Kähler geometry for $su(1,N|M)$ superconformal mechanics

Erik Khastyan,[1,*] Sergey Krivonos,[2,†] and Armen Nersessian[1,2,3,‡]

[1]*Yerevan Physics Institute, 2 Alikhanian Brothers Street, Yerevan 0036, Armenia*
[2]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia*
[3]*Institute of Radiophysics and Electronics, Ashtarak-2, Ashtarak 0203, Armenia*

(Received 27 October 2021; accepted 6 December 2021; published 7 January 2022)

We suggest the $su(1,N|M)$ superconformal mechanics formulated in terms of phase superspace given by the noncompact analogue of complex projective superspace. We parametrized this phase space by the specific coordinates allowing us to interpret it as a higher-dimensional superanalogue of the Lobachevsky plane parametrized by lower half-plane (Klein model). Then we introduced the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables, we demonstrated that the proposed scheme allows us to construct the $su(1,N|M)$ supeconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb-like systems with a $su(1,N|M)$ dynamical superalgebra and found that oscillatorlike systems admit deformed $\mathcal{N}=2M$ Poincaré supersymmetry, in contrast with Coulomb-like ones.

DOI: 10.1103/PhysRevD.105.025007

I. INTRODUCTION

Kähler manifolds are the Hermitian manifolds, which possesses the symplectic structure obeying the specific compatibility condition with the Riemann (and/or complex) structure [1]. Being highly common objects in almost all areas of theoretical physics, these manifolds usually appear as configuration spaces of the particles and fields. Only in a limited number of physical problems, they appear as phase spaces, mostly for the description of various generalizations of tops, the Hall effect (including its higher-dimensional generalizatons, see, e.g. [2] and Refs. therein), etc. Respectively, the number of the known nontrivial (super)integrable systems with Kähler phase spaces is very restricted, and their study does not attract much attention. The widely known integrable model with Kähler phase space extensively studying nowadays is the compactified Ruijsenaars-Schneider model with an excluded center of mass, whose phase space is complex projective space [3].

On the other hand, there are some indications that Kähler phase spaces can be useful for the study of conventional Hamiltonian systems, i.e., for the systems formulated on cotangent bundle of Riemann manifolds. A very simple example of such a system is one-dimensional conformal mechanics formulated in terms of a Lobachevsky plane ("noncompact complex projective plane") treated as a phase space [4]. Such a description, being quite elegant, allows immediate construction of $\mathcal{N}=2M$ superconformal extension associated with $su(1,1|M)$ superalgebra. Recently, the similar formulation of some higher-dimensional systems was given in terms of $su(1,N)$-symmetric Kähler phase space treated as the noncompact version of a complex projective space [5]. In such an approach, all symmetries of the generic superintegrable conformal-mechanical systems acquire interpretation in terms of the powers of the $su(1,N)$ isometry generators. The maximally superintegrable generalizations of the Euclidean oscillator/Coulomb systems has also been considered; all the symmetries of these superintegrable systems were expressed via $su(1,N)$ isometry generators as well. However, the supersymmetrization aspects of that systems was not considered there at all. In the present paper, we construct the $\mathcal{N}$-extended superconformal extensions of the systems considered in [5], as it was done in [4] for an one-dimensional case. Namely, we consider the systems with $su(1,N|M)$-symmetric $(N|M)_{\mathbf{C}}$-dimensional Kähler phase superspace (in what follows, we denote it by $\widetilde{\mathbb{CP}}^{N|M}$) and relate their symmetries with the isometry generators of the super-Kähler structure. We construct this superspace, reducing the $(N+1|M)_{\mathbf{C}}$-dimensional complex pseudo-Euclidean superspace by the $U(1)$-group action and then identify the reduced phase superspace with the noncompact analogue of complex projective superspace constructed in [6]. We parametrize this superspace by the complex bosonic variable

*khastyanerik@gmail.com
†krivonos@theor.jinr.ru
‡arnerses@yerphi.am

w, $\mathrm{Im}\, w < 0$, by the $N-1$ complex bosonic variables $|z^\alpha| \in [0, \infty)$, $\arg z \in [0; 2\pi)$, and by M complex fermionic coordinates η^A. Thus, it can be considered as the N-dimensional extension of the Klein model of Lobachevsky plane [7]. This allows us to connect the complex coordinate w with the radial coordinate and momentum of the conformal-mechanical system spanned by $su(1,1)$ subalgebra and separate the $su(1,1)$ generators interpreting them as Hamiltonian, conformal boosts, and dilatation operators. The rest bosonic generators z^α parametrize the angular part of integrable conformal mechanics with Euclidean configuration spaces.[1] Relating the angular coordinates and momenta with the action-angle variables, we describe all symmetries of the generic superintegrable conformal-mechanical systems in terms of the powers of the $su(1,N)$ isometry generators. An important aspect of the proposed approach is the choice of canonical coordinates where all fermionic degrees of freedom appear only in the angular part of the Hamiltonian.

Furthermore, we construct the superanalogues of the maximally superintegrable generalizations of the Euclidean oscillator/Coulomb systems considered in [5] as follows: we preserve the form of Hamiltonian expressed via generators of $su(1,1)$ subalgebra but extend the phase space $\widetilde{\mathbb{CP}}^N$ to phase superspace $\widetilde{\mathbb{CP}}^{N|M}$. As a result, we find that these superextensions preserve all symmetries of the initial bosonic Hamiltonians and possess a maximal set of functionally independent fermionic integrals; i.e., they remains superintegrable in the sense of the super-Liouville theorem. We also find, that the constructed oscillatorlike systems (in contrast with Coulomb-like ones) possess deformed $\mathcal{N} = 2M, d = 1$ Poincaré supersymmetry (see [9]) and express all the symmetries of these superintegrable systems via $su(1,N)$ isometry generators as well.

The paper organized as follows.

In Sec. II, we present the basic facts on Kähler supermanifolds and construct, by the Hamiltonian reduction, the noncompact complex projective superspace $\widetilde{\mathbb{CP}}^{N|M}$ in the parametrization similar to those of Klein model. In Sec. III, we analyze the symmetry algebra of $\widetilde{\mathbb{CP}}^{N|M}$ and extract from it the $su(1,N|M)$-superconformal systems. In Sec. IV, we introduce the canonical coordinates, which naturally split radial and angular parts of the Hamiltonian and relate the angular part with the systems formulating in terms of action-angle variables. In the Sec. V, we construct superintegrable supergeneralizations of oscillator- and Coulomb-like systems. In Sec. VI, we represent the Kähler structure of phase superspace in the Fubini-Study-like form. We conclude the paper by the outlook and final remarks in Sec. VII.

[1]The convenience of the separation of the radial coordinates from the angular one in the study of conformal mechanics and in their supersymmetrization was demonstrated, e.g., in [8].

II. NONCOMPACT COMPLEX PROJECTIVE SUPERSPACE

The (even) $(N|M)$-dimensional Kähler supermanifold can be defined as a complex supermanifold with a symplectic structure, given by the expression,

$$\Omega = \imath(-1)^{p_I(p_J+1)} g_{I\bar{J}} dZ^I \wedge d\bar{Z}^J, \qquad d\Omega = 0, \quad (1)$$

with Z^I denoting N complex bosonic coordinates and M complex fermionic ones. The $p_I \coloneqq p(Z^I)$ is Grassmanian parity of coordinate: it is equal to zero for bosonic coordinate and to one for the fermionic one. Through the paper, we will use the following conjugation rule: $\overline{Z^I Z^J} = \bar{Z}^I \bar{Z}^J$, $\overline{\bar{Z}^I Z^J} = Z^I \bar{Z}^J, \overline{\bar{Z}^I \bar{Z}^J} = Z^I Z^J$, for both bosonic and fermionic variables.

The "metrics components" $g_{I\bar{J}}$ can then be locally represented in the form,

$$g_{I\bar{J}} = \frac{\partial^L}{\partial Z^I} \frac{\partial^R}{\partial \bar{Z}^J} K(Z, \bar{Z}), \quad (2)$$

where $\partial^{L(R)}/\partial Z^I$ denotes left(right) derivatives and the function K is called Kähler potential.

The Poisson brackets associated with this Kähler structure looks as follows:

$$\{f, g\} = \imath\left(\frac{\partial^R f}{\partial \bar{Z}^I} g^{\bar{I}J} \frac{\partial^L g}{\partial Z^J} - (-1)^{p_I p_J} \frac{\partial^R f}{\partial Z^I} g^{\bar{J}I} \frac{\partial^L g}{\partial \bar{Z}^J}\right), \quad \text{where}$$
$$g^{\bar{I}J} g_{J\bar{K}} = \delta^{\bar{I}}_{\bar{K}}, \qquad \overline{g^{\bar{I}J}} = (-1)^{p_I p_J} g^{\bar{J}I}. \quad (3)$$

As in the pure bosonic case, the isometries of Kähler supermanifolds are given by the holomorphic *Hamiltonian vector fields*,

$$\mathbf{V}_\mu \coloneqq \{h_\mu(Z, \bar{Z}), \ \} = V^I(Z) \frac{\partial^L}{\partial Z^I} + \bar{V}^I(\bar{Z}) \frac{\partial^L}{\partial \bar{Z}^I}, \quad (4)$$

where $h_\mu(Z, \bar{Z})$ are real functions called Killing potentials (see, e.g., [6,10] for the details).

Our goal is to study the systems on the Kähler phase space with $su(1,N|M)$ isometry superalgebra. For the construction of such phase space, it is convenient, at first, to present the linear realization of $u(1,N|M)$ superconformal algebra on the complex pseudo-Euclidean superspace $\mathbb{C}^{1,N|M}$ equipped with the canonical Kähler structure (and thus, by the canonical supersymplectic structure) and then reduce it by the action of $u(1)$ generator.

It is instructive to present this reduction in details. Let us equip, at first, the $(N+1|M)$-dimensional complex superspace with the canonical symplectic structure,

$$\Omega_0 = \imath \sum_{a,b=0}^{N} \gamma_{a\bar{b}} dv^a \wedge d\bar{v}^b + \sum_{A=1}^{M} d\eta^A \wedge d\bar{\eta}^A, \qquad (5)$$

with $v^a, \bar{v}^a$ being bosonic variables, and $\eta^A, \bar{\eta}^A$ being fermionic ones, and with the matrix $\gamma_{a\bar{b}}$ chosen in the form,

$$\gamma_{a\bar{b}} = \left(\begin{array}{cc|ccc} 0 & -i & & & \\ i & 0 & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right), \qquad a,b = N,0,1,\dots,N-1. \tag{6}$$

With this supersymplectic structure, we can associate the Poisson brackets given by the relations,

$$\{v^a, \bar{v}^b\} = -\imath\gamma^{\bar{b}a}, \qquad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{A\bar{B}},$$
$$\gamma^{\bar{a}b}\gamma_{b\bar{c}} = \delta^{\bar{a}}_{\bar{c}}. \tag{7}$$

Equivalently,

$$\{v^0, \bar{v}^N\} = 1, \qquad \{v^N, \bar{v}^0\} = -1,$$
$$\{v^\alpha, \bar{v}^\beta\} = \imath\delta^{\alpha\bar{\beta}}, \qquad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{A\bar{B}}. \tag{8}$$

Here we introduced the indices $\alpha, \beta = 1, \dots, N-1$.

On this superspace, we can define the linear Hamiltonian action of $u(1,N|M) = u(1) \times su(1,N|M)$ superalgebra,

$$\{h_{a\bar{b}}, h_{c\bar{d}}\} = -\imath(h_{a\bar{d}}\gamma^{\bar{c}b} - h_{c\bar{b}}\gamma^{\bar{a}d}),$$
$$\{\Theta_{A\bar{a}}, \bar{\Theta}_{\bar{B}b}\} = h_{b\bar{a}}\delta^{B\bar{A}} - R_{A\bar{B}}\gamma^{\bar{b}a},$$
$$\{\Theta_{A\bar{a}}, h_{b\bar{c}}\} = -\imath\Theta_{A\bar{c}}\gamma^{\bar{b}a}, \tag{9}$$

$$\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath(R_{A\bar{D}}\delta^{B\bar{C}} - R_{C\bar{B}}\delta^{D\bar{A}}),$$
$$\{\Theta_{A\bar{a}}, R_{B\bar{C}}\} = -\imath\Theta_{B\bar{a}}\delta^{C\bar{A}}, \tag{10}$$

where

$$h_{a\bar{b}} = \bar{v}^a v^b, \qquad \Theta_{A\bar{a}} = \bar{\eta}^A v^a, \qquad R_{A\bar{B}} = \imath\bar{\eta}^A\eta^B. \tag{11}$$

The $u(1)$ generator defining the center of $u(1.N|M)$ is given by the expression,

$$J = \gamma_{a\bar{b}} v^a \bar{v}^b + \imath\eta^A\bar{\eta}^A\colon \{J, h_{a\bar{b}}\} = \{J, \Theta_{A\bar{a}}\}$$
$$= \{J, R_{A\bar{B}}\} = 0. \tag{12}$$

Hence, reducing the system by the action of this generator, we will get the "noncompact" projective superspace $\widetilde{\mathbb{CP}}^{N|M}$ (i.e., the supergeneralization of noncompact projective space $\widetilde{\mathbb{CP}}^N$), which is $(2N|2M)$-(real)dimensional space.

For performing the reduction by the action of generator (12), we have to choose, at first, the $2N$ real (N complex) bosonic and $2M$ real (M complex) fermionic functions commuting with J. Then, we have to calculate their Poisson brackets and restrict the latters to the level surface,

$$J = g. \tag{13}$$

As a result we will get the Poisson brackets on the reduced $(2N|2M)$-(real) dimensional space, with that $u(1)$-invariant functions playing the role of the latter's coordinates.

The required functions could be easily found as

$$w = \frac{v^N}{v^0}, \qquad z^\alpha = \frac{v^\alpha}{v^0}, \qquad \theta^A = \frac{\eta^A}{v^0}\colon$$
$$\{w, J\} = \{z^a, J\} = \{\theta^A, J\} = 0, \quad \text{and} \quad \text{c.c.} \tag{14}$$

Calculating their Poisson brackets and having in mind the expression following from Eqs. (12) and (13),

$$A \coloneqq \frac{1}{v^0\bar{v}^0}\bigg|_{J=g} = \frac{1}{g}\left(\imath(w-\bar{w}) - \sum_{\gamma=1}^{N-1} z^\gamma\bar{z}^\gamma + \imath\sum_{C=1}^{M}\theta^C\bar{\theta}^C\right), \tag{15}$$

we get the reduced Poisson brackets defined by the following nonzero relations (and their complex conjugates):

$$\{w,\bar{w}\} = -A(w-\bar{w}), \qquad \{z^\alpha, \bar{z}^\beta\} = \imath A\delta^{\alpha\bar{\beta}}, \qquad \{\theta^A, \bar{\theta}^B\} = A\delta^{A\bar{B}}, \qquad \{w, \bar{z}^\alpha\} = A\bar{z}^\alpha, \qquad \{w, \bar{\theta}^A\} = A\bar{\theta}^A. \tag{16}$$

These Poisson brackets are associated with the supersymplectic structure,

$$\Omega = \frac{\imath}{g}\left[\frac{1}{A^2}dw\wedge d\bar{w} - \frac{\imath z^\alpha}{A^2}dw\wedge d\bar{z}^\alpha - \frac{\theta^A}{A^2}dw\wedge d\bar{\theta}^A + \frac{\imath\bar{z}^\alpha}{A^2}dz^\alpha\wedge d\bar{w} + \left(\frac{g\delta_{\alpha\bar{\beta}}}{A} + \frac{\bar{z}^\alpha z^\beta}{A^2}\right)dz^\alpha\wedge d\bar{z}^\beta\right.$$
$$\left. - \frac{\imath\bar{z}^\alpha\theta^A}{A^2}dz^\alpha\wedge d\bar{\theta}^A - \frac{\bar{\theta}^A}{A^2}d\theta^A\wedge d\bar{w} + \frac{\imath\bar{\theta}^A z^\alpha}{A^2}d\theta^A\wedge d\bar{z}^\alpha - \left(\frac{\imath g\delta_{A\bar{B}}}{A} + \frac{\bar{\theta}^A\theta^B}{A^2}\right)d\theta^A\wedge d\bar{\theta}^B\right]. \tag{17}$$

It is defined by the Kähler potential,

$$\mathcal{K} = -g\log(\imath(w-\bar{w}) - z^\alpha\bar{z}^\alpha + \imath\theta^A\bar{\theta}^A). \tag{18}$$

In what follows, we will call this space "noncompact projective superspace $\widetilde{\mathbb{CP}}^{N|M}$." The isometry algebra of this space is $su(1,N|M)$, which can be easily obtained by the restriction of the generators (9), (10) to the level surface (13). It is defined by the following Killing potentials:

$$H \coloneqq v^N\bar{v}^N|_{J=g} = \frac{w\bar{w}}{A}, \qquad K \coloneqq v^0\bar{v}^0|_{J=g} = \frac{1}{A}, \qquad D \coloneqq (v^N\bar{v}^0 + v^0\bar{v}^N)|_{J=g} = \frac{w+\bar{w}}{A}, \tag{19}$$

$$H_\alpha \coloneqq \bar{v}^\alpha v^N|_{J=g} = \frac{\bar{z}^\alpha w}{A}, \qquad K_\alpha \coloneqq \bar{v}^\alpha v^0|_{J=g} = \frac{\bar{z}^\alpha}{A}, \qquad h_{\alpha\bar{\beta}} \coloneqq \bar{v}^\alpha v^\beta|_{J=g} = \frac{\bar{z}^\alpha z^\beta}{A}, \tag{20}$$

$$Q_A \coloneqq \bar{\eta}^A v^N|_{J=g} = \frac{\bar{\theta}^A w}{A}, \qquad S_A \coloneqq \bar{\eta}^A v^0|_{J=g} = \frac{\bar{\theta}^A}{A}, \qquad \Theta_{A\bar{\alpha}} \coloneqq \bar{\eta}^A v^\alpha|_{J=g} = \frac{\bar{\theta}^A z^\alpha}{A}, \tag{21}$$

$$R_{A\bar{B}} \coloneqq \imath\bar{\eta}^A\eta^B|_{J=g} = \imath\frac{\bar{\theta}^A\theta^B}{A}. \tag{22}$$

Constructed super-Kähler structure can be viewed as a higher dimensional analogue of the Klein model of Lobachevsky space, where the latter is parametrized by the lower half-plane. One can choose, instead of a nondiagonal matrix Eq. (6), the diagonal one, $\gamma_{a\bar{b}} = \mathrm{diag}(1,-1,\ldots,-1)$. In that case, the reduced Kähler structure will have the Fubini-Study-like form (see Sec. VI). In the next section, we will analyze the isometry algebra defined by these generators in detail. Presented choice (6) is motivated by its convenience for the analyzing superconformal mechanics. Indeed, in that case, the generators (19) define conformal subalgebra $su(1,1)$ and are separated from the rest $su(N-1|M)$ generators. Thus, they can be interpreted as the Hamiltonian of conformal mechanics, the generator of conformal boosts and the generator of dilatation.

In the next section, we will analyze in details these superconformal mechanics and their dynamical defined by the generators (19)–(22).

III. $su(1,N|M)$ SUPERCONFORMAL ALGEBRA

The generators (Killing potentials) (19)–(22) form $su(1,N|M)$ superalgebra given by Eqs. (9), (10) with $\gamma_{a\bar{b}}$ defined in Eq. (6). Its explicit expression with separated $su(1,1)$ subalgebra is represented below. For convenience, it is divided into three sectors: "bosonic," "fermionic," and "mixed" ones.

A. "Bosonic" sector: $su(1,N)\times u(M)$ algebra

The bosonic sector is the direct product of the $su(1,N)$ algebra defined by the generators (19), (20), and the $u(M)$ algebra defined by the R-symmetry generators (22). Explicitly, the $su(1,N)$ algebra is given by the relations,

$$\{H,K\} = -D, \qquad \{H,D\} = -2H, \qquad \{K,D\} = 2K, \tag{23}$$

$$\{H,K_\alpha\} = -H_\alpha, \qquad \{H,H_\alpha\} = \{H,h_{\alpha\bar{\beta}}\} = 0, \tag{24}$$

$$\{K,H_\alpha\} = K_\alpha, \qquad \{K,K_\alpha\} = \{K,h_{\alpha\bar{\beta}}\} = 0, \tag{25}$$

$$\{D,K_\alpha\} = -K_\alpha, \qquad \{D,H_\alpha\} = H_\alpha, \qquad \{D,h_{\alpha\bar{\beta}}\} = 0, \tag{26}$$

$$\{K_\alpha,K_\beta\} = \{H_\alpha,H_\beta\} = \{K_\alpha,H_\beta\} = 0, \tag{27}$$

$$\{K_\alpha,\bar{K}_\beta\} = -\imath K\delta_{\alpha\bar{\beta}}, \qquad \{H_\alpha,\bar{H}_\beta\} = -\imath H\delta_{\alpha\bar{\beta}}, \qquad \{h_{\alpha\bar{\beta}},h_{\gamma\bar{\delta}}\} = \imath(h_{\alpha\bar{\delta}}\delta_{\gamma\bar{\beta}} - h_{\gamma\bar{\beta}}\delta_{\alpha\bar{\delta}}), \tag{28}$$

$$\{K_\alpha,h_{\beta\bar{\gamma}}\} = -\imath K_\beta\delta_{\alpha\bar{\gamma}}, \qquad \{H_\alpha,h_{\beta\bar{\gamma}}\} = -\imath H_\beta\delta_{\alpha\bar{\gamma}}, \qquad \{K_\alpha,\bar{H}_\beta\} = h_{\alpha\bar{\beta}} + \frac{1}{2}(I-\imath D)\delta_{\alpha\bar{\beta}}, \tag{29}$$

where

$$I \coloneqq g + \sum_{\gamma=1}^{N-1} h_{\gamma\bar{\gamma}} + \sum_{C=1}^{M} R_{C\bar{C}}. \tag{30}$$

The R-symmetry generators form $u(M)$ algebra and commute with all generators of $su(1,N)$,

$$\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath(R_{A\bar{D}}\delta_{C\bar{B}} - R_{C\bar{B}}\delta_{A\bar{D}}),$$
$$\{R_{A\bar{B}}, (H; K; D; K_\alpha; H_\alpha; h_{\alpha\bar{\beta}})\} = 0. \tag{31}$$

It is clear that the generators H, D, and K form conformal algebra $su(1,1)$, the generators $h_{\alpha\bar{\beta}}$ form the algebra $u(N-1)$, and all together—the $su(1,1)\times u(N-1)$ algebra. Notice, that the generator I in Eq. (30) defines the Casimir of conformal algebra $su(1,1)$,

$$\mathcal{I} \coloneqq \frac{1}{2}I^2 = \frac{1}{2}D^2 - 2HK. \tag{32}$$

Hence, choosing H as a Hamiltonian, we get that H_α, $h_{\alpha\bar{\beta}}, R_{A\bar{B}}$ define its constant of motion. Similarly, choosing the generator K as a Hamiltonian, we get that it has constants of motion $K_\alpha, h_{\alpha\bar{\beta}}, R_{A\bar{B}}$.

B. "Fermionic" sector

The Poisson brackets between fermionic generators (21) have the form,

$$\{S_A, \bar{S}_B\} = K\delta_{A\bar{B}}, \qquad \{Q_A, \bar{Q}_B\} = H\delta_{A\bar{B}}, \qquad \{S_A, \bar{Q}_B\} = -\imath R_{A\bar{B}} + \frac{\imath}{2}(I - \imath D)\delta_{A\bar{B}}, \tag{33}$$

$$\{\Theta_{A\bar{\alpha}}, \bar{\Theta}_{B\bar{\beta}}\} = R_{A\bar{B}}\delta_{\beta\bar{\alpha}} + h_{\beta\bar{\alpha}}\delta_{A\bar{B}}, \qquad \{S_A, \bar{\Theta}_{B\bar{\alpha}}\} = K_\alpha\delta_{A\bar{B}}, \qquad \{Q_A, \bar{\Theta}_{B\bar{\alpha}}\} = H_\alpha\delta_{A\bar{B}}, \tag{34}$$

$$\{S_A, S_B\} = \{Q_A, Q_B\} = \{\Theta_{A\bar{\alpha}}, \Theta_{B\bar{\beta}}\} = \{S_A, Q_B\} = \{S_A, \Theta_{B\bar{\alpha}}\} = \{Q_A, \Theta_{B\bar{\alpha}}\} = 0. \tag{35}$$

Hence, the functions Q_A play the role of supercharges for the Hamiltonian H, and the functions S_A define the supercharges of the Hamiltonian given by the generator of conformal boosts K.

C. "Mixed" sector

The mixed sector is given by the relations,

$$\{H, Q_A\} = \{H, \Theta_{A\bar{\alpha}}\} = 0, \qquad \{H, S_A\} = -Q_A, \tag{36}$$

$$\{K, S_A\} = \{K, \Theta_{A\bar{\alpha}}\} = 0, \qquad \{K, Q_A\} = S_A, \tag{37}$$

$$\{D, S_A\} = -S_A, \qquad \{D, Q_A\} = Q_A, \qquad \{D, \Theta_{A\bar{\alpha}}\} = 0 \tag{38}$$

$$\{Q_A, \bar{K}_\alpha\} = -\Theta_{A\bar{\alpha}}, \qquad \{Q_A, H_\alpha\} = \{Q_A, \bar{H}_\alpha\} = \{Q_A, \bar{K}_\alpha\} = \{Q_A, h_{\alpha\bar{\beta}}\} = 0, \tag{39}$$

$$\{S_A, \bar{H}_\alpha\} = \Theta_{A\bar{\alpha}}, \qquad \{S_A, K_\alpha\} = \{S_A, \bar{K}_\alpha\} = \{S_A, H_\alpha\} = \{S_A, h_{\alpha\bar{\beta}}\} = 0, \tag{40}$$

$$\{\Theta_{A\bar{\alpha}}, K_\beta\} = \imath S_A\delta_{\beta\bar{\alpha}}, \qquad \{\Theta_{A\bar{\alpha}}, H_\beta\} = \imath Q_A\delta_{\beta\bar{\alpha}}, \qquad \{\Theta_{A\bar{\alpha}}, \bar{H}_\alpha\} = \{\Theta_{A\bar{\alpha}}, \bar{K}_\alpha\} = 0, \qquad \{\Theta_{A\bar{\alpha}}, h_{\beta\bar{\gamma}}\} = \imath\Theta_{A\bar{\gamma}}\delta_{\beta\bar{\alpha}}, \tag{41}$$

$$\{S_A, R_{B\bar{C}}\} = -\imath S_B\delta_{A\bar{C}}, \qquad \{Q_A, R_{B\bar{C}}\} = -\imath Q_B\delta_{A\bar{C}}, \qquad \{\Theta_{A\bar{\alpha}}, R_{B\bar{C}}\} = -\imath\Theta_{B\bar{\alpha}}\delta_{A\bar{C}}. \tag{42}$$

Looking to the all Poisson bracket relations together, we conclude that

(i) The bosonic functions H_α, $h_{\alpha\bar{\beta}}$, and the fermionic functions Q_A, $\Theta_{A\bar{\alpha}}$ commute with the Hamiltonian H and thus, provide it by the superintegrability property[2];

(ii) The bosonic functions K_α, $h_{\alpha\bar{\beta}}$ and the fermionic functions S_A,$\Theta_{A\bar{\alpha}}$ commute with the generator K. Hence, the Hamiltonian K defines the superintegrable system as well.

[2]In accord with superanalogue of Liouville theorem [11], the system on $(2N.M)$ phase superspace is integrable if and only if it possess N commuting bosonic integrals (with nonvanishing and functionally independent bosonic parts) and M fermionic ones.

(iii) The triples $(H, H_\alpha, Q_A,)$ and $(K, K_\alpha, S_A,)$ transform into each other under the discrete transformation,

$$(w, z^\alpha, \theta^A) \to \left(-\frac{1}{w}, \frac{z^\alpha}{w}, \frac{\theta^A}{w}\right) \Rightarrow D \to -D, \qquad \begin{cases} (H, H_\alpha, Q_A,) & \to \quad (K, -K_\alpha, -S_A), \\ (K, K_\alpha, S_A) & \to \quad (H, H_\alpha, Q_A,) \end{cases}. \tag{43}$$

(iv) The functions $h_{\alpha\bar{\beta}}, \Theta_{A\bar{\alpha}}$ are invariant under discrete transformation (43). Moreover, they appear to be constants of motion both for H and K. Hence, they remain to be constants of motion for any Hamiltonian being the functions of H, K. In particular, adding to the Hamiltonian H, the appropriate function of K, we get the superintegrable oscillator- and Coulomb-like systems with dynamical superconformal symmetry (see Sec. V).

(v) The superalgebra $su(1, N|M)$ admits five-graded decomposition [12,13],

$$su(1, N|M) = \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+1} \oplus \mathfrak{f}_{+2} \quad \text{with} \quad [\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_{i+j} \quad \text{for } i, j \in \{-2, -1, 0, 1, 2\}, \tag{44}$$

where $\mathfrak{f}_i = 0$ for $|i| > 2$ is understood. The subset $\mathfrak{f}_0$ includes the generators $D, h_{\alpha\bar{\beta}}, \Theta_{A\bar{\alpha}}, \bar{\Theta}A\bar{\alpha}, R_{A\bar{B}}$, the subsets $\mathfrak{f}_{-2}$ and $\mathfrak{f}_2$ contain only generators H and K, respectively, while the subsets $\mathfrak{f}_{-1}$ and $\mathfrak{f}_1$ contain the generators $H_\alpha, \bar{H}_\alpha, Q_A, \bar{Q}_A$ and $K_\alpha, \bar{K}_\alpha, S_A, \bar{S}_A$.

Let us conclude this section by the following remark. It is easy to see, that the generator (30) commutes the generators $H, D, K, S_A, Q_A, R_{A\bar{B}}$. Hence, these generators form superconformal algebra $su(1, 1|M)$ witha central charge $\sqrt{2\mathcal{I}}$ (32) (being the Casimir of $su(1, 1|M)$) as well),

$$\begin{gathered}
\{H, K\} = -D, \qquad \{H, D\} = -2H, \qquad \{K, D\} = 2K, \qquad \{S_A, \bar{S}_B\} = K\delta_{A\bar{B}}, \qquad \{Q_A, \bar{Q}_B\} = H\delta_{A\bar{B}}, \\
\{S_A, \bar{Q}_B\} = -\imath R_{A\bar{B}} + \frac{\imath}{2}(\sqrt{2\mathcal{I}} - \imath D)\delta_{A\bar{B}}, \\
\{H, S_A\} = -Q_A, \qquad \{K, Q_A\} = S_A, \qquad \{H, Q_A\} = \{K, S_A\} = 0, \qquad \{D, S_A\} = -S_A, \qquad \{D, Q_A\} = Q_A, \\
\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath(R_{A\bar{D}}\delta_{C\bar{B}} - R_{C\bar{B}}\delta_{A\bar{D}}), \qquad \{S_A, R_{B\bar{C}}\} = -\imath S_B\delta_{A\bar{C}}, \qquad \{Q_A, R_{B\bar{C}}\} = -\imath Q_B\delta_{A\bar{C}}.
\end{gathered} \tag{45}$$

In the next section, we will express presented $su(1, N|M)$ generators in appropriate canonical coordinates, and in this way, we will relate presented formulas with the superextensions of conventional conformal mechanics.

IV. CANONICAL COORDINATES AND ACTION-ANGLE VARIABLES

To define the canonical coordinates, we pass from the complex bosonic coordinates w, z^α,

$$w = x + \imath y, \qquad z^\alpha = q_\alpha e^{\imath\varphi_\alpha}, \quad \text{where } y < 0, \qquad q_\alpha \geq 0, \qquad \varphi_\alpha \in [0, 2\pi), \qquad q^2 \coloneqq \sum_{\alpha=1}^{N-1} q_\alpha^2 < -2y. \tag{46}$$

Then we redefine fermionic ones such that the new variables will have canonical Poisson brackets.

For this purpose, we write down the symplectic/Kähler one form and identify it with the canonical one,

$$\mathcal{A} = -\frac{g}{2}\frac{dw + d\bar{w} - \imath(z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{\imath(w - \bar{w}) - z^\gamma\bar{z}^\gamma + \imath\theta^C\bar{\theta}^C} \coloneqq p_x dx + \pi_\alpha d\varphi_\alpha + \frac{1}{2}\chi^A d\bar{\chi}^A + \frac{1}{2}\bar{\chi}^A d\chi^A. \tag{47}$$

After some calculations and canonical transformation $(p_x, x) \to (-\frac{r^2}{2}, \frac{p_r}{r})$, one can obtain

$$w = \frac{p_r}{r} - \imath\frac{I}{r^2}, \qquad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r}e^{\imath\varphi_\alpha}, \qquad \theta^A = \frac{\sqrt{2}}{r}\chi^A, \tag{48}$$

where $r, p_r, \pi_\alpha, \varphi_\alpha, \chi^A, \bar{\chi}^A$ are canonical coordinates,

$$\{r, p_r\} = 1, \qquad \{\varphi_\beta, \pi_\alpha\} = \delta_{\alpha\beta}, \qquad \{\chi^A, \bar{\chi}^B\} = \delta^{A\bar{B}}, \qquad \pi_\alpha \geq 0, \qquad \varphi^\alpha \in [0, 2\pi), \qquad r > 0. \tag{49}$$

They expresses via initial ones as follows:

$$p_r = \frac{w+\bar{w}}{2}\sqrt{\frac{2}{A}}, \qquad r = \sqrt{\frac{2}{A}}, \qquad \pi_\alpha = \frac{z^\alpha \bar{z}^\alpha}{A},$$
$$\varphi_\alpha = \arg(z^\alpha), \qquad \chi^A = \frac{\theta^A}{\sqrt{A}}, \qquad \text{c.c.}, \tag{50}$$

where

$$I = g + \sum_{\alpha=1}^{N-1}\pi_\alpha + \sum_{A=1}^{M}\imath\bar{\chi}^A\chi^A,$$
$$A = \frac{\imath(w-\bar{w}) - \sum_{\gamma=1}^{N-1} z^\gamma\bar{z}^\gamma + \imath\sum_{C=1}^{M}\theta^C\bar{\theta}^C}{g} = \frac{2}{r^2}. \tag{51}$$

In these canonical coordinates the isometry generators read

$$H = \frac{p_r^2}{2} + \frac{I^2}{2r^2}, \qquad K = \frac{r^2}{2}, \qquad D = p_r r, \tag{52}$$

$$H_\alpha = \sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}\left(p_r - \imath\frac{I}{r}\right), \qquad K_\alpha = r\sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha},$$
$$h_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath(\varphi_\alpha-\varphi_\beta)}, \tag{53}$$

$$Q_A = \frac{\bar{\chi}^A}{\sqrt{2}}\left(p_r - \imath\frac{I}{r}\right), \qquad S_A = \frac{\bar{\chi}^A}{\sqrt{2}}r,$$
$$\Theta_{A\bar{\alpha}} = \bar{\chi}^A\sqrt{\pi_\alpha}\mathrm{e}^{\imath\varphi_\alpha}, \qquad R_{A\bar{B}} = \imath\bar{\chi}^A\chi^B. \tag{54}$$

Interpreting r as a radial coordinate, and p_r as radial momentum, we get the superconformal mechanics with angular Hamiltonian given by

$$\mathcal{I} = \frac{I^2}{2} \coloneqq \frac{1}{2}(I_0 + (\bar{\chi}\chi))^2, \quad \text{with} \quad I_0 \coloneqq g + \sum_{\alpha=1}^{N-1}\pi_\alpha,$$
$$(\bar{\chi}\chi) \coloneqq \sum_{A=1}^{M}\imath\bar{\chi}^A\chi^A. \tag{55}$$

So, the fermionic part of superconformal Hamiltonian is encoded in its angular part.

The explicit dependence of the Hamiltonian H and the supercharges Q_A on the fermions is as follows:

$$H = H_0 + \frac{I_0(\bar{\chi}\chi)}{r^2} + \frac{(\bar{\chi}\chi)^2}{2r^2},$$
$$Q_A = -\frac{\bar{\chi}^A}{\sqrt{2}}\left(p_r - \imath\frac{I_0}{r} - \imath\frac{(\bar{\chi}\chi)}{r}\right), \tag{56}$$

while the dependence of bosonic integrals H_α on fermions is given by the expression,

$$H_\alpha = H_\alpha^0 - \frac{K_\alpha(\bar{\chi}\chi)}{2K}, \tag{57}$$

where

$$H_0 \coloneqq \frac{p_r^2}{2} + \frac{I_0^2}{2r^2},$$
$$H_\alpha^0 = \sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}\left(p_r - \imath\frac{I_0}{r}\right): \{H_\alpha^0, H^0\} = 0. \tag{58}$$

So, proposed superconformal Hamiltonian H inherits all symmetries of initial Hamiltonian H_0 (given by $H_\alpha^0, h_{\alpha\bar{\beta}}$).

Looking at the functional dependence of the angular Hamiltonian $\mathcal{I}$ from the angular variables $\varphi^\alpha, \pi_\alpha$, one can expect that the set of conformal mechanics admitting proposed $su(1,N|M)$ superconformal extensions seems to be very restricted. However, it is not the case, since it is not necessary to interpret φ^α as a coordinate of the configuration space, and π_α as its canonically conjugated momentum. Instead, since π_α define a constant of motion of the bosonic Hamiltonian H_0 (and of the respective angular Hamiltonian $\mathcal{I}_0 = H_0K/2 - D^2$), we can interpret it as the action variable I_α and consider φ^α as a respective angle variable Φ_α.

Furthermore, suppose that $\pi_\alpha, \varphi_\alpha$ are related with the action-angle variables (I_α, Φ_α) of some $(N-1)$-dimensional angular mechanics by the relations,

$$\pi_\alpha = n_\alpha I_\alpha, \qquad \varphi_\alpha = \frac{\Phi_\alpha}{n_\alpha}, \quad \text{where } n_\alpha \in \mathbb{N},$$
$$\{\Phi_\alpha, I_\beta\} = \delta_{\alpha\beta}, \qquad \Phi_\alpha \in [0, 2\pi). \tag{59}$$

Upon this identification, the bosonic part of the angular Hamiltonian (55) takes a form,

$$\tilde{\mathcal{I}}_0 = \frac{1}{2}\left(g + \sum_{\alpha=1}^{N-1} n_\alpha I_\alpha\right)^2, \quad \text{with} \quad n_\alpha \in \mathbb{N}, \tag{60}$$

but the bosonic generators $H_\alpha, S_\alpha, h_{\alpha\bar{\beta}}$, become locally defined, $\varphi_\alpha \in [0, 2\pi/n_\alpha)$ and fail to be constants of motion. To get the globally defined bosonic generators, we have to take their relevant powers,

$$\tilde{H}_\alpha \coloneqq (H_\alpha)^{n_\alpha}, \quad \tilde{K}_\alpha \coloneqq (K_\alpha)^{n_\alpha}, \quad \tilde{h}_{\alpha\bar{\beta}} \coloneqq (h_{\alpha\bar{\beta}})^{n_\alpha n_\beta}, \tag{61}$$

as well as replace the fermionic generator $\Theta_{A\bar{\alpha}}$ by the following one:

$$\tilde{\Theta}_{A\bar{\alpha}} = (H_\alpha)^{n_\alpha - 1}\Theta_{A\bar{\alpha}}. \tag{62}$$

As a result, the dynamical (super)symmetry algebra becomes a nonlinear deformation of $su(1,N|M)$.

The angular Hamiltonian (60) defines the class the superintegrable generalizations of the conformal mechanics, and of the oscillator- and Coulomb-like systems on the N-dimensional Euclidean spaces [14]. As a particular case, this class of systems includes the "charge-monopole" system

[15], Smorodinsky-Winternitz system [16] (for the explicit expressions of the action-angle variables of these systems, see, respectively, [17,18]), as well as the rational Calogero models.[3] Thus, proposed systems can be considered as their $2M$ superconformal extensions.

Since the generators $Q_A, S_A, R_{A\bar{B}}$ remain unchanged upon above identification [as well as the expression of the angular Hamiltonian (32) via generators H, K, D], we conclude that listed generators form superconformal algebra $su(1,1|N)$ with the central charge (45).

Finally, notice that in Eq. (60), the nonzero constant $g \neq 0$ appears, and the range of validity of the action variables is fixed to be $I_\alpha \in [0,\infty)$. As a result, standard free particle and conformal mechanics cannot be included in the proposed description, since for these systems, we should choose $g = 0, I_\alpha \in [0,\infty)$. To exclude this constant, we should replace the initial generators by the following ones:

$$\mathcal{H} \coloneqq H - \frac{g(g-2I)}{4K}, \qquad \mathcal{H}_\alpha \coloneqq H_\alpha + \imath g \frac{K_\alpha}{2K},$$
$$\mathcal{Q}_A \coloneqq Q_A - \imath g \frac{S_A}{2K}. \tag{63}$$

This deformation will further “nonlinearize” the dynamical supersymmetry algebra $su(1,N|M)$.

V. OSCILLATOR- AND COULOMB-LIKE SYSTEMS

In the previous section, we mentioned that the angular Hamiltonian (60) defines the superintegrable deformations of N-dimensional oscillator and Coulomb system [14], while in [5], the examples of such systems on noncompact projective space $\widetilde{\mathbb{CP}}^N$ playing the role of phase space were constructed. So, one can expect that on the phase superspace $\widetilde{\mathbb{CP}}^{N|M}$, one can construct the supercounterparts of that systems, which presumably, possess (deformed) $\mathcal{N} = 2M$, $d = 1$ Poincaré supersymmetry. Below, we examine this question and show that our claim is correct in some particular cases.

A. Oscillatorlike systems

We define the supersymmetric oscillatorlike system by the phase space $\widetilde{\mathbb{CP}}^{N|M}$ [equipped with the Poisson brackets (16)] by the Hamiltonian,

$$H_{\rm osc} = H + \omega^2 K, \tag{64}$$

where the generators H, K are given by Eq. (19). In canonical coordinates (50), it reads

$$H_{\rm osc} = \frac{p_r^2}{2} + \frac{(g + \sum_{\alpha=1}^{N-1}\pi_\alpha + \sum_{A=1}^{M}\imath\bar{\chi}^A\chi^A)^2}{r^2} + \frac{\omega^2 r^2}{2}. \tag{65}$$

This system possesses the $u(N-1)$ symmetry given by the generators $h_{\alpha\bar{\beta}}$ defined in Eq. (20) (among them $N-1$ constants of motion π_α are functionally independent), the $u(M)$ R symmetry given by the generators $R_{A\bar{B}}$ (22) as well as $N-1$ hidden symmetries given by the generators,

$$M_{\alpha\beta} = (H_\alpha + \imath\omega K_\alpha)(H_\beta - \imath\omega K_\beta)$$
$$= \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2) \colon \{H_{\rm osc}, M_{\alpha\beta}\} = 0. \tag{66}$$

The generators (66) and the $su(N)$ generators $h_{\alpha\bar{\beta}}$ form the following symmetry algebra:

$$\{h_{\alpha\bar{\beta}}, M_{\gamma\delta}\} = \imath(M_{\alpha\delta}\delta_{\gamma\bar{\beta}} + M_{\gamma\alpha}\delta_{\delta\bar{\beta}}), \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0, \tag{67}$$

$$\{M_{\alpha\beta}, \bar{M}_{\gamma\delta}\} = \imath\Bigg(4\omega^2 I h_{\alpha\bar{\delta}}h_{\beta\bar{\gamma}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\alpha\bar{\gamma}}}\delta_{\alpha\bar{\gamma}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\alpha\bar{\delta}}}\delta_{\alpha\bar{\delta}}$$
$$- \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\beta\bar{\gamma}}}\delta_{\beta\bar{\gamma}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\beta\bar{\delta}}}\delta_{\beta\bar{\delta}}\Bigg), \tag{68}$$

with I given by Eq. (30) and summation over repeated indices is not assumed.

Besides, this system has a fermionic constant of motion $\Theta_{A\bar{\alpha}}$ defined in Eq. (21). Hence, it is superintegrable system in the sense of super-Liouville theorem; i.e., it has $2N-1$ bosonic and $2M$ fermionic, functionally independent, constants of motion [11]. Further generalization to the systems with angular Hamiltonian (60) is straightforward.

Let us show, that for the even $M = 2k$ this system possesses the deformed $\mathcal{N} = 2k$ Poincaré supersymmetry, in the sense of papers [9]. For this purpose, we choose the following Ansatz for supercharges:

$$\mathcal{Q}_A = Q_A + \omega C_{AB}\bar{S}_B, \tag{69}$$

with the constant matrix C_{AB} obeying the conditions,

$$C_{AB} + C_{BA} = 0, \qquad C_{AB}\bar{C}_{BD} = -\delta_{A\bar{D}}. \tag{70}$$

For sure, the condition (70) assumes that M is an even number, $M = 2k$.

Calculating Poisson brackets of the functions (69), we get

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{\rm osc}\delta_{AB}, \qquad \{\mathcal{Q}_A, \mathcal{Q}_B\} = -\imath\omega\mathcal{G}_{AB},$$
$$\{\bar{\mathcal{Q}}_A, \bar{\mathcal{Q}}_B\} = \imath\omega\bar{\mathcal{G}}_{AB}, \tag{71}$$

where

[3] To our best knowledge, action-angle variables for the angular part of the rational Calogero models are not yet constructed explicitly. However, we have at hand, the spectrum of the angular part of rational Calogero model [19]. Taking its (semi)classical limit, we can conclude that it has the form Eq. (60); see, e.g., [14].

$$\mathcal{G}_{AB} \coloneqq C_{AC}R_{B\bar{C}} + C_{BC}R_{A\bar{C}},$$
$$\mathcal{G}_{\bar{A}\bar{B}} \coloneqq \bar{\mathcal{G}}_{AB} = \bar{C}_{AC}R_{C\bar{B}} + \bar{C}_{BC}R_{C\bar{A}},$$
$$\bar{\mathcal{G}}_{AB} = \bar{C}_{AC}\bar{C}_{DB}\mathcal{G}_{DC}. \tag{72}$$

Then, we get that the algebra of generators $\mathcal{Q}_A$, $\mathcal{H}_{\rm osc}$, $\mathcal{G}^B_A$ is closed indeed,

$$\{\mathcal{Q}_A, H_{\rm osc}\} = \omega C_{AB}\mathcal{Q}_B, \qquad \{\mathcal{G}_{AB}, H_{\rm osc}\} = 0, \tag{73}$$

$$\{\mathcal{Q}_A, \mathcal{G}_{BC}\} = \imath(C_{AB}\mathcal{Q}_C + C_{AC}\mathcal{Q}_B),$$
$$\{\mathcal{Q}_A, \bar{\mathcal{G}}_{BC}\} = -\imath(\bar{C}_{BD}\mathcal{Q}_D\delta_{A\bar{C}} + \bar{C}_{CD}\mathcal{Q}_D\delta_{A\bar{B}}). \tag{74}$$

Hence, for the $M = 2k$, the above oscillatorlike system (64) possesses deformed $\mathcal{N} = 4k$ supersymmetry. In the particular case $M = 2$, the choice of the matrix C_{AB} is unique (up to unessential phase factor): $C_{AB} \coloneqq e^{\kappa}\varepsilon_{AB}$. In that case, the above relations define the superalgebra $su(1|2)$-deformation of $\mathcal{N} = 4$ Poincaré supersymmetric mechanics studied in details in [9]. For the $k \geq 2$, the choice of matrices C_{AB} is not unique, and we get the family of deformed $\mathcal{N} = 4k$ Poincaré supersymmetric mechanics.

Let us present other deformed $\mathcal{N} = 2M$ Poincaré supersymmetric systems, whose bosonic part is different from those of Eq. (64) but nevertheless, has the oscillator potential.

For this purpose, we choose another ansatz for supercharges (in contrast with previous case M is not restricted to be even number),

$$\tilde{\mathcal{Q}}_A = Q_A + \imath\omega S_A. \tag{75}$$

These supercharges generates the $su(1|M)$ superalgebra, and thus generalizes the systems considered in [9] to arbitrary M,

$$\{\tilde{\mathcal{Q}}_A, \bar{\tilde{\mathcal{Q}}}_B\} = \mathcal{H}_{\rm osc}\delta_{AB} - \omega\mathcal{R}^A_B, \qquad \{\tilde{\mathcal{Q}}_A, \tilde{\mathcal{Q}}_B\} = 0,$$
$$\{\mathcal{R}^B_A, \mathcal{R}^D_C\} = \imath(\mathcal{R}^D_A\delta^B_C - \mathcal{R}^B_C\delta^D_A) \tag{76}$$

$$\{\tilde{\mathcal{Q}}_A, \mathcal{R}^C_B\} = \imath\left(\frac{1}{M}\tilde{\mathcal{Q}}_A\delta_{B\bar{C}} + \tilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right),$$
$$\{\tilde{\mathcal{Q}}_A, \mathcal{H}_{\rm osc}\} = \imath\omega\frac{2M-1}{M}\tilde{\mathcal{Q}}_A, \tag{77}$$

where

$$\mathcal{H}_{\rm osc} \coloneqq H_{\rm osc} - \omega\left(I + \frac{1}{M}\sum_C R_{C\bar{C}}\right),$$
$$\mathcal{R}^B_A \coloneqq R_{A\bar{B}} - \frac{1}{M}\delta^B_A\sum_C R_{C\bar{C}} \tag{78}$$

with I defined by Eq. (30). Hence, the Hamiltonian get the additional bosonic term proportional to the Casimir of conformal group. In canonical coordinates (50), it reads

$$\mathcal{H}_{\rm osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \frac{\omega^2 r^2}{2} - \omega\left(\sqrt{2\mathcal{I}} + \frac{1}{M}(\bar{\chi}\chi)\right). \tag{79}$$

This Hamiltonian, seemingly, describes the oscillatorlike systems specified by the presence of external magnetic field.

So, choosing $\widetilde{\mathbb{CP}}^{N|M}$ as a phase superspace, we can easily construct superintegrable oscillatorlike systems, which possess deformed $\mathcal{N} = 2M$, $d = 1$ Poincaré supersymmetry.

B. Coulomb-like systems

Now, let us construct on the phase space $\widetilde{\mathbb{CP}}^{N|M}$ with the Poisson bracket relations (16), the Coulomb-like system given by the Hamiltonian,

$$H_{\rm Coul} = H + \frac{\gamma}{\sqrt{2K}}, \tag{80}$$

where the generators H, K are defined by Eq. (19).

The bosonic constants of motion of this system are given by the $u(N-1)$ symmetry generators $h_{\alpha\beta}$, and by the $N-1$ additional constants of motion,

$$R_\alpha = H_\alpha + \imath\gamma\frac{K_\alpha}{I\sqrt{2K}}:\ \{H_{\rm Coul}, R_\alpha\} = \{H_{\rm Coul}, h_{\alpha\bar{\beta}}\} = 0, \tag{81}$$

where $H_\alpha, K_\alpha, \eta_{\alpha\bar{\beta}}$ are defined by Eq. (20). These generators form the algebra,

$$\{R_\alpha, \bar{R}_{\bar{\beta}}\} = -\imath\delta_{\alpha\bar{\beta}}\left(H_{\rm Coul} - \frac{\imath\gamma^2}{2I^2}\right) + \frac{\imath\gamma^2 h_{\alpha\bar{\beta}}}{2I^3},$$
$$\{h_{\alpha\bar{\beta}}, R_\gamma\} = \imath\delta_{\gamma\bar{\beta}}R_\alpha, \qquad \{R_\alpha, R_\beta\} = 0. \tag{82}$$

Besides, proposed system has $2M$ fermionic constants of motion given by $\Theta_{A\bar{\alpha}}$ and $u(M)$ R symmetry given by $R_{A\bar{B}}$. Hence, it is superintegrable in the sense of super-Liouville theorem [11]. So, we constructed the maximally superintegrable Coulomb problem with dynamical $su(1,N|M)$ superconformal symmetry, which inherits all symmetries of initial bosonic system.

One can expect, that in analogy with oscillatorlike system, our Coulomb-like system would possess (deformed) $\mathcal{N} = 2M$-super-Poincaré symmetry for $M = 2k$ and $\gamma > 1$. However, it is not the case.

Indeed, let us choose the following Ansatz for supercharges:

$$\mathcal{Q}_A = Q_A + \sqrt{2\gamma}C_{AB}\frac{\bar{S}_B}{(2K)^{3/4}}, \tag{83}$$

with the constant matrix C_{AB} obeying the conditions (70), $M = 2k$ and $\gamma > 0$.

Calculating their Poisson brackets, we find

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{\rm Coul}\delta_{A\bar{B}} + \frac{3}{2}\frac{\sqrt{2\gamma}}{(2K)^{7/4}}(S_A\bar{C}_{BD}S_D + \bar{S}_B C_{AD}\bar{S}_D), \tag{84}$$

$$\{\mathcal{Q}_A, \mathcal{Q}_B\} = -\frac{\imath\sqrt{2\gamma}}{2(2K)^{3/4}}(C_{BD}\mathcal{R}^D_A + C_{AC}\mathcal{R}^D_B),$$
$$\{\mathcal{Q}_A, \mathcal{R}^C_B\} = -\imath\mathcal{Q}_B\delta_{A\bar{C}}, \tag{85}$$

where $\mathcal{R}^A_B$ is defined in Eq. (78).

Further calculating the Poisson brackets of $\mathcal{Q}_A$ with the generators appearing in the rhs of the above expressions, we get that the superalgebra is not closed. For example,

$$\{\mathcal{Q}_A, H_{\rm Coul}\} = \frac{3\gamma}{(2K)^{3/2}}S_A + \frac{\sqrt{2\gamma}}{(2K)^{3/4}}C_{AB}\left(\bar{\mathcal{Q}}_B - \frac{3}{4K}\bar{S}_B D\right). \tag{86}$$

Hence, proposed supercharges do not yield closed deformation of $\mathcal{N}=2M$-super-Poincaré algebra.

Let us choose another ansatz for supercharges (as above we assume that $\gamma > 0$),

$$\tilde{\mathcal{Q}}_A = \mathcal{Q}_A + \imath\sqrt{2\gamma}e^{\imath\frac{\pi}{2}}\frac{S_A}{(2K)^{3/4}}, \tag{87}$$

which yields

$$\{\tilde{\mathcal{Q}}_A, \bar{\tilde{\mathcal{Q}}}_B\} = \mathcal{H}_{\rm Coul}\delta_{A\bar{B}} + \frac{\sqrt{2\gamma}}{2(2K)^{3/4}}\mathcal{R}^B_A, \quad \{\tilde{\mathcal{Q}}_A, \tilde{\mathcal{Q}}_B\} = 0,$$
$$\{\tilde{\mathcal{Q}}_A, \mathcal{R}^C_B\} = \imath\left(\frac{1}{M}\tilde{\mathcal{Q}}_A\delta_{B\bar{C}} - \tilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right), \tag{88}$$

where

$$\mathcal{H}_{\rm Coul} = H_{\rm Coul} - \frac{\sqrt{2\gamma}}{(2K)^{3/4}}\left(I - \frac{1}{2M}\sum_C R_{C\bar{C}}\right), \tag{89}$$

with I and $\mathcal{R}^A_B$ are defined, respectively, in Eqs. (55) and (78). In canonical coordinates (50), this Hamiltonian reads

$$\mathcal{H}_{\rm Coul} = \frac{p_r}{2} + \frac{\mathcal{I}}{r^2} + \frac{\gamma}{r} - \frac{\sqrt{2\gamma}}{r^{3/2}}\left(g + \sum_\alpha \pi_\alpha + \frac{2M-1}{2M}(\bar{\chi}\chi)\right). \tag{90}$$

However, one can easily check that proposed supercharges do not yield closed deformation of Poincaré superalgebra as well, e.g.,

$$\left\{\tilde{\mathcal{Q}}_A, \frac{\mathcal{R}^C_B}{(2K)^{3/4}}\right\} = \frac{\imath}{(2K)^{3/4}}\left(\frac{1}{M}\tilde{\mathcal{Q}}_A\delta_{B\bar{C}} - \tilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right) + \frac{3}{2}\frac{S_A}{(2K)^{7/4}}\mathcal{R}^C_B. \tag{91}$$

So, proposed superextensions of Coulomb-like systems, being well-defined from the viewpoint of superintegrability, do not possess neither $\mathcal{N}=2M$ supersymmetry, nor its deformation. The $su(1,N|M)$ superalgebra plays the role of dynamical algebra of that systems.

VI. FUBINI-STUDY-LIKE KÄHLER STRUCTURE

The above considered super-Kähler structure is obviously the higher-dimensional superanalogue of the Klein model of Lobachevsky space. On the other hand, Lobachevsky space has other common parametrization as well, which is known as Poincaré disc [7]. The higher-dimensional generalization of Poincaré disc parametrizing the noncompact complex projective space is quite similar to the Fubini-Study structure for $\mathbb{CP}^N$. It is defined by the Kähler potential,

$$\mathcal{K} = -g\log\left(1 - \sum_{a=1}^N z^a\bar{z}^a\right). \tag{92}$$

For the obtaining of the superanalogue of this potential from $\mathbb{C}^{1,N|M}$, one should pass from the matrix (6) to the diagonal matrix $\gamma_{a\bar{b}} = {\rm diag}(1,-1,\ldots,-1)$. This can be done by the transformation,

$$v^0 \to \frac{v^0 + v^N}{\sqrt{2}}, \qquad v^N \to \frac{v^0 - v^N}{\imath\sqrt{2}}. \tag{93}$$

On the reduced phase space (93) corresponds to the transformation,

$$w \to \imath\frac{z^N - 1}{z^N + 1}, \qquad z^\alpha \to \sqrt{2}\frac{z^\alpha}{z^N + 1},$$
$$\theta^A \to \sqrt{2}\frac{\theta^A}{z^N + 1}. \tag{94}$$

Thus, we will get the Fubini-Study-like Kähler potential,

$$\mathcal{K} = -g\log(1 - z^c\bar{z}^c + \imath\theta^C\bar{\theta}^C), \tag{95}$$

which defines the following Kähler structure:

$$\Omega = \frac{\imath}{g}\left[\left(\frac{g\delta_{a\bar{b}}}{\tilde{A}} + \frac{\bar{z}^a z^b}{\tilde{A}^2}\right)dz^a\wedge d\bar{z}^b + \frac{\imath\bar{\theta}^A z^a}{\tilde{A}^2}d\theta^A\wedge d\bar{z}^a - \frac{\imath\bar{z}^a\theta^A}{\tilde{A}^2}dz^a\wedge d\bar{\theta}^A - \left(\frac{g\delta_{A\bar{B}}}{\tilde{A}} + \frac{\bar{\theta}^A\theta^B}{\tilde{A}^2}\right)d\theta^A\wedge d\bar{\theta}^B\right], \tag{96}$$

where we have used a similar notation as in Eq. (15),

$$\tilde{A} \coloneqq \frac{1 - z^c\bar{z}^c + \imath\theta^C\bar{\theta}^C}{g}. \qquad (97)$$

The respective Poisson brackets read

$$\{z^a, \bar{z}^b\} = \imath\tilde{A}(\delta^{a\bar{b}} - z^a\bar{z}^b), \qquad \{z^a, \bar{\theta}^A\} = \imath\tilde{A}z^a\bar{\theta}^A,$$
$$\{\theta^A, \bar{\theta}^B\} = \tilde{A}(\delta^{A\bar{B}} + \theta^A\bar{\theta}^B). \qquad (98)$$

Now let us introduce the canonical coordinates, but now taking the symplectic/Kähler one form associated with the Kähler potential (95), i.e., the one that define "Fubini-Study"-like metric. Then, as before, one needs to identify it with the canonical one, and this canonical coordinates will play the role of "Cartesian" coordinates instead of the "spherical" ones discussed above,

$$\tilde{\mathcal{A}} = -\frac{g}{2}\frac{\imath(\bar{z}^a dz^a - z^a d\bar{z}^a) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{1 - z^c\bar{z}^c + \imath\theta^C\bar{\theta}^C}$$
$$\coloneqq p_a d\varphi_a + \frac{1}{2}\chi^A d\bar{\chi}^A + \frac{1}{2}\bar{\chi}^A d\chi^A. \qquad (99)$$

It leads to the relations,

$$z^a = \sqrt{\frac{p_a}{g + p - \imath\chi^C\bar{\chi}^C}} e^{\imath\varphi_a}, \qquad \theta^A = \frac{\sqrt{2}}{r}\chi^A,$$
$$p = \sum_a p_a, \qquad (100)$$

or

$$p_a = \frac{z^a\bar{z}^a}{\tilde{A}}, \qquad \varphi_a = \arg(z^a), \qquad \chi^A = \frac{\theta^A}{\sqrt{\tilde{A}}}, \qquad (101)$$

where $\tilde{A}$ is defined by Eq. (97).

These coordinates are related with Eq. (50) as follows:

$$p_\alpha = \pi_\alpha, \, p_N = \frac{1}{4}\left(p_r^2 + \left(r - \frac{\sqrt{2\mathcal{I}}}{r}\right)^2\right),$$
$$\varphi_N = \arctan\left(\frac{2xy}{(x-y)(x+y)}\right), \qquad (102)$$

where

$$x = 1 - \frac{p_r^2}{r^2} - \frac{2\mathcal{I}}{r^4}, \qquad y = \frac{p_r}{r}, \qquad (103)$$

while χ^A and φ_α remains unchanged after transition from one parametrization to the other.

Finally, let us draw readers attention to the complete similarity of the bosonic part of Eq. (100) with the equations mapping parametrizing compactified Ruijsenaars-Schneider model with an excluded center of mass to the complex projective (phase) space $\mathbb{CP}^N$. This prompt us, at first, to construct the conformal-invariant analogue of that model by replacing the complex projective space by its noncompact analogue $\widetilde{\mathbb{CP}}^N$. Then, one can try to construct its $su(1,N|M)$-supeconformal extension by further replacement of $\widetilde{\mathbb{CP}}^N$ by $\widetilde{\mathbb{CP}}^{N|M}$.

VII. CONCLUDING REMARKS

In this paper, we suggested to construct the $su(1,N|M)$-superconformal mechanics formulating them on phase superspace given by the noncompact analogue of complex projective superspace $\widetilde{\mathbb{CP}}^{N|M}$. The $su(1,N|M)$ symmetry generators were defined there as a Killing potentials of $\widetilde{\mathbb{CP}}^{N|M}$. We parametrized this phase space by the specific coordinates allowing us to interpret it as a higher-dimensional superanalogue of the Lobachevsky plane parametrized by lower half-plane (Klein model). Then, we transited to the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables, we demonstrated that the proposed scheme allows us to construct the $su(1,N|M)$ supeconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb-like systems with a $su(1,N|M)$ dynamical superalgebra, and found that oscillatorlike systems admit deformed $\mathcal{N} = 2M$ Poincaré supersymmetry, in contrast with Coulomb-like ones.

In fact, the proposed scheme demonstrated the effectiveness of the supersymmetrization via formulation of the initial systems in terms of Kähler phase space and further superextension of the latter. In order to relate considered systems with the conventional ones (with Euclidean configuration spaces), we restricted ourself by the noncompact complex projective superspace. So, we are sure that applying the same approach to the conventional (compact) complex projective spaces, we can find many new integrable systems as well and construct their unexpected extended supersymmetric extensions.

The proposed scheme could obviously be extended to the systems on complex Grassmanians (and on their noncompact analogues). In particular, we expect to find, in this way, the $\mathcal{N}$-supersymmetric extensions of compactified spin-Ruijsenaars-Schneider models. Moreover, it seems to be straightforward task to apply proposed approach to the systems with generic $U(N)$-invariant Kähler phase spaces locally defined by the Kähler potential $\mathcal{K}(z^a\bar{z}^a)$. We expect that it can be done in terms of Kähler phase superspace locally defined by the potential,

$$\tilde{\mathcal{K}} = \mathcal{K}(z^a\bar{z}^a + \imath\eta^a\bar{\eta}^A). \qquad (104)$$

In this way, we expect to construct the $\mathcal{N} = 2M$ supersymmetric extensions of the systems with curved (Riemann) configuration space as well, in particular, of the so-called κ-deformations (i.e., spherical/hyperbolic generalizations) of conformal mechanics, oscillator, and Coulomb systems [20,21].

Finally, notice that considered phase superspace is not associated with external algebra of initial bosonic manifold, and thus, it is not related with the superfield approach. Thus, it is interesting to consider the systems with $(N|kM)$-dimensional Kähler phase superspaces defined by the potentials,

$$\tilde{\mathcal{K}} = \mathcal{K}(z^a\bar{z}^a) + F(\imath g_{a\bar{b}}\eta^a_\alpha\bar{\eta}^b_\alpha), \qquad F'(0) = \text{const}, \qquad (105)$$

and construct, in this way, the $\mathcal{N} = kN$ supersymmetric mechanics. A very preliminary attempt in this direction was done in [22], where the $\mathcal{N} = 2$ supersymmetric extensions of the systems with generic Kähler phase space was considered. However, this promising direction was not further developed since that time. We plan to consider listed problems elsewhere.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation of Basic Research Grant No. 20-52-12003 (S. K., A. N.) and by the Armenian Science Committee Projects No. 20RF-023, No. 21AG-1C062 (E. Kh., A. N.), and No. 21AA-1C001(E. Kh.). The work of E. Kh. was completed within the Regional Doctoral Program on Theoretical and Experimental Particle Physics Program sponsored by VolkswagenStiftung, and within ICTP Network Program NT-04.

[1] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley Classics Library, New York, 1996), Vol. 2 .

[2] D. Karabali and V. P. Nair, Quantum Hall effect in higher dimensions, Nucl. Phys. **B641**, 533 (2002); B. P. Dolan and A. Hunter-McCabe, Ground state wave functions for the quantum Hall effect on a sphere and the Atiyah-Singer index theorem, J. Phys. A **53**, 215306 (2020).

[3] S. N. M. Ruijsenaars and H. Schneider, A new class of integrable systems and its relation to solitons, Ann. Phys. (N.Y.) **170**, 370 (1986); S. N. M. Ruijsenaars, Complete integrability of relativistic calogero-moser systems and elliptic function identities, Commun. Math. Phys. **110**, 191 (1987); Action-angle maps and scattering theory for some finite-dimensional integrable systems. III. Sutherland type systems and their duals, Publ. Res. Inst. Math. Sci. Kyoto **31**, 247 (1995); J. F. van Diejen and L. Vinet, The quantum dynamics of the compactified trigonometric Ruijsenaars-Schneider model, Commun. Math. Phys. **197**, 33 (1998); L. Fehér and T. F. Görbe, Trigonometric and elliptic Ruijsenaars–Schneider systems on the complex projective space, Lett. Math. Phys. **106**, 1429 (2016).

[4] T. Hakobyan and A. Nersessian, Lobachevsky geometry of (super)conformal mechanics, Phys. Lett. A **373**, 1001 (2009).

[5] E. Khastyan, A. Nersessian, and H. Shmavonyan, Noncompact $\mathbb{CP}^N$ as a phase space of superintegrable systems, Int. J. Mod. Phys. A **36**, 2150055 (2021).

[6] O. M. Khudaverdian and A. P. Nersessian, Even and odd symplectic and Kahlerian structures on projective superspaces, J. Math. Phys. (N.Y.) **34**, 5533 (1993).

[7] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry-Mrthods and Applications. Part I: The Geometry of Surfaces, Transformation Groups, and Fields* (Springer-Verlag, New York, 1992).

[8] T. Hakobyan, S. Krivonos, O. Lechtenfeld, and A. Nersessian, Hidden symmetries of integrable conformal mechanical systems, Phys. Lett. A **374**, 801 (2010); O. Lechtenfeld, A. Nersessian, and V. Yeghikyan, Action-angle variables for dihedral systems on the circle, Phys. Lett. A **374**, 4647 (2010); T. Hakobyan, A. Nersessian, and V. Yeghikyan, Cuboctahedric Higgs oscillator from the Calogero model, J. Phys. A **42**, 205206 (2009).

[9] E. Ivanov and S. Sidorov, Deformed supersymmetric mechanics, Classical Quantum Gravity **31**, 075013 (2014); Super Kähler oscillator from $SU(2|1)$ superspace, J. Phys. A **47**, 292002 (2014); Long multiplets in supersymmetric mechanics, Phys. Rev. D **93**, 065052 (2016); E. Ivanov, S. Sidorov, and F. Toppan, Superconformal mechanics in $SU(2|1)$ superspace, Phys. Rev. D **91**, 085032 (2015); E. Ivanov, O. Lechtenfeld, and S. Sidorov, $SU(2|2)$ supersymmetric mechanics, J. High Energy Phys. 11 (2016) 031; E. Ivanov, A. Nersessian, S. Sidorov, and H. Shmavonyan, Symmetries of deformed supersymmetric mechanics on Kähler manifolds, Phys. Rev. D **101**, 025003 (2020).

[10] A. Nersessian Elements of (super-)Hamiltonian formalism, Lect. Notes Phys. **698**, 139 (2006).

[11] V. N. Shander, Complete integrability of ordinary differential equations on supermanifolds, Funct. Anal. Appl. **17**, 74 (1983); Darboux and Liouville theorems on supermanifolds, DAN Bulg. **36**, 309 (1983); O. M. Khudaverdian and A. P. Nersessian, Formulation of Hamiltonian mechanics with even and odd poisson brackets, Report No. EFI-1031-81-87-YEREVAN, 1987.

[12] B. Bina and M. Günaydin, Real forms of nonlinear superconformal and quasi-superconformal algebras and their unified realization, Nucl. Phys. **B502**, 713 (1997).

[13] J. Palmkvist, A realization of the Lie algebra associated to a Kantor triple system, J. Math. Phys. (N.Y.) **47**, 023505 (2006).

[14] T. Hakobyan, O. Lechtenfeld, and A. Nersessian, Superintegrability of generalized Calogero models with oscillator or Coulomb potential, Phys. Rev. D **90**, 101701 (2014).

[15] J. Schwinger, A magnetic model of matter Science **165**, 757 (1969).D. Zwanziger, Exactly soluble nonrelativistic model of particles with both electric and magnetic charges Phys. Rev. **176**, 1480 (1968).

[16] I. Fris, V. Mandrosov, Ya. A. Smorodinsky, M. Uhlir, and P. Winternitz, On higher symmetries in quantum mechanics, Phys. Lett. **16**, 354 (1965); P. Winternitz, Ya. A. Smorodinsky, M. Uhlir, and I. Fris, Symmetry groups in classical and quantum mechanics, Sov. J. Nucl. Phys. **4**, 444 (1967); A. A. Makarov, Ya. A. Smorodinsky, Kh. Valiev, and P. Winternitz, A systematic search for non-relativistic system with dynamical symmetries, Nuovo Cimento A **52**, 1061 (1967).

[17] A. Saghatelian, Near-horizon dynamics of particle in extreme Reissner-Nordstróm and Clement-Gal'tsov black hole backgrounds: Action-angle variables, Classical Quantum Gravity **29**, 245018 (2012).

[18] A. Galajinsky, A. Nersessian, and A. Saghatelian, Superintegrable models related to near horizon extremal Myers-Perry black hole in arbitrary dimension, J. High Energy Phys. 06 (2013) 002; Action-angle variables for spherical mechanics related to near horizon extremal MyersPerry black hole, J. Phys. Conf. Ser. **474**, 012019 (2013).

[19] M. Feigin, O. Lechtenfeld, and A. P. Polychronakos, The quantum angular Calogero-Moser model, J. High Energy Phys. 07 (2013) 162.

[20] P. Dombrowski and J. Zitterbarth, On the planetary motion in the 3-Dim standard spaces of constant curvatur, Demonstr. Math. **24**, 375 (1991); A. Ballesteros, F. J. Herranz, M. A. del Olmo, and M. Santander, Quantum structure of the motion groups of the two-dimensional Cayley-Klein geometries, J. Phys. A **26**, 5801 (1993); M. F. Rañada and M. Santander, Superintegrable systems on the two-dimensional sphere S2 and the hyperbolic plane H2, J. Math. Phys. (N.Y.) **40**, 5026 (1999); M. F. Ranada, The Tremblay-Turbiner-Winternitz system on spherical and hyperbolic spaces: Superintegrability, curvaturedependent formalism and complex factorization, J. Phys. A **47**, 165203 (2014).

[21] T. Hakobyan, A. Nersessian, and H. Shmavonyan, Symmetries in superintegrable deformations of oscillator and Coulomb systems: Holomorphic factorization, Phys. Rev. D **95**, 025014 (2017).

[22] S. Bellucci and A. Nersessian, Kahler geometry and SUSY mechanics, Nucl. Phys. B, Proc. Suppl. **102**, 227 (2001).