But there are also many QFTs where we cannot rely on holomorphy considerations. In general, this makes it difficult to study strong coupling dynamics in such systems. Anomalies of symmetries provide a potentially promising route for constraining strong coupling dynamics because they are among the most robust "topological" aspects of a QFT.

From the perspective of string theory, it is natural to ask whether there is a geometric lift of these "bottom-up" considerations which can be used to constrain the dynamics of string compactification geometries with little or no supersymmetry. In the putative effective field theory, strong coupling effects lead to modifications of the classical internal geometry. Conversely, by studying nonperturbative contributions in a string compactification, one can hope to pinpoint the onset of strong coupling effects in a given QFT.

In this paper, we study 3D $\mathcal{N}=1$ vacua as obtained from M-theory on a local $Spin(7)$ space. Such compactifications are closely related to 4D $\mathcal{N}=1$ vacua as obtained from M-theory on local G_2 spaces and F-theory on local elliptically fibered Calabi-Yau fourfolds, where holomorphic structures play a more prominent role [1,2]. These systems are also potentially of interest in the study of "4D $\mathcal{N}=1/2$" F-theory based models of dark energy (see, e.g., Refs. [3–6]). For earlier work on M-theory compactified on a $Spin(7)$ space, see, e.g., Refs. [7–14].

In particular, we focus on the local $Spin(7)$ space defined by a four-manifold M_4 of ADE singularities, with further enhancement in the singularity type along subspaces. As advocated in Ref. [5], a fruitful way to analyze such geometries is via the associated 7D gauge theory of a spacetime filling 6-brane wrapping M_4. Local intersections with other 6-branes can be modeled in terms of background internal profiles for fields of the 7D gauge theory. These background fields satisfy the Vafa-Witten equations on a non-Kähler four-manifold [15] and specify a stable Higgs bundle.

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In these models, the field content descends from modes spread over the entirety of M_4, namely "bulk modes," and those which are trapped along a subspace, i.e., "local matter." Depending on the choice of M_4 and localization profile, one can envision engineering a rich class of possible 3D $\mathcal{N}=1$ gauge theories coupled to matter. A general feature of these QFTs is that the localized matter actually realizes a 3D $\mathcal{N}=2$ sector, and the bulk modes of the system can be packaged in terms of 3D $\mathcal{N}=1$ multiplets. We can also use this geometric starting point to analyze the spectrum of extended objects, as obtained from M2-branes and M5-branes wrapped on noncompact cycles of the geometry, as well as the resulting higher-form symmetries acting on these objects.

Performing a general analysis of localized zero modes, we show that they generically reside on codimension-3 subspaces inside of M_4. This is very much in line with what we would have gotten from the circle reduction of M-theory on a local G_2 space, as obtained from a three-manifold M_3 of ADE singularities, where chiral matter is localized at points of the three-manifold (see, e.g., Refs. [16–19]). It is, however, a bit surprising from the perspective of F-theory compactified on a local Calabi-Yau fourfold given by a Kähler surface of ADE singularities, where chiral matter is obtained from 6D hypermultiplets in the presence of a background gauge field flux [20,21].[1]

We give some general methods of construction centered on building up four-manifolds M_4 from connected sums with summands $M_3^{(i)} \times S^1$, where $M_3^{(i)}$ is a three-manifold. Local G_2 systems with matter at points of $M_3^{(i)}$ give rise to 4D $\mathcal{N}=1$ matter, and this general gluing construction produces 3D $\mathcal{N}=2$ matter coupled together by 3D $\mathcal{N}=1$ bulk modes spread over all of M_4. A related construction involving gluing non-Kähler four-manifolds to Kähler manifolds with matter on holomorphic curves leads to 3D $\mathcal{N}=4$ matter coupled via 3D $\mathcal{N}=1$ matter. We also present a complementary quotient construction based on elliptically fibered Calabi-Yau fourfolds in an Appendix.

Provided one is interested in generating *classical* 3D $\mathcal{N}=1$ field theories, the geometry of singular $Spin(7)$ spaces provides a promising way to engineer examples. In the flow to the deep infrared, however, there can potentially be strong corrections to these classical results due to the absence of any constraint from holomorphy. Indeed, this is just the string compactification analog of the standard difficulties faced in analyzing 3D $\mathcal{N}=1$ QFTs.

Faced with these difficulties, we can ask whether robust quantities such as anomalies can constrain possible quantum corrections. For 3D systems, this has proven to be a powerful way to study the phase structure in the infrared, even in the absence of supersymmetry (see, e.g., Refs. [22,23]). From this perspective, it is tempting to just take the classical 3D system engineered from the large volume $Spin(7)$ geometry and use this as a starting point for a purely 3D analysis. The difficulty here is that, in addition to all of the classical zero modes, there is the entire spectrum of Kaluza-Klein states which need to be integrated out. To perform a proper analysis of such systems, we therefore need to track the higher-dimensional origin (when available) of these 3D symmetries.

Our aim here will be to focus on symmetries which can be analyzed for any candidate 3D theory, namely the action of 3D spatial reflections on physical fields, and its interplay with other symmetry transformations coming from gauge symmetries or global symmetries. Anomalies of these symmetries, including mixed gravitational-parity and gauge-parity anomalies then provide nonperturbative control over some aspects of these systems.[2] The corresponding anomalies are often calculable and provide us with a sharp tool in constraining the resulting 3D $\mathcal{N}=1$ vacua obtained from local $Spin(7)$ spaces. More precisely, we shall be interested in the action of reflections for the 3D theory in a Euclidean spacetime,

$$\mathsf{R}_i^{3d}\colon\ x^i \to -x^i, \quad \text{and} \quad \mathsf{R}_i^{3d}\colon\ x^j \to x^j \quad \text{for } j \neq i, \tag{1.1}$$

and the corresponding transformations on our physical fields. This turns out to be a bit subtle because the physical content of our 3D system descends from a higher-dimensional starting point, so reflections in three dimensions may end up being composed with other internal reflection symmetries.

With this in mind, we first track how reflection assignments of 7D super Yang-Mills (SYM) theory descend from M-theory on an ADE singularity and 10D SYM theory on a T^3. Doing so, we show that the 7D reflection action on the physical fields is determined by a composition of spacetime and internal reflections of the higher-dimensional theory. Similarly, in the 3D effective field theory, the reflection assignments come about as compositions of reflections,

$$\mathsf{R}_i^{3d} = \mathsf{R}_i^D \prod_{\text{int}} \mathsf{R}_{\text{int}}^D, \tag{1.2}$$

for a higher-dimensional theory in D spacetime dimensions, where R_i^D denotes the reflection action on the D-dimensional fields in the 3D spacetime direction and $\mathsf{R}_{\text{int}}^D$ is a reflection in the internal directions. This leads to

[1] In F-theory models, this leads to zero modes which are localized in all four directions of the internal Kähler manifold. The reason this generically does not occur in the $Spin(7)$ setting is that there are three rather than two adjoint valued degrees of freedom in the Higgs field.

[2] These are sometimes referred to as "parity anomalies," but as explained in Ref. [24], it is more appropriate to view them as associated with various reflections. In a reflection symmetric background, the latter must vanish, but the former (when gravity is decoupled) can be nonzero.

additional possible discrete symmetries which can be arranged by tuning the moduli of the internal geometry.

The classical zero modes of a given local model each make contributions to anomalies associated with 3D reflection symmetries which we can explicitly evaluate. Strong coupling effects can potentially gap the system or at least remove some of the candidate zero modes, but a remnant of the "topological order" associated with these anomalies will still persist.

We also develop some examples illustrating these general points. As a preliminary example which we repeatedly return to throughout the paper, we show that matter localized on a one-dimensional subspace of M_4 can, in the limit where M_4 is decompactified, be understood as 4D matter with a position dependent mass, as in the standard topological insulator construction (see, e.g., Refs. [25–27]). As another general class of examples, we show how to take a chiral 4D $\mathcal{N}=1$ system engineered from the Pantev-Wijnholt (PW) system compactified on a further S^1 and glue it to its reflected image, resulting in a reflection symmetric 3D $\mathcal{N}=1$ theory. As an additional set of examples, we consider analogs of grand unified theories (GUTs) in three dimensions engineered from related gluing constructions.

The rest of this paper is organized as follows. In Sec. II, we discuss aspects of 7D super Yang-Mills theory coupled to defects and explain its relation to local $Spin(7)$ spaces. We show in Sec. III that localized matter fields in local $Spin(7)$ geometries generically lie on one-dimensional subspaces. In Sec. IV, we show that local matter can be interpreted as edge modes of a 4D topological insulator. In Sec. V, we give some gluing constructions for how to produce 3D $\mathcal{N}=1$ vacua starting from 3D $\mathcal{N}=2$ building blocks. We then turn to quantum effects, beginning in Sec. VI with a study of reflection symmetries in 7D SYM and their higher-dimensional origins. In Sec. VII, we track the resulting reflection assignments for 4D and 3D field theories obtained from compactification of 7D SYM theory. We then turn to the computation of various parity anomalies in Sec. VIII. Section IX analyzes various quantum corrections to the classical backgrounds. In Sec. X, we turn to some examples illustrating where we engineer various 3D $\mathcal{N}=1$ field theories and compute the associated parity anomalies. Section XI contains our conclusions and directions of further investigation.

We defer several additional technical items to the Appendixes. In Appendix A, we discuss some additional aspects of classical zero mode localization in local $Spin(7)$ systems. In Appendix B, we demonstrate a new construction of $Spin(7)$ manifolds as a quotient of Calabi-Yau four-folds in a local patch that we conjecture extends to compact cases. Appendix C reviews our various conventions for 10D and 7D spinors. In Appendix D, we provide details on a super quantum mechanics construction of Euclidean M2-branes in our local $Spin(7)$ models which illuminates the appearance of a twisted differential operator in the 3D $\mathcal{N}=1$ superpotential. Appendix E covers the dimensional reduction of reflection transformations from seven dimensions to four dimensions and three dimensions. Appendix F covers a G_2-spectral cover construction of 4D $\mathcal{N}=1$ $\mathfrak{so}(10)$ gauge theories, which, after dimensional reduction on a circle, is an example of a building block for constructing 3D $\mathcal{N}=1$ $Spin(7)$ systems in Sec. X. Finally, Appendix G reviews the various 3D parity anomalies discussed in the bulk of this paper and how they arise as the phase ambiguity of a 3D theory's partition function after placing it on an nonorientable Pin^+ manifold.

II. HIGGS BUNDLE APPROACH TO LOCAL $Spin(7)$ GEOMETRIES

As mentioned in the Introduction, our interest in this paper is the study of 3D $\mathcal{N}=1$ vacua as engineered by M-theory on manifolds of $Spin(7)$ holonomy. In particular, we shall be interested in spacetimes given by a (possibly warped) product of 3D Minkowski space with this internal geometry. Early work in this direction primarily focused on the case of smooth $Spin(7)$ spaces; see, e.g., Refs. [7,12,14]. Our general aim in this paper will be to study the comparatively less explored case of a four-manifold M_4 of ADE singularities which give rise to singular noncompact $Spin(7)$ spaces. Much as in Ref. [5], our approach will be to analyze the local M-theory dynamics in terms of the world volume theory of 7D SYM theory filling the 3D spacetime and wrapping the four-manifold M_4. This is just the Vafa-Witten twist of $\mathcal{N}=4$ super Yang-Mills theory on a four-manifold [15].

Recall that M-theory on an ADE singularity gives rise to 7D SYM theory with gauge group of ADE type. The bosonic content of this theory consists of a 7D gauge connection and three scalars ϕ_a which transform in the triplet representation of the $SU(2)$ R-symmetry. Some of this structure can be seen from the classical geometry of the ADE singularity. For example, upon resolving the ADE singularity, we get a collection of S^2's which intersect according to the corresponding ADE Dynkin diagram. Integrating the M-theory 3-form potential over each such S^2 gives rise to a $U(1)$ vector potential. The "off-diagonal" components of the vector potential for the 7D SYM theory come from M2-branes wrapped on the other simple roots obtained from the homology lattice of the resolved space. Fibering this further over a four-manifold M_4 just amounts to considering 7D SYM wrapped on M_4. The assumption that this fibration is a local $Spin(7)$ space means we retain 3D $\mathcal{N}=1$ supersymmetry in the transverse 3D Minkowski directions. In the 7D SYM theory, this is enforced by taking a topological twist of the theory so that we retain at least one real doublet of supercharges in the 3D theory.[3] This turns out to be the same as the Vafa-Witten topological twist [15] for 4D $\mathcal{N}=4$ super Yang-Mills theory on a four-manifold M_4.

After the topological twist, the triplet of scalars assemble into adjoint-valued self-dual 2-forms which we write as $\Phi_{\rm SD}$. The field content of the theory also includes the 7D gauge connection A_{7d} as well as their fermionic superpartners. Observe that $\Lambda^2_{\rm SD}\to M_4$, the bundle of self-dual 2-forms over M_4, gives rise to a local G_2 space (with a possibly incomplete metric). Indeed, we could also consider type IIA string theory on this 7D background, where we wrap D6-branes on M_4. The point is that we expect to arrive at the same 3D theory from either starting point. Because we have a local G_2 space, we know it comes equipped with a distinguished associative 3-form. This also furnishes us with a "cross-product" operation on the self-dual 2-forms, where we have

$$(\Phi\times\Phi)_a=\varepsilon_{abc}\Phi_b\Phi_c, \tag{2.1}$$

where ε_{abc} is the canonical three-index "volume form" of the fiber in $\Lambda^2_{\rm SD}\to M_4$. For adjoint-valued forms, the multiplication on the right-hand side is replaced by a commutator. This implies that the product of Φ with itself may be nonzero.

It is convenient to package the 7D fields in terms of suitable 3D $\mathcal{N}=1$ supermultiplets, sorted according to their Lorentz quantum numbers in the 3D spacetime. For example, the 7D gauge connection decomposes as $A_{7d}\to A_{3d}\oplus A_{M_4}$, so from A_{3d}, we get a 3D $\mathcal{N}=1$ vector multiplet, while from A_{M_4}, we get additional adjoint-valued scalar multiplets. By similar reasoning, the self-dual 2-forms can be viewed as a collection of 3D $\mathcal{N}=1$ scalar multiplets labeled by points of M_4.

The vacua of this compactified system are captured by the critical points of a corresponding 3D $\mathcal{N}=1$ superpotential

$$W_{3d}=\int_{M_4}\mathrm{Tr}\Phi_{\rm SD}\wedge\left(F+\frac{1}{3}\Phi_{\rm SD}\times\Phi_{\rm SD}\right), \tag{2.2}$$

where $\Phi_{\rm SD}\times\Phi_{\rm SD}$ is shorthand for the cross-product of Eq. (2.1), F denotes the superfield associated with the curvature of the field strength on M_4 and by abuse of notation $\Phi_{\rm SD}$ also labels a superfield. The Bogomol'nyi–Prasad–Sommerfield (BPS) equations of motion are

$$F_{\rm SD}+\Phi_{\rm SD}\times\Phi_{\rm SD}=0,\qquad d_A\Phi_{\rm SD}=0, \tag{2.3}$$

and vacua are given by solutions to the BPS equations of motion modulo gauge transformations. In the above, $F_{\rm SD}$ is the self-dual part of the field strength, i.e., $F_{\rm SD}=\frac{1}{2}(F+*F)$. These equations define a stable Higgs bundle on M_4.

[3] More precisely, we start with the $SU(2)_R$ R-symmetry of the 7D theory in flat space and consider a homomorphism $\mathfrak{su}(2)_R\to\mathfrak{so}(2,1)\oplus\mathfrak{so}(4)_{M_4}$ which is trivial on the $\mathfrak{so}(2,1)$ factor.

From this starting point, we can also analyze backgrounds where the singularity type enhances along subspaces of M_4. In general terms, we can consider some "parent gauge theory" with gauge group $\tilde{G}$ and take a background solution which leaves some residual gauge symmetry unbroken. We get massless matter fields in the 3D theory from the first order fluctuations around this background:

$$A=a+\langle A\rangle \tag{2.4}$$

$$\Phi=\varphi+\langle\Phi\rangle. \tag{2.5}$$

The matter fields will transform in representations of the unbroken gauge symmetry $H\subset\tilde{G}$.

For ease of exposition, here we primarily focus on the case where the gauge connection is switched off, so in other words, we take $\langle A\rangle=0$, so that only $\langle\Phi\rangle$ has a nonzero background value. In this case, the BPS equations of motion enforce the condition that $\Phi_{\rm SD}$ takes values in the Cartan subalgebra of $\tilde{G}$. We can speak of a collection of independent self-dual 2-forms which satisfy the equation of motion $d\Phi_{\rm SD}=0$. Now, to get a solution on M_4 which is globally valid, we sometimes need to allow for singularities along various subspaces. In physical terms, these are "sources" which are needed to satisfy a Gauss' law constraint on an otherwise compact space. For example, if we assume that all two-cycle periods of $\Phi_{\rm SD}$ are zero, there must be these defect sources (flavor brane intersections) as otherwise $\Phi_{\rm SD}=0$. In this case, the singularities we consider are of the form

$$d\Phi_{\rm SD}=\sum_i v_i\delta_{L_i}, \tag{2.6}$$

where L_i denotes a one-dimensional subspace of M_4 and δ_{L_i} denotes a delta function 3-form with support on L_i. Treating $\Phi_{\rm SD}$ as an Abelian flux, one can see that Gauss's law requires

$$\sum_i v_i[L_i]=0\in H^1(M_4,\mathbb{R}) \tag{2.7}$$

to be satisfied. Notice that, since we are free to change the orientations of the L_i's, this condition implies that $\sum v_i=0$ mod 2.

This background initiates a breaking pattern to a subgroup $G\times U(1)^n\subset\tilde{G}$ (for now, we shall be a bit cavalier with the global topology of the group). A zero mode in a particular representation of $G\times U(1)^n$ comes from decomposition of the adjoint representation of $\tilde{G}$ into irreducible representations of $G\times U(1)^n$. Focusing on the simplest nontrivial case where we have a single $U(1)$ factor, and *candidate* localized modes with charge q with respect to this $U(1)$, we get the linearized approximation to the BPS equations of motion:

$$(da)_{\mathrm{SD}} + q\Phi_{\mathrm{SD}} \times \varphi = 0, \tag{2.8}$$

$$d\varphi - q\Phi_{\mathrm{SD}} \wedge a = 0. \tag{2.9}$$

We can package this in terms of a twisted differential operator,

$$D_{q\Phi} \equiv \begin{pmatrix} q\Phi_{\mathrm{SD}}\times & D_{\mathrm{sig.}} \\ D_{\mathrm{sig.}} & -q\Phi_{\mathrm{SD}}\wedge \end{pmatrix}, \tag{2.10}$$

where $D_{\mathrm{sig.}} = d + d^{\dagger}$ is the signature operator.[4] For additional discussion on the operator $D_{q\Phi}$ in the context of the associated supersymmetric quantum mechanics theory, see Appendix D. In terms of this, we can speak of a zero mode equation for a given representation,

$$D_{q\Phi}\Psi_q = 0, \tag{2.11}$$

where $\Psi_q = (\varphi, a - *a)$ is a two-component vector with mixed form degree.

At a practical level, we can search for localized solutions by considering the rescaled limit $\Phi_{\mathrm{SD}} \to t\Phi_{\mathrm{SD}}$, and taking t to be large. In this case, our candidate zero modes amount to those loci where Φ_{SD} has a zero. From the point of view of a spectral cover construction, these zeros represent pairwise intersections of sheets. For example, let $\tilde{G} = SU(N)$, and fix a section $v \in \Lambda^2_{\mathrm{SD}}$. The spectral equation in the fundamental representation is then

$$\det(v\mathbf{1}_{N\times N} - \Phi_{\mathrm{SD}}) = 0, \tag{2.12}$$

which defines a four-manifold inside the bundle Λ^2_{SD}, which is a finite cover of M_4. This is of course directly related to the IIA dual picture of the same system which is a local G_2 model on the total space of Λ^2_{SD} with a collection of intersecting D6-branes supported along the spectral cover.

An important caveat here is that zeros of the Higgs field are really just candidate zero modes, and in principle, these solutions can receive small mass terms due to Euclidean M2-branes which stretch between pairs of these loci. This is very analogous to what happens in the PW system [17]. In the rescaling limit $\Phi_{\mathrm{SD}} \to t\Phi_{\mathrm{SD}}$, we can see such corrections by performing an expansion in $1/t$.

With this caveats in mind, let us now turn to the counting of both bulk zero modes and localized zero modes with respect to the background Higgs field profile. Consider first the zero modes with charge $q = 0$. In this case, we find $b^2_{\mathrm{SD}} + b^1 + 1$ zero modes for each representation uncharged under $U(1)$ (we assume the vector bundle is switched off), where the contribution from 1 is just the contribution from the 3D $\mathcal{N}=1$ vector multiplet. These modes will not be localized by the profile of Φ_{SD}. Even if these modes are massless at the classical level in the 3D effective field theory, the actual zero mode count can potentially be different, due to some of these modes pairing up or additional strong coupling effects. Later on, we will show that there is an anomaly capable of "detecting" a contribution from $|b^2_{\mathrm{SD}} - b^1 + 1|$ such zero modes.

Next, consider the zero modes with charge $q \neq 0$. One issue we encounter is that as far as we are aware the counting of zero modes does not cleanly reduce to a cohomology group calculation. One simpler problem is to look for the zero locus of Φ_{SD} given that zero modes localize around this locus in the limit of large vacuum expectation value (vev) of Φ_{SD}. The zero locus of Φ_{SD} can be characterized quite nicely as comprising a set of N_Φ circles ℓ_m where the integer N_Φ is congruent to $(b_2^+ - b_1 + 1)$ mod 2 [28]. As in the case of delocalized zero modes, the mod 2 reduction of N_Φ will be related to the presence of an anomaly in the 3D effective field theory that will be discussed later in the paper. Another useful property is that the zero locus of Φ_{SD} is homologically trivial as it is dual to the Euler class of Λ^2_{SD}, which is zero [28]. Around any zero of Φ_{SD}, we can model its profile locally as

$$\Phi_{\mathrm{SD}}|_{\ell_m} = df_m \wedge dt_m + *_3 df_m. \tag{2.13}$$

Here, t_m is a local coordinate on the circle ℓ_m, f_m is a function that does not depend on t_m, and the Hodge star operation $*_3$ is taken in the directions normal to ℓ_m inside M_4. The fact that the profile of Φ_{SD} can be written in terms of some function on a three-manifold is reminiscent of the PW system [17]. The localized mode will actually behave as a 4D mode on $\mathbb{R}^{1,2} \times \ell_m$ with chirality determined by the sign of the determinant of the Hessian of the function f_m, much as in Ref. [17].

The leading order interaction terms for these candidate zero modes are obtained by expanding our 3D superpotential of Eq. (2.2) about a fixed background. This leads to the interaction terms for candidate zero modes,

$$W_{3d} = \int_{M_4} \sum_{q_i+q_j+q_k=0} (\varphi_i \wedge (a_j \wedge a_k) + \varphi_i \wedge (\varphi_j \times \varphi_k)) + \cdots, \tag{2.14}$$

where the "…" refers to contributions which include both mass terms and additional compactification effects which are suppressed in the large volume limit.

It is also of interest to consider *massive* modes, namely those for which $D_{q\Phi}\Psi_q = m\Psi_q$, which, before twisting, derives from the usual 4D massive Dirac equation written as (turning off the scalars associated to Φ_{SD} for simplicity)

[4] Note that in four Euclidean dimensions $d^{\dagger} = - * d *$.

$$\begin{pmatrix} 0 & \not{D} \\ \not{D} & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = m \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{2.15}$$

where here ψ_R and ψ_L are right-/left-handed Weyl fermions on M_4 which are the fermionic components of massive fluctuations. The important point for us is that this Dirac equation explicitly references the *sign* of a real mass, m. In the context of reducing to a 3D theory, integrating out positive mass and negative mass terms can shift the Chern-Simons level(s) of the 3D gauge theory with a contribution which depends on the sign of the mass term.[5]

A. Defects and higher-form symmetries

Although our main emphasis will be on various aspects of reflection symmetries, it is worthwhile to also discuss how our considerations fit with more global structures of the resulting 3D effective field theory, and in particular various higher-form symmetries which might be present. See, e.g., Refs. [32–40] for additional details on aspects of higher-form symmetries and their relation to string compactification.

For starters, we can ask about the global nature of the 7D SYM gauge group. Indeed, *a priori*, we could have a simply connected G or a quotient by some subgroup $C_G \subset Z_G$, with Z_G the center of G. Geometrically, the center $\mathcal{C} = \Lambda^*/\Lambda$, where $\Lambda = H_2^{cpct}(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{Z})$ is just the homology group for the resolution of the ADE singularity $\mathbb{C}^2/\Gamma$ and Λ^* is the dual lattice. Indeed, we can consider M2-branes stretched on the noncompact two-cycles of Λ^*, and these descend to nondynamical Wilson lines. This is basically just the computation of a "defect group" as in Refs. [33,41] (see also Refs. [36,42]). The specific choice of gauge group is then dictated by boundary conditions for discrete fluxes on a bounding S^3/Γ at infinity in $\mathbb{C}^2/\Gamma$, and the analysis of Sec. 5 in Ref. [33] can be repurposed to recover the different possible options.

Closely related to this is the spectrum of extended objects generated by M5-branes wrapping these same noncompact two-cycles in the fiber direction. For example, we can consider wrapping an M5-brane over all of M_4 and a noncompact two-cycle of Λ^*, and this gives rise to objects which are charged under a magnetic 0-form symmetry in three dimensions. If we have a two-cycle $\Sigma \in H_2(M_4, \mathbb{Z})$ (neglecting torsion), then we get a corresponding stringlike defect, i.e., domain wall from also wrapping on a noncompact two-cycle of Λ^*. In 3D field theory terms, this is an object which is charged under a magnetic 2-form symmetry. The associated extended objects then have charges labeled by a maximal isotropic sublattice of $H_2(M_4, \mathcal{C})$ (see Ref. [33]).

In actual local models, of course, we typically have a more complicated configuration of intersecting 6-branes. For example, with local matter, we often speak of a parent gauge group $\tilde{G}$ and its unfolding to a descendant $H \subset \tilde{G}$. Moreover, this local enhancement can be different at different locations of M_4. Based on this, we need to be able to globally fit together all the different possible choices. While in general this is a challenging problem, the key point is that in spectral cover constructions where we start with a single $\tilde{G}$ we already know that the appropriate notion of Λ and Λ^* is given by $\Lambda = H_2^{cpct}(\widetilde{\mathbb{C}^2/\Gamma_{\tilde{G}}}, \mathbb{Z})$. For example, if we consider adjoint breaking of $E_8 \supset E_6 \times SU(3)/\mathbb{Z}_3 \supset E_6 \times S(U(1)^3)/\mathbb{Z}_3$, we know that we have matter in the 27 of E_6, so the gauge group cannot be $E_6/\mathbb{Z}_3$. The point is that we also need to account for the contribution from the flavor branes of the model. Indeed, since the center of E_8 is trivial (since its Cartan matrix is unimodular), the resulting defect group (s) associated with Λ^*/Λ are all trivial. That being said, we can of course consider other starting points for our $\tilde{G}$ which have a nontrivial center. In such situations, we can use the geometry to quickly ascertain the candidate higher-form symmetries in the 3D effective field theory. A final comment is that there can potentially be an additional contribution from the noncompact four-cycle Poincaré dual to M_4 itself. It would be interesting to study this universal contribution, but we defer this to future work.

III. ULTRALOCAL *Spin*(7) MATTER FIELDS

In the previous section we discussed in general terms the use of Higgs bundles on M_4 as a tool in understanding localized matter in $Spin(7)$ compactifications of M-theory. Our aim in this section will be study in more detail the profile of these zero mode solutions. In particular, we would like to understand to what extent these zero modes can actually be localized.

As is common in the string compactification literature, there are distinct notions of localization. First, we can work in an "ultralocal" patch of the four-manifold diffeomorphic to $\mathbb{R}^4$. We can also work in the context of a local model where M_4 is compact (up to deleting various subspaces to satisfy Gauss's law constraints), but where the $Spin(7)$ geometry is noncompact. Finally, we can consider what happens when we take a compact $Spin(7)$ geometry, in which case the 3D effective field theory will be coupled to gravity.

In this section, we study the ultralocal properties of localized matter fields in $Spin(7)$ spaces. Our analysis in this section will be purely classical, since we neglect all quantum corrections coming from dynamics of the 3D effective field theory [and its lift to nonperturbative instanton corrections to the classical $Spin(7)$ geometry]. Later, we will analyze some features which are robust against such corrections. Throughout, for ease of exposition,

[5]For a related discussion on the relation between 4D anomalies and their circle compactification in the context of M-/F-theory duality, see, e.g., Refs. [29–31].

we assume that the metric on our patch of M_4 is just the standard flat space metric on $\mathbb{R}^4$.

To frame the discussion to follow, recall that the local $Spin(7)$ equations of motion amount to a hybrid of the PW and Beasley-Heckman-Vafa (BHV) Higgs bundle systems [2], so we expect that the profile of localized zero modes will share some similarities with these cases. Starting from our local $Spin(7)$ system on $M_3 \times S^1$ for M_3 a three-manifold, we reach the PW system by assuming all fields are constant along the S^1 direction, and contracting Φ_{SD} with the 1-form on S^1, and also dropping the contribution from the gauge field along the circle. We can reach the BHV system by specializing M_4 to be a Kähler surface, and in this case, Φ_{SD} splits up as an adjoint-valued (2,0)-form, as well as a (1,1)-form proportional to the Kähler form which decouples from the rest of the system. We reference the Higgs fields for these special cases as Φ_{PW} and Φ_{BHV} in what follows.

Now, in the context of local G_2 compactifications, chiral matter is localized along codimension-7 subspaces, namely at points [16,43]. This can also be seen by a direct analysis of the corresponding local Higgs bundle, and the vanishing locus of the PW Higgs field [17–19]. In that case, the Higgs field is an adjoint-valued 1-form on a three-manifold, and since we have three distinct scalar degrees of freedom, we expect to generically localize matter along a codimension-3 subspace, that is, a point. Observe that this is precisely what we also expect in the context of local $Spin(7)$ systems on a four-manifold M_4. Indeed, in that case as well, we have three independent components of Φ_{SD}, and this generates a codimension-3 subspace inside M_4, namely a distinguished set of one-dimensional subspaces.

In the case of compactification on elliptically fibered Calabi-Yau fourfolds, localization of matter actually descends from two separate sources. From geometry, we get localization along complex codimension-3 subspaces. This is in accord with the fact that in the BHV system on a Kähler surface S we have a single (2,0)-form, and the zeros of this cut out complex codimension-1 subspaces. Further localization is possible once we include the effects of flux [21,44–47]. Generically, we get an equation of motion on a complex curve of the form $\overline{\partial}_A \psi = 0$, and so when the curvature associated with the gauge connection is nontrivial, we see a further localization to a real codimension-4 subspace in S.[6] In the context of local $Spin(7)$ systems, we are of course free to consider the special case of BHV backgrounds, and so in this sense, we can cut out a real codimension-2 subspace in M_4 with the Higgs field and then introduce a suitable gauge field flux to produce further localization at a point.

Comparing the generic expectations from the PW and BHV systems, we see that the former would appear to predict matter localization along codimension-7 subspaces, while the BHV system would appear to predict matter localization along codimension-8 subspaces. What then, should we expect in the full-fledged local $Spin(7)$ system?

Our general claim is that in this setting matter is generically localized along codimension-7 subspaces even when gauge field flux is switched on. This is in accord with what we expect from the PW system but appears to be in tension with expectations from the BHV system. The main idea is that if we again attempt to analyze matter localization by first considering the zeros of the Higgs field we get a codimension-3 subspace inside the four-manifold M_4. Now, if we attempt to further localize by switching on a gauge field flux, we face the awkward situation that we are attempting to pull back this flux onto the matter one-cycle. The best that can happen is that we can pullback the gauge field to a holonomy on the matter one-cycle, but this does not produce any further localization in the zero mode, a fact which can be explicitly checked by analyzing the 1D "Dirac equation" on this subspace.[7] The perhaps counterintuitive conclusion is that inside M_4 generic backgrounds will lead to localization along one-cycles.

Our plan in the rest of this section will be to explicitly illustrate how matter localization works in this setting. For our purposes, it suffices to work in a local patch of M_4 which is diffeomorphic to $\mathbb{R}^4$. We first present the explicit local equations of motion and then turn to the special cases of localized matter in BHV and PW systems, illustrating that merging these solutions involves a clash in the choice of complex structure, which in turn obstructs further localization in the zero modes. We then turn to more general local $Spin(7)$ backgrounds. Additional technical details are given in Appendix A.

A. Localization in a patch

Before delving into the details of the examples, let us write down the system of zero mode equations with respect to a local coordinate system $(x^1, x^2, x^3, x^4) = (x, y, u, v)$. In the following, the background fields are $A = \{A_1, A_2, A_3, A_4\}$ and $\Phi = \{\Phi_1, \Phi_2, \Phi_3\}$. The fluctuations are $a = \{a_1, a_2, a_3, a_4\}$ and $\varphi = \{\varphi_1, \varphi_2, \varphi_3\}$, where the subscript on Φ and φ refers to the directions in the bundle of self-dual 2-forms.[8]

[6]We comment here that, although this localization appears to be gauge dependent, this is an artifact of only working in a local patch. On a compact manifold, there are peaks and valleys to the associated zero mode, which are determined by the holonomy of the gauge connection.

[7]The effect of the gauge field background is to produce a nontrivial holonomy around the circle where the mode is localized, and unless the component of the gauge field along the circle $A \in 2\pi\mathbb{Z}$, the mode is lifted with mass of order $\lfloor \frac{1}{2\pi} \iota^*_{\ell_i} A \rfloor$.

[8]One convenient basis of self-dual 2-forms on $\mathbb{R}^4$ is the 't Hooft matrices $\eta^i_{\mu\nu}$.

The zero mode equations for the $Spin(7)$ system are

$$D_1a_4 - D_4a_1 + D_2a_3 - D_3a_2 - [\Phi_2, \varphi_3] + [\Phi_3, \varphi_2] = 0, \tag{3.1}$$

$$D_1a_2 - D_2a_1 + D_3a_4 - D_4a_3 - [\Phi_1, \varphi_2] + [\Phi_2, \varphi_1] = 0, \tag{3.2}$$

$$D_1a_3 - D_3a_1 + D_4a_2 - D_2a_4 - [\Phi_1, \varphi_3] + [\Phi_3, \varphi_1] = 0, \tag{3.3}$$

$$D_1\varphi_1 - [\Phi_1, a_1] + D_2\varphi_2 - [\Phi_2, a_2] + D_3\varphi_3 - [\Phi_3, a_3] = 0, \tag{3.4}$$

$$D_4\varphi_3 - [\Phi_3, a_4] + D_1\varphi_2 - [\Phi_2, a_1] - D_2\varphi_1 + [\Phi_1, a_2] = 0, \tag{3.5}$$

$$D_1\varphi_3 - [\Phi_3, a_1] - D_4\varphi_2 + [\Phi_2, a_4] - D_3\varphi_1 + [\Phi_1, a_3] = 0, \tag{3.6}$$

$$D_2\varphi_3 - [\Phi_3, a_2] - D_3\varphi_2 + [\Phi_2, a_3] + D_4\varphi_1 - [\Phi_1, a_4] = 0. \tag{3.7}$$

In the examples, we take a gauge theory with Lie algebra $\mathfrak{su}(2)$ and align the background fields in the Cartan $P = i/2\sigma_3$. The fluctuations will be in the generator $Q = i/2(\sigma_1 + i\sigma_2)$ of the complexified Lie algebra. In the following, we will omit matrices for the sake of simplicity. Our aim will be to study localization properties of zero modes by looking at some examples.

Let us now expand on the general expectation that we do not expect to find full localization in codimension 4 but will only be able to generically achieve localization in codimension 3. To this end, we separate the localization effect provided by the Higgs field and by the gauge flux. In order to do so, we can take the limit of large value for Φ via the rescaling $\Phi \to t\Phi$. One can solve the equations to zeroth order in $1/t$ by first supposing that (φ, a) is a PW solution localized on the line (actually a circle, so we identify $x_4 \sim x_4 + 2\pi R$) $x_1 = x_2 = x_3 = 0$ but with possible extra dependence on the x_4 direction. In other words, our ansatz will be $(\varphi, a) = (f_1(x_4)\varphi_{\rm PW}(x_i), f_2(x_4)a_{\rm PW}(x_i))$ for $1 \leq i \leq 3$. The equations above then become

$$iA_1a_4 - D_4a_1 + iA_2a_3 - iA_3a_2 = 0, \tag{3.8}$$

$$iA_1a_2 - iA_2a_1 + iA_3a_4 - D_4a_3 = 0, \tag{3.9}$$

$$iA_1a_3 - iA_3a_1 + D_4a_2 - iA_2a_4 = 0, \tag{3.10}$$

$$iA_1\varphi_1 + iA_2\varphi_2 + iA_3\varphi_3 = 0, \tag{3.11}$$

$$D_4\varphi_3 + iA_1\varphi_2 - iA_2\varphi_1 = 0, \tag{3.12}$$

$$iA_1\varphi_3 - D_4\varphi_2 - iA_3\varphi_1 = 0, \tag{3.13}$$

$$iA_2\varphi_3 - iA_3\varphi_2 + D_4\varphi_1 = 0. \tag{3.14}$$

If we now undo the scaling of Φ by a coordinate rescaling, then since F is the same form degree of Φ we get instead $F \to \frac{1}{t}F$. This means that in Eq. (3.8) it is possible to gauge away A_1, A_2, and A_3 in the $t \to \infty$ limit since the connection is becoming flat. Hence, $f_1 = f_2 = Ce^{-iA_4x_4}$ for some constant (vector) C. Effectively, this means that the effect of the gauge background is to produce a nontrivial holonomy around the circle where the mode is localized, but this does not produce any further localization.

We will now discuss localization for the BHV and PW systems separately and then try to combine both and see that no fully localized mode exists.

B. BHV versus PW localization

We now turn to some examples of matter localization in BHV and PW systems and in particular illustrate some of the difficulties in simply combining these solutions to produce further localization in $Spin(7)$ backgrounds.

1. BHV localization

To discuss the BHV mode, let us consider a solution on $\mathbb{R}^4$ with complex coordinates $x + iy$ and $u + iv$ with a Kähler form $-\frac{i}{2}(dx \wedge dy + du \wedge dv)$. We can parametrize a general Abelian background as

$$A_1 = Ny, \qquad A_2 = -Nx, \qquad A_3 = -Nv, \qquad A_4 = Nu,$$
$$\Phi_1 = -\mu x, \qquad \Phi_2 = \mu y, \qquad \Phi_3 = 0. \tag{3.15}$$

This background admits a fully localized mode. The solution is

$$a_1 = -(N + \sqrt{N^2 + \mu^2})e^{-\frac{1}{2}N(u^2+v^2)-\frac{1}{2}(x^2+y^2)\sqrt{N^2+\mu^2}}, \tag{3.16}$$

$$a_2 = i(N + \sqrt{N^2 + \mu^2})e^{-\frac{1}{2}N(u^2+v^2)-\frac{1}{2}(x^2+y^2)\sqrt{N^2+\mu^2}}, \tag{3.17}$$

$$\varphi_1 = i\mu e^{-\frac{1}{2}N(u^2+v^2)-\frac{1}{2}(x^2+y^2)\sqrt{N^2+\mu^2}}, \tag{3.18}$$

$$\varphi_2 = -\mu e^{-\frac{1}{2}N(u^2+v^2)-\frac{1}{2}(x^2+y^2)\sqrt{N^2+\mu^2}}, \tag{3.19}$$

with all the remaining components set to zero. Notice in particular the component $a_1 + ia_2$ is localized and so is $\varphi_1 - i\varphi_2$. This is a familiar situation in F-theory models [21,44–47] where a 6D (in our case 5D) hypermultiplet can be first localized on a complex matter curve and full localization can be provided by threading an Abelian flux through the curve. Here, the matter curve is located at $x = y = 0$.

2. PW localization

Let us now turn to localization of PW matter in our local $Spin(7)$ system. We isolate one of the four directions in our local patch by writing $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. As an example, consider the Abelian background:

$$A_1 = 0, \qquad A_2 = 0, \qquad A_3 = 0, \qquad A_4 = 0,$$
$$\Phi_1 = -\lambda x, \qquad \Phi_2 = -\lambda y, \qquad \Phi_3 = 2\lambda u. \tag{3.20}$$

This admits a localized mode in codimension 3 of the form

$$a_3 = e^{-\frac{1}{2}\lambda(x^2+y^2+2u^2)}, \qquad \varphi_3 = -ie^{-\frac{1}{2}\lambda(x^2+y^2+2u^2)}, \tag{3.21}$$

with all the remaining components set to zero. Notice in particular that this background localizes the component $a_3 + i\varphi_3$. This background realizes a chiral multiplet along the matter line $x = y = u = 0$.

3. Obstructions to further localization

One issue when comparing the two localized set of modes in general is the following: the complex structures of the BHV and PW localized modes seem to be incompatible. This is a stumbling block to further localization. More specifically, BHV tends to localize modes of the form[9] $a_1 \pm ia_2$, $a_3 \pm ia_3$ and $\varphi_1 \mp i\varphi_2$ (the upper sign refers to a 4D chiral mode, and the lower one refers to a 4D antichiral mode). By this, we mean that if, for example, $a_1 + ia_2$ is localized then it must happen that $a_2 = -ia_1$. On the other hand, PW tends to localize one component of the fluctuation of the gauge field and its associated fluctuation of the Higgs field.[10] This means that the PW Higgs field localizes only one component of the gauge field and one component of the Higgs field, whereas BHV needs either two components of the gauge field or two components of the Higgs field, and naively it seems that both conditions are incompatible at the same time.

C. $Spin(7)$ backgrounds

Having dealt with the special cases of BHV and PW backgrounds embedded in the local $Spin(7)$ system of equations, we now turn to a more general class of examples. As a first example, we consider a background in which the Higgs field and gauge field flux are both switched on. This is essentially a "hybrid" of the BHV and PW backgrounds considered previously. As another example, we show that for backgrounds with just the Higgs field switched on localization can always be repackaged in terms of a closely related PW system. In both of these examples, zero modes localize on a one-dimensional subspace; i.e., they are codimension 3 in the four-manifold M_4 or, equivalently, codimension 7 in the local $Spin(7)$ geometry.

1. Hyrbid BHV and PW example

We now construct an example which is a hybrid of a BHV and PW background:

$$A_1 = Ny, \quad A_2 = -Nx, \quad A_3 = -Nv, \quad A_4 = Nu, \quad \Phi_1 = -\mu x - 2\lambda x, \quad \Phi_2 = \mu y + \lambda y, \quad \Phi_3 = \lambda u. \tag{3.22}$$

The solution of the zero mode equations has the form

$$\{a_1, a_2, a_3, a_4, \varphi_1, \varphi_2, \varphi_3\} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \eta_1, \eta_2, \eta_3\} e^{-\frac{1}{2}x.M.x}, \tag{3.23}$$

where we refer to Appendix A for further details of the matrix M in Eq. (A1). We have

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \eta_1, \eta_2, \eta_3\} = \tag{3.24}$$

$$\left\{ \frac{(3\lambda + 2\mu)\left(\sqrt{\lambda^2 + 4N^2}\sqrt{(\lambda+\mu)^2(\frac{4N^2}{(3\lambda+2\mu)^2} + 1)} + \lambda(\lambda+\mu)\right)}{4N} + N(\lambda+\mu), \right. \tag{3.25}$$

[9] Here, BHV is intended to be identified with a solution of the local $Spin(7)$ equations such that $\Phi_3 = 0$. Other choices are equivalent to this one.

[10] Here, PW is a solution that is invariant in one direction t where t can be any of the coordinates on the four-manifold. Given a gauge field A_i, the associated component of the Higgs field is $\iota_t \iota_i \Phi_{SD}$.

$$-\frac{1}{2}i(\lambda+\mu)\sqrt{\lambda^2+4N^2}-\frac{1}{2}i(3\lambda+2\mu)\sqrt{(\lambda+\mu)^2\left(\frac{4N^2}{(3\lambda+2\mu)^2}+1\right)},0,0, \tag{3.26}$$

$$-\frac{i(3\lambda+2\mu)\left(\mu\sqrt{\lambda^2+4N^2}+\lambda\left(\sqrt{\lambda^2+4N^2}+\sqrt{(\lambda+\mu)^2(\frac{4N^2}{(3\lambda+2\mu)^2}+1)}\right)\right)}{4N},(\lambda+\mu)^2,0\bigg\}. \tag{3.27}$$

Note that, since the real part of M has one zero eigenvalue, the modes will not be fully localized in codimension 4, confirming our earlier heuristic result.

2. Pure Higgs field example

The next background we shall discuss is the generic $Spin(7)$ pure Higgs field background; i.e., we leave the gauge field flux switched off. Our aim will be to show that this can be recast in terms of a quite similar analysis in terms of a local PW system. We can write the Higgs field as

$$\Phi_i = L_{ij}x_j. \tag{3.28}$$

The background equations become the conditions

$$L_{11}+L_{22}+L_{33}=0, \tag{3.29}$$

$$L_{12}-L_{21}+L_{34}=0, \tag{3.30}$$

$$L_{13}-L_{31}+L_{24}=0, \tag{3.31}$$

$$L_{23}-L_{32}-L_{14}=0. \tag{3.32}$$

To solve for the zero mode equations, we make the ansatz

$$\{a_1,a_2,a_3,a_4,\varphi_1,\varphi_2,\varphi_3\}=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\eta_1,\eta_2,\eta_3\}e^{-\frac{1}{2}x.M.x}. \tag{3.33}$$

Here, α_i and η_i are complex numbers. The equations are going to fix these coefficients as well as the matrix M. The equations become

$$\alpha_4M_{1j}x_j-\alpha_1M_{4j}x_j+\alpha_3M_{2j}x^j-\alpha_4M_{3j}x_j-iL_{2j}x_j\eta_3+iL_{3j}x_j\eta_2=0, \tag{3.34}$$

$$\alpha_2M_{1j}x_j-\alpha_1M_{2j}x_j+\alpha_4M_{3j}x_j-\alpha_3M_{4j}x_j-iL_{1j}x_j\eta_2+iL_{2j}x_j\eta_1=0, \tag{3.35}$$

$$\alpha_3M_{1j}x_j-\alpha_1M_{3j}x_j+\alpha_2M_{4j}x_j-\alpha_4M_{2j}x_j-iL_{1j}x_j\eta_3+iL_{3j}x_j\eta_1=0, \tag{3.36}$$

$$M_{1j}x_j\eta_1-iL_{1j}x_j\alpha_1+M_{2j}x_j\eta_2-iL_{2j}x_j\alpha_2+M_{3j}x_j\eta_3-iL_{3j}x_j\alpha_3=0, \tag{3.37}$$

$$M_{4j}x_j\eta_3-iL_{3j}x_j\alpha_4+M_{1j}x_j\eta_2-iL_{2j}x_j\alpha_1-M_{2j}x_j\eta_1+iL_{1j}x_j\alpha_2=0, \tag{3.38}$$

$$M_{1j}x_j\eta_3-iL_{3j}x_j\alpha_1-M_{4j}x_j\eta_2+iL_{2j}x_j\alpha_4-M_{3j}x_j\eta_1+iL_{1j}x_j\alpha_3=0, \tag{3.39}$$

$$M_{2j}x_j\eta_3-iL_{3j}x_j\alpha_2-M_{3j}x_j\eta_2+iL_{2j}x_j\alpha_3+M_{4j}x_j\eta_1-iL_{1j}x_j\alpha_4=0. \tag{3.40}$$

One may might wonder if it is possible to rotate this solution to a PW solution. This requires the $\Phi_{\rm SD}$ to be independent of one coordinate. In a coordinate independent manner, we require there to exist a vector X such that the Lie derivative $\mathcal{L}_X\Phi_{\rm SD}=0$. We write generically $X=\alpha\partial_x+\beta\partial_y+\gamma\partial_u+\delta\partial_v$ and compute the Lie derivative. Recall that when acting on a differential form ω one has that

$$\mathcal{L}_X\omega=\iota_Xd\omega+d\iota_X\omega. \tag{3.41}$$

In our case, since $d\Phi_{\rm SD}=0$, the first term drops, so it is only necessary to compute the second one. We will spare the details of the computation and simply say that the vanishing of $\mathcal{L}_X\Phi_{\rm SD}$ can be written as the linear system

$$\begin{bmatrix} L_{31} & L_{32} & -L_2-L_{11} & L_{12}-L_{21}\\ -L_{21} & -L_2 & -L_{23} & L_{13}-L_{31}\\ L_{11} & L_{12} & L_{13} & L_{23}-L_{32}\\ L_{11} & L_{12} & L_{13} & L_{23}-L_{32}\\ L_{21} & L_2 & L_{23} & L_{31}-L_{13}\\ L_{31} & L_{32} & -L_2-L_{11} & L_{12}-L_{21}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{bmatrix}=0. \tag{3.42}$$

Here, we put the system on shell, requiring that the background solves the equations of motion written above. Quite interestingly, there is always a solution to this system (the matrix has rank 3), so it is always possible to find a vector X with $\mathcal{L}_X \phi_{\mathrm{SD}} = 0$, implying that for this background we can always make a choice of coordinates that brings us back to a PW system. This means that a background Φ_{SD} that is linear in the coordinates is always equivalent to a PW background.

IV. LOCAL MATTER AND TOPOLOGICAL INSULATORS

Although our eventual aim is the study of 3D matter fields coupled to dynamical gauge fields, in this section, we show that we can already use what we have developed to engineer a rich class of 3D systems coupled to a 4D topological bulk, which we can think of as various instances of topological insulators (see, e.g., Refs. [25,26,48]). An interesting feature of these systems is that many features can be deduced from primarily topological considerations. Since we anticipate strong quantum corrections in local $Spin(7)$ compactifications with 3D $\mathcal{N}=1$ supersymmetry, our eventual aim in this paper will be to extract robust topological quantities. The case of topological insulators engineered via local $Spin(7)$ compactifications thus serves as a useful "warm-up exercise" for the full problem.

Recall that a simple instance of this sort of configuration arises from a 4D Dirac fermion on $\mathbb{R}^{2,1} \times \mathbb{R}_t$ with a position dependent mass $m(t)$ such that $m > 0$ for $t > 0$ and $m < 0$ for $t < 0$, with $m = 0$ at $t = 0$. In the bulk, we have a $U(1)$ global symmetry, and if we consider the topological term $\theta F \wedge F$, with F the background field strength for this $U(1)$, then we can think of the localized mode as being trapped at the interface between a $\theta = 0$ and $\theta = \pi$ phase.

In this section, we illustrate that matter localization in local $Spin(7)$ systems can be viewed as engineering a class of topological insulators, but where now the global symmetry in the bulk is more general. One way for us to proceed is to actually start with M-theory on a Calabi-Yau threefold given by a curve Σ of ADE singularities. This gives rise to a 5D gauge theory with gauge group G of ADE type. If we allow further enhancements in the singularity type at points of the curve, we get 5D hypermultiplets. To give an explicit example, we engineer a 5D $SU(N)$ gauge theory coupled to a single 5D hypermultiplet in the fundamental representation of $SU(N)$. This is generated by the local Calabi-Yau hypersurface,

$$y^2 = x^2 + z^N(z-u) \tag{4.1}$$

where $z = 0$ denotes the location of the curve Σ and u denotes a local coordinate along the curve. At $u = 0$, we get a matter field trapped along the curve.

Returning to the general case, if we further compactify on a circle, we wind up with a 4D $\mathcal{N}=2$ hypermultiplet in a representation $\mathbf{R}$ of G. In the limit where Σ is noncompact, we just have a global symmetry G coupled to our 4D field. Observe that the fermionic content of this system consists of a 4D Dirac fermion. Switching on a position dependent mass term of the kind used in the topological insulator just discussed, we get further localization to a 3D Dirac fermion.

This localization can be understood in terms of a PW system on the three-manifold $M_3 = \mathbb{R}_t \times \Sigma$ or, equivalently, in terms of a local $Spin(7)$ system on $\mathbb{R}_t \times \Sigma \times S^1$. The Higgs field on the entire M_4 is given by

$$\Phi_{\mathrm{SD}} = \phi_{\mathrm{PW}} \wedge d\theta + *_3 \phi_{\mathrm{PW}}, \tag{4.2}$$

where $*_3$ is the Hodge star operator on M_3, ϕ_{PW} is a harmonic 1-form on $\Sigma \times \mathbb{R}_t$, and $d\theta$ is the volume form on the S^1 factor. We can set up ϕ_{PW} to have a zero-locus at a point localized at $t = 0$ and a point in Σ (which in local coordinates we take to be $x_1 = x_2 = 0$), and it will locally be of the form

$$\phi_{\mathrm{PW}} = d[(1-\kappa)x_1^2 - (1+\kappa)x_2^2 + 2\kappa t^2]. \tag{4.3}$$

Observe that when $\kappa = 0$ the matter field is delocalized on $\mathbb{R}_t \times S^1$ but is still trapped at $x_1 = x_2 = 0$. The relative minus signs on our configuration are necessary to ensure that we can actually solve the equations of motion on Σ in this limit. When $\kappa \neq 0$, we can interpret our construction in terms of a 4D domain wall fermion trapped at $t = 0$. The relative strength of localization in the different directions depends on the metric data. For example, if Σ is very small, then there is a sense in which the mode delocalizes on Σ but remains tightly trapped in the $\mathbb{R}_t$. Explicitly, the zero mode profile for our domain wall fermion is of the form

$$\psi \sim \exp\left(-\frac{1}{2}(|\kappa|t^2 - |1-\kappa|x_1^2 - |1+\kappa|x_2^2)\right), \tag{4.4}$$

where to avoid clutter we have suppressed the explicit form content.[11] We have therefore engineered a 3D $\mathcal{N}=2$ chiral matter multiplet localized along the S^1-factor, or its 4D $\mathcal{N}=1$ analog if we decompactify the S^1 factor. Observe that on $\mathbb{R}_t$ our matter field ψ satisfies the equation of motion

$$(\partial_t + |\kappa|t)\psi(t) = 0, \tag{4.5}$$

which can be seen as a Dirac equation for the localized chiral multiplet with a position-dependent mass, $m(t) = |\kappa|t$.

A further comment here is that because our internal space is noncompact we do not have a 3D dynamical gauge field. The 4D gauge coupling is given by

[11] Recall from Sec. II that the $Spin(7)$ wave functions on $\Sigma \times S^1 \times \mathbb{R}_t$ follow from Eq. (4.4) as $a = \psi$, $\varphi = \psi \wedge d\theta + *_3\psi$.

$$\frac{1}{g_{4d}^2} = \frac{1}{g_{7d}^2} \times \mathrm{Vol}(\Sigma \times S^1); \tag{4.6}$$

namely, we think of M-theory on an ADE singularity as engineering 7D super Yang-Mills theory, and further compactification relates this gauge coupling to its lower-dimensional counterpart. In particular, in the limit where $\mathrm{Vol}(\Sigma) \to \infty$, the 4D bulk dynamics trivializes.

Changing perspective, we can also think of our localized matter as descending from a PW system on the three-manifold $\tilde{M}_3 = \mathbb{R}_t \times \Sigma$, which results in a 4D $\mathcal{N} = 1$ chiral multiplet. By itself, this would produce a gauge anomaly if we had tried to compactify the $\mathbb{R}_t$ direction. In the context of our 4D bulk and 3D edge mode, this is just anomaly inflow from the bulk to the boundary [49], as applied, for example, in Refs. [16,43]. Note also that in the context of the PW system we could localize additional matter fields at other points of $\tilde{M}_3 = \mathbb{R}_t \times \Sigma$. This would amount to introducing additional domain wall fermions which couple to each other via the bulk. Our local $Spin(7)$ system generalizes this further because there is no need for matter to be localized on a common S^1.

Instead of directly proceeding in terms of the equations of motion for localized zero modes, we could have instead phrased our analysis in terms of the bulk topological insulator. Indeed, as emphasized in Refs. [27,50], for example, some features of the bulk / boundary dynamics can be captured in purely topological terms. For example, we can speak of the jump in the θ angle as we pass from one side of the interface to the other. Now, in our context, the θ angle of the 4D bulk descends from the period integral of the M-theory 3-form potential C_3,

$$\theta = \int_{M_3} C_3, \tag{4.7}$$

where $M_3 = \Sigma \times S^1$. If we consider integrating out the 4D Dirac fermion, then we can view this as a system with no bulk matter but instead a position dependent θ angle, and thus we can alternatively view this as a position dependent C_3. This effective G_4-flux would then have delta function support precisely at the location of our matter field:

$$G_4 = \frac{1}{2}\delta(t)dt \wedge \delta_{\{u=0\}} \wedge d\theta. \tag{4.8}$$

Here, $d\theta$ is the unit normalized volume form on the circle and $\delta_{\{u=0\}} = \frac{1}{2i}\delta(u)\delta(\bar{u})du d\bar{u}$. Note that the coefficient here is $1/2$. This is actually required to properly account for the localized matter field, which can be thought of as Chern-Simons theory at "level $1/2$" (which only makes sense if we couple to a 4D bulk).

V. GLUING CONSTRUCTIONS

Up to now, our discussion of the local $Spin(7)$ system has involved working in a local patch where our 7D SYM theory is placed on a noncompact four-manifold M_4. Based on our previous analysis, we expect that the localized matter can be understood as 3D $\mathcal{N} = 2$ matter (at least locally), simply due to the fact that they typically fill out complex representations of the gauge group. Of course, since we are dealing with a local $Spin(7)$ compactification, we only expect to retain 3D $\mathcal{N} = 1$ supersymmetry, a feature which emerges upon working with a compact M_4. Indeed, the matter multiplets which are delocalized over M_4 fill out $\mathcal{N} = 1$ multiplets. Moreover, on a general M_4, we can write down more general interaction terms which would have been forbidden with $\mathcal{N} = 2$ supersymmetry.

With these considerations in mind, our aim in this section will be to develop a gluing construction for building more general solutions. The main idea will be to consider building blocks, either obtained from the local PW system or the local BHV system. In Appendix B, we present a somewhat different method for building examples of local $Spin(7)$ systems based on a quotient construction of BHV solution. This method should lift to compact geometries but as it is somewhat orthogonal to the other developments of the paper, we have placed it in an Appendix.

A. Connected sums of G_2 local models

We now construct a class of local $Spin(7)$ systems by starting with PW backgrounds on four-manifolds of the form $M_4^{(i)} = M_3^{(i)} \times S^1$; i.e., we consider solutions to our local $Spin(7)$ equations which are "trivial" along the S^1 factor. Starting from $M_4^{(i)}$ and $M_{(4)}^{(j)}$, a pair of such building blocks, we can glue these four-manifolds together by cutting out a four-ball from each and then gluing along a neck region. Topologically, this is just the standard connected sum $M_4^{(i)} \# M_4^{(j)}$. In the present context, however, we demand more because each four-manifold is also equipped with a Higgs bundle, and we need to ensure that these solutions can *also* be extended across the neck region. See Fig. 1 for a depiction of this gluing procedure, where we also indicate possible zeros for the Higgs field.

In more detail, the connected sum construction on its own is a topological operation after which there may be one of several prescriptions to assign a metric to the resulting space. We can carry this out for any collection of for manifolds of the form $M_3^{(i)} \times S^1$ on each block, but for ease of exposition, we discuss it in detail in the special case where we have $S^3 \times S^1$ summands. If the building blocks have constant Ricci curvature, one may be interested in requiring the composite space to also have constant Ricci curvature. In Ref. [51], after assuming a certain non-degeneracy condition of a Poisson operator which $S^3 \times S^1$ satisfies on the building blocks, there is a general

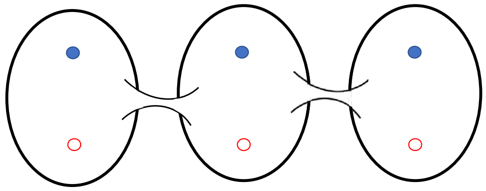

FIG. 1. Depiction of a connected sum construction of M_4 in terms of three summands of the form $S^1 \times M_3^{(i)}$. Each of the three blobs represents a four-manifold $M_3^{(i)} \times S^1$ for $M_3^{(i)}$ a three-manifold. The dots indicate localized matter on circles, where the solid blue dots represent localized chiral matter in $\mathbf{R}$ and the open red dots represent localized chiral matter in $\bar{\mathbf{R}}$ from the point of view of the PW system on the three-manifold $M_3^{(i)}$.

perturbative procedure to prove the existence of such a metric on the total space. To illustrate, we can consider $S^3 \times S^1$ blocks with round metric,

$$ds^2 = d\phi^2 + d\Omega_{S^3}^2 = d\phi^2 + d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2) 0 \le \phi, \varphi \le 2\pi 0 \le \theta, \quad \psi \le \pi, \tag{5.1}$$

by interpolating, with bump functions, to standard spatial wormholes connecting neighboring blocks. More specifically, to glue two neighboring $S^3 \times S^1$ blocks, start by choosing the same point on each $p_0 \equiv (\phi_0, \varphi_0, \psi_0, \theta_0)$ such that it is far away from any zeros or poles of the Higgs field. We can define the radial coordinate for the four-ball neighborhoods $B_\epsilon^4(p_0)$ as

$$\rho^2 \equiv (\phi - \phi_0)^2 + (\psi - \psi_0)^2 + (\theta - \theta_0)^2 \sin^2(\psi - \psi_0) + (\varphi - \varphi_0)^2 \sin^2(\psi - \psi_0) \sin^2(\theta - \theta_0), \tag{5.2}$$

where the positive and negative branches of ρ each represent a copy of $S^3 \times S^1$. Similarly, we can also define a natural angular element $d\tilde{\Omega}$ of the neighborhood as well so that the metric on $B_\epsilon^4(p_0)$ has the form

$$ds^2 = d\rho^2 + \rho^2 d\tilde{\Omega}^2. \tag{5.3}$$

This reproduces the statement that a neighborhood of a point in $S^3 \times S^1$ is diffeomorphic to an open ball in $\mathbb{R}^4$. Let us define a bump function $b(\rho)$ such that $b(0) = b_0 > 0$ and $b(\pm\epsilon) = 0$. Here, $b_0 < \epsilon$ is the inner radius of the wormhole. Then, the metric

$$ds^2 = d\rho^2 + [\rho^2 + b(\rho)^2] d\tilde{\Omega}^2 \tag{5.4}$$

interpolates between the two branches of ρ and hence the two copies of $S^3 \times S^1$. This metric ansatz is symmetric enough such that the resulting four-manifold possesses an orientation reversal: 1) switch the two manifolds, sending $\rho \to -\rho$, and 2) apply an orientation-reversing map of S^3 to each factor.[12] This construction can then be generalized to any sequence of connected sums $\#_i(M_3^{(i)} \times S^1)$.

Giving an exact solution to the Higgs field would in principle be possible but nontrivial. Instead, we can make the assumption that the gluing neck's diameter, ϵ, is very small. By assumption $\Phi(p_0) \neq 0$, the solution is well approximated by simply setting $\Phi = \Phi(p_0)$ on the neck. Note that if the neck were large we would have to calculate the corrections to the original building block Higgs field which alter (in pairs) our count of zeros, and there may be additional pairs of zeros of Φ on the neck. Also note that the existence of such a harmonic 2-form (with singularities) is guaranteed by a (relative) Hodge theorem, so we know there exists a solution arbitrarily close to the $\Phi|_{\text{neck}} = \Phi(p_0)$ approximation.

1. Zero mode counting

An advantage of this connected sum construction is that we can read off the matter content from these local building blocks. First of all, the codimension-3 matter in each individual summand of a connected sum construction is essentially unchanged from what we have in the PW case [17,18]. There is a slight subtlety here because *a priori* there could be additional ways for matter in conjugate representations to pair up, especially due to working in lower dimensions. One can visualize this in terms of Euclidean M2-branes which stretch across the gluing neck region. We revisit this issue in Sec. IX.

Putting this issue aside, we can determine the zero mode content of our newly constructed local $Spin(7)$ system just from analyzing the profile of the Higgs field. For localized matter on an individual building block $M_3 \times S^1$, there is not much difference from what we would get just from studying the PW system on the three-manifold M_3, as in Refs. [17,18]. To quickly review these results, consider the simple Higgsing $\tilde{G} \to G \times U(1)$ from turning on a 1-form Φ_{PW} on M_3. Equations for localized zero modes in the representation $\mathbf{R}_q$ are

$$(d + q\Phi_{\text{PW}})\psi_q = 0 \tag{5.5}$$

$$(d^\dagger + q\iota_\Phi)\psi_q = 0. \tag{5.6}$$

By exponentiating the wave functions $\psi' = e^{-qf}\psi$ ($\Phi_{\text{PW}} = df$), these equations reduce to a local cohomology group. Indeed, we observe that $e^{-qf} d e^{qf} = (d + q\Phi_{\text{PW}})$. Note that this same step is taken when solving for ground states in superquantum mechanics with a target space potential. To properly account for the Φ_{PW} singularities,

[12] A common such example that has two fixed points (the minimal number for S^3) is to send $(x_1, x_2, x_3, x_4) \to (x_1, -x_2, -x_3, -x_4)$ given $\sum_i x_i^2 = R^2$. Note the antipodal map for S^3 is orientation preserving.

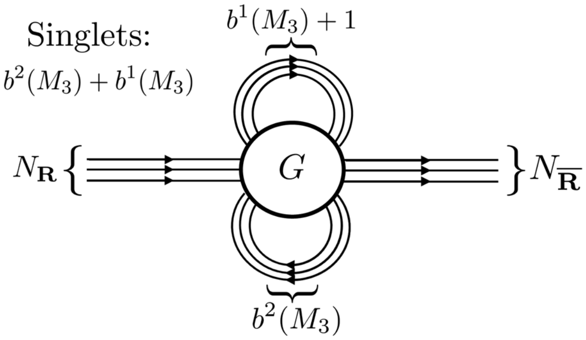

FIG. 2. Matter content of the 3D theory when $M_4 = S^1 \times M_3$. The theory has $\mathcal{N} = 2$ supersymmetry, but the quiver is expressed in terms of $\mathcal{N} = 1$ fields where double lines connected to the gauged node are adjoint fields, single lines denote fields in $\mathbf{R}$ and its conjugate, and no lines denote singlets.

let us define the loci positive/negatively charged loci as $\Delta_\pm$, which consists of $n_\pm$ points on M_3 (formulas for more general singularity sources are given in the aforementioned references). We excise their neighborhoods as $\tilde{M}_3 \equiv M_3 \backslash (\Delta_+ \cup \Delta_-)$ and work in cohomology relative to negative sources. The relevant formulas for our localized zero modes are then given by the following Betti numbers:

$$b^1(\tilde{M}_3, \Delta_-) = b^1(M_3) + n_- - 1$$
$$= \text{\# of chiral modes in representation } \mathbf{R} \quad (5.7)$$

$$b^2(\tilde{M}_3, \Delta_-) = b^2(M_3) + n_+ - 1$$
$$= \text{\# of chiral modes in representation } \bar{\mathbf{R}}. \quad (5.8)$$

See Fig. 2 for a depiction of the resulting quiver gauge theory in the case of compactification of the local $Spin(7)$ system on the four-manifold $M_4 = S^1 \times M_3$.

On the other hand, the bulk modes which are a spread across all of our new four-manifold M_4 will certainly be modified. First of all, given a breaking pattern such as $\tilde{G} \to G \times U(1)^n$, we expect a 3D $\mathcal{N} = 1$ vector multiplet for the unbroken gauge group $G \times U(1)^n$.[13] Additionally, we can expect bulk modes in 3D $\mathcal{N} = 1$ matter multiplets transforming in the adjoint representation of $G \times U(1)^n$ [i.e., they are neutral under the $U(1)$ factors]. These are counted by $b^1(M_4)$ and $b^2_{\mathrm{SD}}(M_4)$, which, respectively, come from the internal vector potential and the self-dual 2-form of the 7D SYM theory. In terms of our connected sum building blocks, we have

$$b^1(M_4) = \sum_i b^1(M_4^{(i)}) = \sum_i (b^1(M_3^{(i)}) + 1) \qquad (5.9)$$

$$b^2_{\mathrm{SD}}(M_4) = \sum_i b^2_{\mathrm{SD}}(M_4^{(i)}) = \sum_i b^2(M_3^{(i)}), \qquad (5.10)$$

in the obvious notation. By inspection, when we have more than one building block, the zero modes do not automatically sort into "complex" 3D $\mathcal{N} = 2$ matter multiplets. In fact, precisely because these modes transform in a real representation, and are spread over the entire four-manifold, we generically expect them to lift in pairs. From the structure of the form content, this involves a pairing between the 1-forms and the self-dual 2-forms. In subsequent sections, we will revisit the precise remnant of these zero modes which can be detected by discrete anomalies in the 3D effective field theory.

A closely related comment is that the localized matter fills out 3D $\mathcal{N} = 2$ matter multiplets, which we can view as the dimensional reduction of 4D $\mathcal{N} = 1$ chiral matter on a circle. For bulk matter, we expect the scalar degrees of freedom to split into two types, namely those which are even under a reflection of a spatial coordinate (i.e., $x^i \to -x^i$) and those which are odd under such a reflection. This in turn will impact how we count various contributions to the discrete anomalies. We defer a full treatment of this important issue to Sec. VI where we discuss reflections on the various kinds of matter fields in our system.

B. Connected sums of CY_4 local models

In the previous subsection, we illustrated how to start with a collection of 4D $\mathcal{N} = 1$ theories engineered in M-theory on local G_2 spaces and, via a suitable gluing construction, build 3D $\mathcal{N} = 1$ systems. The main feature of all these constructions is that localized matter still fills out 3D $\mathcal{N} = 2$ supermultiplets, while bulk modes and vector multiplets only fill out 3D $\mathcal{N} = 1$ supermultiplets.

Now, another way to generate 4D $\mathcal{N} = 1$ vacua would be to start with F-theory on an elliptically fibered Calabi-Yau fourfold. Compactifying on a further circle would result in an M-theory background which retains 3D $\mathcal{N} = 2$ supersymmetry. In this case, chiral matter of the 4D system really results from two related effects. Geometrically, the enhancement in the singularity type along a curve would give us 4D $\mathcal{N} = 2$ hypermultiplets. Switching on a background flux from the 7-branes then results in 4D $\mathcal{N} = 1$ matter [20,21]. From the perspective of an M-theory background, then, the geometrically localized matter will fill out 3D $\mathcal{N} = 4$ supermultiplets, but flux will then lead to 3D $\mathcal{N} = 2$ supermultiplets.

As already mentioned, the local model associated with F-theory on a CY_4 is just the BHV system on a Kähler surface S, as studied in Refs. [20,21,44]. Of course, S is also a four-manifold, so we can build up connected sums of such building blocks to arrive at more general four-manifolds.

[13] Again, we are being cavalier with the global structure of the unbroken group. Additionally, some of these $U(1)$ factors may end up decoupling due to further interactions with a "Green-Schwarz" axion, but this is something we cannot address in the purely local context.

Since we are interested in 3D $\mathcal{N}=1$ vacua, we actually require that the resulting four-manifold is not Kähler. One way to ensure this is to simply glue together Kähler surfaces with a suitable orientation reversal in the gluing process, see Fig. 4. For example, we can glue two copies of $\mathbb{CP}^2$ to arrive at $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, and this has signature zero,[14] and moreover, it is non-Kähler.

Compared with our discussion where we glued PW building blocks, we face an additional complication here in that the orientation reversal of S a Kähler surface does not simply result in another Kähler surface (although S and $\bar{S}$ are homeomorphic). If we must then deal with the local $Spin(7)$ system on $\bar{S}$ anyway, one might ask what has been gained by introducing a gluing construction at all.

The main point is that in our construction we can assume that the Higgs field profile is only nontrivial on the Kähler surface summands and remains trivial on the orientation reversed summands (which are non-Kähler). This sort of construction requires a specific profile for the Higgs field in the gluing region between a Kähler summand S_i and a non-Kähler summand $\overline{S_j}$. In particular, we demand that the Higgs field tends to zero there. That this can be arranged was shown in Ref. [2], although the price we pay is that we only get an approximate BHV solution in the Kähler surface region since all three components of Φ_{SD} are necessarily switched on (although the contribution from the component parallel to the Kähler form on S_i is exponentially suppressed). Proceeding in this way, we can engineer examples where the localized matter is still of the type found in a BHV system, but where the bulk modes are now of the more general kind found in local $Spin(7)$ systems. Observe also that nothing stops us from building up several such Kähler and non-Kähler summands.

With this in mind, we can also proceed to write down the classical zero mode content for our 3D effective field theory. For the BHV system, we have the well-known contributions for matter localized on curves (possibly in the presence of gauge field fluxes), as discussed, for example, in Refs. [20,21]. Let us therefore focus on the contributions from the bulk modes which fill out genuine 3D $\mathcal{N}=1$ matter multiplets in the adjoint representation of $G\times U(1)^n \subset \tilde{G}$, with notation as in Sec. VA. Writing our M_4 as a connected sum of S_i Kähler summands and $\overline{T_k}$ summands with T_k Kähler as in Fig. 3, we have[15]

$$b^1(M_4) = \sum_i b^1(S_i) + \sum_k b^1(\overline{T_k}) = \sum_i 2h^{1,0}(S_i) + \sum_k 2h^{1,0}(T_k) \tag{5.11}$$

[14]Given a four-manifold M with signature $\sigma(M)$, we have $\sigma(M\#\overline{\mathbb{CP}^2}) = \sigma(M) + \sigma(\overline{\mathbb{CP}^2}) = \sigma(M) - 1$.

[15]The precise gluing does not affect these topological quantities.

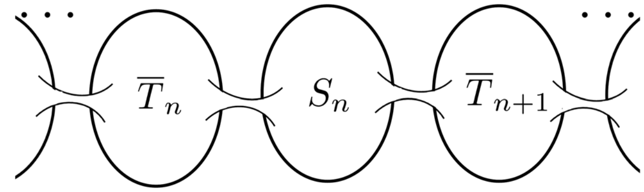

FIG. 3. An illustration of our gluing construction of Kähler manifolds.

$$b^2_{\mathrm{SD}}(M_4) = \sum_i b^2_{\mathrm{SD}}(S_i) + \sum_k b^2_{ASD}(T_k) = \sum_i (2h^{2,0}(S_i)+1) + \sum_k (h^{1,1}(T_k)-1), \tag{5.12}$$

where in the above we used the fact that orientation reversal interchanges the self-dual and anti-self-dual 2-forms. Similar formulas hold for the Betti numbers of vector bundles built in this way.

As a final amusing comment, we note that Hopf surfaces are complex but not Kähler and are diffeomorphic to $S^3\times S^1$, which is quite similar to the PW building block discussed in Sec. VA.

VI. C, R, & T TRANSFORMATIONS

Up to now, our discussion has basically focused on the classical geometry of a local $Spin(7)$ compactification. By this, we simply mean that we have obtained our zero mode content from the linearized approximation to the Vafa-Witten equations our four-manifold M_4. Since we are dealing with 3D $\mathcal{N}=1$ vacua, we can expect many strong coupling effects to enter the low energy effective field theory. On general grounds, the best we can hope for us to is to extract those features of the compactification which are robust against such strong coupling effects.

One such feature is the structure of global anomalies in 3D theories. These involve the study of the 3D theory on a

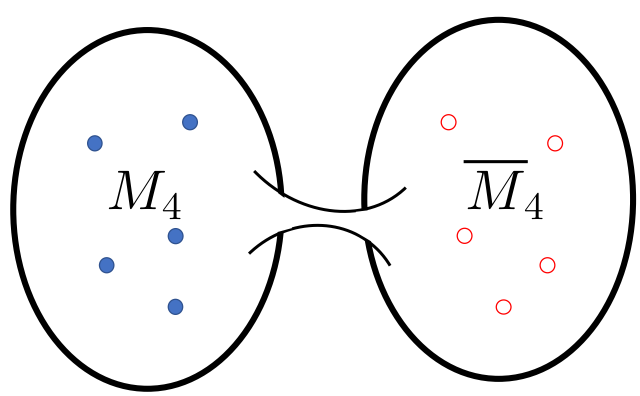

FIG. 4. Depiction of the connected sum construction used in the section. The separate blue and red dots indicate localized modes (on circles) with opposite chiralities/Hessians.

three-manifold N_3 and tracking the response of the partition function $Z[N_3]$ under symmetry transformations. With an eye toward understanding universal aspects of local $Spin(7)$ backgrounds, our aim here will be to extract constraints from various kinds of discrete symmetries which are in some sense universal. The classic examples of this type include reflection of a spatial coordinate $x^i \mapsto -x^i$, as well as a geometric time reversal operator $x^0 \mapsto -x^0$, and in cases where we also have complex representations of a gauge group, we can also speak of charge conjugation operations as well.[16]

Tracking these discrete symmetries from a top-down perspective turns out to be somewhat subtle because the parity assignments in the internal directions often end up impacting the parity assignments in the 3D uncompactified directions.[17]

To illustrate some of the subtleties, consider the compactification of a higher-dimensional vector boson on a torus T^n. We can compose reflections of the 3D spacetime with internal reflections on the torus. Since our vector boson is a 1-form, there can be a nontrivial mixture between these operations which will in turn impact whether we refer to the resulting 3D spin-0 degrees of freedom as scalars or pseudoscalars.

Our aim in this section and the next will be to give a top-down treatment of such 3D discrete transformations by recasting them in terms of operations on the extra-dimensional geometry. This is important both in terms of understanding the geometric origin of these transformations as well as in terms of understanding what geometric constraints are sufficient to ensure the existence of such symmetries.

Since we are primarily interested in local $Spin(7)$ systems, our focus will be on understanding the discrete symmetries of 7D SYM theory. There are several canonical routes to realizing this gauge theory, and it is helpful to consider different ways to engineer this system. In gauge theory terms, one simple way to proceed is to start with 10D SYM theory, as obtained, for example, from heterotic strings in flat space. Compactification on a T^3 then results in 7D SYM. A complementary starting point is M-theory on an ADE singularity. From either starting point, we can ask how geometric reflection operations on the respective 10D and 11D (Euclidean) spacetimes descend to our 7D theory. Further compactification on a four-manifold then provides a general method for tracking the descent of these symmetries to a 3D system.

[16]In some of the literature, these reflection transformations are sometimes referred to as "parity transformations."

[17]From a bottom-up perspective, one might be tempted to assert that once the 3D QFT is *defined* we can simply study the discrete symmetries of this system. Part of the issue with this approach is that it presupposes that we know what 3D QFT we actually got in the first place from compactifying on a local $Spin(7)$ geometry.

As a point of notation, we will often be working with an n-dimensional Euclidean signature spacetime with local coordinates $(x^0, \ldots, x^{n-1})$. We introduce the operation R_i, which acts as

$$\mathsf{R}_i\colon\ x^i \to -x^i, \quad \text{and} \quad \mathsf{R}_i\colon\ x^j \to x^j \quad \text{for } j \neq i. \tag{6.1}$$

The case of $i = 0$ corresponds to a reflection of the Euclidean spacetime "time coordinate." In continuing back to Lorentzian signature via $x^0 \mapsto ix^0$, the action of R_0 would act as the combination CT, which can be specified even when charge conjugation C and time reversal T do not make sense separately (see, e.g., Ref. [52] for further discussion). As nomenclature, we shall also refer to a scalar, vector, and p-form as those which transform with their standard geometric operations. We append the modifier "pseudo" or "twisted" whenever there is an additional minus sign under geometric reflection operations in Euclidean signature.

A. 10D origin of 7D symmetries

In this subsection, we start from the discrete transformations of 10D SYM theory with gauge group G and study the descent of these discrete transformations under compactification on a T^3. This results in discrete symmetry operations for 7D SYM theory in flat space.

To frame the discussion to follow, recall that the field content of 10D SYM consists of a vector boson A_M and a Majorana-Weyl spinor ζ. Our conventions for 10D spinors are summarized in Appendix C. For the vector boson A_M, the action of the various reflection symmetries (in Euclidean signature) is simply

$$\mathsf{R}_M^{10d}\colon\ A_N(x) \mapsto (-1)^{\delta_{M,N}} A_N(\mathsf{R}_M^{10d} x); \tag{6.2}$$

namely, we flip the sign of the component of the gauge field which undergoes reflection. It is also convenient to state this in terms of the transformation on the 1-form $A = A_M dx^M$, which transforms as

$$\mathsf{R}_M^{10d}\colon\ A(x) \mapsto A(\mathsf{R}_M^{10d} x). \tag{6.3}$$

Although all fields transform in the adjoint representation, we can also speak of various "charge conjugation operations" which amount to automorphisms of the gauge group.

Turning next to the fermionic content of the theory, it is convenient for our present purposes to work in terms of a Dirac spinor of $\mathrm{Spin}(9,1)$ subject to various chirality and reality constraints. In terms of the 10D gamma matrices Γ_M^{10d} acting on a Dirac spinor field $\Psi(x)$, we have

$$\mathsf{R}_M^{10d}\colon\ \Psi(x) \mapsto \xi \Gamma_M^{10d} \Psi(\mathsf{R}_M^{10d} x). \tag{6.4}$$

Here, we have introduced an arbitrary complex phase ξ, although physical considerations in any compactified system restrict us to fourth roots of unity, i.e., $\xi^4 = 1$. Of course, in 10D SYM, we have a Majorana-Weyl spinor rather than a Dirac spinor. Consequently, there is no reflection symmetry *per se* in the 10D theory. We can, however, still speak of symmetry transformations such as $\mathsf{R}_i^{10d}\mathsf{R}_j^{10d}$ which compose multiple reflections. It is these composite operations which we expect to descend in various ways to compactified theories.

Let us now compactify on a T^3. We keep all field profiles trivial on the T^3, so we expect to get 7D SYM theory with gauge group G. Recall that the bosonic content consists of a vector boson and an R-symmetry triplet ϕ_a, and the fermionic content consists of a 7D Dirac fermion which we can write as a pair of fermions Ψ_I subject to a symplectic-Majorana constraint. This is just inherited from the 10D Majorana-Weyl condition (see Appendix C for the precise form of this constraint). The flat space action is given by

$$S = \int dt d^6 x \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}D_\mu\phi_a D^\mu\phi_a + \frac{1}{4}[\phi_a,\phi_b][\phi_a,\phi_b] - \frac{i}{2}\overline{\Psi}^I \not{D}\Psi_I - \frac{i}{2}\sigma^a_{IJ}\bar{\Psi}^I[\phi_a,\Psi^J]\right]. \quad (6.5)$$

Let us now turn to the discrete reflection symmetries of the 7D system. We begin by working from a bottom-up perspective, emphasizing only the 7D geometric reflections. We will then look at how these descend from ten dimensions. We have the 7D reflections R_i^{7d}, as well as the "analytic continuation" of R_0^{7d} which we denote as CT^{7d}. To begin, we start with the action of CT^{7d} and R_i^{7d} on the fermions:

$$\mathsf{CT}^{7d}\Psi_I(t,x_i) = i\Gamma_0^{7d}\Psi_I(-t,x_i), \quad (6.6)$$

$$\mathsf{R}_i^{7d}\Psi_I(t,x_j) = \Gamma_i^{7d}\Psi_I(t,(-1)^{\delta_{ij}}x_j). \quad (6.7)$$

Note that these actions are compatible with the symplectic-Majorana condition. The action of CT^{7d} and R_i^{7d} leaves the kinetic term invariant if the action on the gauge field is[18]

$$\mathsf{CT}^{7d}A_\mu = (-1)^{\delta_{\mu 0}}A_\mu, \quad (6.8)$$

$$\mathsf{R}_i^{7d}A_\mu = (-1)^{\delta_{\mu i}}A_\mu. \quad (6.9)$$

The last term that needs to be checked is the coupling between the fermions and the scalars. One can show that

$$\mathsf{CT}^{7d}\bar{\Psi}^I\Psi^J = \bar{\Psi}^I\Psi^J, \quad (6.10)$$

$$\mathsf{R}_i^{7d}\bar{\Psi}^I\Psi^J = -\bar{\Psi}^I\Psi^J. \quad (6.11)$$

[18]It is important to remember that CT^{7d} is antilinear.

Therefore, in order for these transformations to be symmetries, they need to act as

$$\mathsf{CT}^{7d}\phi_a = -\phi_a, \quad (6.12)$$

$$\mathsf{R}_i^{7d}\phi_a = -\phi_a. \quad (6.13)$$

The fact that the R-symmetry triplet ϕ_a transforms as pseudoscalars under the R^{7d} is somewhat counterintuitive. At this point, we need to recognize that compactification on a T^3 can impact the reflection transformation rules. In particular, the 7D *physical* reflection symmetry is related to a composition of several 10D *geometric* reflections. For example, we have

$$\mathsf{R}_i^{7d} = \mathsf{R}_i^{10d}\mathsf{R}_7^{10d}\mathsf{R}_8^{10d}\mathsf{R}_9^{10d}, \quad (6.14)$$

and as already stated, these act on the fields as

$$\mathsf{R}_i^{7d}(\phi_a) = -\phi_a, \mathsf{R}_i^{7d}(A) = A \quad (6.15)$$

$$\mathsf{R}_i^{7d}(\Psi_I) = \mathsf{R}_{789}^{10d}\mathsf{R}_i^{10d}(\Psi_{MW}^{10d}) = +\Gamma_i^{7d}\Psi_I, \quad (6.16)$$

where we have written $A = A_i dx^i$ as a 1-form. The overall sign in the fermion transformation rule follows if we assume we are reducing a positive chirality Majorana-Weyl fermion from ten dimensions with phase $\xi = +1$

An implicit feature of our discussion so far has been a specific choice of reflection symmetry on the 7D fermions:

$$\mathsf{CT}^{7d}\Psi_I(\mathsf{CT}^{7d}x) = i\Gamma_0^{7d}\Psi_I(\mathsf{CT}^{7d}x), \quad (6.17)$$

$$\mathsf{R}_i^{7d}\Psi_I(\mathsf{R}_i^{7d}x) = \Gamma_i^{7d}\Psi_I(\mathsf{R}_i^{7d}x). \quad (6.18)$$

In particular, we see that this means $(\mathsf{CT}^{7d})^2 = (-1)^F$, and $(\mathsf{R}_i^{7d})^2 = 1$, where $(-1)^F$ acts on a single fermion as -1. In other words, we have implicitly chosen to work on a 7D manifold with Pin^+ structure. We can alternatively ask whether we could have specified 7D SYM on a 7D manifold with Pin^- structure. This alternate possibility would have occurred if we had demanded the transformation rules,

$$\mathsf{CT}^{7d,-}\Psi_I(\mathsf{CT}^{7d,-}x) = \Gamma_0^{7d}\Psi_I(\mathsf{CT}^{7d,-}x), \quad (6.19)$$

$$\mathsf{R}_i^{7d,-}\Psi_I(\mathsf{R}_i^{7d,-}x) = i\Gamma_i^{7d}\Psi_I(\mathsf{R}_i^{7d,-}x), \quad (6.20)$$

which would have resulted in $(\mathsf{CT}^{7d,-})^2 = 1$, and $(\mathsf{R}_i^{7d,-})^2 = (-1)^F$.

Indeed, depending on the number of coordinates in ten dimensions that we reflect, we will obtain different 7D structures, either Pin^+ or Pin^-. The rule is the following one: a reflection of $2r$ coordinates in ten dimensions gives a Pin^+ structure if r is even or a Pin^- structure if r is odd. The reason for this is that when doing such a reflection twice, one is performing a 2π rotation in r two-dimensional planes upon which spinors acquire a $(-1)^r$ factor. See, for example, Ref. [53] for more details and examples in other dimensions. This explains the rule for reflections we obtained before: the R_i transformation gives a Pin^+ structure, thus requiring the reflection of all internal coordinates and therefore a flip in sign of the adjoint scalars. A Pin^- structure can be obtained by modifying the transformation rules for the spinors, for example, taking

$$\mathsf{R}_i^{7d,-}\Psi_I(t,x_j) = i(-1)^I\Gamma_i\Psi_I(t,(-1)^{\delta_{ij}}x_j). \tag{6.21}$$

This would force the following transformation on the adjoint scalars,

$$\mathsf{R}_i^{7d,-}\phi_1 = \phi_1, \tag{6.22}$$

$$\mathsf{R}_i^{7d,-}\phi_2 = \phi_2, \tag{6.23}$$

$$\mathsf{R}_i^{7d,-}\phi_3 = -\phi_3, \tag{6.24}$$

meaning that only one internal coordinate is reflected. In this case, a topological twist of 7D SYM would necessarily require us to simultaneously switch on a background $SU(2)$ R-symmetry gauge field as well as a discrete reflection symmetry gauge field. While this would certainly be interesting to study further, in what follows, we exclusively consider the case where our 7D theory is placed on a Pin^+ background.

B. 11D origin of 7D symmetries

In the previous subsection, we focused on reflections of 7D SYM, as generated from compactification of 10D SYM. From an effective field theory standpoint, this is sufficient to understand many aspects of how reflections will descend to 3D vacua of local $Spin(7)$ geometries.

On the other hand, it is somewhat unsatisfactory because it deprecates the role of the original compactification geometry. Since part of our aim is to understand how compactifications of singular $Spin(7)$ spaces can give rise to various 3D $\mathcal{N}=1$ theories, we clearly need to also track the geometric origin of these discrete symmetry transformations from an M-theory perspective. With this in mind, our plan in this subsection will be to study how the same sorts of transformations descend from M-theory compactified on an ADE singularity.

This already leads to a puzzle when considering the "parity assignment" for the adjoint-valued Higgs fields of 7D SYM. Recall that metric moduli of an M-theory compactification descend to scalars, as opposed to pseudoscalars. On the other hand, we also know from the previous subsection that the adjoint-valued Higgs fields of 7D SYM transform (on a Pin^+ background) as pseudoscalars. From the perspective of 10D SYM theory, this comes about because a suitable combination of the 10D geometric reflections descends to the 7D reflection symmetry, namely, $\mathsf{R}_i^{7d} = \mathsf{R}_i^{10d}\mathsf{R}_7^{10d}\mathsf{R}_8^{10d}\mathsf{R}_9^{10d}$. Here, we ask how all of these reflections instead descend from an 11D starting point.

To avoid conflating the various notions of reflection symmetries discussed earlier, we refer to the 11D Euclidean reflection symmetries as $\mathcal{R}_i$ and the analytic continuation of $\mathcal{R}_0$ to Lorentzian signature as $\mathcal{CT}$. Our aim will be to understand how these geometric reflections produce reflections on the fields of 7D SYM.

To frame the discussion to follow, it is actually helpful to first begin with the reflection transformations on the M-theory fields. We expect to be able to consider M-theory on Pin^+ backgrounds because the 11D Majorana condition for the supercharges is compatible with a Pin^+ rather than Pin^- structure (see, e.g., Refs. [54,55]). The field content of M-theory consists of a "3-form" potential C_3 as well as the metric and 11D gravitino.

Consider first the transformation rules for C_3. An important subtlety here is that this is actually a *pseudo-3-form*, or in more technical terms a twisted 3-form,[19] as can be seen by analyzing the reflection symmetry transformation properties of the Chern-Simons coupling $C\wedge G\wedge G$ (see, e.g., Ref. [56]).[20] In components, the reflection transformation is

$$\mathcal{R}_S\colon\ C_{TUV}(X) \mapsto -(-1)^{\delta_{ST}}(-1)^{\delta_{SU}}(-1)^{\delta_{SV}}C_{TUV}(\mathcal{R}_S X), \tag{6.25}$$

where the tensor indices run over the 11D spacetime directions. Again, the additional minus sign compared with a geometric 3-form tells us we are dealing with a twisted differential form. The transformation rules for the metric are straightforward and obey

$$\mathcal{R}_S\colon\ G_{UV}(X) \mapsto (-1)^{\delta_{SU}}(-1)^{\delta_{SV}}G_{UV}(\mathcal{R}_S X). \tag{6.26}$$

Finally, we have the transformation rules for the gravitino:

$$\mathcal{R}_S\colon\ \Psi_U(X) \mapsto (-1)^{\delta_{SU}}\Gamma_S^{11d}\Psi_U(\mathcal{R}_S X). \tag{6.27}$$

We get bosonic degrees of freedom from the dimensional reduction of the 3-form potential and the metric and their fermionic superpartners from the reduction of the gravitino degrees of freedom. In the case where we compactify on a

[19]Recall that a twisted differential form is one that is twisted by the orientation bundle. A twisted differential form ρ on any manifold M behaves as follows: given a map $f\colon M\to M$ that is orientation reversing, then $f^*\rho = -\rho$.

[20]Briefly, consider M-theory on an orientable background Y_{11}. The reflection symmetries act as $\mathcal{R}_i\colon \int_{Y_{11}} CGG \to (-1)^3 \int_{\overline{Y_{11}}} CGG = \int_{Y_{11}} CGG$, where $\overline{Y_{11}}$ denotes the orientation reversal of Y_{11}.

space with singularities, we get additional degrees of freedom from branes wrapped on collapsing cycles.

Indeed, the case of primary interest to us here is where we compactify on an ADE singularity $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$ with $\Gamma_{\mathfrak{g}} \subset SU(2)$ a finite subgroup of ADE type, as indicated by the Lie algebra label $\mathfrak{g}$. Recall that this engineers 7D SYM. Resolving the singularity, the effective divisors intersect according to (minus) the Cartan matrix for the corresponding Lie algebra. The off-diagonal massive W-bosons of the gauge theory come from M2-branes wrapped over the simple roots. It is already instructive to consider the dimensional reduction of the M-theory 3-form potential on the resolved space. Labeling the compactly supported basis of 2-forms as ω_I, we can decompose the 3-form as

$$C_3 = \sum_I A_1^{(I)} \wedge \omega_I, \tag{6.28}$$

so we appear to get a collection of pseudo-1-forms in seven dimensions. Indeed, if we also track the dimensional reduction of the deformation moduli of the ADE singularity, we observe that these are just metric degrees of freedom which fill out an R-symmetry triplet. From this perspective, it would appear that we get a 7D pseudovector multiplet rather than a vector multiplet for a $U(1)^r$ gauge theory.

By inspection, however, we see that if there happened to be a symmetry which acted on the ω_I as $\omega_I \mapsto -\omega_I$, then we could compose this with the geometric reflections $\mathcal{R}_i^{11d}$ to again reach a 7D theory of a vector multiplet. What then is the geometric origin of these transformations? At some level, it is just the statement that our ADE singularity is building up the root space of a Lie algebra, and so we are free to consider the action of the automorphisms of this Lie algebra on the geometry. In some cases, the required automorphisms will just be inner automorphisms (namely, they involve the adjoint action of the Lie algebra), and in other cases, they will be outer automorphisms.

To illustrate, it is already instructive to return to 10D SYM theory compactified on a circle. According to our analysis of the previous subsection, we expect to get 9D vector multiplet, with bosonic field content consisting of a 9D vector and an adjoint-valued pseudoscalar ϕ_{9d}. Giving a vev to this pseudoscalar in a direction of the Cartan subalgebra would break reflection symmetries. Observe, however, that we really have two symmetries; one is reflection R^{9d}, and one is a charge conjugation operation $\mathsf{C}^{9d}\colon \langle\phi_{9d}\rangle \mapsto -\langle\phi_{9d}\rangle$, which descends from an automorphism of the Lie algebra. In this case, the combinations $\mathsf{C}^{9d}\mathsf{R}^{9d}$ are preserved on the moduli space. Of course, if we had started with the pseudovector multiplet theory, we would have instead retained R^{9d} and broken $\mathsf{C}^{9d}\mathsf{R}^{9d}$. Clearly, similar considerations descend to 7D SYM.

Returning to our discussion of M-theory backgrounds, we observe that the proposed transformation $\omega_I \mapsto -\omega_I$ is carrying out the desired operation, but now in geometric terms. Of course, in the full compactification, we need to consider more than just the action of these conjugation operations on the Cartan subalgebra, and so we refer to this class of geometric operations as $\mathcal{A}_{\mathfrak{g}}$ to emphasize their connection to the automorphisms of the gauge algebra/singular geometry.

TABLE I. Table illustrating the generalized charge conjugation operations generated by automorphisms $\mathcal{A}$ on the generators T^A of a Lie algebra. In the $\mathfrak{so}(2N)$ case, $\mathcal{O}$ is a matrix of $O(2N)$ with determinant -1. In the case of $\mathfrak{e}_{6,7,8}$, ι refers to an involution, which is an outer automorphism for $\mathfrak{e}_6$ and an inner automorphism for $\mathfrak{e}_{7,8}$.

	$\mathfrak{u}(1)$	$\mathfrak{su}(N)$	$\mathfrak{so}(2N)$	$\mathfrak{e}_6$	$\mathfrak{e}_{7,8}$
$\mathcal{A}_{\mathfrak{g}}(T^A)$	$-T^A$	$-(T^a)^t$	$\mathcal{O}^{-1}(T^A)\mathcal{O}$	$\iota(F)$	$\iota(F)$.

We summarize in Table I the action on the Lie algebra generators T^A for each automorphism associated with generalized charge conjugation $\mathcal{A}_{\mathfrak{g}}$. In the table, the matrix $T^A \in O(2N)$ has det $= -1$, and ι are different involutions on the $\mathfrak{e}_{6,7,8}$ Lie algebras that act as -1 on the maximal tori components of the fields [thereby extending the $U(1)$ definition of $\mathcal{A}_{\mathfrak{g}}$]. For $\mathfrak{e}_6$, this is an outer automorphism, while for $\mathfrak{e}_7$ and $\mathfrak{e}_8$, it is an inner automorphism.[21]

Before composing $\mathcal{A}_{\mathfrak{g}}$ with $\mathcal{R}_i$, we take note of an interesting property of $\mathcal{R}_i$ on the non-Abelian 7D vector multiplet. Since the M2-brane charge is reflection odd, an M2-brane wrapping S^2 cycles associated to simple roots of $\mathfrak{g}$ has its $U(1)^r$ charge vector transform as $\mathcal{R}_i\colon q_I \mapsto -q_I$, which means that in the singular limit this amounts to a Lie algebra involution. So, to complete the action of $\mathcal{R}_i$ to the off-diagonal field components, for say $\mathfrak{g} = \mathfrak{su}(N)$, requires

$$\mathcal{R}_i\colon A \to -(A)^t \phi_a \to (\phi_a)^t, \tag{6.29}$$

where the overall sign is fixed by knowledge of the action of the maximal tori components. Therefore, after composition with $\mathcal{A}_{\mathfrak{g}}$, we get back R_i^{7d} as before:

$$\mathsf{R}_i^{7d} = \mathcal{A}_{\mathfrak{g}} \circ \mathcal{R}_i. \tag{6.30}$$

It is now convenient to package all of this in terms of a table illustrating how the different symmetries of a higher-dimensional compactification descend to transformations. We collect in Table II the transformation rules for 7D $\mathfrak{su}(N)$ gauge theory, illustrating the origins from M-theory on an A-type singularity. Similar tables can be constructed for the other Lie algebras using Table I.

[21]Even though it may be undone by a gauge transformation for $\mathfrak{e}_{7,8}$, it allows us to define the transformation on the moduli space of Higgsings between various gauge groups (hyper-Kähler moduli space of *ALE* fibers from the M-theory viewpoint).

TABLE II. Table of reflection transformation properties for the fields A_μ, ϕ_a, and Ψ_I of 7D SYM with gauge algebra $\mathfrak{su}(N)$, with all fields presented as $N \times N$ matrices in the adjoint representation. We collect here both the 11D reflection transformations as well as the 7D reflection transformations.

	$\mathcal{R}_i$	$\mathcal{CT}$	R_i^{7d}	CT^{7d}
A_μ	$-(A)^t(-1)^{\delta\mu i}$	$-(-A)^t(-1)^{\delta\mu 0}$	$(-1)^{\delta\mu i}(A)$	$(-1)^{\delta\mu 0}(A)$
ϕ_a	$+(\phi_a)^t$	$+(\phi_a)^t$	$-(\phi_a)$	$-(\phi_a)$
Ψ_I	$-\Gamma_i(\Psi_I)^t$	$-i\Gamma_0(\Psi_I)^t$	$\Gamma_i(\Psi_I)$	$i\Gamma_0(\Psi_I)$

The upshot of this analysis is that it is somewhat a matter of taste whether we refer to our 7D compactified theory as a pseudovector multiplet or as a vector multiplet. Said differently, different duality frames, be it heterotic on a T^3, or M-theory on a K3 surface might "canonically favor" one presentation or the other, but the further composition with an automorphism allows us to pass between the two presentations. In this sense, we are filling in an additional entry in heterotic/M-theory duality.

As an additional comment, our emphasis here has been on realizing 7D SYM theory with a simply laced gauge algebra. One can also ask about the nonsimply laced case, which corresponds to quotienting the Lie algebra by a suitable outer automorphism. In M-theory terms, this amounts to activating a background discrete flux "at infinity" (see, e.g., Refs. [57–59]), and in F-theory terms, it can be obtained by starting with a noncompact elliptically fibered K3 surface and upon compactifying on a further circle, twisting by this outer automorphism. Indeed, a general comment is that 7D SYM theory with a nonsimply laced gauge theory requires the spin-1 degrees of freedom to transform as a vector, simply because of the structure of the kinetic term $\mathrm{Tr}_{\mathfrak{g}} F^2$ under reflection symmetries.

VII. 4D AND 3D REFLECTIONS VIA 7D SYM

In the previous section, we analyzed the reflection symmetries associated with 7D SYM theory. In particular, we saw that compactification of 10D SYM (as obtained from heterotic on a T^3) and M-theory compactification on an ADE singularity lead to different reflection parity assignments for the 7D fields, which are related by a further automorphism action. Since we are ultimately interested in further compactification of our system to 4D and especially 3D vacua, our aim here will be to understand the resulting reflection parity assignments for zero modes generated by our local $Spin(7)$ system. With the analysis of the previous section in mind, our starting point will be 7D SYM theory, with bosonic content given by a 7D vector potential and a triplet of pseudoscalars associated with the Higgs field. Our aim will be to track the impact of further discrete transformations which come from compactifying on a three-manifold M_3 or four-manifold M_4.

This section is organized as follows. As a warmup, we first return to the case of the PW system compactified on an orientable three-manifold M_3 and track how reflections act on the resulting 4D fields. Then, we turn to the case of 3D $\mathcal{N}=1$ vacua as obtained from our local $Spin(7)$ system compactified on an orientable four-manifold M_4 and study the same question. To a certain extent, the reflection assignments for local matter are dictated by effective field theory considerations, since we know that we get 4D $\mathcal{N}=1$ chiral multiplets from the PW system, and compactification on a circle produces 3D $\mathcal{N}=2$ matter multiplets. The situation is far less straightforward in the case of the bulk matter fields, since these fill out genuine 3D $\mathcal{N}=1$ matter multiplets.

A. 4D reflections

As a warmup to the full problem, in this subsection, we treat the case of 7D SYM compactified on an orientable three-manifold M_3. Our task will be to understand the higher-dimensional origin of various reflection/parity assignments for modes of the resulting 4D system.

To a certain extent, we are just producing a 4D $\mathcal{N}=1$ supersymmetric system, and we know how the standard 4D operations referred to as C^{4d}, P^{4d}, and T^{4d} act on any 4D field theory, supersymmetric or not. In terms of our reflection operations, we of course have

$$\mathsf{P}^{4d} = \mathsf{R}_1^{4d}\mathsf{R}_2^{4d}\mathsf{R}_3^{4d} \quad \text{and} \quad \mathsf{CT}^{4d} = \mathsf{R}_0^{4d}, \tag{7.1}$$

where in the last equation we are again analytically continuing from Euclidean to Lorentzian signature. Recall also that on a 4D $\mathcal{N}=1$ chiral multiplet with scalar component $s_{4d} = s_1 + is_2$, s_1 is a scalar, but s_2 is a pseudoscalar (see, e.g., Ref. [60]). Under parity, the chiral multiplet S transforms as $S \to \bar{S}$, which also sends the left-handed Weyl fermion of the supermultiplet to a right-handed Weyl fermion in the conjugate representation.

What complicates the story is that we are now asking how to lift these operations back to statements about a local G_2 system, which as we have already remarked can be analyzed by starting with 10D SYM theory compactified on T^*M_3. To determine the parity transformations for the 4D fields, we observe that, although T^*M_3 is even dimensional, we can (since an oriented M_3 is automatically parallelizable) speak of an orientation reversal operation O_{T^*} on the cotangent directions as well, which locally looks like

$$\mathsf{O}_{T^*} = \mathsf{R}_7^{10d}\mathsf{R}_8^{10d}\mathsf{R}_9^{10d}, \tag{7.2}$$

which we recognize as just the "internal reflections" associated with the compactification of 10D SYM theory on a T^3. Putting these comments together, we recognize that the complexified connection $\mathcal{A} = A + i\phi_{\mathrm{PW}}$ of the PW system breaks up into a 4D scalar and a 4D pseudoscalar.

Summarizing, we conclude that the appropriate reflection symmetries of the 4D effective field theory descend as

$$\mathsf{R}_i^{4d} = \mathsf{R}_i^{7d}, \tag{7.3}$$

where now $i = 0, 1, 2, 3$ for the 4D Euclidean directions. Of course, a particular compactification need not preserve parity, but this depends on the specific matter content and interactions of the resulting effective field theory.

B. 3D reflections

Let us now turn to the reflection assignments for the zero modes generated from compactification of 7D SYM theory on a four-manifold M_4. Here, it is helpful to start with the fermionic content, including the supercharges of the 7D theory.

Now, a 7D Dirac spinor can be decomposed into the fields λ_ℓ and λ_r which transform as $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2})$, respectively, under $\mathrm{Spin}(1,2)\times SU(2)_\ell \times SU(2)_r \times SU(2)_R$, where here $SU(2)_\ell \times SU(2)_r$ denotes the Lorentz group on M_4 and $SU(2)_R$ refers to the R-symmetry. At the level of structure groups, we topologically twist by identifying $SU(2)_{r'} \equiv \mathrm{diag}(SU(2)_r \times SU(2)_R)$ so that in terms of the representations of $\mathrm{Spin}(1,2)\times SU(2)_\ell \times SU(2)_{r'}$ the 3D fermions of interest are

$$\lambda_\ell \equiv \chi \leftrightarrow (\mathbf{2},\mathbf{2},\mathbf{2}), \tag{7.4}$$

$$\lambda_{r'} \equiv (\psi,\lambda) \leftrightarrow (\mathbf{2},\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1}). \tag{7.5}$$

The twisted fields, χ, ψ, and λ are the 3D superpartners for the fluctuations a, φ, and A_{3d}, respectively. From the 7D SYM action, we see under a parity transformation in a 3D spatial direction, R_i^{7d}, that a is even, while φ is odd. Furthermore, since $\Gamma_i = \sigma_i \otimes \gamma_5^{(4d)}$, the 3D fermions transform as (we suppress the action on the spacetime coordinates)

$$\mathsf{R}_i^{7d}(\chi) = +\gamma_i\chi \mathsf{R}_i^{7d}(\psi) = -\gamma_i\psi \mathsf{R}_i^{7d}(\lambda) = -\gamma_i\lambda. \tag{7.6}$$

We note that the reflection transformations for the 3D gaugino λ are the same as for the supercharge. Indeed, the supercharge transforms as $\mathsf{R}_i^{7d}(Q_\alpha) = -\gamma_i Q_\alpha$, and considering $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\gamma_\beta^{\alpha\mu}\partial_\mu$, we see that the $\mathcal{N}=1$ superspace coordinate, θ_α, transforms as $\mathsf{R}_i^{7d}(\theta_\alpha) = -\gamma_i\theta_\alpha$, $\mathsf{R}_i^{7d}(Q_\alpha) = -\gamma_i Q_\alpha$.

It is convenient to package the field content of the 7D SYM theory in terms of 3D $\mathcal{N}=1$ superfields. In the case of A_{3d} and the 3D gaugino, this is just the 3D $\mathcal{N}=1$ vector multiplet, where A_{3d} transforms as a vector (as opposed to a pseudovector) and the 3D gaugino is reflection odd [as in Eq. (7.6)]. As for the bulk modes associated with the internal fluctuations of the vector potential a and the self-dual Higgs field φ, these split up into adjoint valued 3D $\mathcal{N}=1$ pseudoscalar and scalar multiplets:

$$\text{pseudoscalar multiplet}:\ \boldsymbol{\varphi} = \varphi + i\theta^\alpha\psi_\alpha + \frac{i}{2}\theta^\alpha\theta_\alpha F_\varphi \tag{7.7}$$

$$\text{scalar multiplet}:\ \boldsymbol{a} = a + i\theta^\alpha\chi_\alpha + \frac{i}{2}\theta^\alpha\theta_\alpha F_a. \tag{7.8}$$

On a general M_4, there is no natural way to combine these fields into "complexified combinations" as would be appropriate for 3D $\mathcal{N}=2$ matter multiplets.

Let us now turn to the localized matter fields. We treat the "generic situation" where matter is localized on a one-dimensional subspace of M_4. Recall that this matter can be treated locally as a zero mode of the PW system on a three-manifold which is then further compactified on a circle. At this level of analysis, it is therefore enough for us to track the reflection assignments of a chiral and antichiral superfield after reduction on circle. Of course, these zero modes descend from trapped fluctuations of the bulk modes, but we leave this point implicit in what follows.

Now, for a 4D $\mathcal{N}=1$ chiral multiplet reduced on a circle, we start with a 4D real scalar and a pseudoscalar as well as a left-handed Weyl fermion. Reduction on a circle gives us a 3D real scalar and a pseudoscalar as well as a 3D Dirac fermion ψ_D. Since the 4D theory was chiral, it is a bit of a misnomer to speak of reflection in just one direction. We can, however, compose two such reflections to generate a sensible 3D operation,

$$\mathsf{R}_i^{3d}\psi_D = \mathsf{R}_i^{4d}\mathsf{R}_\perp^{4d}\psi_D, \tag{7.9}$$

where we treat ψ_D as a 4D spinor and $\mathsf{R}_\perp^{4d}$ denotes reflection along the circle. So, it does make sense to speak of a definite reflection assignment for this 3D Dirac fermion.

What then, is the reflection assignment for this zero mode? The first comment is that we expect opposite reflection assignments for reduction of a left-handed and right-handed Weyl fermion. The second comment is that we expect that circle reduction of a 4D $\mathcal{N}=1$ gauge theory will lead to Weyl fermions all with the same reflection assignment. Since we have already fixed the reflection assignment for our 3D $\mathcal{N}=1$ vector multiplet to be -1, we conclude that the 3D Dirac fermion ψ_D obtained from reduction of a left-handed Weyl fermion will produce reflection assignment -1. Conversely, if we had started with a right-handed Weyl fermion, the resulting 3D Dirac fermion $\tilde{\psi}_D$, would have had a reflection assignment of $+1$. We can explicitly see this if we compactify a 4D Dirac fermion $\binom{\Psi_L}{\Psi_R}$ to two 3D Dirac fermions. Consider the standard Weyl basis of 4D gamma matrices

$$\gamma_0 = i\sigma_2\otimes\sigma_3, \qquad \gamma_1 = \sigma_2\otimes\sigma_1,$$
$$\gamma_2 = \sigma_2\otimes\sigma_2, \qquad \gamma_\perp = \sigma_1\otimes 1; \tag{7.10}$$

then, we have that $\mathsf{R}_1\mathsf{R}_\perp(\Psi_L) = -i\gamma_1\gamma_\perp\binom{\Psi_L}{\Psi_R} = \gamma_1^{(3d)}\binom{-\Psi_L}{\Psi_R}$ after dimensional reduction which returns our proposed 3D reflection phases.

Geometrically, recall that for the PW system on a three-manifold M_3 we get chiral versus antichiral matter depending on the signature of the Hessian $*_{M_3} d *_{M_3} \phi_{\mathrm{PW}}$. In particular, a Hessian with signature $(+,+,-)$ produces a left-handed Weyl fermion (a chiral mode), while a Hessian with signature $(-,-,+)$ produces a right-handed Weyl fermion (an antichiral mode). Proceeding now to the case of the local $Spin(7)$ system with matter localized on a circle, we see that these two choices of signature are related by an orientation reversal in the directions transverse to the matter circle.

Indeed, in a local neighborhood of the matter circle, we can define the operation $\mathsf{R}_i^{10d}\mathsf{R}_{3456789}^{10d}$. We observe that $\mathsf{R}_{3456789}^{10d}$ is a local orientation reversal operation on $\mathbb{R}^7$, but there is no generic expectation that on $\Lambda^2_{\mathrm{SD}} M_4$ that such an action is a symmetry of the geometry. For local matter, however, where the geometry really is just $S^1 \times \mathbb{R}^6$, we do have such a symmetry action, and this specifies a well-defined notion of how reflections descend to three dimensions.

Let us summarize the reflection assignments for the various bosonic and fermionic matter fields. Starting first with the bulk fluctuations a, φ, as well as the localized "complex" scalars $\tau_{+,+,-}$ and $\tau_{-,-,+}$, we have[22]

$$a \to +a, \qquad \varphi \to -\varphi, \qquad \tau_{+,+,-} \to -\tau_{+,+,-}, \\ \tau_{-,-,+} \to +\tau_{-,-,+}. \tag{7.11}$$

Here, we remark that the signature of the internal Hessian flips sign under this 3D reflection. For the fermionic fluctuations associated with each of these modes, we have

$$\chi^{(a)} \to +\gamma_i \chi^{(a)}, \qquad \psi^{(\varphi)} \to -\gamma_i \psi^{(\varphi)}, \\ \psi_{+,+,-} \to -\gamma_i \psi_{+,+,-}, \qquad \psi_{-,-,+} \to +\gamma_i \psi_{-,-,+}, \tag{7.12}$$

in the obvious notation.

An additional comment here is that if we neglect interaction terms there are a large number of additional symmetries which act on our fields. Indeed, in addition to the 3D reflection symmetry R_i^{3d}, we can also speak of various charge conjugation actions which act on individual localized Dirac fermions as $\psi_D \to \psi_D^*$. From the perspective of compactifying 7D SYM, this provides us with a large set of candidate symmetries. Of course, in an actual compactification, we expect many of these symmetries to wind up being broken. The way to extract the ones which are preserved involves simply working out the 3D effective field theory and checking which symmetries are preserved at the level of the classical interaction terms. Our discussion here has primarily emphasized what a 3D field theorist would see, but *a priori*, there is no guarantee that a given compactification will actually preserve these symmetries. We return to this issue in Sec. IX. For additional discussion of the various symmetries which appear in the free field limit, see Appendix E.

VIII. PARITY ANOMALIES

Having analyzed the reflection symmetry transformations for the matter content of 7D SYM coupled to various lower-dimensional defects, we now turn to the anomalies generated by the matter content of this system. Again, our motivation for doing so is that anomalies are robust against strong coupling effects, and so are particularly important to track in local $Spin(7)$ systems. In Appendix G, we review some additional aspects of parity anomalies of 3D gauge theories which we implicitly assume below. Again, we note that as discussed in Ref. [24] it is more appropriate to view these anomalies as associated with a reflection.

In general terms, we shall be interested in the resulting 3D effective field theory we get after compactification. In this 3D theory, we can consider working on a Euclidean signature three manifold N_3. If we have a symmetry S of the classical action, we can ask about the response to the partition function $Z(N_3)$ under such a symmetry transformation,[23]

$$Z(N_3) \to \exp(2\pi i \nu_{\mathsf{S}}/K_{\mathsf{S}}) Z(\mathsf{S}N_3), \tag{8.1}$$

where here we allow for the possibility that S is a spacetime transformation which acts on N_3. The parameter K_{S} is a positive integer, and so we often refer to ν_{S} as valued in the finite group $\mathbb{Z}_{K_{\mathsf{S}}}$. In our setting, we can consider the action of the 3D reflection symmetries R_i^{3d}, of which there is a corresponding "gravitational-parity anomaly" ν_{R}. We remark that this phase is defined modulo $\mathbb{Z}_{16}$, as explained in Ref. [24] (see also Ref. [74]). Some systems also have an independent charge conjugation symmetry, and so in principle, we could also discuss ν_{CR}, which in Lorentzian

[22] For the zero modes localized on a circle $S^1_\perp$, our 3D reflection $\mathsf{R}_i^{3d} = \mathsf{R}_i^{4d}\mathsf{R}_\perp^{4d}$. This introduces an additional minus sign in the reflection assignment for the pseudoscalar of the 4D $\mathcal{N}=1$ chiral multiplet, so that in 3D we wind up with a pair of scalars or a pair of pseudoscalars. We can also consider composition with a local charge conjugation operation C, and then it is a matter of convention what one wishes to call "reflection," i.e., R versus CR. Here, we have opted to use the same conventions for reflection on 3D Dirac fermions used in Ref. [52].

[23] Here, we assume the existence of a partition function. A common situation, especially in string compactification on a noncompact background, is that our 3D QFT is actually a relative QFT in the sense that we have a vector of partition functions. We ignore this subtlety in what follows. For further discussion, see, e.g., Refs. [33,41,61–73].

TABLE III. Indices for various representations of ADE groups.

G			
$SU(N)$	$\mathbf{T}(\mathbf{N})=1/2$	$\mathbf{T}(\mathbf{adj})=N$	
Spin(N)	$\mathbf{T}(\mathbf{N})=1$	$\mathbf{T}(\mathbf{adj})=N-2$	
E_6	$\mathbf{T}(\mathbf{27})=3$	$\mathbf{T}(\mathbf{adj})=12$	
E_7	$\mathbf{T}(\mathbf{56})=6$	$\mathbf{T}(\mathbf{adj})=18$	
E_8	$\mathbf{T}(\mathbf{248})=30$		
$SU(5)$	$\mathbf{T}(\mathbf{5})=1/2$	$\mathbf{T}(\mathbf{10})=3/2$	$\mathbf{T}(\mathbf{adj})=5$
Spin(10)	$\mathbf{T}(\mathbf{10})=1$	$\mathbf{T}(\mathbf{16})=2$	$\mathbf{T}(\mathbf{adj})=8$

signature is synonymous with ν_{T}, i.e., a time reversal anomaly.

Of course, we can also have mixed parity/gauge symmetry anomalies. For a theory with gauge group H, we can label these as $\nu_{\mathsf{R}H}$. In practical terms, one can compute this by tracking a possible one-loop shift to the Chern-Simons coupling from massless fermions.[24] For a massless fermion in a representation $\mathbf{R}$, we get a shift at one loop given by

$$\delta k = \alpha \mathbf{T}(\mathbf{R}), \tag{8.2}$$

where $\mathbf{T}(\mathbf{R})$ is the index of the representation $\mathbf{R}$ and we implicitly mean a 3D Dirac fermion with $\alpha=1$ if the representation is pseudoreal or complex and a 3D Majorana fermion with $\alpha=1/2$ if the representation is real. The normalization is such that $\mathbf{T}(\mathbf{N})=1/2$ for the fundamental representation of $SU(N)$. As an example, one can see that for a $\mathcal{N}=1$ $SU(N)$ theory with N_f flavors the shift is

$$\delta k = \frac{N_f}{2} + \frac{h^\vee}{2}, \tag{8.3}$$

where $h^\vee$ is the dual Coxeter number of the gauge group [for $SU(N)$, we have that $h^\vee = N$]. Given that the bare Chern-Simons level k is an integer in order to be able to have a zero Chern-Simons level at one loop, one must ensure the condition that $N_f = N$ mod 2 for these theories, matching with, e.g., Ref. [75]. In Table III, we collect our conventions for various indices of representations. Lastly, it is tempting to also consider the parity anomalies associated with placing 7D SYM on a general background which has some reflection symmetries, but we leave this to future work.

A general comment here is that in passing from our local $Spin(7)$ system to our 3D effective field theory we can expect some of the symmetries of the 3D theory to be inherited from "top-down" considerations, while some may only be emergent after some details of the compactification have decoupled. In this section, we work in a specific idealized limit where we assume that the only interactions between zero modes are mediated by the 3D $\mathcal{N}=1$ vector multiplet. In this case, the issue of parity assignments just follows from tracking the overall matter content of the system, an issue we just addressed in Sec. VII. Of course, we expect that in a full compactification there may be various parity breaking effects. For example, integrating out the Kaluza-Klein (KK) modes of the four-manifold can be visualized as starting with a collection of 3D Dirac fermions with mass terms added. Since the sign of these mass terms will shift the bare Chern-Simons coupling, the net impact from such zero modes can also produce an explicit parity breaking interaction (the Chern-Simons term itself). If, however, our M_4 enjoys an internal reflection symmetry, then these KK modes come in pairs, and there will be no overall shift to the level.

With this simplified situation in mind, we just need to track the contributions from the bulk zero modes and the localized zero modes to the various parity anomalies. A general rule of thumb is that for a single 3D Majorana fermion χ with parity $\varepsilon_\chi = \pm 1$ under a reflection,

$$\mathsf{R}_i^{3d}\chi = \varepsilon_\chi \gamma_i^{3d}\chi, \tag{8.4}$$

the contribution to the anomaly ν_{R} is just $\nu_{\mathsf{R}}(\chi) = -\varepsilon_\chi$, where our overall choice of minus sign convention has been introduced to avoid clutter in later equations. By the same reasoning, if we have a 3D Dirac fermion ψ with parity $\varepsilon_\psi = \pm 1$ which transforms in a complex representation $\mathbf{R}$ of the unbroken gauge group $H = G \times U(1)^n \subset \tilde{G}$ (with notation as in Sec. II), then the contribution to the anomaly is $\nu_{\mathsf{R}}(\psi) = \varepsilon_\psi \dim_{\mathbb{R}} \mathbf{R}$; i.e., we can just count the number of real degrees of freedom. There are bulk and local contributions to ν_{R}:

$$\nu_{\mathsf{R}} = \nu_{\mathsf{R}}^{\text{bulk}} + \nu_{\mathsf{R}}^{\text{local}}. \tag{8.5}$$

The contribution from the bulk modes comes from the 3D gaugino of the vector multiplet and the superpartners of the internal 1-form and self-dual 2-form,

$$\nu_{\mathsf{R}}^{\text{bulk}} = \dim_{\mathbb{R}} H \times (1 - b^1 + b^2_{\text{SD}}) \mod 16, \tag{8.6}$$

where the overall factor of $\dim_{\mathbb{R}} H$ is due to the fact that all zero modes in the bulk transform in the adjoint representation. Note that this also includes the singlets associated with the "adjoint" of the $U(1)$ factors of $H = G \times U(1)^n \subset \tilde{G}$. An important feature of Eq. (8.7) is that we can express it in terms of the Euler character and signature of M_4; namely we have

$$\nu_{\mathsf{R}}^{\text{bulk}} = \dim_{\mathbb{R}} H \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)) \mod 16, \tag{8.7}$$

[24] One way to think about this has to do with introducing a Pauli-Villars regulator for the loop integral. Picking a sign for the Pauli-Villars mass then produces a consequent shift from a single massless mode.

where $\chi(M_4) = 2 - 2b^1 + b^2$ and $\sigma(M_4) = b^2_{\mathrm{SD}} - b^2_{\mathrm{ASD}}$.[25] So, in other words, $\nu^{\mathrm{bulk}}_{\mathsf{R}}$ detects a topological structure on M_4, at least modulo $\mathbb{Z}_{16}$. Another comment is that this provides some *a posteriori* justification for the reflection assignments previously found in Sec. VII.

Turning next to the localized zero modes, we can have different complex representations contributing. Returning to our reflection assignments from Eq. (7.12) which we reproduce here:

$$\chi^{(a)} \to +\gamma_i \chi^{(a)}, \qquad \psi^{(\varphi)} \to -\gamma_i \psi^{(\varphi)},$$
$$\psi_{+,+,-} \to -\gamma_i \psi_{+,+,-} \psi_{-,-,+} \to +\gamma_i \psi_{-,-,+}. \tag{8.8}$$

Additionally, there is an overall minus sign having to do with whether we have a $(+,+,-)$ mode or a $(-,-,+)$ mode. Recall our reflection assignments from Eq. (7.12). Splitting up the zero modes into these two sets, we get

$$\nu^{\mathrm{local}}_{\mathsf{R}} = \sum_{i\in(+,+,-)} \dim_{\mathbb{R}} \mathbf{R}_i - \sum_{i'\in(-,-,+)} \dim_{\mathbb{R}} \mathbf{R}_{i'} \mod 16. \tag{8.9}$$

Consider next the mixed gauge-parity anomaly $\nu_{\mathsf{R}H}$. For H a non-Abelian simply connected gauge group, this is a mod $\mathbb{Z}_2$ phase, and for $H = U(1)^n$, this is a mod $\mathbb{Z}_4$ phase. For more general gauge group choices, this depends on the reduced Pin^+ bordism groups, a point we discuss further in Appendix G. In any event, we again just need to count the different contributions from the bulk and localized zero modes. In general terms, for a 3D Majorana fermion χ with reflection assignment ε_χ in a real representation $\mathbf{R}$ of H, the contribution to $\nu_{\mathsf{R}H}$ is $\nu_{\mathsf{R}H}(\chi) = \varepsilon_\chi \mathbf{T}(\mathbf{R})$, while for a 3D Dirac fermion ψ with reflection assignment ε_ψ in a complex or pseudoreal representation $\mathbf{R}$ of H, the contribution to $\nu_{\mathsf{R}H}$ is $\nu_{\mathsf{R}H}(\psi) = 2\varepsilon_\psi \mathbf{T}(\mathbf{R})$. Consequently, the contributions to $\nu_{\mathsf{R}H}$ from the bulk and localized zero modes assemble in the expected way:

$$\nu_{\mathsf{R}H} = \nu^{\mathrm{bulk}}_{\mathsf{R}H} + \nu^{\mathrm{local}}_{\mathsf{R}H} \mod 2, \tag{8.10}$$

$$\nu^{\mathrm{bulk}}_{\mathsf{R}H} = h^\vee(H) \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)) \mod 2, \tag{8.11}$$

$$\nu^{\mathrm{local}}_{\mathsf{R}H} = \sum_{i\in(+,+,-)} 2\mathbf{T}(\mathbf{R}_i) - \sum_{i'\in(-,-,+)} 2\mathbf{T}(\mathbf{R}_{i'}) \mod 2. \tag{8.12}$$

As a brief comment, we see that the contribution to the Chern-Simons level (working modulo the integers) is just $2\nu_{\mathbf{R}H}$.

Having spelled out the general contributions to anomalies from our bulk and localized matter, we are now ready to use this to track various quantum corrections which appear in local $Spin(7)$ systems. We turn to this next.

[25] Note that we met this topological quantity previously in Sec. II in the context of determining the number of zero-circles of Φ_{SD} modulo 2 (see Ref. [28] for some discussion on the topological significance of this quantity).

IX. QUANTUM CORRECTIONS TO *Spin*(7) MATTER

As already mentioned in previous sections, part of our goal in studying various reflection symmetries of 7D SYM is to use this additional structure to potentially constrain the effects of quantum corrections. Now, an important issue we face is that *a priori* there is no reason to expect M-theory on a generic $Spin(7)$ background to preserve reflection symmetries, and there is a remnant of this same issue in compactification of 7D SYM compactified on a four-manifold M_4. The way this shows up in the resulting 3D effective field theory is that various candidate discrete symmetry transformations will end up being broken by interaction terms.

On the other hand, we can also consider tuning the moduli of the compactification so that a reflection symmetry is preserved. This is actually more common than one might initially anticipate. For example, since M-theory also includes objects such as orientifold M2-planes (i.e., OM2-planes), we expect that for a suitable choice of moduli we have internal reflection symmetries which act on the $Spin(7)$ space. Indeed, there are also well-known compact examples of $Spin(7)$ spaces which exhibit such symmetries (see, e.g., Refs. [76,77]). In the context of a local model, this can also be arranged; for example, the local G_2 space $\Lambda^2_{\mathrm{SD}} S^4$ with a round metric for the four-sphere has many such symmetries.

We also expect that it is possible to compactify M-theory on various nonorientable backgrounds, as long as they admit a Pin^+ structure. From an effective field theory point of view, this involves gauging some combination of reflection symmetries, but for this to be so, one must have a reflection symmetry in the first place.

With these considerations in mind, we adopt the following strategy. We begin by building a local model which has an internal reflection symmetry. From what we have worked out in the previous sections, we know that this will then descend to a reflection symmetry of the 3D effective field theory. This in turn puts strong constraints on possible quantum corrections to the matter content. In particular, we can then use 't Hooft anomaly matching to track the value of ν_{S} for any spacetime symmetry S. So, even if a particular localized or bulk zero mode does not survive, it can still leave a remnant in its contribution to ν_{S}.

At a more generic point of the moduli space where compactification effects explicitly break a reflection symmetry, it is still in principle possible for the resulting 3D effective field theory to be reflection symmetric. For example, if we add an irrelevant reflection breaking operator to an effective field theory, then in the deep infrared, one can still recover such a symmetry. Of course, relevant interaction terms will impact the surviving symmetries.

Of particular significance is the Chern-Simons interaction term. This can receive a "bare" contribution from the ambient 4-form flux of an M-theory compactification as well as from integrating out KK modes of the compactification. One of our aims in this section will be to determine the overall value of this coupling. Our general claim is that we can use the zero mode content of the 3D effective field theory as a proxy for determining the presence (or absence) of such a contribution. We note that in some cases the appearance of a background G-flux is sometimes required by a quantization condition [12,14], and we will see a sharp analog of this in the context of our local gauge theory analysis.

Additionally, we also must pay attention to nonperturbative instanton corrections. In the closely related context of 4D $\mathcal{N}=1$ theories generated from local G_2 geometries, such effects are responsible for generating mass terms for vectorlike pairs as well as Yukawa interaction terms. The former is a parity invariant contribution, while the latter typically breaks parity. Since we have emphasized that our local $Spin(7)$ system can be built up from PW building blocks, we can expect similar effects to be present in our system as well.

In the process of compactifying, then, we encounter several different mass scales. First of all, we have the UV cutoff of the 7D SYM theory $\Lambda_{\mathrm{UV}}^{7d}$. Then, we have a generic radius of compactification for the four-manifold, which introduces a Kaluza-Klein mass scale Λ_{KK}. Below this, we expect to have a 3D effective field theory. The 3D gauge coupling will flow to strong coupling since the gauge field kinetic term is a relevant operator, and this occurs at a strong coupling scale $\Lambda_{\mathrm{strong}}^{3d}\sim\Lambda_{KK}(\Lambda_{KK}/\Lambda_{\mathrm{UV}}^{7d})^3$. Euclidean M2-brane instanton effects can generate masses Λ_{inst} for vectorlike pairs but are exponentially suppressed by a volume factor. On an isotropic M_4, this will be small relative to $\Lambda_{\mathrm{strong}}^{3d}$. Finally, we have the "deep infrared" $\Lambda_{\mathrm{IR}}^{3d}$ where even these massive states have been integrated out. All told, then, we have the hierarchy of mass scales:

$$\Lambda_{\mathrm{UV}}^{7d}\gg\Lambda_{\mathrm{KK}}\gg\Lambda_{\mathrm{strong}}^{3d}\gg\Lambda_{\mathrm{inst}}\gg\Lambda_{\mathrm{IR}}^{3d}. \tag{9.1}$$

Our plan in the remainder of this section will be to track these compactification effects in more detail. We begin by analyzing the impact of compactification effects on the Chern-Simons level of the 3D gauge theory. We then turn to an analysis of quantum corrections. We then turn to some explicit geometric examples illustrating these general considerations. Appendix D contains additional details on instanton corrections to the local $Spin(7)$ system of equations.

A. Chern-Simons level contributions

In this subsection, we turn to a more a detailed analysis of how various contributions to the compactification can shift the Chern-Simons level. In particular, we discuss the conditions necessary for such effects to cancel out so that a 3D reflection symmetry might still be present.

In a general compactification, there are various sources of contributions to the Chern-Simons level. First of all, we can study the contribution from the zero modes, and as we discussed in Sec. VIII, these contribute to various parity anomalies. Second, we have the contributions from the Kaluza-Klein states. At first glance, it would appear that such modes cannot contribute, simply because they are heavy states, and we will integrate them out anyway. There is a subtlety here, because in three dimensions we also know that for a single Dirac fermion ψ_D coupled to a background $U(1)$ gauge field adding a mass term $m\overline{\psi_D}\psi_D$ and integrating out the fermion produces a shift in the Chern-Simons level of the background $U(1)$, with the sign of the shift dictated by the sign of m. *A priori*, then, it might happen that a massive KK state could shift the background Chern-Simons level.

To better understand this issue, we return to the 7D SYM theory coupled to defects. As we have already mentioned, one way for us to study this system of intersecting branes is to begin with a parent gauge theory with gauge group $\tilde{G}$ and then consider adjoint Higgsing to $G\times U(1)^n\subset\tilde{G}$. In general then, we need to pay attention to mass terms generated by the internal components of the 7D gauge field and the background Higgs field. Writing out the 7D Dirac equation (and including the background Higgs fields as well), we have

$$D_{3d}\Psi_I+\mathcal{D}\Psi_I=0, \tag{9.2}$$

where we have introduced the twisted differential operator:

$$\mathcal{D}=\begin{pmatrix}\Phi_{\mathrm{SD}}\times & D_{\mathrm{sig.}}\\ D_{\mathrm{sig.}} & -\Phi_{\mathrm{SD}}\wedge\end{pmatrix}. \tag{9.3}$$

The point is that, even though our original 7D SYM theory is reflection symmetric (when the overall 7D Chern-Simons level is zero[26]), activating vevs for the Higgs fields introduces an explicit parity breaking term precisely because these modes transform as pseudoscalars. So, from a 3D perspective, we can think of D_{sig} as generating a parity preserving mass term for Kaluza-Klein modes, while the action of Φ_{SD} will produce a parity breaking mass term. A related comment is that there is a coupling between localized modes and the bulk Higgs field given by the superpotential term [5],

$$W_{3d}\supset\int_L\Sigma^{\dagger}\Phi_{\mathrm{SD}}\Sigma, \tag{9.4}$$

for $\mathcal{N}=2$ matter Σ trapped on a line L.

In the presence of a background position dependent profile for Φ, then, we have an explicit parity breaking contribution to the background. Integrating out the

[26]At least in type IIA backgrounds, the 7D Chern-Simons term requires a Romans mass background, and we are assuming this is absent.

corresponding massive states generated by this profile, we see that the background value of Φ_{SD} can impact the Chern-Simons level of the 3D effective field theory. To a certain extent, we have already seen how Φ_{SD} impacts reflection assignments in the 3D effective field theory because each localized zero mode contributes to ν_{R}; i.e., we get a contribution to the gravitational-parity anomaly. So, generically, we expect Kaluza-Klein states which pick up a mass from Φ_{SD} to induce a shift in the 3D Chern-Simons level. On the other hand, we have already explained that a background profile for Φ_{SD} can produce trapped zero modes, which also induces a contribution to the 3D parity anomaly; i.e., we can view it as producing a shift in the background Chern-Simons level.

Our main claim is that these two contributions always cancel each other out. The main idea is to consider an initially aligned stack of 6-branes and to gradually switch on a background value of Φ_{SD}. Initially, we can pair up positive and negative parity KK modes through mass terms such as $\overline{\psi_+}\psi_-$. Once we tilt the 6-branes, however, one of these modes becomes trapped (i.e., it is either a $\tau_{+,+,-}$ or $\tau_{-,-,+}$ zero mode). This also shifts the mass matrix for the KK modes. Since this is a small perturbation, we expect the value of ν_{R} to be the same before and after; i.e., we can locally split up the contributions to the effective Chern-Simons level as

$$k(\mathrm{KK}) + k(\mathrm{Loc.}) = 0, \tag{9.5}$$

where the second term corresponds to all the contributions from "local matter." More precisely, we have two sources for local contributions. First of all, we have the contributions from the local matter. Second, we have contributions which are associated with singularities for the Higgs field which are generically localized on one-dimensional subspaces. Recall, however, that such singularities are associated with giving vevs to the scalars of matter localized on a line of some bigger parent gauge theory [5,21]. So, we break this up further as a schematic contribution of the form

$$k(\mathrm{KK}) + k(\mathrm{Loc.\,Matt.}) + k(\mathrm{Loc.\,Sing.}) = 0. \tag{9.6}$$

So, at least from the perspective of the 3D effective field theory, integrating the KK states generates an explicit parity breaking contribution, but this is *exactly opposite* to the contribution from local matter. From this perspective, the classical action (after integrating out the KK modes) breaks parity, but the one loop contribution from the local zero modes restores it.

One might view this as slightly undesirable, since the classical action now breaks parity. Of course, we can also restore classical parity by considering a broader class of M-theory backgrounds. Indeed, in a general M-theory background on an eight-manifold X_8, it is well known that the 4-form flux G_4 can induce further shifts in the effective Chern-Simons level through the 7D term (see, e.g., Refs. [12,78]),

$$\int_{7d} \frac{1}{4\pi} CS_3(A) \wedge \frac{i^* G_4}{2\pi}, \tag{9.7}$$

where i^*G_4 is the pullback of the 4-form flux onto $M_4 \subset X_8$.

In fact, sometimes a 4-form flux *must* be switched on since the quantization condition is [79]

$$\left[\frac{G_4}{2\pi}\right] - \frac{p_1(X_8)}{4} \in H^4(X_8, \mathbb{Z}). \tag{9.8}$$

We would like to understand whether a half-quantized flux needs to be switched on when we have M_4 a four-manifold of ADE singularities. Rather than attempt to directly define and compute the Pontryagin class on the singular space X_8, we can instead contemplate what happens with pure 7D SYM theory with gauge group H compactified on M_4. From our previous analysis, we know that the bulk zero modes generate a contribution to the mixed gauge-parity anomaly given by (treated as an integer)

$$\nu_{\mathsf{R}H} = h^\vee(H) \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)). \tag{9.9}$$

Observe that when this is an odd number we get a corresponding half-integer shift in the Chern-Simons level. Since $1/2(\chi(M_4) + \sigma(M_4))$ is an integer, we conclude that this happens precisely when $h^\vee(H)$ is an odd number, as can happen for $SU(N)$ gauge theory with N odd.[27] In this case, we see that, much as we had in Eq. (9.5), the mixed gauge-parity anomaly allows us to detect the presence of a half-integer G_4-flux. In particular, we have

$$k(\mathrm{G\text{-}flux}) + k(\mathrm{Bulk\ Zero\ Modes}) \in \mathbb{Z}, \tag{9.10}$$

in the obvious notation.

B. Quantum corrections

Let us now turn to a more explicit analysis of quantum corrections. To a certain extent, we have already detailed the robust observables as captured by anomalies. Indeed, supersymmetry provides only mild protection against such effects because the system we have engineered has only 3D $\mathcal{N}=1$ supersymmetry. That said, we note that the localized matter fields really fill out 3D $\mathcal{N}=2$ multiplets, while the bulk modes are in 3D $\mathcal{N}=1$ multiplets. In passing from 7D SYM down to 3D, we can ask about the impact of various loop corrections, as

[27] We can also get $h^\vee(H)$ odd if H is not simply laced.

generated both by the zero modes and by the KK modes. For ease of exposition, we primarily focus on Yukawa-like interactions of the form

$$c\phi\overline{\psi_-}\psi_+ + \mathrm{H.c.}, \tag{9.11}$$

where ϕ is a pseudoscalar, ψ_- is a reflection odd Dirac fermion, and ψ_+ is a reflection even fermion. Here, c is a dimensionless parameter, and we have taken our scalars to have scaling dimension 1. This is natural in our setting because the kinetic term for such scalars directly descends from compactification of 7D SYM, and so they come with a kinetic term of the form

$$\frac{\mathrm{Vol}(M_4)}{g_{7d}^2}(\partial\phi)^2 \sim \frac{1}{g_{3d}^2}(\partial\phi)^2. \tag{9.12}$$

Suppose now that ϕ and ψ_- are actually KK modes. This occurs, for example, in considering a bulk mode coupling to the modes localized on a circle L via the term

$$W_{3d} \supset \int_L \Sigma_{\mathrm{KK}}^\dagger \Phi_{\mathrm{SD}}^{\mathrm{KK}} \Sigma_{(0)}, \tag{9.13}$$

where $\Sigma_{(0)}$ is a zero mode, Σ_{KK} is a KK mode localized on the line L, and Φ_{SD}^{KK} is a bulk KK Mode. If we consider loop diagrams with internal KK modes, we see that generically we might pick up an additional mass term for some of our candidate zero modes, provided such interaction terms are compatible with the reflection (and other discrete) symmetries retained by the compactification. We observe that the overall mass scale generated in this way is

$$m_{3d} \sim g_{3d}^2(\Lambda_{\mathrm{KK}}) \sim \Lambda_{\mathrm{KK}} \times \left(\frac{\Lambda_{\mathrm{KK}}}{\Lambda_{\mathrm{UV}}^{7d}}\right)^3 \ll \Lambda_{\mathrm{KK}}, \tag{9.14}$$

i.e., we can generate mass terms, but these only become prominent as we approach the scale of strong coupling in the 3D theory.

In addition to effects from integrating out KK states, we also expect contributions from Euclidean M2-branes and M5-branes. Let us begin with the Euclidean M2-brane contributions. Recall that in the PW system these instanton effects can, for example, lift vectorlike pairs of matter fields, as follows from a direct analysis of the associated supersymmetric quantum mechanics for the 4D theory compactified further on a T^3. Similar considerations apply in local $Spin(7)$ systems, where we "smear" the M2-brane over the matter one-cycles. We comment on this in further detail in Appendix D. The general qualitative point is that we can suppress the size of mass terms for local matter by keeping the corresponding zero mode circles geometrically sequestered. This produces a mass scale $\Lambda_{\mathrm{inst}} \ll \Lambda_{\mathrm{strong}}^{3d}$. Additionally, M2-brane instanton corrections can generate Yukawa-like interactions for our 3D $\mathcal{N}=2$ matter. This is essentially the same mechanism for Yukawa interactions generated in 4D $\mathcal{N}=1$ vacua of local G_2 models.[28]

We can also have M5-brane instanton effects. In the related context of M-theory on a Calabi-Yau fourfold, it is well-known that M5-branes wrapped on holomorphic divisors D with arithmetic genus $\chi(D, \mathcal{O}_D) = 1$ can contribute to the resulting superpotential [86], and in the presence of localized singularities, there is a natural extension of these considerations which can be interpreted as a strong coupling contribution in 3D gauge theories [87]. In our setting where we have a four-manifold of ADE singularities generating a local $Spin(7)$ space, we can clearly identify six-cycles coming from one of the collapsing S^2's in the fiber, over M_4. Note also that there can be further enhancements in the singularity type when we have local matter, and this in turn means that these instantons can generate additional nonperturbatively induced interaction terms for the matter.

An important comment here is that we have also emphasized that at least in some cases we can equivalently study the resulting 3D effective field theory by working with type IIA string theory on the local G_2 space $\Lambda_{\mathrm{SD}}^2 M_4$, with spacetime filling D6-branes wrapping M_4. Here, we expect that M2-brane instantons descend to worldsheet instantons and that M5-brane instanton corrections descend to Euclidean D4-branes. In the D4-brane case, it wraps all of the M_4, as well as a finite interval stretched between the different sheets of the associated spectral cover. This is compatible with the standard way in which M-theory on an ALE space reduces to type IIA when we reduce along the circle direction of the ALE space.[29]

X. EXAMPLES

In this section, we turn to some explicit examples. Our operating theme will be to construct specific four-manifolds and Higgs field profiles which offer some level of protection against various quantum corrections.

A. Topological insulator revisited

As a first example, let us begin by revisiting the case of a topological insulator considered in Sec. IV. Recall that in this case we took a PW system on the three-manifold $M_3 = S^1 \times \Sigma$ with a single 4D $\mathcal{N}=2$ hypermultiplet localized at a point of Σ and spread over S^1. Introducing a position dependent profile for the Higgs field in a transverse

[28]This is another distinction with expectations from the BHV system, where Yukawas can be generated by classical geometry with textures generated by further intersections and/or fluxes and instantons (see, e.g., Refs. [21,44,45,81–85]).

[29]See, for example, Refs. [88,89] for further discussion on such instanton corrections in the context of intersecting D6-brane models in type IIA compactified on a Calabi-Yau threefold.

direction $\mathbb{R}_\perp$, we have a single 3D $\mathcal{N}=2$ matter field trapped at the point $x_\perp = 0$ of $\mathbb{R}_\perp$. The reflection assignment for the associated 3D Dirac fermion is then dictated by the relative change in sign for this Higgs field, and it of course transforms in some representation $\mathbf{R}$ of the unbroken gauge group H.

In the present setup where our M_4 is noncompact, the unbroken gauge group H is actually just a flavor symmetry of the 3D field theory. We can calculate the anomalies associated with reflections, as well as mixed reflection-gauge transformation anomalies. Indeed, this 3D mode will generate a shift in the background Chern-Simons level for H given by (with one choice of reflection assignment),

$$\delta k = \mathbf{T}(\mathbf{R}), \tag{10.1}$$

so, for example, in the case where the Lie algebra of H is $\mathfrak{su}(N)$, we get matter in fundamental representation from the unfolding $\mathfrak{su}(N+1) \to \mathfrak{su}(N) \times \mathfrak{u}(1)$, and the two-index antisymmetric from $\mathfrak{so}(2N) \to \mathfrak{su}(N) \times \mathfrak{u}(1)$ (see, e.g., Ref. [90]). The contribution to δk is a half-integer shift for the fundamental representation $\mathbf{N}$s, and for a two-index antisymmetric representation $\mathbf{\Lambda^2 N}$, we get $(N-2)/2$, which is also a half-integer when N is odd. The corresponding gravitational-parity anomalies are just $\nu_{\mathsf{R}}(\mathbf{N}) = 2N$ and $\nu_{\mathsf{R}}(\mathbf{\Lambda^2 N}) = N(N-1)$.

Another interesting case to consider is matter transforming in a representation of $\mathfrak{so}(10)$. We get matter in the spinor representation 16 from unfolding $\mathfrak{e}_6 \to \mathfrak{so}(10) \times \mathfrak{u}(1)$. We get matter in the vector representation 10 from unfolding $\mathfrak{so}(12) \to \mathfrak{so}(10) \times \mathfrak{u}(1)$. A general comment here is that, even though the 10 is a real representation of $\mathfrak{so}(10)$, we are still dealing with a 3D Dirac fermion; i.e., it contributes as ten complex fermions.[30] For the spinor, we note that $\mathbf{T}(\mathbf{16}) = 2$; i.e., there is no mixed parity-Spin(10) anomaly. Note also that $\nu_{\mathsf{R}}(\mathbf{16}) = 0$ mod 16. For the vector representation, we have $\mathbf{T}(\mathbf{10}) = 1$, so again we do not get a mixed gauge-parity anomaly. In contrast to the spinor representation, now we have $\nu_{\mathsf{R}}(\mathbf{10}) = 20 = 4$ mod 16.

In the case where we just have a localized mode on a noncompact M_4, we also see that the KK modes of the 7D SYM theory have actually decoupled from the 3D boundary mode. This just follows from the fact that the corresponding Yukawa interactions discussed around Eq. (9.12) are set by g_{3d}^2, but this is zero in the case where H is a flavor symmetry.

It is also natural to ask about what happens if we compactify $\mathbb{R}_\perp$ to a circle, which we write as $S^1_\perp$. In this case, we expect the Higgs field to generically have additional zeros. One way to argue for this is to observe that on the three-manifold specified by $\tilde{M}_3 = \Sigma \times S^1_\perp$ having a single zero mode would have generated a 4D $\mathcal{N}=1$ field theory with a gauge anomaly. The simplest resolution is the assumption that there is an additional zero mode somewhere else on $S^1_\perp \times \Sigma$, which in 4D terms would give rise to a "vectorlike pair." More precisely, a Gauss law type constraint on the PW system tells us that in this case we should expect a single $(+,+,-)$ mode and a single $(-,-,+)$ mode in the same representation $\mathbf{R}$ (see, e.g., Ref. [18]). From a 3D perspective, we get two 3D Dirac fermions, one with reflection assignment $+1$ and the other with reflection assignment -1. We can again switch off all mass term contributions to this pair of localized modes by decompactifying Σ. Of course, in this case, we really just have two decoupled 3D fields, but we can reintroduce interactions by taking Σ large, but of finite volume.

B. Vectorlike models

Let us now turn to some constructions with vectorlike matter spectra, i.e., constructions in which the local matter is reflection symmetric. Our plan will be to take a 3D $\mathcal{N}=2$ QFT and its "reflection conjugate" R(QFT) and then geometrically glue them together to build a new 3D theory, i.e., QFT#R(QFT), through the operation discussed in Sec. V A.

To frame the discussion to follow, suppose we have successfully engineered a 4D $\mathcal{N}=1$ *chiral* model using the PW system on a three-manifold M_3. In this case, we have various matter fields localized at points of M_3 transforming in complex representations $\mathbf{R}_i$ of the unbroken gauge group H. Denote by ε_i the parity assignment for each such local matter field, i.e., $\varepsilon = -1$ for $(+,+,-)$ matter and $\varepsilon = +1$ for $(-,-,+)$ matter. If we compactify this chiral theory on a circle, we just get a 3D $\mathcal{N}=2$ theory. One way for us to build more elaborate examples is to use our gluing construction from Sec. V A. This will produce 3D $\mathcal{N}=2$ matter coupled to the reduction of 3D $\mathcal{N}=1$ "bulk matter" as well as the 3D $\mathcal{N}=1$ vector multiplet.

Now, since we are dealing with 4D chiral theories, there is no *a priori* guarantee that a reflection symmetry will survive in the resulting 3D effective field theory. One way to ensure this, however, is to take an individual building block $M_4 \simeq M_3 \times S^1$ and its associated Higgs bundle as well as its orientation reversed counterpart $\overline{M_4}$. We can accomplish this by performing an orientation reversal on just the M_3 factor, while leaving the S^1 factor untouched. When we do this, all of the $(+,+,-)$ matter will now become $(-,-,+)$ matter; i.e., we get exactly the opposite reflection assignments for our 3D $\mathcal{N}=2$ matter. Gluing together M_4 and $\overline{M_4}$, we then get a 3D $\mathcal{N}=1$ theory where all the localized matter is reflection symmetric; i.e., there is no net contribution to the various parity anomalies. Consider next the bulk matter of the glued theory. Recall that in general we have

[30] From the perspective of the unfolding breaking pattern, observe that these modes are still charged under a $\mathfrak{u}(1)$, so in that sense, they are automatically still complex representations.

$$\chi(M_4\#\overline{M_4}) = 2\chi(M_4) - 2 \quad \text{and} \quad \sigma(M_4\#\overline{M_4}) = 0, \tag{10.2}$$

so, for example, the contribution to the gravitational-parity anomaly written in Eq. (8.8) M_4; namely, we have

$$\nu_{\mathsf{R}}^{\text{bulk}}(M_4\#\overline{M_4}) = \dim_{\mathbb{R}} H \times (\chi(M_4) - 1). \tag{10.3}$$

Using the fact that our building block was just $M_4 = M_3 \times S^1$, we also have $\chi(M_4) = 0$, so we can also write

$$\nu_{\mathsf{R}}^{\text{bulk}}(M_4\#\overline{M_4}) = -\dim_{\mathbb{R}} H. \tag{10.4}$$

More generally, if we had ℓ such chiral building blocks, we can build more general backgrounds. Depending on the details of this, we can engineer additional discrete symmetries just from permuting these different chiral factors. The further gluing to a orientation reversed $\overline{M_4}$ will then produce another reflection symmetric spectrum, but with additional discrete symmetries. Note that, independent of the details of these gluing operations, the contribution to the gravitational-parity anomaly obeys Eq. (10.4). Indeed, if we can decompose $M_4 = (Q_4)^{\#\ell}$, for some four-manifold Q_4, then we can also write[31]

$$\nu_{\mathsf{R}}^{\text{bulk}}(M_4\#\overline{M_4}) = \dim_{\mathbb{R}} H \times (\ell\chi(Q_4) - 2\ell + 1), \tag{10.5}$$

and again, if we further specialize to the case $Q_4 = Q_3 \times S^1$ for some three-manifold Q_3, we get

$$\nu_{\mathsf{R}}^{\text{bulk}}(M_4\#\overline{M_4}) = \dim_{\mathbb{R}} H \times (-2\ell + 1). \tag{10.6}$$

C. GUT-like models

In the previous section, we discussed a general method for building examples with vectorlike matter. In particular, we could start with a PW background which would realize a 4D $\mathcal{N}=1$ theory with matter spectrum a GUT, and then we could add to it the corresponding mirror reflection. On the other hand, it is natural to consider more general possibilities where we may not have a purely geometric reflection symmetry but which may nevertheless have vanishing parity anomalies.

To build an example of this sort, suppose we have already engineered a PW system on a three-manifold M_3 which has a bulk $SO(10)$ gauge group and local matter in the 16 and 10 (see Appendix F for some additional details). For ease of exposition, we assume all the local matter is of $(+,+,-)$ type, and so we also require singularities in the Higgs field to satisfy the corresponding Gauss law constraint for the spectral cover construction. Now, we can simply take this model and compactify on a circle to arrive at a 3D $\mathcal{N}=2$ theory, as obtained from compactifying on $M_3 \times S^1$. To get a 3D $\mathcal{N}=1$ theory, we can glue our $M_3 \times S^1$ to another four-manifold N_4 equipped with a trivial Higgs bundle (i.e., no local matter present), e.g., $M_4 = (M_3 \times S^1)\#N_4$. This results in bulk matter fields in 3D $\mathcal{N}=1$ supermultiplets. A general comment here is that we can engineer a model with 10s and 16s starting from either the parent gauge group E_8 or from E_7. In the latter case, we have a $\mathbb{Z}_2$ center, and this descends to nontrivial higher-form symmetries in the 3D effective field theory. If we build a model with just 16s, we could in principle start with an E_6 gauge theory, which has a $\mathbb{Z}_3$ center.

Recall from Sec. X A that a single 16 of $\mathfrak{so}(10)$ produces no anomaly, while a single 10 contributes $\nu_{\mathsf{R}SO(10)} = 0$ mod 2. From the bulk modes in the adjoint of $\mathfrak{so}(10)$, the contributions to the gauge-parity anomaly and gravitational-parity anomalies are

$$\nu_{\mathsf{R}SO(10)}^{\text{bulk}} = h^\vee(SO(10)) \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)) \equiv 0 \mod 2 \tag{10.7}$$

$$\begin{aligned}\nu_{\mathsf{R}}^{\text{bulk}} &= \dim_{\mathbb{R}}(SO(10)) \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)) \\ &\equiv -\frac{3}{2}(\chi(M_4) + \sigma(M_4)) \mod 16. \end{aligned} \tag{10.8}$$

All told, then, the contribution to ν_{R} just depends on the number of 10's and the adjoint-valued bulk modes:

$$\nu_{\mathsf{R}} \equiv 4N_{\mathbf{10}} - \frac{3}{2}(\chi(M_4) + \sigma(M_4)) \mod 16. \tag{10.9}$$

As we already mentioned previously, the anomalies provide robust information which survives into the deep infrared. In particular, even if we classically localize a matter field, it might happen that strong quantum corrections still gap the system, leaving only some topological order behind. As an interesting special case, suppose we have engineered a model with local matter in the 16 of $SO(10)$, and where the only bulk matter comes from the 3D $\mathcal{N}=1$ vector multiplet. There is no gauge-parity anomaly in this system, and the contribution to the gravitational-parity anomaly can be set to zero by taking $M_4 = (S^3 \times S^1)^{\#\ell}$ with $\ell \equiv 1$ mod 16. In this case, the 3D theory in the deep infrared has trivial topological order. At a general level, we expect this to occur due to strong coupling effects in the 3D effective field theory, as we already discussed in Sec. IX B.

We can also engineer other examples of GUT-like modes by again simply borrowing construction techniques from the PW system on M_3. Consider, for example, an E_6 gauge theory with $N_{\mathbf{27}}$ $(+,+,-)$ matter fields in the 27. We can get this starting from unfolding E_7, which has a $\mathbb{Z}_2$ center. In this case, the $\mathbb{Z}_2$ descends to the 3D effective field theory as various higher-form

[31] This is just because $\chi((Q_4)^{\#\ell}) = \ell\chi(Q_4) - 2(\ell - 1)$.

symmetries. A single 27 contributes $\nu_{\mathsf{R}_{E_6}}(\mathbf{27}) = 0$ mod 2; i.e., there is no gauge-parity anomaly from the local matter. The contribution to the gravitational-parity anomaly is $\nu_{\mathsf{R}}(\mathbf{27}) = 6$ mod 16. From the bulk modes in the adjoint of $\mathfrak{e}_6$, the contributions to the gauge-parity anomaly and gravitational-parity anomalies are

$$\nu^{\mathrm{bulk}}_{\mathsf{R}_{E_6}} = h^{\vee}(E_6) \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)) \equiv 0 \mod 2 \quad (10.10)$$

$$\nu^{\mathrm{bulk}}_{\mathsf{R}} = \dim_{\mathbb{R}}(E_6) \times \frac{1}{2}(\chi(M_4) + \sigma(M_4)) \equiv 7\chi(M_4) \mod 16, \quad (10.11)$$

where in the second equation we dropped the contribution from $\sigma(M_4)$, since it is divisible by 16. All told, then, the contribution to ν_{R} just depends on the number of 27's and the adjoint-valued bulk modes:

$$\nu_{\mathsf{R}} \equiv 6N_{\mathbf{27}} + 7\chi(M_4) \mod 16. \quad (10.12)$$

XI. CONCLUSIONS

In this paper, we have studied topologically robust quantities associated with M-theory compactified on a local $Spin(7)$ space. In particular, we have analyzed the local matter present in configurations of intersecting 6-branes as generated by the unfolding of singularities associated with M_4 a four-manifold of ADE singularities. Our primary tool in carrying out this analysis has been the study of 7D SYM theory coupled to defects and the implications for the resulting 3D effective field theory. At the classical level, we showed that matter in such systems is generically localized along codimension-3 subspaces of the four-manifold. We also explained how to calculate various observables which are robust against quantum corrections. In particular, we have shown how reflections and automorphisms of 10D SYM theory on a T^3 and 11D M-theory on an ADE singularity descend to reflections in 7D SYM and, moreover, how these symmetries further descend to the 3D effective field theory. We used this to extract the corresponding contributions to anomalies which are robust against quantum corrections. We also presented various examples illustrating these general themes. We also took some preliminary steps in determining the spectrum of extended objects in these theories and their transformations under higher-form symmetries. In the remainder of this section, we discuss some avenues for future investigation.

Our primary emphasis has been on extracting some robust calculable features of local $Spin(7)$ systems. It is natural to ask whether we can say more about the resulting 3D effective field theories. For example, depending on the specific matter content, Chern-Simons level, and presence (or absence) of a higher-form symmetry, we might expect to generate a rich class of possible 3D field theories. It would clearly be instructive to analyze these possibilities and also use this as a starting point for a geometrization of proposed field theoretic dualities. Along these lines, we anticipate that flop transitions in the associated local $Spin(7)$ space will provide important topological insights on these issues.

Our emphasis in this paper has been on the study of 3D effective field theories generated by working with M-theory on a noncompact $Spin(7)$ space. It is also natural to consider the resulting theories obtained from taking perturbative superstring theory on the same backgrounds. For type IIB and IIA, this would give rise to 2D $\mathcal{N} = (0,2)$ and $\mathcal{N} = (1,1)$ theories, respectively, while for heterotic strings, this would generate 2D $\mathcal{N} = (0,1)$ theories. The case of type IIB backgrounds is particularly interesting because here we recover a notion of holomorphy.

The main tool of analysis we have used in the study of local $Spin(7)$ systems is 7D SYM theory and, in particular, the associated partial topological twist of $\mathcal{N} = 4$ SYM theory. We also used some general gluing construction techniques to build various examples of 3D systems. It is natural to ask how these gluing operations equipped with Higgs bundles lift to a local $Spin(7)$ geometry and, more globally, to possibly compact $Spin(7)$ spaces. As noted in Ref. [2], these sorts of operations can often be interpreted as a generalization of the standard connected sums construction of Refs. [91,92] [see also Ref. [1] for a proposed extension to $Spin(7)$ spaces]. Related to this, it would be interesting to better understand the possible constraints which come from coupling our 3D theories to gravity.

A more long-term goal in this direction would be to use the results obtained here to start building more explicit F-theory backgrounds on a $Spin(7)$ space. Indeed, we anticipate that the analysis of anomalies and topological structures found here will provide additional insight into the "$\mathcal{N} = 1/2$" backgrounds considered in Refs. [5,6] (see also Refs. [93,94]).

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APPENDIX A: MORE DETAILS ON CLASSICAL ZERO MODE LOCALIZATION

In this Appendix, we provide further details on the zero mode localization analysis of Sec. III, with particular emphasis on the case where a flux turned is turned on. First off all, the matrix M used in the localization argument is

$$M=\begin{pmatrix} \frac{(2\lambda+\mu)\sqrt{\lambda^2+\frac{4N^2(\lambda+\mu)^2}{(3\lambda+2\mu)^2}+2\lambda\mu+\mu^2}}{\lambda+\mu} & -\frac{i\lambda N}{3\lambda+2\mu} & 0 & 0 \\ -\frac{i\lambda N}{3\lambda+2\mu} & \sqrt{\lambda^2+\frac{4N^2(\lambda+\mu)^2}{(3\lambda+2\mu)^2}+2\lambda\mu+\mu^2} & 0 & 0 \\ 0 & 0 & \sqrt{\lambda^2+4N^2} & iN \\ 0 & 0 & iN & 0 \end{pmatrix}. \tag{A1}$$

Now, to expedite the analysis of the equations, we can write them collectively as $\mathbf{D}\Psi=0$, where Ψ is the column vector

$$\Psi=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \tag{A2}$$

and $\mathbf{D}$ is the matrix operator

$$\mathbf{D}=\begin{bmatrix} 0 & -\Phi_3 & \Phi_2 & -\Phi_1 & D_4 & -D_3 & D_2 \\ \Phi_3 & 0 & -\Phi_1 & -\Phi_2 & D_3 & D_4 & -D_1 \\ -\Phi_2 & \Phi_1 & 0 & -\Phi_3 & -D_2 & D_1 & D_4 \\ \Phi_1 & \Phi_2 & \Phi_3 & 0 & -D_1 & -D_2 & -D_3 \\ -D_4 & -D_3 & D_2 & D_1 & 0 & \Phi_3 & -\Phi_2 \\ D_3 & -D_4 & -D_1 & D_2 & -\Phi_3 & 0 & \Phi_1 \\ -D_2 & D_1 & -D_4 & D_3 & \Phi_2 & -\Phi_1 & 0 \end{bmatrix}. \tag{A3}$$

Here, when writing Φ_i, we mean $[\Phi_i,\cdot]$. To get a better understanding of the equations, we can take the square of $\mathbf{D}$ and focus on its antisymmetric part

$$
\begin{aligned}
&\mathbf{D}^2|_{\mathrm{asym}} = i\mathbf{F} \\
&= \left(\begin{array}{cccc}
0 & -F_{12}+2F_{34}-2F_{56} & -F_{13}-2\left(F_{24}+F_{57}\right) & -F_{14}+2F_{23}-2F_{67} \\
F_{12}-2F_{34}+2F_{56} & 0 & 2F_{14}-F_{23}-2F_{67} & -2F_{13}-F_{24}+2F_{57} \\
F_{13}+2\left(F_{24}+F_{57}\right) & -2F_{14}+F_{23}+2F_{67} & 0 & 2F_{12}-F_{34}-2F_{56} \\
F_{14}-2F_{23}+2F_{67} & 2F_{13}+F_{24}-2F_{57} & -2F_{12}+F_{34}+2F_{56} & 0 \\
F_{15}-2\left(F_{26}+F_{37}\right) & 2F_{16}+F_{25}+2F_{47} & 2F_{17}+F_{35}-2F_{46} & -2F_{27}+2F_{36}+F_{45} \\
F_{16}+2F_{25}-2F_{47} & -2F_{15}+F_{26}-2F_{37} & 2F_{27}+F_{36}+2F_{45} & 2F_{17}-2F_{35}+F_{46} \\
F_{17}+2\left(F_{35}+F_{46}\right) & F_{27}+2F_{36}-2F_{45} & -2F_{15}-2F_{26}+F_{37} & -2F_{16}+2F_{25}+F_{47}
\end{array}\right. \\
&\qquad \left.\begin{array}{ccc}
2\left(F_{26}+F_{37}\right)-F_{15} & -F_{16}-2F_{25}+2F_{47} & -F_{17}-2\left(F_{35}+F_{46}\right) \\
-2F_{16}-F_{25}-2F_{47} & 2F_{15}-F_{26}+2F_{37} & -F_{27}-2F_{36}+2F_{45} \\
-2F_{17}-F_{35}+2F_{46} & -2F_{27}-F_{36}-2F_{45} & 2F_{15}+2F_{26}-F_{37} \\
2F_{27}-2F_{36}-F_{45} & -2F_{17}+2F_{35}-F_{46} & 2F_{16}-2F_{25}-F_{47} \\
0 & -2F_{12}-2F_{34}-F_{56} & -2F_{13}+2F_{24}-F_{57} \\
2F_{12}+2F_{34}+F_{56} & 0 & -2F_{14}-2F_{23}-F_{67} \\
2F_{13}-2F_{24}+F_{57} & 2F_{14}+2F_{23}+F_{67} & 0
\end{array}\right).
\end{aligned}
\tag{A4}
$$

Here, we wrote everything in terms of the curvature of the gauge bundle. The notation F_{ij} is clear for $1 \leq i, j \leq 4$. For $i \leq 4$ and $5 \leq j \leq 7$, we have that $F_{ij} = D_i \phi_{j-3}$. For $5 \leq i, j \leq 7$, we have that $F_{ij} = [\phi_{i-3}, \phi_{j-3}]$. The matrix can be further simplified using the equations of motion that relate the various components of the curvature; however, we will not do it for the moment. The reason we introduced this flux matrix is that if fluxes are constant it is possible to build solutions efficiently by just considering the eigenvectors of $\mathbf{F}$.

As an example, we can take the first $Spin(7)$ background written above. One can show that one eigenvector has the form

$$
\begin{bmatrix}
\frac{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}+3\lambda^2+2\lambda\mu+4N^2}{4N} \\
-i\,\frac{\sqrt{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}+5\lambda^2+6\lambda\mu+2\mu^2+4N^2}}{\sqrt{2}} \\
0 \\
0 \\
-i\,\frac{2\sqrt{2}N(2\lambda+\mu)\sqrt{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}+5\lambda^2+6\lambda\mu+2\mu^2+4N^2}}{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}-3\lambda^2-2\lambda\mu+4N^2} \\
\lambda+\mu \\
0
\end{bmatrix}.
\tag{A5}
$$

To find a solution, we write

$$\Psi = \begin{bmatrix} \frac{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}+3\lambda^2+2\lambda\mu+4N^2}{4N} \\ -i\frac{\sqrt{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}+5\lambda^2+6\lambda\mu+2\mu^2+4N^2}}{\sqrt{2}} \\ 0 \\ 0 \\ -i\frac{2\sqrt{2}N(2\lambda+\mu)\sqrt{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}+5\lambda^2+6\lambda\mu+2\mu^2+4N^2}}{\sqrt{(\lambda^2+4N^2)((3\lambda+2\mu)^2+4N^2)}-3\lambda^2-2\lambda\mu+4N^2} \\ \lambda+\mu \\ 0 \end{bmatrix} e^{-\frac{1}{2}\mathbf{x}.M.\mathbf{x}}. \tag{A6}$$

Plugging this back in the equations of motion, the problem becomes a linear system in the matrix M, and the solution can be found rather quickly (we will not copy it here for the sake of brevity).

APPENDIX B: QUOTIENT CONSTRUCTION

In this Appendix, we present a quotient construction for generating local $Spin(7)$ systems. Our starting point is BHV solutions with a suitable $\mathbb{Z}_2$ action which can actually be lifted to a compact Calabi-Yau fourfold. This provides a way to generate examples of compact spaces. Since it is somewhat orthogonal to the main developments of the text, we have placed it here in an Appendix.

The main idea here is similar in spirit to Joyce's construction method [76], which involves starting with a Calabi-Yau fourfold with an antiholomorphic $\mathbb{Z}_2$ action. After quotienting by this action, one obtains an eight-manifold with possible fixed points, and it can be shown that in favorable circumstances this results in a $Spin(7)$ space. Our emphasis will be somewhat different since we focus on the local Higgs bundle construction, but which can in principle lift to a compact geometry.

At the level of eight-manifolds, the idea will be to start from a Calabi-Yau fourfold and implement a $\mathbb{Z}_2$ action which leaves the Kähler form J invariant but which sends the holomorphic 4-form to its complex conjugate, possibly twisted by a phase:

$$\Omega \mapsto e^{-i\theta}\bar{\Omega}. \tag{B1}$$

We can then consider a Cayley 4-form for a candidate $Spin(7)$ geometry, as given by

$$\Lambda = \mathrm{Re}(e^{i\theta}\Omega) + \frac{1}{2}J \wedge J. \tag{B2}$$

Since we are interested in building local models, we begin by studying possible quotients of four-manifolds and then ask how these can be lifted to suitable eight-manifolds. Our plan will be to construct examples using two basic geometric operations. The first operation is an antipodal map on a $\mathbb{CP}^1$. In homogeneous coordinates $[u_i, v_i]$, it acts as

$$\tau_i\colon\ [u_i, v_i] \mapsto [-\bar{v}_i, \bar{u}_i], \tag{B3}$$

or in terms of a local affine coordinate $z_i = u_i/v_i$, we have $z_i \mapsto -1/\bar{z}_i$.

The second operation is a permutation of coordinates on a product of $\mathbb{CP}^1$'s. Given local coordinates $(z_1, \ldots, z_n)$, this acts as

$$\sigma\colon\ (z_1, \ldots, z_n) \mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(n)}), \tag{B4}$$

where σ is some permutation on n letters.

Let us now turn to some local model considerations. Our starting point is the standard one in local F-theory constructions; namely, we begin with a Kähler surface S wrapped by a stack of 7-branes. In the local model, the moduli space of the spectral cover makes reference to the noncompact Calabi-Yau threefold $K_S \to S$, where K_S is the canonical bundle.

A particularly important special case is given by $S = \mathbb{CP}^1 \times \mathbb{CP}^1$. We will be interested in generating new local models by taking various quotients, using the procedure of Massey (see, e.g., Refs. [95,96]). In this case, we have two antipodal maps, which we denote as τ_i, and $\langle\sigma\rangle$ is isomorphic to $\mathbb{Z}_2$. We also introduce the combination $\tau \equiv \tau_1\tau_2$. Observe that the τ_i and σ's generate the dihedral group D_8, the symmetries of a square. Letting $J = \langle\sigma\rangle$, $K = \langle\tau_1, \tau_2\rangle$, $H = \langle\sigma, \tau\rangle$, and $G = \langle\sigma, \tau_1, \tau_2\rangle$, we have

$$\mathbb{CP}^1 \times \mathbb{CP}^1/J = \mathbb{CP}^2 \tag{B5}$$

$$\mathbb{CP}^1 \times \mathbb{CP}^1/K = \mathbb{RP}^2 \times \mathbb{RP}^2 \tag{B6}$$

$$\mathbb{CP}^1 \times \mathbb{CP}^1/H = S^4 \tag{B7}$$

$$\mathbb{CP}^1 \times \mathbb{CP}^1/G = \mathbb{RP}^4. \tag{B8}$$

We remark that some of these group actions have fixed point loci, but this simply serves to define a branched covering [97].

These quotients are topological in nature since quantities such as the curvature invariants of the underlying four-manifold change. As such, one must exercise some caution in using this to construct backgrounds which solve the corresponding supergravity equations of motion. On the other hand, generating a candidate Cayley four-form provides evidence that the quotient does make sense.

To proceed further, we write down the local presentation of the holomorphic 3-form on $X = K_S \to S$. Introduce local coordinates $[u_i, v_i]$ for each $\mathbb{CP}^1$ factor, and let n denote the normal coordinate direction, as given by a section of K_S^{-1}. Then, the holomorphic 3-form of the Calabi-Yau is

$$\Omega_X = dz_1 \wedge dz_2 \wedge dn. \tag{B9}$$

The Kähler form is given in these local coordinates by

$$J_X = i\frac{dz_1 \wedge d\overline{z_1}}{(1+z_1\overline{z_1})^2} + i\frac{dz_2 \wedge d\overline{z_2}}{(1+z_2\overline{z_2})^2} + i\frac{dn \wedge d\bar{n}}{(1+z_1\overline{z_1})^{-2}(1+z_2\overline{z_2})^{-2}} + \cdots, \tag{B10}$$

where we have only displayed the "diagonal elements."

Let us consider the action under the various symmetry group actions, where we do not change the normal coordinate. We denote the corresponding symmetry generators as σ_S and $\tau_S^{(i)}$, in the obvious notation. The action of σ_S is

$$\sigma_S\colon\ dz_1 \wedge dz_2 \wedge dn \mapsto dz_2 \wedge dz_1 \wedge dn = -dz_1 \wedge dz_2 \wedge dn, \tag{B11}$$

which we summarize as

$$\sigma_S\colon\ \Omega_X \mapsto -\Omega_X. \tag{B12}$$

By similar considerations, we have

$$\sigma_S\colon\ J_X \mapsto J_X. \tag{B13}$$

Consider next the action of $\tau_S^{(i)}$. In this case, we must remember that, since n is a section of $K_S^{-1} = O(2h_1 + 2h_2)$ with h_1 and h_2 the hyperplane classes of $\mathbb{CP}^1 \times \mathbb{CP}^1$, dn will also transform under a coordinate change such as $z_1 \mapsto 1/z_1$ as $dn \mapsto (z_1)^2 dn$. We then determine

$$\tau_S^{(1)}\colon\ dz_1 \wedge dz_2 \wedge dn \mapsto -\left(\frac{z_1}{\overline{z_1}}\right)^2 d\overline{z_1} \wedge dz_2 \wedge dn, \tag{B14}$$

and similar considerations apply for $\tau_S^{(2)}$. Clearly, if we want to extend our group actions to the Calabi-Yau, we must consider a more general transformation law for the normal coordinate. Another issue is that our coordinate transformation picks out only one of the coordinates, leaving us with a mixed holomorphic/antiholomorphic structure.

To solve both problems, we propose to primarily focus on the subgroup H generated by σ and $\tau \equiv \tau_1\tau_2$. In this case, the corresponding action on the holomorphic 3-form is

$$\tau_S\colon\ dz_1 \wedge dz_2 \wedge dn \mapsto \left(\frac{z_1}{\overline{z_1}}\right)^2 \left(\frac{z_2}{\overline{z_2}}\right)^2 d\overline{z_1} \wedge d\overline{z_2} \wedge dn. \tag{B15}$$

To get a sensible extension of the τ action, we propose to extend by demanding

$$\tau_X\colon\ n \mapsto (\overline{z_1})^2(\overline{z_2})^2\bar{n}, \quad \text{and} \quad dn \mapsto (\overline{z_1})^2(\overline{z_2})^2 d\bar{n}; \tag{B16}$$

that is, we consider complex conjugation on the normal coordinate as well. In the case of the extension of σ, we also introduce a sign flip:

$$\sigma_X\colon\ n \mapsto -n. \tag{B17}$$

This produces a sensible action on the holomorphic 3-form for all generators of the subgroup H. Indeed, we have

$$\sigma_X\colon\ \Omega_X \mapsto \Omega_X \tag{B18}$$

$$\tau_X\colon\ \Omega_X \mapsto \overline{\Omega_X}. \tag{B19}$$

Consider next the group action on the Kähler form. For σ_X, we clearly have $\sigma_X(J_X) = J_X$. Consider next the action of τ_X. Plugging in our definitions, we have

$$\tau_X(J_X) = i\frac{d\overline{z_1} \wedge dz_1}{(1+z_1\overline{z_1})^2} + i\frac{d\overline{z_2} \wedge dz_2}{(1+z_2\overline{z_2})^2} + i\frac{d\bar{n} \wedge dn}{(1+z_1\overline{z_1})^{-2}(1+z_2\overline{z_2})^{-2}} + \cdots = -J_X, \tag{B20}$$

so we see that the Kähler form J_X flips sign. Summarizing the action on the Kähler form of the local model, we have

$$\sigma_X\colon\ J_X \mapsto J_X \tag{B21}$$

$$\tau_X\colon\ J_X \mapsto -J_X. \tag{B22}$$

Based on this, we conclude that a quotient by the group G will project out the Kähler form. Not all is lost, however, because we can append to X an additional factor of $\mathbb{R}$. Denote the corresponding space as $Y = X \times \mathbb{R}$, where we let n' denote this direction in $\mathbb{R}$. There is a natural extension of the aforementioned group actions, given by

$$\sigma_Y\colon\ n' \to n' \tag{B23}$$

$$\tau_X\colon\ n' \to -n'. \tag{B24}$$

We now observe that the combination

$$\Phi_{(3)} = \mathrm{Re}(\Omega_X) + J_X \wedge dn' \tag{B25}$$

is invariant under both σ_Y and σ_X. Here, the order-1 coefficients are fixed by the demand that $\Phi_{(3)}$ is a calibration 3-form.

As such, we can take the quotient and arrive at a local model with reduced supersmmetry. The holonomy group is not quite G_2, but for physical applications, it is "close enough."[32] The upshot of this analysis is that we can start from a local BHV construction and then pass to a local $Spin(7)$ model.

Note that after performing such a quotient our $\mathbb{CP}^1 \times \mathbb{CP}^1$ has become an S^4. In cases where we have additional matter fields, we can expect that some matter curves of the original $\mathbb{CP}^1 \times \mathbb{CP}^1$ model will also be identified.

Let us now ask whether these local considerations extend to actual compactification geometries. Along these lines, we consider lifting X to an elliptically fibered Calabi-Yau fourfold. To be concrete, we take our elliptic Calabi-Yau fourfold $Z \to B$ to have base $B = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, with the Weierstrass model

$$y^2 = x^3 + f_{8,8,8}x + g_{12,12,12}, \tag{B26}$$

in the obvious notation. Since we are aiming for a quotient which produces a $Spin(7)$ space, we seek out a quotient which preserves a candidate Cayley 4-form constructed from the holomorphic 4-form and Kähler form of the Calabi-Yau fourfold.

Based on the symmetries of the system, our plan will be to consider a single combined antipodal action given by $\tau = \tau_1\tau_2\tau_3$. To maintain contact with our previous considerations, we opt to consider pairwise permutations of the $\mathbb{CP}^1$ factors, as denoted by σ_{ij}, which ends up generating all of the symmetric group on three letters. Observe that quotienting $(\mathbb{CP}^1)^3$ by the symmetric group on three letters results in $\mathbb{CP}^3$. The further action by τ (which commutes with these pairwise permutations) is then a six-manifold $\mathbb{CP}^3/\langle\tau\rangle$, with $\mathbb{CP}^3$ specifying a two-sheeted branched cover over it.

We would now like to understand whether we can extend this quotient to a Calabi-Yau fourfold. At least locally, there is no issue. To see why, let (by abuse of notation) n denote the normal coordinate for B inside Z. Then, the proposed action on all of the holomorphic coordinates is as follows:

$$\tau\colon\ (z_1, z_2, z_3, n) \mapsto (-1/\overline{z_1}, -1/\overline{z_2}, -1/\overline{z_3}, -\overline{z_1}^2\overline{z_1}^2\overline{z_1}^2\bar{n}) \tag{B27}$$

$$\sigma_{12}\colon\ (z_1, z_2, z_3, n) \mapsto (z_2, z_1, z_3, -n) \tag{B28}$$

$$\sigma_{23}\colon\ (z_1, z_2, z_3, n) \mapsto (z_1, z_3, z_2, -n) \tag{B29}$$

$$\sigma_{13}\colon\ (z_1, z_2, z_3, n) \mapsto (z_3, z_2, z_1, -n). \tag{B30}$$

To extend this in a way compatible with the local geometry, we then require that dx/y, the meromorphic 1-form of the Weierstrass model, transforms as

$$\tau\colon\ dx/y \mapsto -\overline{z_1}^2\overline{z_2}^2\overline{z_3}^2 d\bar{x}/d\bar{y} \tag{B31}$$

$$\sigma_{ij}\colon\ dx/y \mapsto -dx/y. \tag{B32}$$

For now, we assume that an appropriate f and g in the Weierstrass model have been chosen so that such a symmetry is available. We now ask about the action on the holomorphic 4-form and Kähler form. The holomorphic 4-form is given in local coordinates as

$$\Omega_Z = dz_1 \wedge dz_2 \wedge dz_3 \wedge \frac{dx}{y}, \tag{B33}$$

which transforms as

$$\tau(\Omega_Z) = \overline{\Omega_Z} \tag{B34}$$

$$\sigma_{ij}(\Omega_Z) = \Omega_Z, \tag{B35}$$

much as we had in the case of the local model.

The transformation on the Kähler form is a bit more challenging to track since we do not know the explicit Kähler metric of the model. Nevertheless, at least in a local patch, we expect to get a fair approximation by considering the noncompact geometry $\mathcal{O}(2h_1 + 2h_2 + 2h_3) \to B$. The Kähler form is then of Fubini-Study type,

$$J_Z = i\frac{dz_1 \wedge d\overline{z_1}}{(1+z_1\overline{z_1})^2} + i\frac{dz_2 \wedge d\overline{z_2}}{(1+z_2\overline{z_2})^2} + i\frac{dz_3 \wedge d\overline{z_3}}{(1+z_3\overline{z_3})^2} + i\frac{dn \wedge d\bar{n}}{(1+z_1\overline{z_1})^{-2}(1+z_2\overline{z_2})^{-2}(1+z_3\overline{z_3})^{-2}} + \cdots, \tag{B36}$$

where again we have suppressed the display of the off-diagonal elements, and we observe that, much as we had in our local Calabi-Yau threefold example, we have

$$\tau(J_Z) = -J_Z \quad \text{and} \quad \sigma_{ij}(J_Z) = +J_Z. \tag{B37}$$

[32] As an additional comment, we remark that we are exploiting the fact that we are dealing with a noncompact space and the fact that we allow for discrete group factors. For example, nothing stops us from starting with X a Calabi-Yau $(2m+1)$-fold, and from this starting point forming a $(4m+3)$-dimensional manifold with a distinguished $2m+1$ form as given by $\mathrm{Re}(\Omega) + J^m \wedge dn'$, in the obvious notation.

We now observe that the following 4-form is invariant under these combined actions,

$$\Lambda_{(4)} = \mathrm{Re}(\Omega_Z) + \frac{1}{2} J_Z \wedge J_Z, \tag{B38}$$

where the order-1 coefficients are fixed by the demand that $\Lambda_{(4)}$ is a calibration 4-form. So, provided we can specify suitable Weierstrass coefficients compatible with our quotient, we can obtain a reasonable construction of 3D $\mathcal{N} = 1$ backgrounds.

One way to ensure that the group action is a symmetry of the geometry is to impose additional conditions on the Weierstrass coefficients. Explicitly, writing out $f_{8,8,8}$ as a polynomial in the homogeneous coordinates, we have

$$f_{8,8,8} = \sum_{0\leq i,j,k\leq 8} f_{ijk}(u_1)^i(v_1)^{8-i}(u_2)^j(v_2)^{8-j}(u_3)^k(v_3)^{8-k}. \tag{B39}$$

Under the proposed transformations, we have

$$\tau(f_{8,8,8}) = \sum_{0\leq i,j,k\leq 8} \overline{f_{ijk}}(-\overline{v_1})^i(\overline{u_1})^{8-i}(-\overline{v_2})^j(\overline{u_2})^{8-j}(-\overline{v_3})^k(\overline{u_3})^{8-k} \tag{B40}$$

$$\sigma_{12}(f_{\mathrm{symm}}) = \sum_{0\leq i,j,k\leq 8} f_{ijk}(u_2)^i(v_2)^{8-i}(u_1)^j(v_1)^{8-j}(u_3)^k(v_3)^{8-k} \tag{B41}$$

$$\sigma_{23}(f_{\mathrm{symm}}) = \sum_{0\leq i,j,k\leq 8} f_{ijk}(u_1)^i(v_1)^{8-i}(u_3)^j(v_3)^{8-j}(u_2)^k(v_2)^{8-k} \tag{B42}$$

$$\sigma_{13}(f_{\mathrm{symm}}) = \sum_{0\leq i,j,k\leq 8} f_{ijk}(u_3)^i(v_3)^{8-i}(u_2)^j(v_2)^{8-j}(u_1)^k(v_1)^{8-k}. \tag{B43}$$

As an example of how we can enforce a symmetry in this system, consider the special case

$$f_{\mathrm{special}} = f_4((u_1^2v_1^6 + v_1^2u_1^6)(u_2^2v_2^6 + v_2^2u_2^6)(u_3^2v_3^6 + v_3^2u_3^6)), \tag{B44}$$

with f_4 real. This leads to a collection of $SO(8)$ 7-branes on various loci. There are also additional collisions of these 7-branes which lead to conformal matter and "conformal Yukawas" [98–100], but these singularities can all be blown up. The corresponding blown-up geometries admit a natural extension of the group action, so we conclude that these Calabi-Yau fourfolds provide a simple class of examples exhibiting the main features. As another comment, we note that enforcing the condition that the various coefficients in the original model are real does not exclude any possible choice of singular fiber; see, e.g., Ref. [101] for further discussion on this point.

Clearly, there are many choices available, even with such symmetry constraints.

APPENDIX C: 10D AND 7D SPINOR CONVENTIONS

In this Appendix, we summarize our conventions for 10D spinors. We work in a mostly $+'s$ Lorentzian signature spacetime. Our conventions follow those of Appendix B of Ref. [102]. Our 10D and 7D gamma matrices are denoted Γ_M^{10d} and Γ_i^{7d}, where $M = 0, \ldots, 9$ and $\mu = 0, \ldots, 6$. These gamma matrices satisfy the usual anticommutation relations:

$$\{\Gamma_M^{10d}, \Gamma_N^{10d}\} = 2\eta_{MN} \qquad \{\Gamma_\mu^{7d}, \Gamma_\nu^{7d}\} = 2\eta_{\mu\nu}. \tag{C1}$$

We choose a 10D basis that decomposes in terms of the 7D one as

$$\Gamma_{M=\mu}^{10d} = \sigma_1 \otimes \Gamma_\mu^{7d} \otimes \mathbf{1}, \tag{C2}$$

$$\Gamma_{M=7,8,9}^{10d} = \sigma_2 \otimes \mathbf{1} \otimes \sigma_{1,2,3}, \tag{C3}$$

where we define the 7D gamma matrices as

$$\begin{aligned} &\Gamma_0 = i\sigma_2 \otimes \sigma_3 \otimes \mathbf{1} \qquad \Gamma_1 = \sigma_1 \otimes \sigma_3 \otimes \mathbf{1} \\ &\Gamma_2 = \sigma_3 \otimes \sigma_3 \otimes \mathbf{1} \end{aligned} \tag{C4}$$

$$\begin{aligned} &\Gamma_3 = \mathbf{1} \otimes \sigma_1 \otimes \sigma_1 \qquad \Gamma_4 = \mathbf{1} \otimes \sigma_1 \otimes \sigma_2 \\ &\Gamma_5 = \mathbf{1} \otimes \sigma_1 \otimes \sigma_3 \qquad \Gamma_6 = \mathbf{1} \otimes \sigma_2 \otimes \mathbf{1}. \end{aligned} \tag{C5}$$

These conventions then determine that the 10D chirality matrix is $\Gamma_{11}^{10d} = \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1}$. A 10D Weyl spinor ζ of positive chirality that can be thus be written under the decomposition $\mathbb{R}^{9,1} = \mathbb{R}^{6,1} \times \mathbb{R}^3$ as

$$\zeta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \varepsilon_{7d} \otimes \eta_{3d}. \tag{C6}$$

To study the Majorana condition on this spinor, and thus make contact with the gaugino of 10 SYM, we need to understand the action of charge conjugation on these spinors. We will also see how this condition will lead to the symplectic Majorana condtion on 7D spinors after dimensional reduction. We start by introducing the 10D charge conjugation matrix, B, which has the property

$$B\Gamma_M^{10d} B^{-1} = (\Gamma_M^{10d})^*. \tag{C7}$$

Checking with the explicit representation given in Ref. [102], one can see that $B = B^\dagger$ and $BB^* = 1$. The Majorana condition on ζ is

$$\zeta^* = B\zeta. \tag{C8}$$

It is also convenient to define the matrix C that satisfies

$$C\Gamma_M^{10d} C^{-1} = -(\Gamma_M^{10d})^T. \tag{C9}$$

Specifically,$C = B\Gamma_0^{10d}$. One can choose the matrix C as

$$C = \sigma_1 \otimes C_7 \otimes \sigma_2, \tag{C10}$$

where C_7 satisfies $C_7 \Gamma_\mu C_7^{-1} = -\Gamma_\mu^T$. This means that

$$B = -C\Gamma_0^{10d} = -\mathbf{1} \otimes C_7\Gamma_0 \otimes \sigma_2 = -\mathbf{1} \otimes B_7 \otimes \sigma_2. \tag{C11}$$

and to see how this acts on 7D spinors, it is convenient to write

$$\varepsilon_{7d} \otimes \eta_{3d} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}. \tag{C12}$$

Then, the 10D Majorana condition reduces to

$$\varepsilon_1^* = iB_7 \varepsilon_2, \tag{C13}$$

$$\varepsilon_2^* = -iB_7 \varepsilon_1, \tag{C14}$$

which is nothing but the symplectic-Majorana condition in seven dimensions. Note that $B_7 B_7^* = -1$.

APPENDIX D: QUANTUM MECHANICS OF EUCLIDEAN M2-BRANES

In this Appendix, we motivate the appearance of the twisted differential operator in the $Spin(7)$ superpotential. This is done by formulating it as a supercharge in the supersymmetric quantum mechanics (SQM) of Euclidean M2-branes in $Spin(7)$ spaces. This is similar in spirit to Refs. [17,18,103] for local G_2 models. We mainly focus on the case where we have the most concrete results, i.e., where $\Phi_{\mathrm{SD}} = 0$. That being said, we believe this analysis is also helpful in understanding the zero modes of the operator,

$$\mathcal{D}_{\Phi_{\mathrm{SD}}} = \begin{pmatrix} \Phi_{\mathrm{SD}}\times & D_{\mathrm{sig.}} \\ D_{\mathrm{sig.}} & -\Phi_{\mathrm{SD}}\wedge \end{pmatrix}, \tag{D1}$$

which we will refer to as the $Spin(7)$ operator. Our aim will be to show that the SQM picture determines the zero mode count when $\Phi_{\mathrm{SD}} = 0$ (i.e., it will count the bulk zero modes).

Before diving into the details, we address the conceptual puzzle: what does it mean for these M2-branes to wrap calibrated three-cycles in a $Spin(7)$ manifold, X_8, when there canonically is no such notion? The point is that their contributions from wrapping volume-minimizing three-cycles can be still be given (at a formal level) by the spectrum of the above operator, even though for $\Phi_{\mathrm{SD}} \neq 0$ on a general M_4 there is not enough worldline supersymmetry to be able to extract much information.

We now study the case of a trivial background, $\Phi_{\mathrm{SD}} = 0$, starting with the standard Witten SQM on a target (M_4,g) with supercharges $Q = d$ and $Q = d^\dagger$ (i.e., no superpotential)[104]; the Hilbert space is $\mathcal{H} = \oplus_{i=0}^4 \Omega^i(M_4, \mathbb{C})$, and the space of supersymmetric ground states is simply $\oplus_{i=0}^4 H^i(M_4, \mathbb{C})$. In a given M_4 coordinate patch, the theory has four bosons, X^i, and four complex fermions, ψ^i, and in particular has a $\mathbb{Z}_2$ symmetry we call τ: $\psi^i \to \bar{\psi}^i, \bar{\psi}^i \to -\psi^i$. In terms of differential forms, τ is the signature operator in the above Hilbert space basis (recall $\bar{\psi}^i \leftrightarrow dx^i\wedge$), which acts as $\tau(\omega) = -i^{p(p-1)} * \omega$ on a p-form ω, and note also that $\tau^2 = (-1)^F$. Since $[\tau, \hat{H}] = 0$, we can project this SQM onto the $+1$-eigenspace of τ, leaving us with a Hilbert space $\mathcal{H} = \Omega_+^{0|4} \oplus \Omega_+^{1|3} \oplus \Omega_+^2$, where $\Omega_+^{1|3} \equiv \{a - *a; a \in \Omega^1\}$ and $\Omega_+^{0|4} \equiv \{f + f\mathrm{Vol}_{M_4}; f \in \Omega^0\}$. This projected SQM is now the SQM of the M2-branes in a trivial background. From now on, we will often refer to $\Omega_+^{1|3}$ and $\Omega^{0|4}$ as Ω^1 and Ω^0, respectively.

Since $[Q - \bar{Q}, \tau] = 0$ and $\{Q + \bar{Q}, \tau\} = 0$, we seem to have one remaining real supercharge $\mathcal{Q}_1 \equiv i\sqrt{2}(Q - \bar{Q}) = i\sqrt{2}(d - d^\dagger)$ with the Hamiltonian $\hat{H} = \frac{1}{2}\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\} = \mathcal{Q}_1^2 = \frac{1}{2}\Delta$. However, now the operator $\mathcal{Q}_2 \equiv (-1)^F \sqrt{2}(d - d^\dagger)$ is an independent supercharge that commutes with τ, so we have the same amount of supersymmetry as Witten's as SQM. Note also that $\mathcal{Q}_2^2 = \frac{1}{2}\Delta$ and $\{\mathcal{Q}_1, \mathcal{Q}_2\} = 0$.

Consider next the ground states. These are specified by the equation $\mathcal{Q}_1|\psi\rangle = 0$ for the $Spin(7)$ bulk zero modes. For example, given a state $(a - *a) \in \Omega^1$, we have

$$\mathcal{Q}_1(a - *a) = 0 \Leftrightarrow (d - d^\dagger)(a - *a) = 0 \Rightarrow (da)^+ = 0, d^\dagger a = 0. \tag{D2}$$

The $\mathbb{Z}$-grading associated to fermion number in Witten's SQM is broken to $\mathbb{Z}_2$ fermion parity in our SQM as evidenced by the fact that $(da)_{\mathrm{SD}} \in \Omega^2$ and $d^\dagger a \in \Omega^0$ do not have the same fermion number but do have the same fermion parity.

The action of the $Spin(7)$ SQM will be the same as that of Witten's SQM on (M_4, g_{M_4}), where it is understood that the boundary conditions of the path integral will restrict to $\tau = +1$ states, and the allowed operators in correlation functions commute with τ. The Lagrangian is (see, e.g., Ref. [105]):

$$L = \frac{1}{2} g_{ij}\dot{X}^i\dot{X}^j + \frac{i}{2} g_{ij}(\bar{\psi}^i D_t\psi^j - D_t\bar{\psi}^i\psi^j) - \frac{1}{2} R_{ijkl}\psi^i\bar{\psi}^j\psi^k\bar{\psi}^l, \tag{D3}$$

where $D_t\psi^i \equiv \dot{\psi}^i + \Gamma^i_{jk}\dot{X}^j\psi^k$. The field transformations under $\mathcal{Q}_1$ are

$$\delta X^i = \epsilon(\psi^i + \bar{\psi}^i) \tag{D4}$$

$$\delta\psi^i = \epsilon(i\dot{X}^i - \Gamma^i_{jk}\bar{\psi}^j\psi^k) \tag{D5}$$

$$\delta\bar{\psi}^i = \epsilon(i\dot{X}^i + \Gamma^i_{jk}\bar{\psi}^j\psi^k). \tag{D6}$$

We now briefly comment on the case when $\Phi_{\mathrm{SD}} \neq 0$. In typically sigma-model SQM, we can add a superpotential W in the usual way by deforming the supercharge to $-\frac{i}{\sqrt{2}}\mathcal{Q}_1 = d + dW \wedge -d^\dagger - \iota_{dW}$, but this of course is not the same as turning on Φ_{SD} in our $Spin(7)$ background since dW is a 1-form. In particular, if we were to rescale $\Phi_{\mathrm{SD}} \to t\Phi_{\mathrm{SD}}$, we expect perturbative (in $1/t$) ground states that are labeled by circles. Implementing this at the level of SQM seems challenging since $\mathcal{Q}$ has odd-fermion parity $[(-1)^F = -1]$, but Φ_{SD} is parity even, indicating that we have only one real supersymmetry generator. We leave a careful study of this case to future work.

APPENDIX E: 4D AND 3D REFLECTIONS AND THEIR GEOMETRIC ORIGINS

Having already discussed the transformation R_i in 7D SYM in Sec. VI, our goal in this Appendix will be to fill in some of the details of Sec. VII in tracking how this reflection (possibly composed with internal geometric symmetries) reduces to 4D and 3D reflection symmetries.

Starting with the 4D case, we first briefly review the topological twisting of 7D SYM compactified on a three-manifold M_3. Observe that the 7D Dirac spinor decomposes as

$$\psi^{7d}_{\mathrm{Dirac}} = \psi^{4d}_L \otimes \psi^{3d}_{\mathrm{Dirac}} \quad \text{or} \quad \psi^{4d}_{\mathrm{Maj}} \otimes \psi^{3d}_{\mathrm{Dirac}}, \tag{E1}$$

where ψ^{4d}_L is a left-handed Weyl fermion and $\psi^{4d}_{\mathrm{Maj.}}$ is a Majorana fermion. Note that in three Euclidean dimensions the minimal spinor is Dirac due to the lack of a reality condition. As in seven dimensions, we can think of this 3D Dirac spinor as a symplectic-Majorana fermion and intuitively as filling out the representation (2, 2) of $SU(2)_{M_3} \times SU(2)_R$. After the topological twist, this becomes $\mathbf{3} \oplus \mathbf{1}$. The 7D spinor then decomposes into two pieces in four dimensions. The piece in the trivial representation of the twisted M_3 structure group is the 4D gaugino, and the piece that is a 1-form along M_3 is the fermionic component of chiral multiplets. Note that the 4D scalars a and φ naturally arrange themselves into a the complex scalar components of chiral multiplets $S \equiv a + i\varphi$. The supercharge clearly takes values in 1 internally. Since we expect R^{7d}_1 to be preserved in the compactification, we can appeal to the fact that a reflection symmetry in four dimensions acts on the supercharge as (see Ref. [60]) $\mathsf{R}^{7d}_1(Q^{4d}_L) \propto (Q^{4d}_L)^*$ and on chiral superfields as $\mathsf{R}^{7d}_1(\mathbf{S}) = \bar{\mathbf{S}}$. In terms of on-shell components S and $\psi^{(S)}_L$, we have

$$\mathsf{R}^{7d}_1(\psi^{(S)}_L) = i\sigma_1 C_{4d}(\psi^{(S)}_L)^* \tag{E2}$$

$$\mathsf{R}^{7d}_1(S) = S^*. \tag{E3}$$

The transformation of the scalar immediately agrees with expectations since φ is a pseudoscalar while a is a scalar in 7D SYM. The transformation of the Weyl spinor component follows from 7D as well, but we leave the details to a footnote.[33]

In the 3D case, we have already performed the topological twisting in Sec. VII B, as well as the action of R^{7d}_1 on the 3D fields [see Eq. (7.6)]. What remains to be spelled out is the precise geometric origin of Eq. (7.9), which we (essentially) reproduce here,

$$\mathsf{R}^{3d}_i = \mathsf{R}^{4d}_i \mathsf{R}^{4d}_\perp, \tag{E4}$$

[33] Let us define a custom 7D gamma matrix basis as follows: $\Gamma_0 = i\sigma_1 \otimes \mathbf{1} \otimes \mathbf{1}$, $\Gamma_{1\le i\le 3} = \sigma_2 \otimes \sigma_i \otimes \mathbf{1}$, $\Gamma_{4\le i\le 6} = \sigma_3 \otimes \mathbf{1} \otimes \sigma_i$. Since the 7D gaugino is in the adjoint, we can employ the decomposition $\psi^{4d}_{\mathrm{Maj}} \otimes \psi^{3d}_{\mathrm{Dirac}}$ and simply act by Γ_1. Writing the 4D Majorana in terms of left-handed Weyl fermions $\psi^{4d}_{\mathrm{Maj}} = \binom{\psi_L}{C_{4d}\psi^*_L}$, acting by Γ_1 induces $\psi_L \to i\sigma_1 C_{4d}\psi^*_L$. After Higging, the adjoint decomposes into various (possibly complex) representations, where this transformation still stands.

along a compact M_4 that is more general that a product $M_3\times S^1$. Recall that the motivation for such a definition was so that the localized matter fermions transform under reflection as $\psi\to\pm\gamma_i\psi$, i.e., without a complex conjugation, which is more commonly met in 3D field theory literature. Our proposal is that we can define such a symmetry geometrically if M_4 possess an orientation reversing isotropy σ_{M_4} so that

$$\mathsf{R}_i^{3d}\equiv\mathsf{R}_i^{7d}\circ\sigma_{M_4}. \tag{E5}$$

Looking first at what happens to the scalars, we see that a and φ are untouched, leaving the combination that appears for localized matter $a+i\varphi$ without the unwanted conjugation. As for their fermionic partners, first recall that the 7D Dirac gaugino decomposes as

$$\psi^{7d}_{\text{Dirac}}=\psi^{3d}_{\text{Maj}}\otimes(\psi^{4d}_L+\psi^{4d}_R), \tag{E6}$$

where, in the notation of Sec. VII, the 3D Majorana fermion χ is a left-handed Weyl fermion on the internal M_4, while ψ is right-handed. We can quickly find the transformation (up to a phase) by noting that σ_{M_4} exchanges left-/right-handed fermions and recalling $\Gamma_1^{7d}\otimes\gamma_5^{M_4}$, which implies $\psi^{3d}_{\text{Dirac}}=\chi+i\psi\to\gamma_1(\psi-i\chi)=-i\gamma_1\psi^{3d}_{\text{Dirac}}$. The overall phase can be fixed to $+1$ a Pin^+ action by working out the gamma matrix algebra in flat 7D space. Note that deriving the action on bulk modes involves knowing the action of σ_{M_4} on various cohomology classes.

We have thankfully seen that there is no complex conjugation in the reflection transformations of 3D matter fields if we use the definition of Eq. (E5). The puzzle remains, however, of how to derive the fact that localized modes with Hessians $(+,+,-)$ and $(-,-,+)$ should transform as $\psi^{3d}_{\text{Dirac}}\to-\gamma_i\psi^{3d}_{\text{Dirac}}$ and $\psi^{3d}_{\text{Dirac}}\to+\gamma_i\psi^{3d}_{\text{Dirac}}$, respectively. This was motivated in Secs. VII and VIII by several means, one being that a vectorlike pair in four dimensions should not contribute to parity anomalies because one can write a parity conserving mass term. The resolution to this puzzle can be seen at the level of the superpotential. Under σ_{M_4}, the operator $D_{q\Phi_{SD}}\to-D_{q\Phi_{SD}}$ because Φ_{SD} is a pseudoscalar and the signature operator flips sign under orientation reversal. This eigenvalue of this operator is the mass M that couples vectorlike pairs in the $Spin(7)$ superpotential

$$M\Phi^{(a)}_{\bar{\mathbf{R}}}\Phi^{(b)}_{\mathbf{R}}, \tag{E7}$$

where $\Phi^{(a)}$ and $\Phi^{(b)}$ are complex multiplets in conjugate representations. The previous paragraph would suggest that these superfields both transform under R_1^{3d} as $\Phi^{(a,b)}\to+\Phi^{(a,b)}$. The important detail to note is that the above term is only parity odd (i.e., conserves parity) if $M\to-M$, which, as we have just shown, is natural from the $Spin(7)$ perspective, although quite strange from a bottom-up 3D perspective. The trick is to simply redefine $\Phi^{(a)}_{\mathbf{R}}$ and M such that the former is parity odd and the latter is parity even.[34] A similar statement can be made for the parity assignments of bulk mode states, as neutral bulk modes (before expanding in terms of KK eigenstates) enter into the 3D $\mathcal{N}=1$ superpotential as

$$\int_{M_4}\varphi\wedge da, \tag{E8}$$

which again produced "odd" 3D masses due to the transformation σ_{M_4}: $\int_{M_4}\to\int_{\overline{M_4}}=-\int_{M_4}$. By transferring this assignment to the superfield φ, this reproduces the charge assignments used in the main text [see Eq. (7.12)].

APPENDIX F: PANTEV-WIJNHOLT SYSTEM WITH $\mathfrak{so}(10)$ GUT MATTER

Here, we look at how to engineer various (possibly chiral) matter representations in the PW system with gauge algebra $\mathfrak{so}(10)$ via an E_8 unfolding:

$$\mathfrak{e}_8\supset\mathfrak{so}(10)\oplus\mathfrak{su}(4)\supset\mathfrak{so}(10)\oplus\mathfrak{u}(1)^3. \tag{F1}$$

The Higgs bundle $\Phi_{i=1,2,3}$ lives on a three-manifold M_3 and satisfies $d\Phi_i=0$ and $d^\dagger\Phi_i=\sum_I v_{i,I}*\delta_{p_I}$ (provided $\sum_I v_{i,I}=0$). We consider the solution that is defined by the three essentially unique harmonic functions (with singularities) f_i that are solutions to the electrostatic problem $\Delta f=\sum_I v_{i,I}*\delta_{p_I}$ and define the Higgs fields as $df_i=\Phi_i$.

Now, taking a look at the adjoint breaking

$$\begin{aligned}\mathbf{248}\to\mathbf{45}_{\vec{0}}&+\sum_{i=1}^{4}\mathbf{16}_i+\sum_{i=1}^{4}\overline{\mathbf{16}}_i+\sum_{i<j}\mathbf{10}_{ij}\\&+\sum_{i<j}\mathbf{1}_{ij}+3\times\mathbf{1}_{\vec{0}},\end{aligned} \tag{F2}$$

we see that we might have several species of localized matter in addition to the usual bulk modes $\mathbf{24}_{\vec{0}}$ and $\mathbf{1}_{\vec{0}}$. The subscript notation for the localized representations is a shorthand for the $U(1)$ charge vectors; let $\vec{q}_1=(1,0,0)$, $\vec{q}_{2,3}$ be defined accordingly, and $\vec{q}_4=(-1,-1,-1)$. Then, our Abelian charge assignments for the various $\mathfrak{so}(10)$ multiplets are $\mathbf{16}_i=\mathbf{16}_{\vec{q}_i}$, $\overline{\mathbf{16}}_i=\overline{\mathbf{16}}_{-\vec{q}_i}$, $\mathbf{10}_{ij}\equiv\mathbf{10}_{(\vec{q}_i+\vec{q}_j)}$, and $\mathbf{1}\equiv\mathbf{1}_{\pm(\vec{q}_i-\vec{q}_j)}$. Based on the $U(1)$ charges, these various representations are localized around the following loci,

[34] Said more pedantically, smooth orientable manifolds possess a notion of a constant global section of their determinant line bundles, which, for example, when multiplying scalars turns them to pseudoscalars and vice versa. In the product $M\Phi^{(a)}_{\mathbf{R}}$, we are simply redefining who that constant global section belongs to.

$$\{\Phi_i = 0\} \leftrightarrow \mathbf{16}_i \quad \text{or} \quad \overline{\mathbf{16}}_i \tag{F3}$$

$$\{\Phi_i + \Phi_j = 0\} \leftrightarrow \mathbf{10}_{ij} \tag{F4}$$

$$\{\Phi_i - \Phi_j = 0\} \leftrightarrow \mathbf{1}_{ij}, \tag{F5}$$

where the representations on the left/right are localized at Morse index ± 1 zeros. Note that this has a nice spectral cover description in terms of intersections of various combinations of sheets. We can now generalize Eqs. (5.7) and (5.8) to count the number of charged $SO(10)$ modes on the PW building block

$$N_{\mathbf{16}} = \sum_i n_i^- + b^1(M_3) - 1,$$
$$N_{\overline{\mathbf{16}}} = \sum_i n_i^+ + b^2(M_3) - 1 \tag{F6}$$

$$N_{\mathbf{10}} = \sum_{i<j}(n_i^- + n_j^-) + \sum_{i<j}(n_i^+ + n_j^+) + 2b^1(M_3) - 2, \tag{F7}$$

where $n_i^\pm$ denote the number of singular loci for the ith Higgs field with positive/negative residue with respect to the $U(1)_i$-charge.

APPENDIX G: PARITY ANOMALIES OF 3D GAUGE THEORIES

In this Appendix, we provide further details on the obstructions to placing 3D theories that are classically invariant under reflections and/or time reversal, on a nonorientable manifold by reviewing the conditions that were spelled out in Ref. [24]. Namely, we will give a detailed characterization of the anomalies ν_{R} (the gravitational-parity anomaly) and $\nu_{\mathsf{R}G}$ (mixed gauge-parity anomaly) discussed in the main text. In the following, we will always focus on the case where $\mathsf{R}^2 = +1$ [or alternatively $\mathsf{T}^2 = (-1)^F$ where F is the fermion number]; that is, we consider the case of whether we are able to define the theory on a nonorientable manifold with a Pin^+ structure.[35]

1. Warm-up: Gauge-parity anomaly

As a warm-up for our more technical discussion of anomalies via bordism groups, we discuss a convenient way of understanding gauge-parity anomalies. This is to simply start with the 3D theory of N Majorana fermions, whose global symmetry group is $SO(N)$, from which various gauge-parity anomalies may be derived [at least when the gauge group is a subgroup of $SO(N)$],[36] and to check whether there is a gauge-parity anomaly according to Ref. [24], one should check whether any gauge bundle $V_{\mathcal{R}}$ on a four-manifold is "stably trivial," a condition that is equivalent to the vanishing of the Stiefel-Whitney classes

$$w_1(V_{\mathcal{R}}) = w_2(V_{\mathcal{R}}) = w_4(V_{\mathcal{R}}) = 0. \tag{G1}$$

Here, $\mathcal{R}$ is the representation of the fermions under the gauge group G. The vanishing of the Stiefel-Whitney classes depends on the representation $\mathcal{R}$, and so the anomaly can be present or absent depending on the matter content.

In the following, we will take an $\mathcal{N} = 1$ gauge theory with $G = SU(N)$ and N_f multiplets in the fundamental. In this case, the matter representation $\mathcal{R} = \mathbf{fund}^{\oplus N_f} \oplus \mathbf{adj}$. We would like to take an $SU(N)$ bundle as a particular case of an $SO(2N)$ bundle and compute the Stiefel-Whitney classes. Luckily for any complex vector bundle V, one can show that (see, for example, exercise 14-B of Ref. [106])

$$c_i(V) \mod 2 = w_{2i}(V). \tag{G2}$$

All odd degree Stiefel-Whitney classes vanish. Clearly, for an $SU(N)$ bundle, $c_1(V) = 0$, meaning that $w_2(V) = 0$. This means that we need to compute $c_2(V_{\mathcal{R}})$ for our example and take its mod 2 reduction to obtain $\nu_{\mathsf{R}SU(N)}$. In the following, it is important to take the correct normalization, and we will follow the conventions of Ref. [107]. For an $SU(N_c)$ bundle V_ρ, we have that[37]

$$c_2(V_\rho) = \frac{1}{2}\mathrm{tr}_\rho(F \wedge F). \tag{G3}$$

In general, it is convenient to convert the trace in a representation ρ to a normalized trace Tr with the property

$$\frac{1}{4}\int \mathrm{Tr}(F \wedge F) \in \mathbb{Z}. \tag{G4}$$

For $SU(N_c)$, the trace identities are

$$\mathrm{tr}_{\mathbf{fund}}(F \wedge F) = \frac{1}{2}\mathrm{Tr}(F \wedge F),$$
$$\mathrm{tr}_{\mathbf{adj}}(F \wedge F) = N_c \mathrm{Tr}(F \wedge F). \tag{G5}$$

Therefore, for our case,

$$c_2(V_{\mathcal{R}}) = \frac{1}{4}\mathrm{Tr}(F \wedge F)[N_f + 2N_c]. \tag{G6}$$

This means that the fourth Stiefel-Whitney class is

[35] The condition for a manifold X to admit a Pin^+ is that $w_2(TX) = 0$. To have a Pin^- structure, the condition is that $w_2(TX) + w_1^2(TX) = 0$.

[36] To be precise, the global symmetry group is $O(N)$, but in the following, it will not make a difference.

[37] As in Ref. [107], we absorbed a 2π factor in the definition of the Yang-Mills field strength.

$$\int w_4(V_{\mathcal{R}}) = [N_f + 2N_c]m \mod 2, \tag{G7}$$

where $m \in \mathbb{Z}$ is the instanton number. Clearly, if we want the fourth Stiefel-Whitney class to be zero for any $SU(N_c)$ bundle, we need to take $N_f + 2N_c = 0 \bmod 2$ or equivalently $N_f = 0 \bmod 2$.

It is possible to generalize the previous discussion to other simply connected gauge groups for generic matter spectra. Given a theory with simply connected gauge group G, and fermions in the (possibly reducible) representation $\mathcal{R}$, the absence of an anomaly can be written as

$$\nu_{\mathsf{RG}} \frac{1}{2} \int \mathrm{tr}_{\mathcal{R}}(F \wedge F) = 0 \mod 2, \tag{G8}$$

for any G-bundle. Using the index of the representation $\mathcal{R}$, it is possible to rephrase the condition as $2\mathbf{T}(\mathcal{R}) = 0 \bmod 2$. The condition is equivalent to asking whether the number of fermionic zero modes in the background is even or odd. For example, taking $G = E_6$, and since $2\mathbf{T}(\mathbf{27}) = 6$, there are no possible anomalies coming from a gauge bundle. In the next section, we will discuss a different method to detect possible anomalies, which includes purely geometrical contributions and generalizes to other spacetime dimensions.

2. Anomalies and bordisms

A different way to understand the presence of an anomaly when trying to define a theory on an unorientable manifold X is to use the Dai-Freed theorem [108] to define the partition function of a spinor on such a manifold. The proper definition of the phase of the partition function of a spinor is $\exp(2\pi i \eta_Y)$, where Y is a manifold that satisfies $\partial Y = X$ and extends all structures (like pin/spin structures and gauge bundle) defined on X. The quantity η_Y is the eta invariant of the Dirac operator on Y. While this provides a sensible definition of the phase of the partition function of a spinor, this definition may depend on the choice of Y or the extension chosen on Y of structures of X. This is a reflection of an anomaly in the definition of the phase of the partition function. In order to detect such an anomaly, one can compare the phase of the partition function with two different extensions Y_1 and Y_2. Geometrically, this can be accomplished as follows: since Y_1 and Y_2 have the same boundary X, it is possible to glue them along the boundary if one reverses the orientation of one of them, say Y_2. The gluing produces a closed manifold $\hat{Y} = Y_1 \sqcup \overline{Y_2}$, where $\overline{Y_2}$ is the orientation reversed Y_2. Given the properties of the eta invariant $\eta_{Y_1 \sqcup Y_2} = \eta_{Y_1} + \eta_{Y_2}$ and that $\eta_{\bar{Y}} = -\eta_Y$, one finds that

$$\eta_{\hat{Y}} = \eta_{Y_1} - \eta_{Y_2}. \tag{G9}$$

In order to ensure that the partition function is sensible, it is necessary to require that $\eta_Y \in \mathbb{Z}$ for any closed four-dimensional manifold with a Pin^+ structure. Another property of the eta invariant significantly simplifies the computations; it is invariant under bordisms between manifolds. Given two d-dimensional manifolds X_1 and X_2, we say that there is a bordism between them if there exists a $d+1$-dimensional manifold Z such that $\partial Z = X_1 \sqcup \bar{X}_2$. If the manifolds X_i carry additional structures [like (s)pin structures or gauge bundles], we require that they extend to Z. Therefore, in the following, we will need to check what are the allowed values of the eta invariant for different gauge groups to see if any anomaly for time reversal is present when we define our 3D theories on a Pin^+ manifold. In particular, we will consider the bordism groups $\Omega_4^{\mathsf{Pin}^+}(M)$ where four-dimensional manifolds come equipped with a function to some space M. We will take $M = BG$, where BG is the classifying space for a group G. The classifying space of a group G is an infinite-dimensional space that has a principal G-bundle with total space EG, the so-called universal bundle, with the following property: any principal G-bundle over any manifold X is the pullback of EG via some continuous function $f\colon X \to BG$. Therefore, by considering manifolds equipped with maps to the classifying space of some group G, we end up considering all possible principal G-bundles over such manifolds. We shall call $\Omega_d^{\mathsf{Pin}^+}(\mathrm{pt})$ the case without gauge group, that is the case where the function goes to a single point. This case classifies the possible anomalies that come only from geometry. In general, there exists a map $\Phi\colon \Omega_d^{\mathsf{Pin}^+}(M) \to \Omega_d^{\mathsf{Pin}^+}(\mathrm{pt})$. The map Φ deletes the details of the gauge bundle from equivalence classes in the bordism group $\Omega_d^{\mathsf{Pin}^+}(M)$. Given that this map is surjective, it is possible to form a short exact sequence,

$$0 \to \ker\Phi \to \Omega_d^{\mathsf{Pin}^+}(M) \xrightarrow{\Phi} \Omega_d^{\mathsf{Pin}^+}(\mathrm{pt}) \to 0. \tag{G10}$$

We will call $\tilde{\Omega}_d^{\mathsf{Pin}^+}(M) \equiv \ker\Phi$ the reduced bordism group. Due to the existence of a map $\Psi\colon \Omega_d^{\mathsf{Pin}^+}(\mathrm{pt}) \to \Omega_d^{\mathsf{Pin}^+}(M)$ which satisfies $\Phi \cdot \Psi = \mathbb{I}$, the above short exact sequence splits, implying that $\Omega_d^{\mathsf{Pin}^+}(M) \simeq \Omega_d^{\mathsf{Pin}^+}(\mathrm{pt}) \oplus \tilde{\Omega}_d^{\mathsf{Pin}^+}(M)$. The splitting of the bordism group into a purely geometric and a reduced part means that anomalies due to geometry can not be cured by introducing suitable gauge bundles and vice versa. For more details on this map and a proof of the splitting, see, for example, Appendix A of Ref. [109]. For the case at hand, $\Omega_4^{\mathsf{Pin}^+}(\mathrm{pt}) = \mathbb{Z}_{16}$ with the generator being realized by a single Majorana fermion on $\mathbb{RP}^4$. This matches the computation of the eta invariant for this system done in Ref. [52]. This group classifies all possible gravitational-parity anomalies a 3D theory may have, i.e., $\nu_{\mathsf{R}} \in \Omega_4^{\mathsf{Pin}^+}(\mathrm{pt})$.

We will discuss now the result of the computation of the reduced bordism groups for some classes of gauge groups, which is what ν_{RG} take values in. The computation uses the Atiyah-Hirzebruch spectral sequence, which we choose not to review here; see, for example, Ref. [109] for the necessary mathematical background to perform such computations.

For the case of $G = SU(N)$, the computation is rather easy, and one obtains

$$\tilde{\Omega}_4^{\mathsf{Pin}^+}(BSU(N)) = \mathbb{Z}_2. \tag{G11}$$

This confirms the computation done in the previous section where the anomaly is a mod 2 condition. The result for $N = 3$ has already been obtained in Ref. [110] using the Adams spectral sequence. The generator for this reduced bordism group is S^4 with one instanton.

The case of $G = U(1)$ is a bit more tricky, and we obtain

$$\tilde{\Omega}_4^{\mathsf{Pin}^+}(BU(1)) = e(\mathbb{Z}_2, \mathbb{Z}_2), \tag{G12}$$

where $e(G, H)$ is an extension of G by H. Specifically, we have the short exact sequence

$$0 \to H \to e(G, H) \to G \to 0. \tag{G13}$$

Such extensions are classified by the group $\mathrm{Ext}(G, H)$, and for our purposes, $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{\gcd(m,n)}$. This implies that the bordism group can be either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if the extension is trivial or $\mathbb{Z}_4$ if the extension is not trivial. To gain more information, it would be necessary to compute the η-invariant for a fermion. Luckily for the case of $\mathbb{RP}^4$ with some $U(1)$ bundle, this was done in Ref. [24], finding that the phase of the partition function is a root of 16 times a root of 4. This means that

$$\tilde{\Omega}_4^{\mathsf{Pin}^+}(BU(1)) = \mathbb{Z}_4. \tag{G14}$$

The same result for $\tilde{\Omega}_4^{\mathsf{Pin}^+}(BU(1))$ was obtained in Ref. [111] using the Adams spectral sequence.

Finally, we can ask what is the anomaly cancellation condition for any group G on a Pin^+ manifold. The computation of the bordism group $\Omega_4^{\mathsf{Pin}^+}(BG)$ is identical for any simply connected group giving

$$\tilde{\Omega}_4^{\mathsf{Pin}^+}(BG) = \mathbb{Z}_2, \quad \text{if } \pi_1(G) = 0. \tag{G15}$$

The generator is again S^4 with an instanton.

For the case of nonsimply connected gauge groups, the results can vary due to the presence of additional gauge bundles allowed in the path integral. The computations become more complicated, and we will simply quote some results in the literature:

(i) For $G = SO(3)$, it was found in Ref. [112] that $\tilde{\Omega}_4^{\mathsf{Pin}^+}(BSO(3)) = \mathbb{Z}_4$. The invariant measuring the anomaly in this case is $q(w_2)$ where w_2 is the second Stiefel-Whitney class of the $SO(3)$ bundle and $q\colon H^2(X, \mathbb{Z}_2) \to H^4(X, \mathbb{Z}_4)$ is a quadratic refinement. Such refinements depend on the choice of a Pin^+ structure; one choice of such refinement is the Pontryagin square. See footnote 7 of Ref. [112] for more details on such a refinement.

(ii) For $G = PSU(3)$, it was found in Ref. [112] that $\tilde{\Omega}_4^{\mathsf{Pin}^+}(BPSU(3)) = \mathbb{Z}_2$. The generator is the same as in the case of $SU(3)$.

[1] A. P. Braun and S. Schäfer-Nameki, $Spin(7)$-manifolds as generalized connected sums and 3d $\mathcal{N} = 1$ theories, J. High Energy Phys. 06 (2018) 103.

[2] M. Cvetič, J. J. Heckman, T. B. Rochais, E. Torres, and G. Zoccarato, Geometric unification of Higgs bundle vacua, Phys. Rev. D **102**, 106012 (2020).

[3] E. Witten, Is supersymmetry really broken? Int. J. Mod. Phys. A **10**, 1247 (1995).

[4] C. Vafa, Evidence for F theory, Nucl. Phys. **B469**, 403 (1996).

[5] J. J. Heckman, C. Lawrie, L. Lin, and G. Zoccarato, F-theory and dark energy, Fortschr. Phys. **67**, 1900057 (2019).

[6] J. J. Heckman, C. Lawrie, L. Lin, J. Sakstein, and G. Zoccarato, Pixelated dark energy, Fortschr. Phys. **67**, 1900071 (2019).

[7] K. Becker, A note on compactifications on $Spin(7)$-holonomy manifolds, J. High Energy Phys. 05 (2001) 003.

[8] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, New complete noncompact $Spin(7)$ manifolds, Nucl. Phys. **B620**, 29 (2002).

[9] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, Cohomogeneity one manifolds of $Spin(7)$ and G_2 holonomy, Phys. Rev. D **65**, 106004 (2002).

[10] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, New cohomogeneity one metrics with $Spin(7)$ holonomy, J. Geom. Phys. **49**, 350 (2004).

[11] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, Orientifolds and slumps in G_2 and $Spin(7)$ metrics, Ann. Phys. (Amsterdam) **310**, 265 (2004).

[12] S. Gukov and J. Sparks, M theory on $Spin(7)$ manifolds. 1, Nucl. Phys. **B625**, 3 (2002).

[13] S. Gukov and D. Tong, D-brane probes of special holonomy manifolds, and dynamics of $N = 1$ three-dimensional gauge theories, J. High Energy Phys. 04 (2002) 050.

[14] S. Gukov, J. Sparks, and D. Tong, Conifold transitions and five-brane condensation in M theory on $Spin(7)$ manifolds, Classical Quantum Gravity **20**, 665 (2003).

[15] C. Vafa and E. Witten, A strong coupling test of S duality, Nucl. Phys. **B431**, 3 (1994).

[16] B. S. Acharya and E. Witten, Chiral fermions from manifolds of G(2) holonomy, arXiv:hep-th/0109152.

[17] T. Pantev and M. Wijnholt, Hitchin's equations and M-theory phenomenology, J. Geom. Phys. **61**, 1223 (2011).
[18] A. P. Braun, S. Cizel, M. Hübner, and S. Schäfer-Nameki, Higgs bundles for M-theory on G_2-manifolds, J. High Energy Phys. 03 (2019) 199.
[19] R. Barbosa, M. Cvetič, J. J. Heckman, C. Lawrie, E. Torres, and G. Zoccarato, T-branes and G_2 backgrounds, Phys. Rev. D **101**, 026015 (2020).
[20] R. Donagi and M. Wijnholt, Model building with F-theory, Adv. Theor. Math. Phys. **15**, 1237 (2011).
[21] C. Beasley, J. J. Heckman, and C. Vafa, GUTs and exceptional branes in F-theory—I, J. High Energy Phys. 01 (2009) 058.
[22] O. Aharony, Baryons, monopoles and dualities in Chern-Simons-matter theories, J. High Energy Phys. 02 (2016) 093.
[23] C. Córdova, P.-S. Hsin, and N. Seiberg, Time-reversal symmetry, anomalies, and dualities in $(2+1)d$, SciPost Phys. **5**, 006 (2018).
[24] E. Witten, The "parity" anomaly on an unorientable manifold, Phys. Rev. B **94**, 195150 (2016).
[25] C. L. Kane and E. J. Mele, Quantum Spin Hall Effect in Graphene, Phys. Rev. Lett. **95**, 226801 (2005).
[26] B. A. Bernevig and S.-C. Zhang, Quantum Spin Hall Effect, Phys. Rev. Lett. **96**, 106802 (2006).
[27] N. Seiberg and E. Witten, Gapped boundary phases of topological insulators via weak coupling, Prog. Theor. Exp. Phys. (**2016**) 12C101.
[28] T. Perutz, Zero-sets of near-sympletic forms, J. Symplectic Geom. **4**, 237 (2007).
[29] T. W. Grimm and H. Hayashi, F-theory fluxes, chirality and Chern-Simons theories, J. High Energy Phys. 03 (2012) 027.
[30] M. Cvetič, T. W. Grimm, and D. Klevers, Anomaly cancellation and abelian gauge symmetries in F-theory, J. High Energy Phys. 02 (2013) 101.
[31] P. Corvilain, T. W. Grimm, and D. Regalado, Chiral anomalies on a circle and their cancellation in F-theory, J. High Energy Phys. 04 (2018) 020,
[32] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, J. High Energy Phys. 02 (2015) 172.
[33] M. Del Zotto, J. J. Heckman, D. S. Park, and T. Rudelius, On the defect group of a 6D SCFT, Lett. Math. Phys. **106**, 765 (2016).
[34] J. Eckhard, H. Kim, S. Schafer-Nameki, and B. Willett, Higher-form symmetries, Bethe vacua, and the 3d-3d correspondence, J. High Energy Phys. 01 (2020) 101.
[35] D. R. Morrison, S. Schafer-Nameki, and B. Willett, Higher-form symmetries in 5d, J. High Energy Phys. 09 (2020) 024.
[36] F. Albertini, M. Del Zotto, I. Garcia-Etxebarria, and S. S. Hosseini, Higher form symmetries and M-theory, J. High Energy Phys. 12 (2020) 203.
[37] C. Closset, S. Schafer-Nameki, and Y.-N. Wang, Coulomb and Higgs branches from canonical singularities: Part 0, J. High Energy Phys. 02 (2021) 003.
[38] M. Del Zotto, I. Garcia-Etxebarria, and S. S. Hosseini, Higher form symmetries of Argyres-Douglas theories, J. High Energy Phys. 10 (2020) 056.
[39] L. Bhardwaj and S. Schäfer-Nameki, Higher-form symmetries of 6d and 5d theories, J. High Energy Phys. 02 (2021) 159.
[40] M. Buican and H. Jiang, 1-Form symmetry, isolated $N = 2$ SCFTs, and Calabi-Yau threefolds, J. High Energy Phys. 12 (2021) 024.
[41] Y. Tachikawa, On the 6d origin of discrete additional data of 4d gauge theories, J. High Energy Phys. 05 (2014) 020.
[42] M. Cvetič, M. Dierigl, L. Lin, and H. Y. Zhang, Higher-form symmetries and their anomalies in M-/F-theory duality, Phys. Rev. D **104**, 126019 (2021).
[43] B. S. Acharya, X. de la Ossa, and S. Gukov, G flux, supersymmetry and $Spin(7)$ manifolds, J. High Energy Phys. 09 (2002) 047.
[44] C. Beasley, J. J. Heckman, and C. Vafa, GUTs and exceptional branes in F-theory—II: Experimental predictions, J. High Energy Phys. 01 (2009) 059.
[45] J. J. Heckman and C. Vafa, Flavor hierarchy from F-theory, Nucl. Phys. **B837**, 137 (2010).
[46] A. Font, L. E. Ibanez, F. Marchesano, and D. Regalado, Non-perturbative effects and Yukawa hierarchies in F-theory SU(5) Unification, J. High Energy Phys. 03 (2013) 140; Erratum, J. High Energy Phys. 07 (2013) 036.
[47] A. Font, F. Marchesano, D. Regalado, and G. Zoccarato, Up-type quark masses in SU(5) F-theory models, J. High Energy Phys. 11 (2013) 125.
[48] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. **83**, 1057 (2011).
[49] C. G. Callan Jr. and J. A. Harvey, Anomalies and fermion zero modes on strings and domain walls, Nucl. Phys. **B250**, 427 (1985).
[50] N. Seiberg, T. Senthil, C. Wang, and E. Witten, A duality web in $2+1$ dimensions and condensed matter physics, Ann. Phys. (Amsterdam) **374**, 395 (2016).
[51] R. Mazzeo, D. Pollack, and K. Uhlenbeck, Connected sum constructions for constant scalar curvature metrics, Topol. Methods Nonlinear Anal. **6**, 207 (1995).
[52] E. Witten, Fermion path integrals and topological phases, Rev. Mod. Phys. **88**, 035001 (2016).
[53] M. Montero and C. Vafa, Cobordism conjecture, anomalies, and the string lamppost principle, J. High Energy Phys. 01 (2021) 063.
[54] M. Berg, C. DeWitt-Morette, S. Gwo, and E. Kramer, The pin groups in physics: C, P, and T, Rev. Math. Phys. **13**, 953 (2001).
[55] Y. Tachikawa and K. Yonekura, Why are fractional charges of orientifolds compatible with Dirac quantization? SciPost Phys. **7**, 058 (2019).
[56] E. Cremmer, B. Julia, and J. Scherk, Supergravity theory in eleven-dimensions, Phys. Lett. **76B**, 409 (1978).
[57] E. Witten, Toroidal compactification without vector structure, J. High Energy Phys. 02 (1998) 006.
[58] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. R. Morrison, and S. Sethi, Triples, fluxes, and strings, Adv. Theor. Math. Phys. **4**, 995 (2000).
[59] Y. Tachikawa, Frozen singularities in M- and F-theory, J. High Energy Phys. 06 (2016) 128.
[60] S. Weinberg, *The Quantum Theory of Fields. Vol. 3: Supersymmetry* (Cambridge University Press, Cambridge, England, 2013).

[61] E. Witten, Five-brane effective action in M-theory, J. Geom. Phys. **22**, 103 (1997).
[62] O. Aharony and E. Witten, Anti-de Sitter space and the center of the gauge group, J. High Energy Phys. 11 (1998) 018.
[63] E. Witten, AdS/CFT correspondence and topological field theory, J. High Energy Phys. 12 (1998) 012.
[64] G. W. Moore, Anomalies, gauss laws, and page charges in M-theory, C R Phys. **6**, 251 (2005).
[65] D. Belov and G. W. Moore, Holographic action for the self-dual field, arXiv:hep-th/0605038.
[66] D. S. Freed, G. W. Moore, and G. Segal, Heisenberg groups and noncommutative fluxes, Ann. Phys. (Amsterdam) **322**, 236 (2007).
[67] E. Witten, Conformal field theory in four and six dimensions, arXiv:0712.0157.
[68] E. Witten, Geometric langlands from six dimensions, arXiv:0905.2720.
[69] M. Henningson, The partition bundle of type A_{N-1} (2, 0) theory, J. High Energy Phys. 04 (2011) 090.
[70] D. S. Freed and C. Teleman, Relative quantum field theory, Commun. Math. Phys. **326**, 459 (2014)..
[71] S. Monnier, The global anomalies of (2,0) superconformal field theories in six dimensions, J. High Energy Phys. 09 (2014) 088.
[72] S. Monnier, Topological field theories on manifolds with Wu structures, Rev. Math. Phys. **29**, 1750015 (2017).
[73] S. Monnier, The anomaly field theories of six-dimensional (2,0) superconformal theories, Adv. Theor. Math. Phys. **22**, 2035 (2018).
[74] C.-T. Hsieh, G. Y. Cho, and S. Ryu, Global anomalies on the surface of fermionic symmetry-protected topological phases in $(3+1)$ dimensions, Phys. Rev. B **93**, 075135 (2016).
[75] D. Gaiotto, Z. Komargodski, and J. Wu, Curious aspects of three-dimensional $\mathcal{N}=1$ SCFTs, J. High Energy Phys. 08 (2018) 004.
[76] D. Joyce, A new construction of compact 8-manifolds with holonomy $Spin(7)$, J. Diff. Geom. **53**, 89 (1999).
[77] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs (Oxford University Press, New York, 2000).
[78] S. Gukov, C. Vafa, and E. Witten, CFT's from Calabi-Yau four folds, Nucl. Phys. **B584**, 69 (2000); Erratum, Nucl. Phys. **B608**, 477 (2001).
[79] E. Witten, On flux quantization in M theory and the effective action, J. Geom. Phys. **22**, 1 (1997).
[80] R. L. Bryant and S. M. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. **58**, 829 (1989).
[81] V. Bouchard, J. J. Heckman, J. Seo, and C. Vafa, F-theory and neutrinos: Kaluza-Klein dilution of flavor hierarchy, J. High Energy Phys. 01 (2010) 061.
[82] S. Cecotti, M. C. N. Cheng, J. J. Heckman, and C. Vafa, Yukawa couplings in F-theory and non-commutative geometry, arXiv:0910.0477.
[83] F. Marchesano and L. Martucci, Non-Perturbative Effects on Seven-Brane Yukawa Couplings, Phys. Rev. Lett. **104**, 231601 (2010).
[84] F. Marchesano, D. Regalado, and G. Zoccarato, Yukawa hierarchies at the point of E_8 in F-theory, J. High Energy Phys. 04 (2015) 179.
[85] M. Cvetič, L. Lin, M. Liu, H. Y. Zhang, and G. Zoccarato, Yukawa hierarchies in global F-theory models, J. High Energy Phys. 01 (2020) 037.
[86] E. Witten, Non-perturbative superpotentials in string theory, Nucl. Phys. **B474**, 343 (1996).
[87] S. H. Katz and C. Vafa, Geometric engineering of $N=1$ quantum field theories, Nucl. Phys. **B497**, 196 (1997).
[88] R. Blumenhagen, M. Cvetič, and T. Weigand, Spacetime instanton corrections in 4D string vacua: The Seesaw mechanism for D-Brane models, Nucl. Phys. **B771**, 113 (2007).
[89] R. Blumenhagen, M. Cvetič, S. Kachru, and T. Weigand, D-brane instantons in type II orientifolds, Annu. Rev. Nucl. Part. Sci. **59**, 269 (2009).
[90] S. H. Katz and C. Vafa, Matter from geometry, Nucl. Phys. **B497**, 146 (1997).
[91] A. Kovalev, Twisted connected sums and special Riemannian holonomy, arXiv:math/0012189.
[92] M. Haskins, H.-J. Hein, and J. Nordstrm, Asymptotically cylindrical Calabi-Yau manifolds, arXiv:1212.6929.
[93] F. Bonetti, T. W. Grimm, and T. G. Pugh, Non-supersymmetric F-theory compactifications on $Spin(7)$ manifolds, J. High Energy Phys. 01 (2014) 112.
[94] F. Bonetti, T. W. Grimm, E. Palti, and T. G. Pugh, F-theory on $Spin(7)$ manifolds: Weak-coupling limit, J. High Energy Phys. 02 (2014) 076.
[95] W. Massey, The quotient space of the complex projective plane under conjugation is a 4-sphere, Geometriae Dedicata **2**, 371 (1973).
[96] T. Lawson, Splitting S^4 on RP^2 via the branched cover of CP^2 over S^4, Proc. Am. Math. Soc. **86**, 328 (1982).
[97] A. Andreotti, On a theorem of Torelli, Am. J. Math. **80**, 801 (1958).
[98] M. Del Zotto, J. J. Heckman, A. Tomasiello, and C. Vafa, 6D conformal matter, J. High Energy Phys. 02 (2015) 054.
[99] J. J. Heckman, More on the matter of 6D SCFTs, Phys. Lett. B **747**, 73 (2015); Erratum, Phys. Lett. B **808**, 135675 (2020).
[100] F. Apruzzi, J. J. Heckman, D. R. Morrison, and L. Tizzano, 4D gauge theories with conformal matter, J. High Energy Phys. 09 (2018) 088.
[101] M. Dierigl, J. J. Heckman, T. B. Rochais, and E. Torres, Geometric approach to 3D interfaces at strong coupling, Phys. Rev. D **102**, 106011 (2020).
[102] J. Polchinski, *String Theory. Vol. 2: Superstring Theory and Beyond*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2007).
[103] M. Hubner, Local G_2-manifolds, Higgs bundles and a colored quantum mechanics, J. High Energy Phys. 05 (2021) 002.
[104] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. **17**, 661 (1982).
[105] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror Symmetry*, Clay Mathematics Monographs Vol. 1 (AMS, Providence, 2003).

[106] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies (Princeton University Press, Princeton, 1974).
[107] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, Anomaly polynomial of general 6d SCFTs, Prog. Theor. Exp. Phys. **2014**, 103B07 (2014).
[108] X. Dai and D. S. Freed, Eta invariants and determinant lines, J. Math. Phys. **35**, 5155 (1994); Erratum, J. Math. Phys. **42**, 2343 (2001).
[109] I. Garcia-Etxebarria and M. Montero, Dai-freed anomalies in particle physics, J. High Energy Phys. 08 (2019) 003.
[110] M. Guo, P. Putrov, and J. Wang, Time reversal, SU(N) Yang-Mills and cobordisms: Interacting topological superconductors/insulators and quantum spin liquids in $3+1$D, Ann. Phys. (Amsterdam) **394**, 244 (2018).
[111] J. Davighi and N. Lohitsiri, Omega vs pi, and 6d anomaly cancellation, J. High Energy Phys. 05 (2021) 267.
[112] Z. Wan and J. Wang, Higher anomalies, higher symmetries, and cobordisms I: Classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory, Ann. Math. Sci. Appl. **4**, 107 (2019).