Newton-like equations for a radiating particle: The nonrelativistic limit

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A broad class of forces, P, is identified, for which the Abraham-Lorentz-Dirac (ALD) and Newton-like equations have solutions in common. Moreover, these solutions do not present preacceleration or escape into infinity (runaway behavior). Any continuous or piecewise continuous force can be represented in terms of functions belonging to this class P. It was also argued that the set of common solutions of both sets of equations is wider, and it was proved that these solutions could be formulated in terms of generalized functions. The existence of such generalized functions motions is explicitly demonstrated for the relevant example of the instantaneously applied constant force, for which the respective solution of the ALD equation exhibits a lack of causality and runaway motion. In this case, the expressions for the position and velocity of the particle are formulated in terms of generalized functions of time, having only a point support at the time that the force is applied. Thus, both the velocity and the position are discontinuous at the instant that the force was applied. The solution for a particle moving between the plates of a capacitor reproduces the one obtained by A. Yaghjian from his discussion for extended particles. This outcome suggests a possible link or equivalence between both studies. A solution, common to the Newton-like and the ALD equations, for a constant homogeneous magnetic field is also presented. The extension of the results obtained to a relativistic context will be studied in future works.

I. INTRODUCTION

It has been known for more than 100 years that an accelerated charge radiates, and that the effects of this radiation on the charge itself can be described as a reaction force that acts as a friction on the particle. Surprisingly, also during this last century, great effort has been made to find an appropriate equation to describe the motion of a charged particle. Great physicists, starting with Abraham and Lorentz, have worked on this issue and already today there are hundreds of articles in the literature dealing with the topic [1–31]. The Abraham-Lorentz-Dirac (ALD) equations were introduced more than a century ago to describe the interaction of a charged particle with its own electromagnetic field and its relativistic version was developed by Dirac in 1938 [5].

The treatment of this problem by classical electromagnetism has been plagued with difficulties, although this theory has been used to obtain equations that very accurately describe radiation for the most important applications. Starting from Maxwell's equations for the particle's field and Newton's equations for the particle, an equation of motion was obtained, one that included the forces produced by the particle's radiated field. This system of equations proved to be suitable for the description of a macroscopic particle, yet not for a point particle or even for small enough particles. In the low speed limit, these equations are called the nonrelativistic ALD equations. These equations incorporate the effects of the radiation produced by this classical charged particle on the particle itself; that is, they include a term for the reaction force due to the radiation. The fact that these equations of motion are not of second order, that is, there is a third order derivative of position involved, leads to fundamental issues for the solution, such as noncausality, runaway behavior, or contradictions with special relativity. One of the main reasons behind such problems is that the punctual charge's field, supposedly Coulombian, diverges around the position of the particle itself. This situation causes the need for a renormalization of the mass and the existence of the *runaway* solutions. Nevertheless, the works of Poincaré and Dirac solved the contradiction with special relativity but the other problems remained [5].

Some authors maintain that these difficulties can be avoided if the assumption of the electron as a point particle is eliminated, because if the gravitational effects are negligible, then a point particle with finite charge and mass is impossible in classical physics. For example, in Refs. [9,10] the authors affirm that the particle should be

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considered as extended and they derive a second-order equation of motion, which they claim to be exact, assuming the existence of an internal structure within the charged particle. Reference [8] is another work that sustains this line of thought, arguing that for forces that do not depend analytically on time, the ALD equations cannot be derived exactly from the coupled equations for the particle reacting to its own electromagnetic field. Likewise, in Ref. [25] the authors introduce a special form of the extended structure that also removes the *runaway* behavior and preacceleration.

Also, in a different line of thought, some authors maintain that certain physically coherent initial conditions should be imposed upon the solutions of the equation of motion. Some of them state that this procedure could provide an exact equation. This idea was originally proposed by Dirac himself [5]. In the article [11], and some of its references, this point of view has been thoroughly examined. In these works, special constraints have been proposed in order to prevent undesirable properties to show up in the solutions of the ALD equations.

Another variation of the ALD equations consists in developing a solution through several iterations, and generalizing each term into a series. This equation is known as the Landau-Lifshitz (LL) one, which has been carefully studied by some authors.

However, an also numerous group of authors consider the ALD equations as rigorously valid. Therefore, it can be concluded that there is still a heated discussion about the ALD equation.

In Ref. [31], a discussion was presented that can be considered within the just mentioned line of thought. A Newton-like set of equations was examined, whose solutions also solved the ALD equations. In addition, a covariant generalization of these Newton-like equations was proposed, whose motions also satisfy the ALD equations. Although the proposed Newton-like equations suggest that there is no preacceleration or runaway in their solutions, the relationship between the two kinds of solutions, those of the Newton-like equations and those of the ALD ones, was not clear from the beginning. In particular, it was not understood if the ALD equations have or not more physically coherent solutions than the Newton-like ones. This question is still open and is one of the main issues that will be addressed in further works. Its understanding will define if the solutions of the Newton like equations contain or not the whole spectrum of possible motions of the radiating particle.

The present work begins the study of the connections between the solutions of the two types of equations. We start here with their nonrelativistic version, which deals with fewer conceptual difficulties. The results of this first study are expected to be helpful in understanding the relationship between the relativistic forms of both equations. In addition, we will concentrate on generalizing the discussion to include solutions in terms of generalized functions. Then, we will study the motion for an instantaneously applied force described by Heaviside's unitary step function, which is the most relevant case that shows noncausal and runaway behavior in this context.

Our first goal is to introduce a class of forces for which the Newton-like equations are convergent, and therefore, well defined. This set consists of all forces $f^i(t)$ represented by polynomials of the time t. These functions then form the space P. Therefore, by virtue of the Weierstrass approximation theorem, it follows that with functions of this kind, it becomes possible to approximate any continuous or piecewise continuous force as much as it is desired in any closed time interval.

Next, another goal of this study is to formulate non-relativistic solutions common to both equations in terms of generalized functions. For that, we will employ functions of the space P. Then, we will explicitly verify that these solutions are causal and do not show runaway behavior. The solvability of these equations is checked in the sense of generalized functions, according to the theoretical framework introduced by Colombeau and Egorov in [32–34]. This modern approach to generalized functions can be applied even to nonlinear problems, in contrast to the more classical theory of Schwartz and Sobolev [32].

Further, we focus on explicitly solving the particular case of the instantaneously applied constant force. An exact formula is obtained for the coordinates and velocities, once the limits of the sequences that define the solutions are calculated for all times except the instant the force is applied. It follows that the coordinates and the velocities of the particle have discontinuous jumps at the instant in which the force is applied. The absence in the literature of this kind of solution is justified, since it violates the usual rule that the coordinates of the particle do not suddenly change. In fact, this is not the case, since forces depending on derivatives of the Dirac Delta function show up. These kinds of forces clearly lead to discontinuous jumps in the initial coordinates of the particle. Moreover, this property represents a contradiction with special relativity, since it requires speeds greater than the speed of light. However, in the framework of the nonrelativistic movement under consideration, this property still does not represent a conflict. Then, it is in our best interest to verify that there exists a relativistic solution whose limit, when the speed of light becomes infinite, leads to the nonrelativistic solution. This is expected to be studied in subsequent works.

It can be helpful here to further insist in the physical relevance of the forces represented by Heaviside's function. For example, they are relevant in particle accelerators, in all kinds of experiments in which many charged particles suddenly enter regions where electric or magnetic forces exist. Thus, the effective forces acting on them can be accurately represented as instantaneously applied and constant for short time periods. For these reasons this simple example has been frequently discussed when the ALD equations are studied. As a matter of fact, it is

precisely for these forces that some solutions of the ALD equations predict nonphysical results for the motion: runaway solutions and lack of causality. It is mainly because of this, that this example will have a central role in this work.

However, as it was stated before, we intend to present a general discussion about how to define a solution in terms of generalized functions, specifically to address the problem of the force represented by a Heaviside function. Moreover, the solution we developed was causal and did not show runaway behavior.

Actually, the aforementioned problems were not discussed in Ref. [31], as no solutions were formulated in the sense of generalized functions. However, that initial analysis was required to proceed to the study of the problem with a force represented by a Heaviside function.

We estimate that this study explores from a new perspective the difficult problem of finding consistent solutions to the ALD equations that do not show any kind of noncausal or runaway behavior. These solutions should be easier to find in a nonrelativistic context, as a first approach to this problem. This problem has already been solved, at least for the nonrelativistic ALD and the Newton-like equations. Moreover, since we now included solutions formulated in terms of generalized functions, there are now more solutions available for both equations and so, it seems more likely to find properly consistent solutions.

It is of utter importance in this work that the solutions found for a particle moving between the two plates of an electric capacitor that produces a constant electric field coincide with the ones in [8]. The developed nonrelativistic solution allows us to directly evaluate the energy and momentum radiated by the particle at the input and the output of the capacitor. This result suggests that the analysis presented here is perhaps equivalent to the general discussion developed in Ref. [8]. Also, in that work, it was proven that if the spatial extension of the particle is taken into account, the ALD equation ceases to be valid. We expect to assess the possible equivalence between the analysis done here and that of Ref. [8] once the results of this work are extended to the relativistic context.

We also present the exact solution of the Newton-like equations for the case of an homogeneous magnetic field. This solution also satisfies the nonrelativistic ALD equation.

The work is structured as follows. In Sec. II, it is checked that well-defined solutions of the Newton-like equations are also exact solutions of the nonrelativistic ALD equation. Thereafter, in Sec. III, a new class of solutions for the Newton-like equation is identified, that is, a kind of solution that can be represented by polynomial functions of time. Then, it is in Sec. IV that the main result of this work is presented: there are solutions for both ALD and the Newton-like equations that can be expressed in terms of successions of functions defined by these aforementioned polynomials of the time t, in the modern sense of generalized functions according to Colombeau and

Egorov [32–34]. In Sec. V, the existence of this kind of solution is proven with an example relevant to the subject, that of the suddenly applied constant force. Further, in Sec. VI, the solution for a charge moving between the plates of an electric capacitor is developed. Moreover, this solution turned out to be the very same that A. Yaghjian developed in [8] for a radiating extended charge. Finally, in Sec. VII, the solution of the Newton-like equation for an homogeneous magnetic field is found. Then, it is shown that this solution satisfies the ALD equations as well.

II. THE NEWTON-LIKE EQUATIONS

Let us consider a nonrelativistic motion of a particle through space-time described by a trajectory C defined by the curve $x(t) = (x^1(t), x^2(t), x^3(t))$ and parametrized by time t. All along this work we will employ natural units, in which distance and time are measured in cm and the mass in cm^{-1}.

In this case, the ALD equations are

$$ma^i(t) - f^i(t) = k\frac{da^i(t)}{dt}, \tag{1}$$

$$v^i(t) = \dot{x}^i(t) = \frac{dx^i(t)}{dt}, \tag{2}$$

$$a^i(t) = \dot{v}^i(t) = \frac{dv^i(t)}{dt}, \tag{3}$$

where the index i has the three values $i = 1, 2, 3$.

Then, a class of exact solutions of Eq. (1) is defined by the so-called Landau-Lifshitz series [13,31]. Below, we will refer to these equations as the Newton-like equations, explicitly defined as

$$a^i(t) = \frac{1}{m}\sum_{n=0}^{\infty}\frac{d^n}{dt^n}f^i(t)\left(\frac{k}{m}\right)^n. \tag{4}$$

That the solutions of Eq. (4) also solve the ALD ones can be easily checked considering that

$$\begin{aligned}
\dot{a}^i(t) &= \frac{1}{m}\sum_{n=0}^{\infty}\frac{d^{n+1}}{dt^{n+1}}f^i(t)\left(\frac{k}{m}\right)^n \\
&= \frac{1}{k}\sum_{n=0}^{\infty}\frac{d^{n+1}}{dt^{n+1}}f^i(t)\left(\frac{k}{m}\right)^{n+1} \\
&= \frac{1}{k}\left[\sum_{n=0}^{\infty}\frac{d^n}{dt^n}f^i(t)\left(\frac{k}{m}\right)^n - f^i(t)\right] \\
&= \frac{1}{k}\left[ma^i(t) - f^i(t)\right]. \tag{5}
\end{aligned}$$

This form of the Landau-Lifshitz series is a valid solution for accelerations small enough to be considered nonrelativistic, as it is stated in Refs. [13,29]. However, the

Landau-Lifshitz series may diverge and constitutes an asymptotic series of the solution of the ALD equation. It should be remarked, that, up to our knowledge, a proof does not exist requiring that the Landau-Lifshitz series should converge for a force to be physically meaningful. If such a condition results are in fact valid, it constitutes a proof of the equivalence between the ALD and Newton-like equations. The important point for us in what follows is that, assuming that the series is well defined, the path $x^i(t)$ exactly satisfies the nonrelativistic ALD equation:

$$ma^i(t) - f^i(t) = k\dot{a}(t). \qquad (6)$$

III. A WIDE CLASS OF SOLUTIONS FOR THE NEWTON-LIKE EQUATIONS

The aforementioned Newton-like equation for the non-relativistic ALD equation has the form

$$a^i(t) = \frac{1}{m} \sum_{n=0}^{\infty} \frac{d^n}{dt^n} f^i(t) \left(\frac{k}{m}\right)^n.$$

Note that the name, Newton-like, means that the equation has the form of a Newtonian equation in which the acceleration is defined by a series. However, the operator $\hat{O}$

$$\hat{O}f^i(t) = \frac{1}{m} \sum_{n=0}^{\infty} \frac{d^n}{dt^n} f^i(t) \left(\frac{k}{m}\right)^n, \qquad (7)$$

may not converge for a wide class of functions. Therefore, the answer to the following question is not clear: Can any force $f^i(t)$, for example, a continuous one, be linearly approximated by functions belonging to the class for which the series converges?.

The presence of arbitrary time derivatives in the Newton-like equation directly indicates that if the components of the force $f^i(t)$ are polynomial of the time variable t, the series will always be convergent and the finite expression obtained will be an exact solution of the ALD equation. We will call the set of all polynomial functions of time defined in an arbitrary and closed interval (a, b) as P.

The number of possible solutions in this class of functions is large. This is a direct consequence of the so-called Weierstrass approximation theorem: *Any continuous function on the interval (a, b) can be uniformly approximated by polynomials on this interval.*

This theorem implies that it is possible to arbitrarily approximate continuous forces by functions belonging to P. In addition to this, the theorem has been generalized for piecewise continuous functions like the Heaviside's unitary step function.

IV. FORMAL SOLUTION IN TERMS OF GENERALIZED FUNCTIONS

As the class of functions P seems to be general enough, the formulation of solutions common to the Newton-like and the ALD equations within the set of continuous (or piecewise continuous) forces seems feasible. The solutions appear as successions of functions that satisfy the ALD equations in the sense of the generalized functions and belong to the set P. That is, the ALD equations after being multiplied by a test function (infinitely differentiable and with compact support) and integrated, vanish, after taking the limit of the index of their respective associated succession to infinity.

Let us first define a succession of functions, belonging to the previously defined set P, as follows. We will assume that the force $f_i(t)$ is a continuous (or piecewise continuous) function of time and that it can have a divergent LL series for certain values of the time. Then, $f^i(t)$ can be approximated around the instant in which the LL series diverges, with arbitrary precision due to the Weierstrass approximation theorem as

$$\lim_{n \to \infty} f_n^i(t) = f^i(t), \qquad (8)$$

where the expansion $f_n^i(t)$ is a polynomial function in t pertaining to P. Let us now consider a force with a LL series showing divergent behavior at time t and, as well, let us assume that the interval $a \leq t \leq b$ is completely included in the larger interval $0 < t < 1$.

Let us further impose that α and β are such that

$$0 < \alpha < a < b < \beta < 1. \qquad (9)$$

Then, the arbitrarily precise polynomial approximation of the force $f^i(t)$ can be expressed in the form defined in Ref. [35] by the formula

$$f_n^i(t) = \frac{\int_\alpha^\beta f^i(s)[1 - (s - t)^2]^n ds}{\int_{-1}^1 [1 - (s - t)^2]^n ds}. \qquad (10)$$

This representation can be conceived as the integral of the force multiplied by very localized contributions to the force over all the space in which the force is defined. Intuitively, this can actually be interpreted as an integral over all the punctual components of the force over the range of definition, ergo, as the sum of infinitely many contributions of Dirac Delta functions of the argument $(t - s)$. The elements of this succession are plotted for $n = 500$, 1000, 2000, 10000 in the Fig. 1 for the one-dimensional case $i = 1$ and $f^i = 1$.

A. The existence of generalized solutions

Let us now consider the aforementioned formal proof that solutions in the sense of the generalized functions of
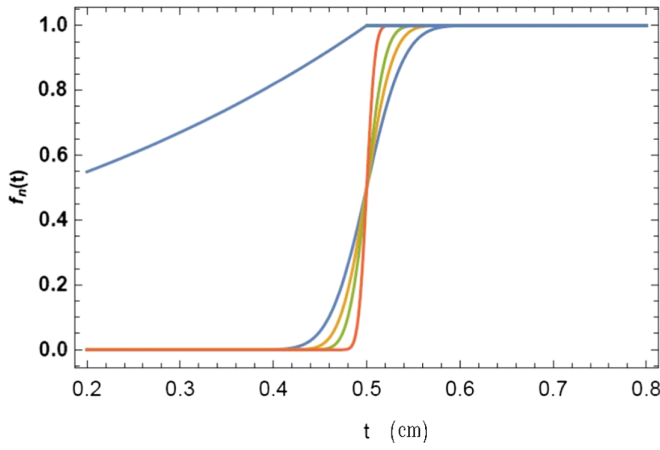
FIG. 1. The figure shows four elements of the sequence that approximates the constant force applied at time $t = \frac{1}{2}$ cm (the time and distance are measured in natural units cm), for the index values: $n = 500, 1000, 2000, 10000$. The effective force associated with the noncausal but controlled solution of the ALD equation is also plotted in blue.

the ALD equations can exist as sequences of functions defined in the space P.

The satisfaction of the ALD equation in the sense of the Colombeau-Egorov theory of generalized functions [32–34] in the internal region of the interval (a, b) can be shown if it is possible to define in P a succession of coordinates, $x_n^i(t)$, $n = 1, 2, \ldots \infty$ for which the following limit vanishes

$$E = \lim_{n \to \infty} \int_a^b \phi(t)[m(1 - \tau \partial_t) a_n^i(t) - f_n^i(t)] dt$$
$$= 0, \tag{11}$$

for arbitrary values of the test functions of time $\phi(t)$ [infinitely differentiable and with compact support contained in the interval (a, b)]. The other quantities related to the equations above are the velocities and accelerations

$$v_n^i(t) = \frac{d}{dt} x_n^i(t),$$

$$a_n^i(t) = \frac{d}{dt} v_n^i(t).$$

Then, we can define the following sequence of functions:

$$a_n^i(t) = \frac{1}{m} \sum_{l=0}^{N_n} \tau^i \frac{d^l}{dt^l} f_n^i(t), \qquad n = 1, 2, 3, \ldots \tag{12}$$

formulated in terms of the force functions $f_n^i(t)$. Also, N_n is an integer that tends to infinity when n also does and it fulfills the condition:

$$N_n + 1 > 2n. \tag{13}$$

Below, we will argue that the succession (12) satisfies the ALD equations in the sense of the generalized functions [32–34] in the limit $n \to \infty$.

Substituting (12) in the solvability condition, we obtain

$$E = \lim_{n \to \infty} \int \phi(t) \left[(1 - \tau \partial_t) \sum_{m=0}^{N_n} \tau^m \frac{d^m}{dt^m} f_n^i(t) - f_n^i(t) \right] dt$$

$$= \lim_{n \to \infty} \int -\phi(t) \tau^{Nn+1} \frac{d^{N_n+1}}{dt^{N_n+1}} f_n^i(t) dt$$

$$= 0, \tag{14}$$

which vanishes due to relation (13), since the only derivative appearing acting on the polynomials $f_n^i(t)$ is zero. That is, the number of derivatives is greater than the order of the polynomial on which the derivatives act in the integrand of (14). Therefore, the defined sequence satisfies the ALD equations in the sense of generalized functions.

Let us now show that the Newton-like equations satisfy their corresponding solvability condition in the sense of generalized functions. After substituting (12) into that equivalent condition, we get

$$E = \lim_{n \to \infty} \int \phi(t) \left[\sum_{m=0}^{N_n} \tau^m \frac{d^m}{dt^m} f_n^i(t) - a_n^i(t) \right] dt$$

$$= 0. \tag{15}$$

which vanishes directly due to the same definition of the sequences (12). In this case the integrand in (15) is exactly zero if the sequences turn out to be properly defined for all the sequence indices and parameters.

Thus, the existence of solutions to the ALD and the Newton-like equations in the form of infinite successions of functions defined in the space P seems plausible. Although, for them to effectively exist, it is required that the infinite sequences should be also well defined for all the sequence indices and parameters before taking the limit. This condition seems to be assured by the Weierstrass theorem.

Finally, we will prove in the next section that such generalized functions actually exist. To support this claim, we will study the particular case of a suddenly applied constant force. It can be underlined that this system had been a relevant primer in connection with the discussions of the properties of radiating particles along the times.

V. THE INSTANTANEOUSLY APPLIED CONSTANT FORCE

Let us now consider the example of the instantaneously applied constant force. As commented before, the resolution of this problem will conclude our analysis in the present paper on the generalized functions of the space P.

Let us assume that the force is applied at a time $t = \frac{1}{2}$. Then, this force can be represented by a Heaviside unitary step function. Now, let us shift the argument of that unitary step function by $\tau = \frac{1}{2}$, then:

$$f^i(t) = f^i \bar{\theta}\left(t - \frac{1}{2}\right)$$
$$= f^i \theta(t),$$

where $\bar{\theta}(t)$ is the usual Heaviside function.

$$\bar{\theta}(t) = \begin{cases} 0 & \text{for } t < 01 \\ 1 & \text{for } t \geq 0 \end{cases}. \tag{16}$$

The constants a, b, α, β will be chosen with the values

$$a = 0.001, \tag{17}$$

$$\alpha = 0.002, \tag{18}$$

$$\beta = 0.998, \tag{19}$$

$$b = 0.999. \tag{20}$$

Then, integrating the expression of the acceleration succession, the formula for the velocity associated with the solution $a_n^i(t)$, in the form of a succession in the interval (a, b) (after taking as initial condition the velocity v_0^i at an instant $t_0 < \frac{1}{2}$) becomes

$$v_n^i(t) = v_0^i + \int_{t_0}^t a_n^i(t') dt'$$

$$= v_0^i + \frac{1}{m} \sum_{l=0}^{N_n} \tau^l \int_{t_0}^t \frac{d^l}{ds^l} f_n^i(s) ds$$

$$= v_0^i + \frac{f^i}{m} \int_{t_0}^t \theta_n(t') dt' + \frac{f^i}{m} \tau[\theta_n(t) - \theta_n(t_0)]$$

$$+ \frac{f^i}{m} \tau^2 \delta_n\left(t - \frac{1}{2}\right) + \frac{f^i}{m} \sum_{l=3}^{N_n} \tau^l \left[\frac{d^{l-2}}{dt'^{l-2}} \delta_n\left(t' - \frac{1}{2}\right)\right]\Big|_{t'=t}, \tag{21}$$

where we defined

$$\frac{d}{dt} \theta_n(t) = \frac{d}{dt} \bar{\theta}_n\left(t - \frac{1}{2}\right)$$

$$= \delta_n\left(t - \frac{1}{2}\right), \tag{22}$$

in which $\delta_n(t - \frac{1}{2})$ is a polynomial regularization of the Dirac Delta centered at $t = \frac{1}{2}$.

Integrating again and assuming the initial condition for the coordinate as x_0^i we have for the coordinates the expression

$$x_n^i(t) = x_0^i + v_0^i(t - t_0) + \int_{t_0}^t v_n^i(t') dt'$$

$$= x_0^i + v_0^i(t - t_0) + \frac{f^i}{m} \sum_{l=0}^{N_n} \tau^l \int_{t_0}^t dt'' \int_{t_0}^{t''} \frac{d^l}{dt'^l} \theta_n(t') dt'$$

$$= x_0^i + v_0^i(t - t_0) + \frac{f^i}{m} \int_{t_0}^t dt'' \int_{t_0}^{t''} \theta_n(t') dt'$$

$$+ \frac{f^i}{m} \tau \int_{t_0}^t [\theta_n(t') - \theta_n(t_0)] dt' + \frac{f^i}{m} \tau^2 [\theta_n(t) - \theta_n(t_0)]$$

$$+ \frac{f^i}{m} \sum_{l=3}^{N_n} \tau^l \left[\frac{d^{l-3}}{dt'^{l-3}} \delta_n\left(t' - \frac{1}{2}\right)\right]\Big|_{t'=t}. \tag{23}$$

The terms of higher order in the derivatives in the respective equations for the velocity and the coordinates have the forms

$$T_v^n(t) = \frac{f^i}{m} \sum_{l=3}^{N_n} \tau^l \left[\frac{d^{l-2}}{dt'^{l-2}} \delta_n\left(t' - \frac{1}{2}\right)\right]\Big|_{t'=t}, \tag{24}$$

$$T_x^n(t) = \frac{f^i}{m} \sum_{l=3}^{N_n} \tau^l \left[\frac{d^{l-3}}{dt'^{l-3}} \delta_n\left(t' - \frac{1}{2}\right)\right]\Big|_{t'=t}, \tag{25}$$

where N_n can be substituted by infinity, because for every $l > N_n$ the number of derivatives present in both relations acting on the polynomials, that define the functions $\delta_n(t')$, make all these terms vanish. But, taking then the limit $n \to \infty$, for all times t different than the moment in which the force was applied $t = \frac{1}{2}$, we have

$$\lim_{n\to\infty} T_v^n(t) = \frac{f^i}{m} \sum_{l=3}^{\infty} \tau^l \left[\frac{d^{l-2}}{dt^{l-2}} \lim_{n\to\infty} \delta_n\left(t - \frac{1}{2}\right)\right]\Big|_{t\neq\frac{1}{2}}$$

$$= \frac{f^i}{m} \sum_{l=3}^{\infty} \tau^l \left[\frac{d^{l-2}}{dt^{l-2}} \delta\left(t - \frac{1}{2}\right)\right]\Big|_{t\neq\frac{1}{2}}$$

$$= 0, \tag{26}$$

$$\lim_{n\to\infty} T_x^n(t) = \frac{f^i}{m} \sum_{l=3}^{\infty} \tau^l \left[\frac{d^{l-3}}{dt^{l-3}} \lim_{n\to\infty} \delta_n\left(t - \frac{1}{2}\right)\right]\Big|_{t\neq\frac{1}{2}},$$

$$= \frac{f^i}{m} \sum_{l=3}^{\infty} \tau^l \left[\frac{d^{l-3}}{dt^{l-3}} \delta\left(t - \frac{1}{2}\right)\right]\Big|_{t\neq\frac{1}{2}},$$

$$= 0. \tag{27}$$

That is, except for the time in which the force was applied $t = \frac{1}{2}$, the corrections to the velocities and coordinates determined by these two terms are zero, because all the derivatives of the Delta function vanish outside their point support. Therefore, the following expressions for the velocity and the coordinates fully determine the values of these two magnitudes at all times except for the moment in which the force is applied $t = \frac{1}{2}$

$$v^i(t) = v_0^i + \frac{f^i}{m}\int_{t_0}^t \theta(t')dt' + \frac{f^i}{m}\tau\theta(t) + \frac{f^i}{m}\tau^2\delta\left(t - \frac{1}{2}\right),$$

$$(28)$$

$$= v_0^i + \frac{f^i}{m}t\theta(t) + \frac{f^i}{4m}\delta\left(t - \frac{1}{2}\right), \qquad (29)$$

$$x^i(t) = x_0^i + v_0^i(t - t_0) + \frac{f^i}{m}\int_{t_0}^t dt'' \int_{t_0}^{t''} \theta(t')dt'$$

$$+ \frac{f^i}{m}\tau\int_{t_0}^t [\theta(t') - \theta(t_0)]dt' + \qquad (30)$$

$$\frac{f^i}{m}\tau^2[\theta(t) - \theta(t_0)] \qquad (31)$$

$$= x_0^i + v_0^i(t - t_0) + \frac{f^i}{2m}\left(t - \frac{1}{2}\right)^2\theta(t)$$

$$+ \frac{f^i}{2m}\left(t - \frac{1}{2}\right)\theta(t) + \frac{f^i}{4m}\theta(t), \qquad (32)$$

$$= x_0^i + v_0^i(t - t_0) + \frac{f^i}{2m}\theta(t)\left(t^2 + \frac{1}{4}\right) \qquad (33)$$

in which it was considered that $\theta(t_0) = 0$.

Hence, the solution defined by the velocity $v^i(t)$ and coordinates $x^i(t)$, which are taken in the limit $n \to \infty$ at all times $t \neq \frac{1}{2}$, is well defined at every instant except for the moment the force was applied.

Let us illustrate the above result with an example, in order to clarify how the solutions in terms of generalized functions can be approximated by successions at every instant of time, except for the moment the force was applied. As it is difficult to plot the sum of all the terms, let us take the following approximation for the acceleration:

$$a_n^{(3)i}(t) = \frac{1}{m}\sum_{l=0}^3 \tau^i \frac{d^l}{dt^l} f_n^i(t), \qquad (34)$$

and to simplify the numbers, the ratio between the force and the mass will take the value

$$\frac{f}{m} = 1. \qquad (35)$$

Also, we will assume that the motion is in one dimension; therefore, we will only consider the component $i = 1$.

That is, to illustrate this solution we will omit the terms of order higher than 3 in the powers of τ and in the derivatives. More terms could be included, but what they produce is a greater complexity in the curves in the area very close in time to the point of connection of the force. Nevertheless, these terms vanish in the limit $n \to \infty$ for every instant $t \neq \frac{1}{2}$.

Figure 2 shows the expressions for the velocity that are obtained by integrating over time the terms of the successions of accelerations $a_n^{(3)i}(t)$ for a one-dimensional
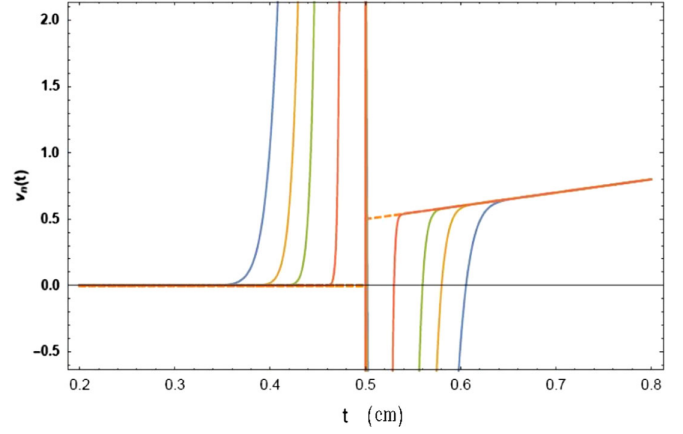

FIG. 2.   The iterative solutions for the velocity $v_n(t)$ in the approximation of the acceleration taken $a_n^{(3)}(t)$ in (34) as functions of the time in cm (the time and distance are measured in natural units cm). Notice how, as the index n of the succession associated to the solution in the form of a generalized function grows, the coordinates get closer and closer to the expression (41). The curves are reducing their width corresponding to the growth of their index n in the values $n = 500, 1000, 2000, 10000$.

problem (i takes a single value $i = 1$). The values of the indices of the sequence and the set of parameters are

$$n = 500, 1000, 2000, 10000, \qquad (36)$$

$$m = 1, \qquad f^i = 1, \qquad i = 1. \qquad (37)$$

For the initial conditions for the velocity and the coordinate the following values were chosen:
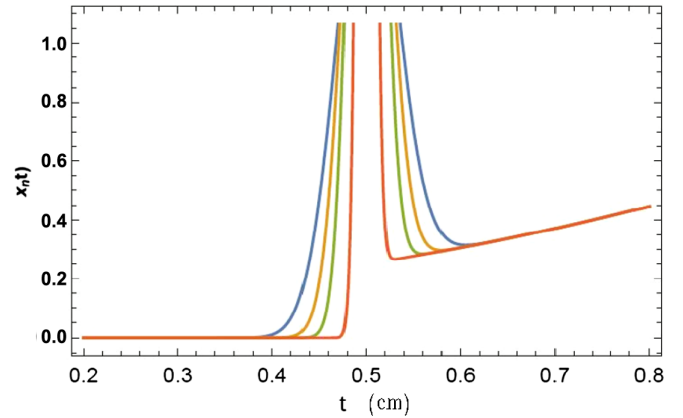

FIG. 3.   Iterative solutions for the coordinate $x_n(t)$ for the approximation of the acceleration $a_n^{(3)}(t)$. Notice how as the index n of the succession associated with the solution in the form of a generalized function grows, the coordinates get closer and closer to the expression (41). The curves are reducing their width corresponding to the growth of their index n in the values $n = 500, 1000, 2000, 10000$. The time and distance are measured in natural units cm.

$$v(-1) = 0, \tag{38}$$

$$x(-1) = 0. \tag{39}$$

As it can be seen in Fig. 2, for times different than the moment the force was applied $t = \frac{1}{2}$ and as the value of the index n grows, the elements of the succession of velocities $v_n(t), n = 500, 1000, 2000, 10000$ are getting closer and closer to the values of the velocities of the solution (28)

$$v(t) = t\bar{\theta}\left(t - \frac{1}{2}\right). \tag{40}$$

The same behavior can be seen in Fig. 3 for the succession of coordinates $x_n(t)$, $n = 500, 1000, 2000, 10000$, which approximates the coordinate values also given by the solution (31)

$$x(t) = \left(\frac{t^2}{2} + \frac{1}{8}\right)\bar{\theta}\left(t - \frac{1}{2}\right). \tag{41}$$

VI. ELECTRON MOVING BETWEEN THE PLATES OF A CAPACITOR

Next, we will proceed to compare our solution for the case of an instantaneously applied constant force with the one obtained by A. Yaghjian in Ref. [8]. In this discussion the author poses the problem of a charged particle that accelerates between the parallel plates of a capacitor that produces a constant electric field E_0. The particle is released at $t = 0$ from a capacitor plate and leaves the capacitor through a small hole in the opposite plate at time $t = t_2$. The causal and controlled solutions for acceleration and velocity proposed by Yaghjian for this problem are as follows:

$$\nu'(t) = \frac{eE_0}{m} \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } 0^+ < t < t_2, \\ 0 & \text{for } t_2^+ < t \end{cases} \tag{42}$$

$$\nu(t) = \begin{cases} 0 & \text{for } t < 0 \\ \Delta\nu_1 + \frac{eE_0}{m}t & \text{for } 0^+ < t < t_2, \\ \Delta\nu_{21} + \frac{eE_0}{m} & \text{for } t_2^+ < t \end{cases} \tag{43}$$

where $\Delta\nu_1 = \nu(0^+)$, $\Delta\nu_2 = \nu(t_2)$, and $\Delta\nu_{21} = \Delta\nu_1 + \Delta\nu_2$. On the other hand, the causal and controlled solutions previously presented by us for velocity and acceleration for every instant of time except for the instant in which the force was applied becomes

$$a^i(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{f^i}{m} & \text{for } t > 0 \end{cases}, \tag{44}$$

$$v^i(t) = \begin{cases} v^i(t_0) & \text{for } t < 0 \\ v^i(t_0) + \frac{f^i}{m}\left(t + \frac{k}{m}\right) & \text{for } t > 0 \end{cases}. \tag{45}$$

From here we see that if we take $f^i = eE_0$ and make $v^i(t_0) = 0$, $\frac{f^i k}{m^2} = \Delta\nu_1$ then both solutions coincide for every instant of time in which the constant force acts except for that instant in which the force was applied. This constitutes an important result if we take into account that in Ref. [8] this conclusion is obtained after assuming the finite spatial extension of the particle. Therefore, the coincidence suggests the possible equivalence between the description presented in this work and the one proposed by A. Yaghjian. Thus, we consider it of great interest to verify this possible equivalence in further works once the results obtained are extended to a relativistic context.

VII. SOLUTION FOR A CONSTANT MAGNETIC FIELD

In this section we want to present the solution of the nonrelativistic Newtonian equation for the important case of a particle moving in a constant magnetic field. Our objective is to verify that the solution of this problem reproduces the movement of the particle in the same magnetic field but for the nonrelativistic ALD equation. The solution to this problem with the ALD equations was found in Ref. [14]; the resulting motion is described by a spiral that tends to a point for sufficiently long times.

The solution that we will show below reproduces this behavior, and then provides another example of motion that simultaneously satisfies the Newtonian-like equations and the ALD ones [13,29,31].

Consider the force vector in the form $f^i(t)$ as the Lorentz force in a constant magnetic field

$$f^i(t) = q\epsilon^{ijk}v^j(t)B^k, \tag{46}$$

where the indices $i, j, k = 1, 2, 3$, $\epsilon^{i,j,k}$ are the Levi-Civita symbol, which stands for the vector product, and q is the charge of the particle. For simplicity the magnetic field will be written in the form

$$B^k = B\delta^{k3}, \tag{47}$$

where δ^{ki} is the Kronecker Delta with indices k, i.

After substituting the expression for the Lorentz force in the Newtonian equations they take the form

$$a^i(t) = \frac{1}{m} \sum_{l=0}^{\infty} \left(\frac{\kappa}{m}\right)^l \frac{d^l}{dt^l} f^i(t),$$

$$\frac{d}{dt} v^j(t) = \frac{qB}{m} \epsilon^{ij} \sum_{l=0}^{\infty} \left(\frac{\kappa}{m}\right)^l \frac{d^l}{dt^l} v^j(t), \qquad (48)$$

where the two-index tensor ϵ^{ij} is defined in terms of the Levi-Civita symbol as

$$\epsilon^{ij} = \epsilon^{ij3}. \qquad (49)$$

Let us separate Eq. (48) into two equations, one for the motion in the plane orthogonal to the magnetic field and the other for the motion parallel to it

$$\frac{d}{dt} v^j(t) = \frac{qB}{m} \epsilon^{ij} \sum_{l=0}^{\infty} \left(\frac{\kappa}{m}\right)^l \frac{d^l}{dt^l} v^j(t), \qquad i = 1, 2, \quad (50)$$

$$\frac{d}{dt} v^3(t) = 0. \qquad (51)$$

From here we observe that the movement parallel to the field is uniform

$$v^3(t) = v_0^3, \qquad (52)$$

$$x^3(t) = v_0^3 t + x_0^3. \qquad (53)$$

For the movement in the orthogonal plane, it will be useful to define the complex variable

$$v(t) = v^1(t) + iv^2(t). \qquad (54)$$

But, since multiplying a complex number by the imaginary unit is equivalent to rotating the corresponding 2D vector to $\frac{\pi}{2}$ from the x_1 axis towards the x_2 axis, we can establish the following equivalence:

$$V'^i = \epsilon^{ij} V^j \equiv V' = iV. \qquad (55)$$

Then, the equation for the motion in the plane orthogonal to the field (50) can be written in a complex expression as follows:

$$\frac{d}{dt} v(t) = \frac{iqB}{m} \sum_{l=0}^{\infty} \tau^l \frac{d^l}{dt^l} v(t), \qquad (56)$$

$$\tau = \frac{\kappa}{m}. \qquad (57)$$

Considering that the equation is invariant to a shift in time $t \rightarrow t + a$, we will look for exponential solutions with the form

$$v(t) = \exp(iwt) v_0. \qquad (58)$$

Substituting this expression in (56) we have

$$iwv_0 = \frac{iqB}{m} \sum_{l=0}^{\infty} (i\tau w)^l v_0 \qquad (59)$$

$$= \frac{iqB}{m} \frac{1}{1 - i\tau w} v_0. \qquad (60)$$

This formula can be written as

$$\left(w^2 + i\frac{w}{\tau} - \frac{iqB}{\tau m}\right) v_0 = 0. \qquad (61)$$

Therefore, this relation implies that the parameter w satisfies

$$w^2 + i\frac{w}{\tau} - \frac{iqB}{\tau m} = 0. \qquad (62)$$

The two solutions of this equation are

$$w = -\frac{i}{2\tau} \pm \frac{i}{2\tau} \sqrt{1 - \frac{4qB\tau}{m} i}. \qquad (63)$$

But, at the zero magnetic field limit the equation takes the form

$$\frac{d}{dt} v(t) = iwv(t) = 0,$$

implying that the physical solution of Eq. (62) is

$$w = -\frac{i}{2\tau} + \frac{i}{2\tau} \sqrt{1 - \frac{4qB\tau}{m} i}. \qquad (64)$$

The complex representation of the solution for the velocity is

$$v(t) = \exp\left[i\left(-\frac{i}{2\tau} + \frac{i}{2\tau}\sqrt{1 - \frac{4qB\tau}{m} i}\right) t\right] v_0. \qquad (65)$$

Therefore, for a real parameter v_0 the solutions for the velocity components are

$$v^1(t) = \mathrm{Re}\left\{ \exp\left[i\left(-\frac{i}{2\tau} + \frac{i}{2\tau}\sqrt{1 - \frac{4qB\tau}{m} i}\right) t\right] \right\} v_0, \quad (66)$$

$$v^2(t) = \mathrm{Im}\left\{ \exp\left[i\left(-\frac{i}{2\tau} + \frac{i}{2\tau}\sqrt{1 - \frac{4qB\tau}{m} i}\right) t\right] \right\} v_0, \quad (67)$$

$$v^3(t) = v_0^3. \qquad (68)$$

The components of the coordinate are obtained after integrating the expressions for the velocity
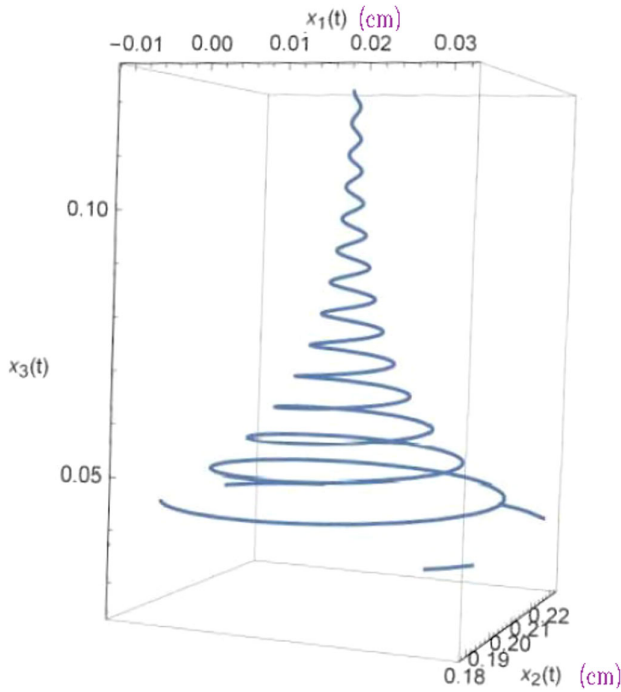
FIG. 4. The graph shows the resulting spiral motion for a charged particle in a constant magnetic field. The movement in the direction parallel to the field is uniform. The kinetic energy of the particle in the plane orthogonal to the field is constantly radiated and the motion becomes uniform at the end. The solution was found first in Ref. [14] after solving the ALD equation. The derivative here after solving the Newtonian equation was obtained in order to verify the exact equivalence between both equations. The spatial and time axis units are in the natural system of units *cm*.

$$x^1(t) = \text{Re}\left\{\frac{\exp\left[i\left(-\frac{i}{2\tau}+\frac{i}{2\tau}\sqrt{1-\frac{4qB\tau}{m}i}\right)t\right]-1}{i\left(-\frac{i}{2\tau}+\frac{i}{2\tau}\sqrt{1-\frac{4qB\tau}{m}i}\right)}\right\}v_0, \quad (69)$$

$$x^2(t) = \text{Im}\left\{\frac{\exp\left[i\left(-\frac{i}{2\tau}+\frac{i}{2\tau}\sqrt{1-\frac{4qB\tau}{m}i}\right)t\right]-1}{i\left(-\frac{i}{2\tau}+\frac{i}{2\tau}\sqrt{1-\frac{4qB\tau}{m}i}\right)}\right\}v_0, \quad (70)$$

$$x^3(t) = v_0^3 t + x_0^3. \quad (71)$$

The evolution of the coordinates in time is shown in Fig. 4. In this picture it can be observed how the particle radiates its kinetic energy while it advances, which determines the spiral shape of the trajectory in the plane perpendicular to the magnetic field.

A. Equivalence between Newtonian equations and ALD equations

Let us now show that the solution found satisfies the ALD equation as well. Then, the ALD equation associated to this case is

$$a^i(t) - \frac{eB}{m}\epsilon^{ij}v^j(t) = \tau\frac{d}{dt}a^i(t), \quad (72)$$

which in complex notation becomes

$$a(t) - i\frac{eB}{m}v(t) = \tau\frac{d}{dt}a(t). \quad (73)$$

But, the derivative of the acceleration, after taking into account the expression (64), becomes

$$\begin{aligned}
\tau\frac{d}{dt}a(t) &= \tau\frac{d^2}{dt^2}v(t) \\
&= \tau(iw)^2\exp(iwt)v_0 \\
&= \tau(iw)^2 v(t) \\
&= \left(\frac{1}{2\tau} - i\frac{qB}{m} - \sqrt{1-\frac{4qB\tau}{m}i}\right)v(t). \quad (74)
\end{aligned}$$

On the other hand, the left side of the ALD equation becomes

$$\begin{aligned}
a(t) - i\frac{eB}{m}v(t) &= \frac{d}{dt}v(t) - i\frac{eB}{m}v(t) \\
&= \left(iw - i\frac{eB}{m}\right)v(t) \\
&= \left(\frac{1}{2\tau} - i\frac{qB}{m} - \sqrt{1-\frac{4qB\tau}{m}i}\right)v(t). \quad (75)
\end{aligned}$$

Therefore, the solution found also satisfies the ALD equation.

VIII. SUMMARY

In the present work, we sought to formulate a causal and controlled solution for the nonrelativistic Abraham-Lorentz-Dirac equation. For this purpose, we focused on the search for a class of functions, which we called *P*, that would make the Landau-Lifshitz series convergent. Then, it turned out that the solutions formulated in the space spanned by such functions were equivalent to the solutions of the ALD equation. The following results were obtained:

(1) A wide class of forces was identified, for which exact solutions to both the ALD and the Newton-like equation could be formulated. This set of forces is composed of polynomial functions of the time *t* defined in an open interval of time. Also, these functions satisfy the Weierstrass theorem, which states that any continuous or piecewise continuous function can be approximated with arbitrary precision in the aforementioned interval. Thus, almost any force can be represented by the elements of this class.

(2) We argue the existence of a wider set of solutions common to the ALD and the Newton-like equations,

in the terms of generalized functions which were defined by sequences of functions of *P*. These solutions did not show any kind of noncausal behavior or uncontrolled growth.

(3) An important solution in terms of generalized functions was found. It was the solution for the instantaneously applied force, which might be considered the most relevant result of this work. As it is known, it is in this case that the ALD equations show the most distinctive difficulties associated with this equation, namely, runaway and preaccelerated behavior. It was explicitly shown how the successions defining the generalized functions get closer and closer to the solution for every instant different than the moment the force was applied when the index of the successions tends to infinity.

(4) Another important conclusion of the discussion is that the solution exactly reproduces the one obtained by A. Yaghjian [8] for the problem of a particle moving between the plates of a capacitor that produces a constant electric field. This conclusion suggests the possible equivalence between the description presented and Yaghjian's proposal in terms of the finite spatial extent of the particle.

(5) Similarly, the nonrelativistic Newtonian equation was applied to the important case of a charged particle moving in a constant magnetic field; the solution obtained reproduces the particle's motion determined by G. N. Plass in [14]. In that reference, the solution was derived after solving the ALD equation for this case. The motion derived here was shown to simultaneously satisfy the Newton-like as well as the ALD equations.

The conclusions drawn from this work, as well as the information acquired and questions that arose during the writing process, motivated new objectives to direct our efforts. A few points of interest for future studies are

(i) To extend the study to the relativistic case, thus formulating a more general discussion that allows verifying that the solution obtained here constitutes a particular limit case of a more general one.

(ii) Afterwards, to study how the relativistic solutions allows us to explain why, as it was seen in this work, a discontinuous jump appear in the initial conditions for the coordinates of the particle, for the instantaneously applied constant force. The idea is that the relativistic solutions can show a continuous time variation of the coordinates that translates into a discontinuous variation of the coordinates once the speed of light is taken in the infinite limit.

(iii) The most ambitious result that is expected to be obtained is to prove that a relativistic version of the Newton-like equation would be equivalent to the corresponding relativistic version of the ALD equations, yet avoiding the aforementioned drawbacks.

(iv) Investigate the possible equivalence between the analysis presented here and the one presented by A. Yaghjian in Ref. [8].

ACKNOWLEDGMENTS

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