Loop quantum deparametrized Schwarzschild interior and discrete black hole mass

Cong Zhang,[1,*] Yongge Ma,[2,†] Shupeng Song,[3,‡] and Xiangdong Zhang[4,§]

[1]*Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland*
[2]*Department of Physics, Beijing Normal University, Beijing 100875, China*
[3]*School of Physics, Beijing Institute of Technology, Beijing 100081, China*
[4]*Department of Physics, South China University of Technology, Guangzhou 510641, China*

(Received 22 July 2021; accepted 10 January 2022; published 26 January 2022)

We present the detailed analyses of a model of loop quantum Schwarzschild interior coupled to a massless scalar field and extend the results in our previous rapid communication [C. Zhang, Y. Ma, S. Song, and X. Zhang, Loop quantum Schwarzschild interior and black hole remnant, Phys. Rev. D **102**, 041502 (2020)] to more general schemes. It is shown that the spectrum of the black hole mass is discrete and does not contain zero. This supports the existence of a black hole remnant after Hawking evaporation due to loop quantum gravity effects. Besides to show the existence of a nonvanishing minimal black hole mass in the vacuum case, the quantum dynamics for the nonvacuum case is also solved and compared with the effective one.

DOI: 10.1103/PhysRevD.105.024069

I. INTRODUCTION

As a prediction of general relativity (GR), the existence of black holes has a broad base of support from observations [1]. However, our understanding of the black hole (BH) is still far from the end. Among those challenging topics on BH, its quantum nature is particularly interesting. By studying quantum BHs, one could not only solve puzzles originating from the classical theory, but also achieve more understanding on the theory of quantum gravity.

As a background-independent approach to quantum gravity, loop quantum gravity (LQG) has been widely studied in the past 30 years [2–7]. Although some important breakthroughs have been made in LQG [8–16], its dynamics is still an open issue. The obstacle of LQG can be bypassed through applying the loop quantization techniques to the symmetry-reduced sectors of GR, where the expression of the Hamiltonian constraint becomes much simpler than that in the full theory. The resulting quantum models are expected to reflect some quantum features of full LQG, in spite of the fact that they might not be equivalent to the direct symmetric sector of full LQG. An improved treatment of quantum-reduced LQG has also been proposed to study the symmetric sectors [17]. These ideas were applied to study loop quantum Schwarzschild BH recently with different perspectives [18–39]. However, most of these studies focused on the effective dynamics, where one considered the Hamiltonian constraint with the holonomy correction and solved the effective Hamilton's equations. This treatment resulted in several important achievements. In particular, it resolves the singularity inside the Schwarzschild BH and predicts certain extensions of the Schwarzschild interior beyond the singularity (see, for instance, [26,27,34–36]). However, in the effective prescription one cannot see more intrinsic quantum natures of BH, such as the ground state of quantum BHs and the discreteness of the spectrum of Dirac observables. After all, it is necessary to consider the quantum dynamics.

There are several crucial topics on the quantum dynamics of BH. One is the issue of the final state of BH evaporation which is related to the constituent of dark matter and the puzzle of information loss. According to the Hawking radiation [40], the primordial mini BHs in the very early Universe should be completely evaporated by now. However, if the BH evaporation is halted at some stable state by some quantum gravity effect, which is called the BH remnant, these remnants would result in important cosmological consequences [41–43]. Remarkably, the remnants originating from these primordial BHs could even comprise the entire dark matter in the Universe [41,42]. Moreover, thanks to the remnant, one could argue that the information fallen into a BH with matter could be stored in the remnant after its evaporation. This provides a possible approach to solve the puzzle of information loss [44,45]. Furthermore, the distortion of the semiclassical Hawking spectrum resulted from certain discreteness of the BH mass was studied [46–49]. It was argued that in certain cases, the

*czhang@fuw.edu.pl
†mayg@bnu.edu.cn
‡songsp@bit.edu.cn
§scxdzhang@scut.edu.cn

distortion could be observable even for macroscopic BHs [46]. Although these debates are crucial and long-standing, there was no systematic study by quantum gravity to lay a solid theoretical foundation for the arguments until the prediction of a BH remnant in LQG models by [50]. In [50], the treatment of loop quantum cosmology (LQC) is employed to give a minisuperspace model based on the Kantowski-Sachs spacetime which has only two degrees of freedom, even though the precise relation between LQC and LQG remains open. The purpose of this paper is to provide the detailed constructions in [50] and extend the results to more general schemes. Moreover, the quantum dynamics of the model will be further studied in detail.

We study the model of a loop quantum Schwarzschild interior coupled to a massless scalar field. The quantum Schwarzschild BH, as the vacuum case of this model, can be resulted by vanishing the scalar field. The phase space of this system contains three pairs of canonical variables: (b, p_b) and (c, p_c) for gravity and (φ, p_φ) for the scalar field. By deparametrizing this model, one gets the physical Hamiltonian $\sqrt{\mathfrak{h}}$ of the relational evolution with respect to the scalar field φ [51–53]. In the classical theory, the Poisson bracket between $\mathfrak{h}$ and the mass M of the Schwarzschild BH vanishes. This indicates a classical Dirac observable $m = cp_c$ proportional to M. However, this commutativity may no longer be kept by the corresponding operators $\hat{\mathfrak{h}}$ and $\hat{m}$ in the quantum theory, which is relevant to the choice of schemes for the quantization of $\mathfrak{h}$. We only focus on the schemes such that $\hat{\mathfrak{h}}$ and $\hat{m}$ are still commutative, since the commutativity means the existence of a Dirac observable $\hat{m}$. Note that a general class of schemes adopted for the loop quantization of the current model can meet our requirement. In particular, it is valid for the μ_o scheme [18,19] and the new scheme balancing the μ_o scheme and $\bar{\mu}$ scheme [24,26,27]. However, it cannot be met by the $\bar{\mu}$ scheme [20,22].

We will first construct the Hamiltonian operator $\hat{\mathfrak{h}}$ and study its properties analytically. Thanks to these analytical results, a numerical method to diagonalize $\hat{\mathfrak{h}}$ is proposed so that the dynamics is computable. Then the quantum dynamics of the model can be solved for the nonvacuum case, and it can be compared with the effective one. For the vacuum case, the Hilbert space consisting of the physical states of the Schwarzschild BH is built up. In this Hilbert space m is promoted to an operator which has discrete spectrum $\overline{\sigma_\xi}$ with $0 \notin \overline{\sigma_\xi}$. This result supports the existence of a stable BH remnant.

This paper is organized as follows. In Sec. II, the theory of a loop quantum Schwarzschild BH interior coupled to a massless scalar field is briefly reviewed, including the deparametrization and the polymer quantization of this model. In Sec. III, we construct the physical Hamiltonian operator and study its properties analytically. Then the quantum dynamics for both nonvacuum and vacuum cases are solved in Sec. V. Finally, in Sec. VI, our results are summarized and discussed.

II. PRELIMINARIES

A. Deparametrization of the model

Given a spatially homogeneous 3-manifold Σ of topology $\mathbb{R} \times S^2$, because of the homogeneity, Σ is endowed with a fiducial metric

$$\mathring{q}_{ab} \mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}x^2 + r_o^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{2.1}$$

where (x, θ, ϕ) are the natural coordinates adapted to the topology and r_o is a constant with dimension of length. Since Σ is noncompact in the x direction, we introduce an elementary cell $\mathcal{C} \cong (0, L_0) \times \mathbb{S}^2$ in Σ and restrict all integrals to this elemental cell to avoid the divergence of integrations.

The classical phase space of gravity coupled to a massless scalar field contains the Ashtekar-Barbero canonical conjugate pairs $(A_a^i(x), E_i^a(x))$ for gravity and $(\varphi(x), \pi(x))$ for scalar field. As far as the homogeneous states are concerned, the scalar field φ is reduced to a constant and the fields $A_a^i(x)$, $E_i^a(x)$ and $\pi_\varphi(x)$ take the forms [53]

$$\begin{aligned} A_a^i \tau_i \mathrm{d}x^a &= \frac{c}{L_0} \tau_3 \mathrm{d}x + b\tau_2 \mathrm{d}\theta - b\tau_1 \sin\theta \mathrm{d}\phi + \tau_3 \cos\theta \mathrm{d}\phi, \\ E_i^a \tau^i \partial_a &= p_c \tau_3 \sin\theta \partial_x + \frac{p_b}{L_0} \tau_2 \sin\theta \partial_\theta - \frac{p_b}{L_0} \tau_1 \partial_\phi, \\ \pi_\varphi &= \frac{p_\varphi}{4\pi r_0^2} \sqrt{\mathring{q}} = \frac{p_\varphi}{4\pi} \sin\theta, \end{aligned} \tag{2.2}$$

where $\tau_j = -i\sigma_j/2$ $(j = 1, 2, 3)$ with σ_j being the Pauli matrix and c, b, p_c, p_b and p_φ are all constants. According to (2.2), the symmetry-reduced phase space is coordinatized by the pairs (c, p_c), (b, p_b) and (φ, p_φ). The nonvanishing Poisson brackets read

$$\{c, p_c\} = 2G\gamma, \quad \{b, p_b\} = G\gamma, \quad \{\varphi, p_\varphi\} = 1. \tag{2.3}$$

By the symmetry-reduced expression (2.2), the Gaussian and diffeomorphism constraints vanish automatically. The dynamics of this model is encoded in the Hamiltonian constraint. In the full theory of gravity coupled to a massless scalar field, the Hamiltonian constraint can be deparametrized as [54]

$$C(x) = \pi_\varphi(x) \pm \sqrt{h(x)} = 0, \tag{2.4}$$

where $h(x) = -2\sqrt{|\det(E(x))|} C_{\mathrm{gr}}(x)$, with C_{gr} being the vacuum-gravity Hamiltonian constraint. The so-called physical Hamiltonian can be written as

$$h_{\rm phy} = \frac{\sqrt{G}\gamma}{\sqrt{4\pi}} \int_\Sigma \mathrm{d}^3x \sqrt{h(x)}$$
$$= \frac{\gamma}{4\sqrt{2}\pi} \int_\Sigma \mathrm{d}^3x \sqrt{E^a_i E^b_j (-F^k_{ab}\epsilon_{ijk} + 2(1+\gamma^2) K^i_{[a} K^j_{b]})}, \tag{2.5}$$

where F^k_{ab} denotes the curvature of the connection A^i_a, K^i_a is the extrinsic curvature and the prefactor is adapted for convenience. It generates the relational evolution with respect to the scalar field. Substituting (2.2) into (2.4) and integrating both sides in the elementary cell $\mathcal{C}$, one finally obtains the Hamiltonian constraint of our model as

$$p_\varphi \pm \frac{\sqrt{4\pi}}{\sqrt{G}L_0\gamma} \sqrt{\mathfrak{h}} = 0, \tag{2.6}$$

where $\mathfrak{h}$ is given by

$$\mathfrak{h} \coloneqq p_b((b^2+\gamma^2)p_b + 2bcp_c). \tag{2.7}$$

In this work, we will also refer to the physical Hamiltonian as $\mathfrak{h}$ though it is related to the true physical Hamiltonian $h_{\rm phy}$ by $\sqrt{\mathfrak{h}} = h_{\rm phy}$.

The quantum states of the model are described by vectors in the Hilbert space $\mathcal{H}_t = \mathcal{H}_{\rm mat} \otimes \mathcal{H}_{\rm gr}$, where $\mathcal{H}_{\rm mat}$ and $\mathcal{H}_{\rm gr}$ are the Hilbert spaces describing the matter and gravity, respectively. The physical states $|\psi\rangle_{\rm phy}$ should satisfy the quantum version of the Hamiltonian constraint, i.e.,

$$\left(\widehat{p}_\varphi \pm \frac{\sqrt{4\pi}}{L_0\gamma\sqrt{G}} \widehat{\sqrt{\mathfrak{h}}}\right)|\psi\rangle_{\rm phy} = 0. \tag{2.8}$$

It gives us a Schrödinger-like equation. Therefore, in the case that the operator $\widehat{\sqrt{\mathfrak{h}}}$ is self-adjoint, the physical states, i.e., the solutions of (2.8), can be expressed as

$$|\psi\rangle_{\rm phy} = e^{\mp i \frac{\sqrt{4\pi G}}{L_0\gamma\ell_{\rm p}^2}\varphi \widehat{\sqrt{\mathfrak{h}}}} |\psi\rangle_{\rm gr}, \tag{2.9}$$

where $|\psi\rangle_{\rm gr} \in \mathcal{H}_{\rm gr}$ is a state of pure gravity and $\ell_{\rm p} \coloneqq \sqrt{G\hbar}$ is the Planck length. A Dirac observable $\mathcal{O}_{\rm phy}$ takes the form

$$\mathcal{O}_{\rm phy} = e^{\mp i \frac{\sqrt{4\pi G}}{L_0\gamma\ell_{\rm p}^2}\varphi \widehat{\sqrt{\mathfrak{h}}}} \mathcal{O}_{\rm gr} e^{\pm i \frac{\sqrt{4\pi G}}{L_0\gamma\ell_{\rm p}^2}\varphi \widehat{\sqrt{\mathfrak{h}}}} \tag{2.10}$$

with an operator $\mathcal{O}_{\rm gr}$ in $\mathcal{H}_{\rm gr}$.

To carry out the above deparametrization procedure, in next subsection we will introduce the polymer quantization for the gravity and obtain its Hilbert space $\mathcal{H}$. Then a self-adjoint operator $\widehat{\mathfrak{h}}$ on $\mathcal{H}$ is proposed by the loop quantization procedure. Finally we restrict ourselves to the subspace consisting of non-negative spectra of $\widehat{\mathfrak{h}}$ to define the Hilbert space

$$\mathcal{H}_{\rm gr} \coloneqq \widehat{P}_{[0,\infty)}\mathcal{H}, \tag{2.11}$$

where $\widehat{P}_{[0,\infty)} \coloneqq \chi_{[0,\infty)}(\widehat{\mathfrak{h}})$ is the projection operator with respect to the spectrum decomposition of $\widehat{\mathfrak{h}}$.

B. The polymer quantization

The polymer quantization of the gravity in this model leads to the Hilbert space [18]

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{\mathfrak{b}} \otimes \tilde{\mathcal{H}}_{\mathfrak{c}} = L^2(\mathbb{R}_{\rm Bohr}, \mathrm{d}\mu_0) \otimes L^2(\mathbb{R}_{\rm Bohr}, \mathrm{d}\mu_0), \tag{2.12}$$

where $\mathrm{d}\mu_0$ is the Haar measure on the Bohr compactification $\mathbb{R}_{\rm Bohr}$ of the real line $\mathbb{R}$ (see Chap. 28 in Ref. [6]). The two spaces $\tilde{\mathcal{H}}_{\mathfrak{b}}$ and $\tilde{\mathcal{H}}_{\mathfrak{c}}$ correspond to the canonical conjugate pairs (b, p_b) and (c, p_c), respectively. The standard basis of these two Hilbert spaces are denoted by $|\mu\rangle \in \tilde{\mathcal{H}}_{\mathfrak{b}}$ and $|\tau\rangle \in \tilde{\mathcal{H}}_{\mathfrak{c}}$. Their inner products read

$$\langle\mu'|\mu\rangle = \delta_{\mu',\mu}, \qquad \langle\tau'|\tau\rangle = \delta_{\tau',\tau}, \tag{2.13}$$

where the Kronecker-δ symbol is employed.

There are two types of basic operators in the Hilbert spaces. One is the momentum-variable operators $\widehat{p}_b$ and $\widehat{p}_c$, whose actions on $|\mu\rangle$ and $|\tau\rangle$ are given by

$$\widehat{p}_b|\mu\rangle = \frac{\gamma\ell_{\rm p}^2}{2}\mu|\mu\rangle,$$
$$\widehat{p}_c|\tau\rangle = \gamma\ell_{\rm p}^2\tau|\tau\rangle. \tag{2.14}$$

The other is the configuration-variable operators $\widehat{e^{i\lambda b}}$ and $\widehat{e^{i\lambda c}}$, whose actions read

$$\widehat{e^{i\lambda b}}|\mu\rangle = |\mu + 2\lambda\rangle,$$
$$\widehat{e^{i\lambda c}}|\tau\rangle = |\tau + 2\lambda\rangle. \tag{2.15}$$

As in the model of LQC, the operators $\widehat{e^{i\lambda b}}$ and $\widehat{e^{i\lambda c}}$ correspond to the holonomies along the edges parallel to the $\mathbb{R}$ direction and the equator (or the longitude because of the homogeneity) of $\mathbb{S}^2$, respectively. Moreover, $\widehat{e^{i\lambda b}}$ and $\widehat{e^{i\lambda c}}$ are not strongly continuous with respect to λ. Therefore there are no operators corresponding to the configuration variables b and c in our model. This respect to the fact that there does not exist an operator corresponding to the connection itself in the full theory.

III. THE PHYSICAL HAMILTONIAN OPERATOR

A. A separable subspace $\mathcal{H}$ of $\tilde{\mathcal{H}}$

According to (2.13), the Hilbert space $\tilde{\mathcal{H}}$ possesses a noncountable orthonormal basis, and hence it is nonseparable. The problem that the kinematical Hilbert space is nonseparable has appeared in both LQG and LQC. In LQG, the nonseparability is caused by the uncountability of the graphs based on which the Hilbert space is defined. In LQC, it is caused by the polymer quantization procedure which leads to an uncountable orthonormal basis as the case of the present model. In LQG, one proposed to employ the diffeomorphism invariance to identify the diffeomorphism equivalent graphs in order to obtain a separable diffeomorphism invariant Hilbert space [4]. In LQC, the problem is tackled by the superselection feature of the Hamiltonian constraint. More precisely, because the Hamiltonian constraint operator preserves some separable Hilbert subspaces, one can confine the study in a certain separable subspace [55,56]. Since we use the LQC treatment in the current model, the situation is very similar to that in LQC. As shown below, a separable subspace $\mathcal{H}\subset\tilde{\mathcal{H}}$ can be selected, which is preserved by the Hamiltonian operator.

Now we adopt some results in [18] to get a quantum physical Hamiltonian of the model. Classically, the physical Hamiltonian (2.7), redenoted by $\mathfrak{h}_c$, can be rewritten as

$$\mathfrak{h}_c = 2p_b b p_c c + p_b^2 b^2 + \gamma^2 p_b^2, \tag{3.1}$$

where $p_c c$ is a Dirac observable. Actually, for vacuum gravity, one has $p_c c = L_0\gamma GM$ with M being the mass of the Schwarzschild BH [53]. Since in the Hilbert space $\tilde{\mathcal{H}}$ there are no operators corresponding to b and c, the expression (3.1) cannot be promoted to an operator directly. We thus return to the integral expression (2.5) of the full deparametrized theory and follow [18] to express $C_{\rm gr}$ in terms of $\overset{\circ}{F} = \mathrm{d}(\gamma K) + [\gamma K, \gamma K]$ and the spatial curvature $\Omega = -\sin(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi\tau_3$, with

$$K = \frac{1}{\gamma}\left(\frac{c}{L_0}\tau_3\mathrm{d}x + b\tau_2\mathrm{d}\theta - b\tau_1\sin\theta\mathrm{d}\phi\right) \tag{3.2}$$

denoting the extrinsic curvature. Then the physical Hamiltonian $h_{\rm phy}$ is expressed by

$$h_{\rm phy} = \frac{1}{4\sqrt{2}\pi}\int_{\mathcal{C}}\mathrm{d}^3y\, h(y), \tag{3.3}$$

where

$$h(y) = \sqrt{E_i^a(y)E_j^b(y)(\overset{\circ}{F}{}^k_{ab}(y) - \gamma^2\Omega^k_{ab}(y))\epsilon_{ijk}}. \tag{3.4}$$

Note that, in comparison with the expression (2.2) of A, the expression (3.2) does not contain the term $\tau_3\cos(\theta)\mathrm{d}\phi$. Thus the "holonomy" of K along the φ direction is much simpler than that of A. Moreover, since Ω does not depend on dynamical variables, one needs not to regularize it by holonomies. To regularize (3.3), one fixes three edges intersecting at a point $y_0\in\mathcal{C}$, where the first edge e_1 is along the $\mathbb{R}$ direction of Σ taking length $\ell_1 = \tilde{\delta}_c L_0$ and the other two edges e_2 and e_3 are along the equator and the longitude of $\mathbb{S}^2$ with the same radians $\ell_2 = \ell_3 = \tilde{\delta}_b$. By defining the "holonomies"

$$h_i = \exp\left(\int_{e_i} K\right), \tag{3.5}$$

one can regularize $\overset{\circ}{F}{}^k_{ab}$ as [18]

$$\overset{\circ}{F}{}^k_{ab}(y_0) = -\sum_{i,j}\frac{2}{\ell_i\ell_j}\mathrm{tr}(h_i h_j h_i^{-1}h_j^{-1}\tau_k)\overset{\circ}{\omega}{}^i_a(y_0)\overset{\circ}{\omega}{}^j_b(y_0) + O(\sqrt{\ell_a\ell_b}), \tag{3.6}$$

where $\overset{\circ}{\omega}{}^i_a$ denotes the cotriad of the fiducial metric (2.1), adapting to the three edges e_i. It should be noted that in the treatment to obtain (3.6) in [18] the edges along the equator and a longitude fail in forming a closed loop. However, it is easy to show that (3.6) still holds for $\overset{\circ}{F}_{\theta,\varphi}$ in the symmetric model. It is worth mentioning that in an improved treatment proposed in [37] for the quantum-reduced model of LQG, one could integrate the connection over a really closed loop on the sphere to regularize the curvature. Substituting (3.6) and the expression (2.2) of E_i^a into (3.4), we thus get a regularized expression of $h(y_0)$ at some point y_0 in the equator as

$$h(y_0) = \frac{1}{4\pi L_0}\sqrt{2p_b\frac{\sin(\tilde{\delta}_b b)}{\tilde{\delta}_b}p_c\frac{\sin(\tilde{\delta}_c c)}{\tilde{\delta}_c} + p_b^2\frac{\sin^2(\tilde{\delta}_b b)}{\tilde{\delta}_b^2} + \gamma^2 p_b^2}. \tag{3.7}$$

Note that the homogeneity indicates that (3.4) takes the same value up to a $\sin(\theta)$ factor due to its density weight at different points on the same sphere. Thus one can obtain the regularized expression of $h(y)$ at any point y as

$$h(y) = \frac{\sin(\theta)}{4\pi L_0}\sqrt{2p_b\frac{\sin(\tilde{\delta}_b b)}{\tilde{\delta}_b}p_c\frac{\sin(\tilde{\delta}_c c)}{\tilde{\delta}_c} + p_b^2\frac{\sin^2(\tilde{\delta}_b b)}{\tilde{\delta}_b^2} + \gamma^2 p_b^2}, \tag{3.8}$$

regardless of the limitation that (3.8) is obtained by using the homogeneity of the 2-sphere rather than doing the regularization point by point on the whole sphere.

Substituting (3.8) into (3.3), one thus gets a regularized version of $h_{\rm phy} = \sqrt{\mathfrak{h}^{(\tilde{\delta}_b,\tilde{\delta}_c)}}$ for our model, where

$$\mathfrak{h}^{(\tilde{\delta}_b,\tilde{\delta}_c)} = 2p_b\frac{\sin(\tilde{\delta}_b b)}{\tilde{\delta}_b}p_c\frac{\sin(\tilde{\delta}_c c)}{\tilde{\delta}_c} + p_b^2\frac{\sin^2(\tilde{\delta}_b b)}{\tilde{\delta}_b^2} + \gamma^2 p_b^2 \tag{3.9}$$

satisfies

$$\mathfrak{h}_c = \lim_{\substack{\tilde{\delta}_b\to 0\\ \tilde{\delta}_c\to 0}}\mathfrak{h}^{(\tilde{\delta}_b,\tilde{\delta}_c)}. \tag{3.10}$$

Then it is straightforward to write down the operator $\widehat{\mathfrak{h}}^{(\tilde{\delta}_b,\tilde{\delta}_c)}$ by replacing the basic variables in (3.9) by their quantum analogs. Note that the limit $\tilde{\delta}_b\to 0$ and $\tilde{\delta}_c\to 0$ of $\widehat{\mathfrak{h}}^{(\tilde{\delta}_b,\tilde{\delta}_c)}$ does not exist for the polymerlike quantization. To obtain certain definite values of $\tilde{\delta}_b$ and $\tilde{\delta}_c$, we will employ the ideas from LQC [55,56] to make use of some physical result of quantum geometry in LQG, which enable us to define a physical regularized Hamiltonian $\mathfrak{h}$ as

$$\mathfrak{h} = \lim_{\substack{\tilde{\delta}_b\to\delta_b\\ \tilde{\delta}_c\to\delta_c}}\mathfrak{h}^{(\tilde{\delta}_b,\tilde{\delta}_c)} = 2p_b\frac{\sin(\delta_b b)}{\delta_b}p_c\frac{\sin(\delta_c c)}{\delta_c} + p_b^2\frac{\sin^2(\delta_b b)}{\delta_b^2} + \gamma^2 p_b^2. \tag{3.11}$$

Then the physical Hamiltonian operator could be obtained by quantizing (3.11). In the loop quantization of the model, various strategies have been proposed in choosing the two quantum parameters δ_b and δ_c. Roughly speaking, the choices can be classified into three schemes. The first one is the μ_0 scheme where δ_b and δ_c are chosen as constants [18,19,21]. The second one is the $\bar{\mu}$ scheme which allows δ_b and δ_c to be any functions of p_b and p_c [20,22]. The third one, referred to as the modified scheme, was developed recently where δ_b and δ_c are phase space dependent only through Dirac observables [24,26,27]. Since the expression $p_c\sin(\delta_c c)/\delta_c$ in (3.11) corresponds to the classical Dirac observable $p_c c = L_0\gamma GM$ in (3.1), we are motivated to assume that the quantum operator $\widehat{h}$ is composed of a Dirac observable $p_c\widehat{\sin(\delta_c c)}/\delta_c$, which is self-adjoint and commutes with $\widehat{\mathfrak{h}}$. This assumption rules out certain strategies such as the $\bar{\mu}$ scheme where δ_c depends on $\widehat{p}_b$ and δ_b depends on $\widehat{p}_c$, since the resulting operator would no longer commute with $\widehat{\mathfrak{h}}$. Furthermore, we assume that δ_b is a constant or any function of the Dirac observable $p_c\widehat{\sin(\delta_c c)}/\delta_c$ for the following analysis. Similar assumptions were adopted in μ_o scheme and the modified scheme. They are sufficient but not necessary to obtain a Dirac observable $p_c\widehat{\sin(\delta_c c)}/\delta_c$. In summary, the current paper focuses on the schemes such that

(i) a separable Hilbert subspace $\mathcal{H}_{\mathfrak{c}}\subset\tilde{\mathcal{H}}_{\mathfrak{c}}$ can be chosen to define an operator $p_c\widehat{\sin(\delta_c c)}/\delta_c$ corresponding to $p_c\sin(\delta_c c)/\delta_c$, which is self-adjoint, commutes with the physical Hamiltonian $\widehat{\mathfrak{h}}$;

(ii) the quantum parameter δ_b is a constant or any function of $p_c\sin(\delta_c c)/\delta_c$.

It should be noted that in the μ_0 scheme the Hamiltonian operator which we obtained coincides with the Hamiltonian constraint operator constructed in [18] up to the inverse volume and some operator orderings. Denote $\sigma_{\mathfrak{c}}$ as the spectrum of the Dirac observable $p_c\widehat{\sin(\delta_c c)}/\delta_c$. Then $\mathcal{H}_{\mathfrak{c}}$ is isometric to the Hilbert space $L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}})$ with the spectral measure $\mathrm{d}\mu_{\mathfrak{c}}$. The Hilbert space $\tilde{\mathcal{H}}_{\mathfrak{b}}\otimes\mathcal{H}_{\mathfrak{c}}\subset\tilde{\mathcal{H}}$ can be represented by $L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\tilde{\mathcal{H}}_{\mathfrak{b}})$ in which each element is a $\tilde{\mathcal{H}}_{\mathfrak{b}}$-valued function on $\sigma_{\mathfrak{c}}$. The representation is defined by

$$U:\tilde{\mathcal{H}}_{\mathfrak{b}}\otimes L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}})\ni\psi_b\otimes\psi_c \mapsto \psi_c(\cdot)\psi_b\in L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\tilde{\mathcal{H}}_{\mathfrak{b}}). \tag{3.12}$$

The inner product in $L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu;\tilde{\mathcal{H}}_{\mathfrak{b}})$ is given by

$$(\psi^{(1)},\psi^{(2)}) = \int_{\sigma_{\mathfrak{c}}}\mathrm{d}\mu_{\mathfrak{c}}\langle\psi^{(1)}(x)|\psi^{(2)}(x)\rangle, \tag{3.13}$$

where $\langle\psi^{(1)}(x)|\psi^{(2)}(x)\rangle$ denotes the inner product of $\psi^{(1)}(x),\psi^{(2)}(x)\in\tilde{\mathcal{H}}_{\mathfrak{b}}$.

For convenience, the elements in the spectrum space can be denoted by $L_o\gamma m\in\sigma_{\mathfrak{c}}$ with $m\in\mathbb{R}$ in analogy with their classical correspondence $p_c c = L_o\gamma GM$. We also define

$$\widehat{\beta}_\lambda \coloneqq p_b\widehat{\sin(\lambda b)}. \tag{3.14}$$

For a state $\psi\in L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\tilde{\mathcal{H}}_{\mathfrak{b}})$, $\psi(L_o\gamma m)\in\tilde{\mathcal{H}}_{\mathfrak{b}}$ is abbreviated to $\psi(m)$. Then the action of $\widehat{\mathfrak{h}}$ on ψ reads

$$(\widehat{\mathfrak{h}}\psi)(m) = \left(\frac{2L_o\gamma m}{\delta_b^{(m)}}\widehat{\beta}_{\delta_b^{(m)}} + \frac{1}{(\delta_b^{(m)})^2}\widehat{\beta}^2_{\delta_b^{(m)}} + \gamma^2\widehat{p}_b^2\right)\psi(m), \tag{3.15}$$

where $\delta_b^{(m)}\equiv\delta_b(L_o\gamma m)$ due to the dependence of δ_b on $p_c\widehat{\sin(\delta_c c)}/\delta_c$. By (3.15), the separable Hilbert subspace $\mathcal{H}\subset L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\tilde{\mathcal{H}}_{\mathfrak{b}})$ preserved by $\widehat{\mathfrak{h}}$ is constructed as follows. Given $L_o\gamma m\in\sigma_{\mathfrak{c}}$, let $\mathcal{H}_{\mathfrak{b}}(m)\subset\tilde{\mathcal{H}}_{\mathfrak{b}}$ be a separable Hilbert space preserved by $\widehat{\beta}_{\delta_b^{(m)}}$. Then the Hilbert space $\mathcal{H}\subset L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\tilde{\mathcal{H}}_{\mathfrak{b}})$ contains the state ψ such that $\psi(m)\in\mathcal{H}_{\mathfrak{b}}(m)$ for all $L_o\gamma m\in\sigma_{\mathfrak{c}}$. For convenience, the Hilbert space $\mathcal{H}$ constructed by $\mathcal{H}_{\mathfrak{b}}(\cdot): m \mapsto \mathcal{H}_{\mathfrak{b}}(m)$ through this procedure will be denoted by $L^2(\sigma_{\mathfrak{c}}, \mathrm{d}\mu_{\mathfrak{c}}; \mathcal{H}_{\mathfrak{b}}(\cdot))$.

B. The operators $\widehat{\beta}_\lambda$

According to (3.15), the action of $\widehat{\mathfrak{h}}$ is determined by the property of $\widehat{\beta}_{\delta_b^{(m)}}$. Since $\delta_b^{(m)}$ is constant for a given $m \in \mathbb{R}$, we will drop the superscript in $\delta_b^{(m)}$ and consider $\widehat{\beta}_{\delta_b}$ with a constant δ_b. Classically, one has

$$p_b \sin(\delta_b b) = \frac{1}{2i}(p_b e^{i\delta_b b} - p_b e^{-i\delta_b b}). \tag{3.16}$$

A well-known ambiguity in the quantization procedure is caused by the operator ordering. By definition, one has

$$\widehat{e^{i\delta_b b}}\widehat{p}_b = (\widehat{p}_b - \delta_b \gamma \ell_{\mathrm{p}}^2)\widehat{e^{i\lambda b}}. \tag{3.17}$$

We hereby introduce a parameter $\mathfrak{b}$ to parametrize various operator-ordering strategies and define

$$\widehat{p_b e^{i\delta_b b}} = (\widehat{p}_b + \gamma \ell_{\mathrm{p}}^2 \mathfrak{b})\widehat{e^{i\delta_b b}}. \tag{3.18}$$

Then, we can define a symmetric operator corresponding to (3.16) as

$$\widehat{\beta}_{\delta_b} := \frac{1}{2i}((\widehat{p}_b + \gamma \ell_{\mathrm{p}}^2 \mathfrak{b})\widehat{e^{i\delta_b b}} - \widehat{e^{-i\delta_b b}}(\widehat{p}_b + \gamma \ell_{\mathrm{p}}^2 \mathfrak{b})). \tag{3.19}$$

Its action is given by

$$\begin{aligned}\widehat{\beta}_{\delta_b}|\mu\rangle = \frac{1}{2i}\frac{\gamma \ell_{\mathrm{p}}^2}{2}((\mu + 2\delta_b + 2\mathfrak{b})|\mu + 2\delta_b\rangle \\ - (\mu + 2\mathfrak{b})|\mu - 2\delta_b\rangle).\end{aligned} \tag{3.20}$$

Thus the separable Hilbert subspaces of $\tilde{\mathcal{H}}_{\mathfrak{b}}$ preserved by $\widehat{\beta}_{\delta_b}$ are given by

$$\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)} = \overline{\{\psi \in \tilde{\mathcal{H}}_{\mathfrak{b}}, \psi(\mu) \neq 0 \text{ only for } \mu = \varepsilon_b + 2n\delta_b, n \in \mathbb{Z}\}} \tag{3.21}$$

for some constant $\varepsilon_b \in [0, 2\delta_b)$. We will show below that the separable Hilbert subspace preserved by the physical Hamiltonian $\widehat{\mathfrak{h}}$ can be constructed with $\mathcal{H}_{\mathfrak{b}}^{\varepsilon_b}$.

Denote the restriction of $\widehat{\beta}_{\delta_b}$ on $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ by $\widehat{\beta}_{\delta_b} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$. Given $\mathcal{H}_{\mathfrak{b}}^{\varepsilon_b}$ and $\mathcal{H}_{\mathfrak{b}}^{\tilde{\varepsilon}_b}$, a natural isomorphism between them can be defined by

$$i: \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)} \ni |\varepsilon_b + 2n\delta_b\rangle \mapsto |\tilde{\varepsilon}_b + 2n\delta_b\rangle \in \mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}. \tag{3.22}$$

It introduces an operator $i(\widehat{\beta}_{\delta_b} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1}$ in $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ whose action reads

$$\begin{aligned}&i(\widehat{\beta}_{\delta_b} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1}|\tilde{\varepsilon}_b + 2n\delta_b\rangle \\ &\quad = \frac{1}{2i}\frac{\gamma \ell_{\mathrm{p}}^2}{2}((2n\delta_b + \tilde{\varepsilon}_b + 2\delta_b + 2\tilde{\mathfrak{b}})|\tilde{\varepsilon}_b + 2n\delta_b + 2\delta_b\rangle \\ &\qquad - (2n\delta_b + \tilde{\varepsilon}_b + 2\tilde{\mathfrak{b}})|\tilde{\varepsilon}_b + 2n\delta_b - 2\delta_b\rangle),\end{aligned} \tag{3.23}$$

where $\tilde{\mathfrak{b}} := \mathfrak{b} + (\varepsilon_b - \tilde{\varepsilon}_b)/2$. For clarity, let us use $\widehat{\beta}_{\delta_b}^{(x)}$ to denote the operator $\widehat{\beta}_{\delta_b}$ with respect to the constant $\mathfrak{b} = x$. According to (3.23), to study the operator $\widehat{\beta}_{\delta_b}^{(x)} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ with respect to $x \neq 0$, one can use the operator $i(\widehat{\beta}_{\delta_b}^{(x)} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1}$ on $\mathcal{H}_{\mathfrak{b}}^{\tilde{\varepsilon}_b}$ with $\tilde{\varepsilon}_b = \varepsilon_b + 2x$ and, thus, have

$$i(\widehat{\beta}_{\delta_b}^{(x)} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1} = \widehat{\beta}_{\delta_b}^{(0)} \restriction \mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}. \tag{3.24}$$

Therefore, without loss of generality, we can set $\mathfrak{b} = 0$ and study properties of $\widehat{\beta}_{\delta_b} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ for any given ε_b.

From now on, let us refer to the restriction $\widehat{\beta}_{\delta_b} \restriction \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ for some $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ as $\widehat{\beta}_{\delta_b}$ unless specially noted. Because $\widehat{\beta}_{\delta_b}$ is unbounded, one has to assign certain domain to complete its definition. A natural choice of the domain $D(\widehat{\beta}_{\delta_b})$ reads

$$D(\widehat{\beta}_{\delta_b}) = \{\psi \in \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}, |\mathrm{supp}(\psi)| < \infty\}, \tag{3.25}$$

where $|\mathrm{supp}(\psi)|$ denotes the cardinality of the support of ψ. This implies that $D(\widehat{\beta}_{\delta_b})$ only consists of a finite linear combination of the basis $|\mu\rangle \in \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$. The essential self-adjointness of $\widehat{\beta}_{\delta_b}$ with the domain $D(\widehat{\beta}_{\delta_b})$ can be proven. Define a self-adjoint operator $\widehat{N}(\geq 1)$ as $\widehat{N}|\mu\rangle = (1 + \mu^2)|\mu\rangle$. Then it is straightforward to verify that there exist numbers $c, d \in \mathbb{R}$ such that

$$\begin{aligned}\|\widehat{\beta}_{\delta_b}\psi\| \leq c\|N\psi\|, \quad \forall\ \psi \in D, \\ |\langle\widehat{\beta}_{\delta_b}\psi, \widehat{N}\psi\rangle - \langle\widehat{N}\psi, \widehat{\beta}_{\delta_b}\psi\rangle| \leq d\|\widehat{N}^{1/2}\psi\|^2, \quad \forall\ \psi \in D,\end{aligned} \tag{3.26}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$. Therefore $\widehat{\beta}_{\delta_b}$ is essentially self-adjoint with the domain $D(\widehat{\beta}_{\delta_b})$ according to Theorem X.37 in [57].

By expanding the eigenequation $\widehat{\beta}_{\delta_b}|\psi\rangle = \omega|\psi\rangle$ with the basis $|\mu\rangle$, one has

$$\omega\psi(\mu) = \frac{1}{2i}\frac{\gamma \ell_{\mathrm{p}}^2}{2}((\mu + 2\delta_b)\psi(\mu + 2\delta_b) - \mu\psi(\mu - 2\delta_b)). \tag{3.27}$$

As shown in the Appendix A, $\psi(\mu)$ behaves asymptotically as

$$\psi(\mu) = \begin{cases} \frac{1}{\sqrt{|\mu|}}(\chi_+ e^{ik\ln(|\mu|)} + \chi_- e^{-ik\ln(|\mu|)}) + O(|\mu|^{-1}), & \mu = \varepsilon_b + 4n\delta_b, \\ \frac{1}{\sqrt{|\mu|}}(\chi_+ e^{ik\ln(|\mu|)} - \chi_- e^{-ik\ln(|\mu|)}) + O(|\mu|^{-1}), & \mu = \varepsilon_b + (4n+2)\delta_b, \end{cases} \tag{3.28}$$

where $k = \frac{\omega}{\gamma\ell_{\rm p}^2\delta_b}$ and $n \in \mathbb{Z}$. This implies that $\int|\psi(\mu)|^2 \mathrm{d}\mu = \infty$, and hence

$$\sum_{\mu=\varepsilon_b+2n} |\psi(\mu)|^2 = \infty.$$

Thus $\psi(\mu)$ is unnormalizable. Moreover, since $\psi(\mu)$ behaves asymptotically as a plane wave of $\ln(|\mu|)$, we can show by using the same techniques as in [58] and the Weyl's criterion [57] that the spectrum of $\widehat{p_b\sin(\delta_b b)}$ is the entire real line $\mathbb{R}$. Furthermore, the two leading-order functions in (3.28), denoted by

$$\psi_0^{\pm}(\mu) \coloneqq \frac{1}{\sqrt{|\mu|}}(\chi_+ e^{ik\ln(|\mu|)} \pm \chi_- e^{-ik\ln(|\mu|)}),$$

satisfy

$$-i\gamma\ell_{\rm p}^2\delta_b \mathrm{sgn}(\mu)\sqrt{|\mu|}\frac{\mathrm{d}}{\mathrm{d}\mu}\sqrt{|\mu|}\psi_0^{\pm}(\mu) = \omega\psi_0^{\mp}(\mu), \tag{3.29}$$

where $\mathrm{sgn}(\mu)$ denotes the sign function of μ. This implies that the operator $\widehat{\beta}_{\delta_b}$, in the large μ limit, returns to the Schrödinger quantization of $p_b b$:

$$\widehat{p_b b} = -i\gamma\ell_{\rm p}^2 \mathrm{sgn}(\mu)\sqrt{|\mu|}\frac{\mathrm{d}}{\mathrm{d}\mu}\sqrt{|\mu|}. \tag{3.30}$$

Therefore, the classical limit of $\widehat{\beta}_{\delta_b}$ is correct. This finishes the quantization procedure of $p_b\sin(\delta_b b)$ and the study of the properties of the corresponding operator $\widehat{\beta}_{\delta_b}$. The same discussion can be transported analogously to $p_c\sin(\delta_c c)$ for constant δ_c. The resulting operator $p_c\widehat{\sin(\delta_c c)}/\delta_c$ is self-adjoint and possesses the entire line as its spectrum. These properties of $p_c\widehat{\sin(\delta_c c)}/\delta_c$ are compatible with our general assumptions.

C. The operator $\widehat{\mathfrak{h}}$

Let us denote the operator in the rhs of (3.15) as

$$\widehat{\mathfrak{h}}^{(m)} \coloneqq \frac{2L_o\gamma m}{\delta_b}\widehat{\beta}_{\delta_b} + \frac{1}{\delta_b^2}\widehat{\beta}_{\delta_b}^2 + \gamma^2\widehat{p}_b^2. \tag{3.31}$$

It should be noted that the operator $\widehat{\beta}_{\delta_b}$ in (3.31) is the original one defined by (3.20) with nonvanishing $\mathfrak{b}$. However, as discussed below (3.23), the identity map i from $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ to the specific Hilbert space $\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}$ with $\tilde{\varepsilon}_b = \varepsilon_b + 2\mathfrak{b}$ ensures that we can set $\mathfrak{b} = 0$. Thus, by this treatment we define an operator in $\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}$ corresponding to $\widehat{\mathfrak{h}}^{(m)}$ as

$$i(\widehat{\mathfrak{h}}^{(m)}\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1} = \frac{2L_o\gamma m}{\delta_b}i(\widehat{\beta}_{\delta_b}\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1} + \frac{1}{\delta_b^2}i(\widehat{\beta}_{\delta_b}^2\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1} + \gamma^2 i(\widehat{p}_b^2\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1}, \tag{3.32}$$

where the map i, defined by (3.22), identifies the Hilbert spaces $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ and $\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}$ with $\tilde{\varepsilon}_b = \varepsilon_b + 2\mathfrak{b}$. For the first two terms in this operator, one has

$$i((\widehat{\beta}_{\delta_b})^n\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1} = (\widehat{\beta}_{\delta_b}^{(0)})^n\restriction\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)} \tag{3.33}$$

where $\widehat{\beta}_{\delta_b}^{(0)}$ is defined by (3.24). For the last term, we have

$$\begin{aligned} &i(\widehat{p}_b^2\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1}|\tilde{\varepsilon}_b + 2n\delta_b\rangle \\ &\quad = [(\widehat{p}_b - \gamma\ell_{\rm p}^2\mathfrak{b})^2\restriction\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}]|\tilde{\varepsilon}_b + 2n\delta_b\rangle. \end{aligned} \tag{3.34}$$

Therefore, (3.32) can be expressed as

$$i(\widehat{\mathfrak{h}}^{(m)}\restriction\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)})i^{-1} = \frac{2L_o\gamma m}{\delta_b}(\widehat{\beta}_{\delta_b}^{(0)}\restriction\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}) + \frac{1}{\delta_b^2}[(\widehat{\beta}_{\delta_b}^{(0)})^2\restriction\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}] + \gamma^2[(\widehat{p}_b - \gamma\ell_{\rm p}^2\mathfrak{b})^2\restriction\mathcal{H}_{\mathfrak{b}}^{(\tilde{\varepsilon}_b)}]. \tag{3.35}$$

That is, by setting $\mathfrak{b} = 0$ for the $\widehat{\beta}_{\delta_b}$ in $\widehat{\mathfrak{h}}^{(m)}$, $\widehat{p}_b$ has to be replaced by $\widehat{p}_b - \gamma\ell_{\rm p}^2\mathfrak{b}$. Thus we redefine $\widehat{\mathfrak{h}}^{(m)}$ as

$$\widehat{\mathfrak{h}}^{(m)} = \frac{2L_o\gamma m}{\delta_b}\widehat{\beta}_{\delta_b} + \frac{1}{\delta_b^2}\widehat{\beta}_{\delta_b}^2 + \gamma^2(\widehat{p}_b - \gamma\ell_{\rm p}^2\mathfrak{b})^2, \tag{3.36}$$

where $\widehat{\beta}_{\delta_b}$ is defined by (3.20) with $\mathfrak{b} = 0$. As a consequence, the operator $\widehat{\mathfrak{h}}$ is changed correspondingly to

$$(\widehat{\mathfrak{h}}\psi)(m) = \widehat{\mathfrak{h}}^{(m)}\psi(m) \tag{3.37}$$

with the redefined $\widehat{\mathfrak{h}}^{(m)}$.

Now let us construct the separable Hilbert $L^2(\sigma_{\mathfrak{c}}, \mathrm{d}\mu_{\mathfrak{c}}; \mathcal{H}_{\mathfrak{b}}(\cdot))$ as mentioned below (3.15). To do this, we only need to assign to each m the Hilbert space $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b(m))}$, i.e., to define $\mathcal{H}_{\mathfrak{b}}(\cdot)$ as

$$\mathcal{H}_{\mathfrak{b}}(\cdot)\colon m\mapsto \mathcal{H}_{\mathfrak{b}}(m)\coloneqq \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b(m))}, \tag{3.38}$$

where $\varepsilon_b\colon m\mapsto\varepsilon_b(m)\in[0,2\delta_b)$ is assumed to be some sufficiently well-behaved function. Then the resulting Hilbert space $L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\mathcal{H}_{\mathfrak{b}}(\cdot))$ is supposed to be acted by $\widehat{\mathfrak{h}}$. To define the domain of the unbounded operator $\widehat{\mathfrak{h}}$, we first define the domain $D_{\mathfrak{b}}(m)$ of $\widehat{\mathfrak{h}}^{(m)}\upharpoonright\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b(m))}$ for each m by

$$D_{\mathfrak{b}}(m)\coloneqq\{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b(m))},\langle\mu|\psi\rangle\neq 0 \text{ for finite numbers of } \mu\}. \tag{3.39}$$

Then the domain of $\widehat{\mathfrak{h}}$, denoted by $D(\widehat{\mathfrak{h}})$, reads

$$D(\widehat{\mathfrak{h}})=\{\psi\in L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}};\mathcal{H}_{\mathfrak{b}}(\cdot)),\psi(m)\in D_{\mathfrak{b}}(m)\ \ \forall\ \ m\}. \tag{3.40}$$

We now complete the rigorous definition of the operator $\widehat{\mathfrak{h}}$. Since $\widehat{\mathfrak{h}}$ is identical to the operator-valued function $\widehat{\mathfrak{h}}^{(\cdot)}\colon m\mapsto\widehat{\mathfrak{h}}^{(m)}$, instead of investigating $\widehat{\mathfrak{h}}$ itself, it is sufficient to study $\widehat{\mathfrak{h}}^{(m)}$ for all m. In the following, to simplify our notions, we will abbreviate $\varepsilon_b(m)$ to ε_b if the dependence of ε_b on m does not matter. Moreover, the restricted operator $\widehat{\mathfrak{h}}^{(m)}\upharpoonright\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ with the domain $D_{\mathfrak{b}}(m)$ is denoted simply by $\widehat{\mathfrak{h}}^{(m)}$. Furthermore, when $\widehat{\mathfrak{h}}$ is mentioned, it refers to the operator $\widehat{\mathfrak{h}}$ with domain $D(\widehat{\mathfrak{h}})$.

The operator $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^{(m)}$ are necessary to be self-adjoint to govern a well-defined dynamics. By using the Kato-Rellich theorem [59], we can prove that both $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^{(m)}$ are essentially self-adjoint (see Appendix B for details). Because of the essential self-adjointness of $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^{(m)}$, their closure, denoted by $\overline{\widehat{\mathfrak{h}}^{(m)}}$ and $\bar{\widehat{\mathfrak{h}}}$, respectively, can be regarded as the Hamiltonian operator of the current model with the desired properties. We use $\overline{D_{\mathfrak{b}}(m)}$ and $D(\bar{\widehat{\mathfrak{h}}})$ to denote the domains of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ and $\bar{\widehat{\mathfrak{h}}}$, respectively.

IV. DYNAMICS GOVERN BY $\bar{\widehat{\mathfrak{h}}}$

To solve the dynamics governed by $\bar{\widehat{\mathfrak{h}}}$, one needs to diagonalize $\bar{\widehat{\mathfrak{h}}}$ or, equivalently, to diagonalize $\overline{\widehat{\mathfrak{h}}^{(m)}}$ for all m. To begin with, we first study the discreteness of the spectrum of $\overline{\widehat{\mathfrak{h}}^{(m)}}$. By definition, $\widehat{\mathfrak{h}}^{(m)}$ can be reexpressed as

$$\widehat{\mathfrak{h}}^{(m)}=-L_0^2m^2\gamma^2+\left(L_0m\gamma+\frac{1}{\delta_b}\widehat{\beta}_{\delta_b}\right)^2+\gamma^2(\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2. \tag{4.1}$$

Thus, one has

$$\begin{aligned}\langle\psi|\overline{\widehat{\mathfrak{h}}^{(m)}}|\psi\rangle &\geq\gamma^2\langle\psi|(\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2|\psi\rangle\\ &\quad -L_0^2m^2\gamma^2\langle\psi|\psi\rangle,\quad\forall\ |\psi\rangle\in\overline{D_{\mathfrak{b}}(m)}.\end{aligned} \tag{4.2}$$

Hence $\overline{\widehat{\mathfrak{h}}^{(m)}}$ is bounded from below. By (4.2), we can obtain

$$\mu_n(\overline{\widehat{\mathfrak{h}}^{(m)}})\geq -L_0^2m^2\gamma^2+\gamma^2\mu_n((\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2), \tag{4.3}$$

where, for an operator $\widehat{A}$, $\mu_n(\widehat{A})$ denotes

$$\mu_n(\widehat{A})\coloneqq\sup_{\varphi_1,\ldots,\varphi_{n-1}\in\mathcal{H}_{\mathfrak{b}}^{\varepsilon_b}}\ \inf_{\substack{\psi\in D_{\mathfrak{b}}(m);\|\psi\|=1;\\ \langle\varphi_i,\psi\rangle=0,\ \forall\ i=1,\ldots,n-1}}\langle\psi,\widehat{A}\psi\rangle. \tag{4.4}$$

Note that $(\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2$ can be defined on $\overline{D_{\mathfrak{b}}(m)}$ by definition. Hence $\mu_n((\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2)$ is well defined. Since $\mu_n((\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2)\to\infty$ as $n\to\infty$, one has that

$$\lim_{n\to\infty}\mu_n(\overline{\widehat{\mathfrak{h}}^{(m)}})=\infty. \tag{4.5}$$

Hence, according to the min-max principle (see, e.g., Theorem XIII.1 in [60]), $\overline{\widehat{\mathfrak{h}}^{(m)}}$ has purely discrete spectrum. In other words, each element in the spectrum of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, denoted by $\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})$, is an eigenvalue of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ with finite multiplicity.

Given the significance of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, it is desirable to understand the properties of $\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})$. In particular, one may ask whether the eigenvalue $\omega(m)$ as a function of m is analytic or not. This issue is closely related to the analyticity of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ on m in the sense of Kato [60,61]. To overcome the technical difficulty that $\overline{\widehat{\mathfrak{h}}^{(m)}}$ for different m are defined in different Hilbert spaces, we employ the following unitary map for a given m_o:

$$\begin{aligned}\mathfrak{i}_m\colon\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b(m))}&\ni|\varepsilon_b(m)+2n\delta_b^{(m)}\rangle\\ &\mapsto|\varepsilon_b(m_o)+2n\delta_b^{(m_o)}\rangle\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b(m_o))},\end{aligned} \tag{4.6}$$

where the dependence of ε_b and δ_b on m is written explicitly. The issue on the analyticity of $\omega(m)$ can be equivalently discussed by that of $\mathfrak{i}_m\overline{\widehat{\mathfrak{h}}^{(m)}}\mathfrak{i}_m^{-1}$ which is defined on $\mathfrak{i}_m\overline{D_{\mathfrak{b}}(m)}=\overline{D_{\mathfrak{b}}(m_o)}$. For simplicity, let us use δ_b^o and ε_b^o to denote $\delta_b^{(m_o)}$ and $\varepsilon_b(m_o)$, respectively.

A sesquilinear form $T_m(\cdot,\cdot)$ associated to $\mathfrak{i}_m\overline{\widehat{\mathfrak{h}}^{(m)}}\mathfrak{i}_m^{-1}$ can be defined on $D_{\mathfrak{b}}(m_o)$ by

$$T_m(\psi,\phi) = \langle\psi|\mathfrak{i}_m\widehat{\mathfrak{h}}^{(m)}\mathfrak{i}_m^{-1}|\phi\rangle, \quad \psi,\phi\in D_{\mathfrak{b}}(m_o). \tag{4.7}$$

Then (4.2) implies that T_m is semibounded, i.e.,

$$T_m(\psi,\psi) \geq -L_0^2m^2\gamma^2\|\psi\|^2, \quad \forall\ \psi\in D_{\mathfrak{b}}(m_o). \tag{4.8}$$

Hence, T_m is closable. Its closure, denoted by $\overline{T_m}$, is the extension of T_m on the closure of $D_{\mathfrak{b}}(m_o)$ with respect to the norm

$$\|\psi\|_m = \sqrt{T_m(\psi,\psi) + (L_0^2m^2\gamma^2+1)\langle\psi|\psi\rangle}. \tag{4.9}$$

Since the norm $\|\psi\|_m$ depends on m, the resulting closures $Q(m)$ of $D_{\mathfrak{b}}(m_o)$ could depend on m in general. Suppose that $\delta_b^{(m)}$ and $\varepsilon_b(m)$ are analytic functions of m at m_o and $\delta_b^{(m_o)}\neq 0$. It turns out that $\overline{T_m}$ is an analytic family of type (a) in the sense of Kato (see Sec. VII 4 in [61]). This, by definition, is ensured by that (i) $Q(m)=Q(m_o)$ for all m sufficiently close to m_o, and (ii) $\overline{T_m}(\psi,\psi)$ is analytic on m for all $\psi\in Q(m_o)$. The detailed proof of this conclusion is presented in Appendix C.

Because $\overline{T_m}$ is symmetric and closed, there is a self-adjoint operator $\widehat{\mathfrak{t}}_m$ associated to it, which is indeed the Friedrichs extension of $\widehat{\mathfrak{h}}^{(m)}$. By the analyticity of $\overline{T_m}$, $\widehat{\mathfrak{t}}_m$ forms an analytic family of type (B) in the sense of Kato (see Sec. VII 4 in [61]). As a consequence, $\mathfrak{i}_m\overline{\widehat{\mathfrak{h}}^{(m)}}\mathfrak{i}_m^{-1}$ and thus $\overline{\widehat{\mathfrak{h}}^{(m)}}$ carry the same property because of $\widehat{\mathfrak{t}}_m = \mathfrak{i}_m\overline{\widehat{\mathfrak{h}}^{(m)}}\mathfrak{i}_m^{-1}$, which can be proven as follows. Firstly, the domain $\overline{D_{\mathfrak{b}}(m_o)}$ of $\mathfrak{i}_m\overline{\widehat{\mathfrak{h}}^{(m)}}\mathfrak{i}_m^{-1}$ is the closure of $D_{\mathfrak{b}}(m_o)$ with respect to the graph norm

$$\|\psi\|_g \coloneqq \sqrt{\|\mathfrak{i}_m\widehat{\mathfrak{h}}^{(m)}\mathfrak{i}_m^{-1}\psi\|^2 + \|\psi\|^2}.$$

Secondly, applying the same techniques in the proof of Lemma C.1 in Appendix C, one can show straightforwardly that $\|\cdot\|_g$ is equivalent to the norm $\|\cdot\|_g'$ defined by

$$\|\psi\|_g' \coloneqq \sqrt{\langle\psi|\widehat{p}_b^4|\psi\rangle + \langle\psi|\psi\rangle}.$$

Finally, because of $\|\psi\|_+ \leq \|\psi\|_g'$ for all $\psi\in D_{\mathfrak{b}}(m_o)$, one concludes $\overline{D_{\mathfrak{b}}(m_o)}\subset Q(m)$, which results in $\widehat{\mathfrak{t}}_m = \mathfrak{i}_m\overline{\widehat{\mathfrak{h}}^{(m)}}\mathfrak{i}_m^{-1}$ according to the uniqueness of the Friedrichs extension.

Due to the analyticity of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, the analyticity of $\omega(m)$ can obtained directly (see, e.g., Chap. XII of [60] and Theorem VII.1.8 in [61]), which is summarized precisely as the following theorem.

Theorem IV.1.—Suppose that $\delta_b^{(m)}$ and $\varepsilon_b(m)$ are analytic functions of m at $\mathring{m}$ and $\delta_b^{(\mathring{m})}\neq 0$. Given an eigenvalue $\mathring{\omega}$ of $\overline{\widehat{\mathfrak{h}}^{(\mathring{m})}}$ which is of an algebraic multiplicity k, for each m which is sufficiently close to $\mathring{m}$, $\overline{\widehat{\mathfrak{h}}^{(m)}}$ has exactly k eigenvalues (counting multiplicity) near $\mathring{\omega}$. These eigenvalues are given by $p(\leq k)$ distinct, single-valued and analytic functions $\omega_1(m),\ldots,\omega_p(m)$.

While this theorem is valid for the general cases that the eigenvalue ω possesses the algebraic multiplicity $k\geq 1$, it can be seen from the numerical results in the next section that each eigenspace of $\widehat{\mathfrak{h}}^{(m)}$ for all m is exactly one-dimensional.

A. Numerical approach to compute the eigenvalues and eigenstates

Since $\overline{\widehat{\mathfrak{h}}^{(m)}}$ has only a discrete spectrum, the eigenvector $|\psi\rangle$ associated to each $\omega\in\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})$ is normalizable. Hence the function $\psi(\mu)\coloneqq\langle\mu|\psi\rangle$ decreases rapidly for sufficiently large values of $|\mu|$. This fact motivates us to use the finite-dimensional cutoff approximation to collect eigenvalues of $\overline{\widehat{\mathfrak{h}}^{(m)}}$. More precisely, we consider a finitely dimensional subspace, denoted by $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}$, spanned by $|\mu_\eta\rangle = |\varepsilon_b+2\eta\delta_b\rangle\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}$ with $|\eta|\leq k$ for some large k. Let $\widehat{P}^{(k)}$ be the projection operator to $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}$ such that

$$\widehat{P}^{(k)}|\mu_\eta\rangle = \begin{cases} |\mu_\eta\rangle, & |\eta|\leq k,\\ 0, & \text{otherwise.}\end{cases} \tag{4.10}$$

Given an eigenvector $|\psi\rangle$ of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ with respect to an eigenvalue ω, we have

$$\begin{aligned}(\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}-\omega)|\psi\rangle &= (\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}-\overline{\widehat{\mathfrak{h}}^{(m)}})|\psi\rangle\\ &= -(1-\widehat{P}^{(k)})\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}|\psi\rangle\\ &\quad -\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}(1-\widehat{P}^{(k)})|\psi\rangle\\ &\quad -(1-\widehat{P}^{(k)})\overline{\widehat{\mathfrak{h}}^{(m)}}(1-\widehat{P}^{(k)})|\psi\rangle.\end{aligned} \tag{4.11}$$

Since $\psi(\mu)\coloneqq\langle\mu|\psi\rangle$ rapidly decreases for sufficiently large values of $|\mu|$, the term $1-\widehat{P}^{(k)}$ in the rhs of (4.11) indicates that $(\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}-\omega)|\psi\rangle$ should be very small for large k. Thus it is reasonable to expect that ω and $|\psi\rangle$ can be approximated by a certain eigenvalue and its corresponding eigenvector of $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}$, respectively, for large k. This is the reason for our finite-dimensional cutoff approximation method. To apply this method, we need to identify those eigenvalues of $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}$ suitable for approximating the eigenvalues of $\widehat{\mathfrak{h}}^{(m)}$ and check whether all eigenvalues of $\widehat{\mathfrak{h}}^{(m)}$ can be approximated by this way.

Since $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}$ is a $(2k+1)$-dimensional symmetric matrix under the basis $|\mu\rangle$, it has $(2k+1)$ eigenvalues. We denote them as $\lambda_i^{(k)}$ with $\lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \cdots \leq \lambda_{2k+1}^{(k)}$, where the subscript i denoted that $\lambda_i^{(k)}$ is the ith eigenvalue from the least and the superscript k corresponds to the superscript of $\widehat{P}^{(k)}$. Obviously, we have $i \leq 2k+1$ in $\lambda_i^{(k)}$. Thus, once the ith eigenvalue of $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}$ for some k is mentioned, k should satisfy $k \geq k_o(i)$ with $k_o(i) \coloneqq (i-1)/2$. Moreover, because $\overline{\widehat{\mathfrak{h}}^{(m)}}$ is semibounded, it has the minimal eigenvalue. Thus its eigenvalues can be denoted by ω_i with $i = 1, 2, \ldots$ such that $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n \leq \cdots$. Then by using the Rayleigh-Reitz technique (see, e.g., Theorem XIII.3 in [60] and Appendix D), one can obtain

$$\lambda_i^{(k_o(i))} \geq \lambda_i^{(k_o(i)+1)} \geq \cdots \geq \lambda_i^{(k_o(i)+n)} \geq \cdots \geq \omega_i. \tag{4.12}$$

As a consequence, the limit of $\lambda_i^{(k)}$ as $k \to \infty$ exists, i.e.,

$$\lambda_i \coloneqq \lim_{k\to\infty} \lambda_i^{(k)} < \infty. \tag{4.13}$$

Consider the sequence $\{\psi_i^{(k_o(i))}, \psi_i^{(k_o(i)+1)}, \ldots, \psi_i^{(k_o(i)+n)}, \ldots\}$, where, for each k, $\psi_i^{(k)}$ is an eigenvector of $\widehat{P}^{(k)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(k)}$ with the eigenvalue $\lambda_i^{(k)}$ and satisfies $\|\psi_i^{(k)}\| = 1$. Because of $\|\psi_k^{(k)}\| = 1$, the sequence $\{\psi_i^{(k_o(i))}, \psi_i^{(k_o(i)+1)}, \ldots, \psi_i^{(k_o(i)+n)}, \ldots\}$ contains a subsequence by the Banach-Alaoglu theorem; i.e., there exists $\psi_i \in \mathcal{H}_{\mathfrak{b}}^{(\epsilon_b)}$ such that

$$\lim_{l\to\infty} \langle \psi_i^{(n_l)} | \varphi \rangle = \langle \psi_i | \varphi \rangle, \quad \forall\ \varphi \in \mathcal{H}_{\mathfrak{b}}^{(\epsilon_b)}. \tag{4.14}$$

Consider all weakly convergent subsequences of $\{\psi_i^{(k_o(i))}, \psi_i^{(k_o(i)+1)}, \ldots, \psi_i^{(k_o(i)+n)}, \ldots\}$ and collect their limits defined by (4.14). Denote the space spanned by these limits as Λ_i. It turns out that each λ_i is an eigenvalue of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ and all the elements in Λ_i are eigenvectors corresponding to the eigenvalue λ_i (see Theorem E.1 in Appendix E for proof). Therefore, the ith eigenvalue and its corresponding eigenvectors of $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}$ with $k \gg i$ approximate some eigenvalue and eigenvectors of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, respectively. To check whether all eigenvalues of $\widehat{\mathfrak{h}}^{(m)}$ can be approximated by the finite-dimensional cutoff approximation, we can show that

$$\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}}) \cap (\lambda_i, \lambda_{i+1}) = \emptyset, \tag{4.15}$$

if $\lambda_i \neq \lambda_{i+1}$ (see Theorem E.2 in Appendix E for more details). In other words, each eigenvalue of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ is a limit point of the sequence $\{\lambda_j^{k_o(j)}, \lambda_j^{k_o(j)+1}, \ldots, \lambda_j^{k_o(j)+n}, \ldots\}$ for some j.

The accuracy of the above approximation can be discussed by the following procedure as in [62,63]. Let $|\psi_i^{(k)}\rangle$ be a normalized eigenvector of $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}$ corresponding to the eigenvalue $\lambda_i^{(k)}$. One has

$$\langle \psi_i^{(k)} | \overline{\widehat{\mathfrak{h}}^{(m)}} | \psi_i^{(k)} \rangle = \lambda_k^{(k)}. \tag{4.16}$$

Defining

$$\epsilon_i^{(k)} \coloneqq \|(\overline{\widehat{\mathfrak{h}}^{(m)}} - \lambda_i^{(k)})\psi_i^{(k)}\|, \tag{4.17}$$

we will show

$$\lambda_i \in (\lambda_i^{(k)} - \epsilon_i^{(k)}, \lambda_i^{(k)} + \epsilon_i^{(k)}). \tag{4.18}$$

Let $\{|\omega, \delta\rangle\}$ be an orthonormal basis of the Hilbert space, consisting of eigenvectors of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, where ω denotes the eigenvalue and δ represents other quantum numbers. Given $\lambda_i^{(k)}$, for arbitrary real numbers α and β with $\alpha < \lambda_i^{(k)} < \beta$, we have

$$\begin{aligned} &\langle \psi_i^{(k)} | (\overline{\widehat{\mathfrak{h}}^{(m)}} - \alpha)(\overline{\widehat{\mathfrak{h}}^{(m)}} - \beta) | \psi_i^{(k)} \rangle \\ &\quad = \sum_{\omega} \sum_{\delta} (\omega - \alpha)(\omega - \beta) |\langle \psi_i^{(k)} | \omega, \delta \rangle|^2. \end{aligned} \tag{4.19}$$

Then supposing $(\alpha, \beta) \cap \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}}) = \emptyset$, one will get

$$\langle \psi_i^{(k)} | (\overline{\widehat{\mathfrak{h}}^{(m)}} - \alpha)(\overline{\widehat{\mathfrak{h}}^{(m)}} - \beta) | \psi_i^{(k)} \rangle \geq 0. \tag{4.20}$$

Because of $\langle \psi_i^{(k)} | (\widehat{\mathfrak{h}}^{(m)} - \lambda_i^{(k)}) | \psi_i^{(k)} \rangle = 0$, $\langle \psi_i^{(k)} | (\overline{\widehat{\mathfrak{h}}^{(m)}} - \alpha)(\overline{\widehat{\mathfrak{h}}^{(m)}} - \beta) | \psi_i^{(k)} \rangle$ can be expanded as

$$\begin{aligned} &\langle \psi_i^{(k)} | (\overline{\widehat{\mathfrak{h}}^{(m)}} - \alpha)(\overline{\widehat{\mathfrak{h}}^{(m)}} - \beta) | \psi_i^{(k)} \rangle \\ &\quad = (\epsilon_i^{(k)})^2 + (\lambda_i^{(k)} - \alpha)(\lambda_i^{(k)} - \beta). \end{aligned} \tag{4.21}$$

Substituting (4.21) into (4.20), we get

$$\beta \leq \lambda_i^{(k)} + \frac{(\epsilon_i^{(k)})^2}{\lambda_i^{(k)} - \alpha}. \tag{4.22}$$

Note that (4.22) holds under the assumption $(\alpha, \beta) \cap \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}}) = \emptyset$. Thus, if $\beta > \lambda_i^{(k)} + (\epsilon_i^{(k)})^2/(\lambda_i^{(k)} - \alpha)$, one will get $(\alpha, \beta) \cap \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}}) \neq \emptyset$, which, together with the fact that $\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})$ is closed, ensures

$$\left(\alpha, \lambda_i^{(k)} + \frac{(\epsilon_i^{(k)})^2}{\lambda_i^{(k)} - \alpha}\right] \cap \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}}) \neq \emptyset. \tag{4.23}$$

Choosing α as $\alpha = \lambda_i^{(k)} - \epsilon_i^{(k)}$, we get

$$(\lambda_i^{(k)} - \epsilon_i^{(k)}, \lambda_i^{(k)} + \epsilon_i^{(k)}] \cap \sigma(\widehat{\overline{\mathfrak{h}}^{(m)}}) \neq \emptyset. \tag{4.24}$$

Noting that (4.24) is true for all $k \geq k_o(i)$ and $\lim_{k\to\infty} \lambda_i^{(k)} \pm \epsilon_i^{(k)} = \lambda_i$, one finally has

$$\lambda_i \in (\lambda_i^{(k)} - \epsilon_i^{(k)}, \lambda_i^{(k)} + \epsilon_i^{(k)}]. \tag{4.25}$$

Similar to the derivation of (4.22), one can also obtain

$$\alpha \geq \lambda_i^{(k)} + \frac{(\epsilon_i^{(k)})^2}{\lambda_i^{(k)} - \beta}. \tag{4.26}$$

By choosing $\beta = \lambda_i^{(k)} + \epsilon_i^{(k)}$, (4.26) will finally lead to

$$\lambda_i \in [\lambda_i^{(k)} - \epsilon_i^{(k)}, \lambda_i^{(k)} + \epsilon_i^{(k)}). \tag{4.27}$$

Then (4.18) is obtained by combining (4.25) and (4.27).

The above discussion can also be applied to estimate the accuracy of approximating $|\psi_i\rangle$ by $|\psi_i^{(k)}\rangle$. Given an interval $(\alpha_o, \beta_o) \ni \lambda_i$ such that λ_i is the only eigenvalue contained in it, we have

$$\begin{aligned}
&\langle\psi_i^{(k)}|(\widehat{\mathfrak{h}}^{(m)} - \alpha_o)(\widehat{\mathfrak{h}}^{(m)} - \beta_o)|\psi_i^{(k)}\rangle \\
&- (\lambda_i - \alpha_o)(\lambda_i - \beta_o)\sum_\delta |\langle\psi_i^{(n)}|\lambda_i, \delta\rangle|^2 \\
&= \sum_{\lambda\neq\lambda_i}\sum_\delta (\lambda - \alpha)(\lambda - \beta)\|\langle\psi_k^{(n)}|\lambda, \delta\rangle\| \geq 0.
\end{aligned} \tag{4.28}$$

Substituting (4.21) into (4.28), one obtains

$$\begin{aligned}
&(\epsilon_i^{(k)})^2 + (\lambda_i^{(k)} - \alpha)(\lambda_i^{(k)} - \beta) \\
&\geq (\lambda_i - \alpha_o)(\lambda_i - \beta_o)\sum_\delta |\langle\psi_i^{(n)}|\lambda_i, \delta\rangle|^2,
\end{aligned} \tag{4.29}$$

which leads to

$$1 - \sum_\delta |\langle\psi_i^{(k)}|\lambda_i, \delta\rangle|^2 \leq \frac{(\varepsilon_i^{(k)})^2}{(\lambda_i^{(k)} - \alpha_o)(\beta_o - \lambda_i^{(k)})}. \tag{4.30}$$

Now let us assume $\lambda_{i-1}^{(k)} \neq \lambda_i^{(k)} \neq \lambda_{i+1}^{(k)}$ without loss of generality. Then one can choose a sufficiently large k such that

$$\lambda_{i-1}^{(k)} + \epsilon_{i-1}^{(k)} < \lambda_i^{(k)} - \epsilon_i^{(k)}, \quad \lambda_i^{(k)} + \epsilon_i^{(k)} < \lambda_{i+1}^{(k)} - \epsilon_{i+1}^{(k)}, \tag{4.31}$$

which, together with (4.18), implies that the interval $(\lambda_{i-1}^{(k)} + \epsilon_{i-1}^{(k)}, \lambda_{i+1}^{(k)} - \epsilon_{i+1}^{(k)})$ contains the single eigenvalue λ_i. Then, according to (4.30), we get

$$\begin{aligned}
&1 - \sum_\delta |\langle\psi_i^{(k)}|\lambda_i, \delta\rangle|^2 \\
&\leq \frac{(\epsilon_i^{(k)})^2}{(\lambda_i^{(k)} - \lambda_{i-1}^{(k)} - \epsilon_{i-1}^{(k)})(\lambda_{i+1}^{(k)} - \lambda_i^{(k)} - \epsilon_{i+1}^{(k)})}.
\end{aligned} \tag{4.32}$$

The left-hand side of (4.32) measures the accuracy of approximating $|\psi_i\rangle$ by $|\psi_i^{(k)}\rangle$ because $|\psi_i^{(k)}\rangle$ is normalized.

B. Finite-dimensional cutoff approximation for the numerical computation

In our model, there are two free parameters ε_b and $\mathfrak{b}$. By mimicking the derivation of Theorem C.1 in Appendix C, one can show that $\widehat{\overline{\mathfrak{h}}^{(m)}}$, as an operator-valued function of ε_b and $\mathfrak{b}$, forms an analytic family of type (B) in the sense of Kato. Therefore, we can set $\varepsilon_b = 0 = \mathfrak{b}$ in our computation, and the results for case with either $\varepsilon_b \neq 0$ or $\mathfrak{b} \neq 0$ can be obtained by perturbing that for the case of $\varepsilon_b = 0 = \mathfrak{b}$. In other words, the former can be expanded as some power series of ε_b and $\mathfrak{b}$, and the convergences of the series are ensured by the analyticity of $\widehat{\mathfrak{h}}^{(m)}$ on ε_b and $\mathfrak{b}$.

By setting $\varepsilon_b = 0$, we work in the specific Hilbert space define by (3.21) with $\varepsilon_b = 0$. This Hilbert space is denoted by $\mathcal{H}^{(0)}$ where a state is given by a wave function $\psi(2\eta\delta_b) \equiv \psi(\eta)$ with $\eta \in \mathbb{Z}$. Then the action of $\widehat{\mathfrak{h}}^{(m)}$ reads

$$\begin{aligned}
(\widehat{\overline{\mathfrak{h}}^{(m)}}\psi)(\eta) = &-\frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4(\eta+2)(\eta+1)\psi(\eta+2) - i\gamma^2\ell_{\mathrm{p}}^2 L_0 m(\eta+1)\psi(\eta+1) \\
&+ \left(\frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4(\eta+1)^2 + \frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4(1+4\delta_b^2\gamma^2)\eta^2\right)\psi(\eta) + i\gamma^2\ell_{\mathrm{p}}^2 L_0 m\eta\psi(\eta-1) - \frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4\eta(\eta-1)\psi(\eta-2).
\end{aligned} \tag{4.33}$$

Thus the eigenequation $(\widehat{\overline{\mathfrak{h}}^{(m)}}\psi)(\eta) = \omega\psi(\eta)$ results in

$$\begin{aligned}
\psi(\eta+2) = \frac{4}{\gamma^2\ell_{\mathrm{p}}^4(\eta+2)(\eta+1)}\Bigg(&-\omega\psi(\eta) - i\gamma^2\ell_{\mathrm{p}}^2 L_0 m(\eta+1)\psi(\eta+1) \\
&+ \left(\frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4(\eta+1)^2 + \frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4(1+4\delta_b^2\gamma^2)\eta^2\right)\psi(\eta) + i\gamma^2\ell_{\mathrm{p}}^2 L_0 m\eta\psi(\eta-1) - \frac{1}{4}\gamma^2\ell_{\mathrm{p}}^4\eta(\eta-1)\psi(\eta-2)\Bigg).
\end{aligned} \tag{4.34}$$

For $\eta = 0$, (4.34) becomes

$$\psi(2) = \left(-\frac{2\omega}{\gamma^2 \ell_{\mathrm{p}}^4} + \frac{1}{2}\right)\psi(0) - i\frac{2L_0 m}{\ell_{\mathrm{p}}^2}\psi(1), \tag{4.35}$$

which implies that the value of $\psi(2)$ depends only on $\psi(0)$ and $\psi(1)$. Once $\psi(2)$ is obtained by (4.35), the values of $\psi(\eta)$ for all $\eta \geq 3$ can be obtained by (4.34). Thus, the values of $\psi(\eta)$ for all $\eta \geq 2$ are determined completely by $\psi(0)$ and $\psi(1)$ by (4.34).[1] Similarly, one can rewrite (4.34) to express $\psi(\eta - 2)$ in terms of $\psi(\eta - 1)$, $\psi(\eta)$, $\psi(\eta + 1)$ and $\psi(\eta + 2)$. Then, by setting $\eta = -1$, it is easy to see that the values of $\psi(\eta)$ for all $\eta \leq -3$ are completely determined by $\psi(-1)$ and $\psi(-2)$. Thus, for a given eigenvector of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, its values for $\eta < 0$ decouple from its values for $\eta \geq 0$. Hence, the eigenvectors of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ can be classified into two superselected sectors. The first sector consists of those ψ vanishing for $\eta < 0$, while the second sector consists of the ones vanishing for $\eta \geq 0$. Consider a transformation T defined as

$$(T\psi)(\eta) = \psi(-\eta - 1). \tag{4.36}$$

Then T relates the eigenvectors in the two sectors. It should be noted that this classification of the eigenvectors is valid only for the case of $\varepsilon_b = 0 = \mathfrak{b}$. However, for the case of $\varepsilon_b \neq 0$ or $\mathfrak{b} \neq 0$, the eigenvectors can be divided into two sectors, satisfying $\sum_{\eta \geq 0} |\psi(\eta)|^2 \ll 1$ or $\sum_{\eta < 0} |\psi(\eta)|^2 \ll 1$, respectively.

Given an eigenvector ψ_- in the second sector with the eigenvalue ω_-, a straightforward calculation gives that

$$(\overline{\widehat{\mathfrak{h}}^{(m)}} + \widehat{\epsilon})T\psi_- = \omega_- T\psi_-, \tag{4.37}$$

where $\widehat{\epsilon}$ satisfies

$$(\widehat{\epsilon}\psi)(\eta) = -\gamma^4 \ell_{\mathrm{p}}^4 \delta_b^2 (2\eta + 1)\psi(\eta). \tag{4.38}$$

Thus $T\psi_-$ is an eigenvector of $\overline{\widehat{\mathfrak{h}}^{(m)}} + \widehat{\epsilon}$ with respect to the eigenvalue ω_-.

Define $\widehat{H}(\lambda) = (\overline{\widehat{\mathfrak{h}}^{(m)}} + \widehat{\epsilon}) + \lambda\widehat{\epsilon}$ on $\overline{D_{\mathfrak{b}}(m)}$. Similar to the discussion on $\overline{\widehat{\mathfrak{h}}^{(m)}}$, we can show that $\widehat{H}(\lambda)$ is self-adjoint with the domain $\overline{D_{\mathfrak{b}}(m)}$ independent of λ. Moreover, $\widehat{H}(\lambda)$ forms an analytic family of type (A) in the sense of Kato. This can be verified easily by showing that $\widehat{H}(\lambda)|\psi\rangle$ for all $|\psi\rangle \in \overline{D_{\mathfrak{b}}(m)}$ is a vector-valued analytic function of λ (see, e.g., Chap. XII.1 in [60]). Therefore, the Rayleigh-Schrödinger perturbation theory can be applied to expand the eigenvalues $\Omega^{(\lambda)}$ and the eigenvectors $\Psi^{(\lambda)}$ of $\widehat{H}(\lambda)$ as the Rayleigh-Schrödinger series

$$\Omega^{(\lambda)} = \omega_- + \sum_{n=1}^{\infty} c_n (\lambda\gamma^4 \ell_{\mathrm{p}}^4 \delta_b^2)^n,$$
$$\Psi^{(\lambda)} = T\psi_- + \sum_{k=1}^{\infty} (\lambda\gamma^4 \ell_{\mathrm{p}}^4 \delta_b^2)^n \phi_n, \tag{4.39}$$

where c_n and ϕ_n denotes the coefficients (see, e.g., [60]) and the convergence of these series for all $\lambda \in \mathbb{R}$ is ensured by the analyticity of $\widehat{H}(\lambda)$ on λ. By setting $\lambda = -1$, one gets the eigenvalue $\Omega^{(-1)} =: \omega_+$ and the corresponding eigenvector $\Psi^{(-1)} =: \psi_+$ of the operator $\widehat{H}(-1) = \overline{\widehat{\mathfrak{h}}^{(m)}}$. By substituting the explicit expression of ϕ_n into (4.39), one can get $\psi_+(\eta) = 0$ for all $\eta < 0$. Thus ψ_+ are in the first sector. Therefore, for each eigenvector of $\widehat{\mathfrak{h}}^{(m)}$ in the second sector with eigenvalue ω_-, there always exists an adjoint eigenvector ψ_+ of $\widehat{\mathfrak{h}}^{(m)}$ in the first section with eigenvalue ω_+ nearby ω_-.

By (4.39), for a very small value of $\gamma^4 \ell_{\mathrm{p}}^4 \delta_b^2$, the difference between ω_+ and ω_- would be very tiny so that very high computational cost is needed to separate their values numerically. To overcome this difficulty we consider the Hilbert space $\mathcal{H}_{\mathfrak{b}}^{(0+)} \subset \mathcal{H}_{\mathfrak{b}}^{(0)}$ defined by

$$\mathcal{H}_{\mathfrak{b}}^{(0+)} = \overline{\{\psi \in \mathcal{H}_{\mathfrak{b}}^{(0)}, \psi(\eta) = 0 \ \forall \ \eta < 0\}}, \tag{4.40}$$

where the symbol $\overline{\{\cdot\}}$ represents the completion with respect to the inner product of $\mathcal{H}_{\mathfrak{b}}^{(0)}$. Then, we diagonalize the operators $\widehat{\mathfrak{h}}^{(m)} \restriction \mathcal{H}_{\mathfrak{b}}^{(0+)}$ and $(\widehat{\mathfrak{h}}^{(m)} + \widehat{\epsilon}) \restriction \mathcal{H}_{\mathfrak{b}}^{(0+)}$, denoting the restrictions of $\widehat{\mathfrak{h}}^{(m)}$ and $\widehat{\mathfrak{h}}^{(m)} + \widehat{\epsilon}$ on $\mathcal{H}_{\mathfrak{b}}^{(0+)}$, respectively, by the finite-dimensional cutoff approximation method. Let ψ_+ be an eigenvector of $\widehat{\mathfrak{h}}^{(m)} \restriction \mathcal{H}_{\mathfrak{b}}^{(0+)}$ with eigenvalue ω_+, and $\tilde{\psi}_-$ the eigenvector of $(\widehat{\mathfrak{h}}^{(m)} + \widehat{\epsilon}) \restriction \mathcal{H}_{\mathfrak{b}}^{(0+)}$ with eigenvalue ω_-. Then the vectors ψ_+ and $T^{-1}\tilde{\psi}_- =: \psi_-$ in $\mathcal{H}_{\mathfrak{b}}^{(0)}$ are the eigenvectors of $\widehat{\mathfrak{h}}^{(m)}$ with eigenvalues ω_+ and ω_-, respectively. Moreover, (4.39) implies

$$\psi_+(\eta) \cong (T\psi_-)(\eta). \tag{4.41}$$

V. THE QUANTUM DYNAMICS

We now study the dynamics of the model for the cases of $\widehat{p}_\varphi \neq 0$ and $\widehat{p}_\varphi = 0$, respectively. For $\widehat{p}_\varphi \neq 0$, the corresponding classical solution is an extension of the Schwarzschild interior with an extra minimally coupled massless scalar field. This extension is referred to as the Janis-Newman-Winicour (JNW) interior which differs from the usual JNW spacetime as an extension of Schwarzschild exterior [64]. In the classical JNW interior [53], characterized by a parameter $B = 2\sqrt{m^2 + Gp_\varphi^2/4\pi}$,

[1]Note that the values of $\psi(0)$ and $\psi(1)$ have to be chosen suitably so that the resulting $\psi(\eta)$ is normalizable.

there are two singularities at $r=0$ and $r=B$, respectively, where r is the radial coordinate. Once p_φ vanishes, the singularity at $r=B$ will disappear and is replaced by the Schwarzschild horizon, so that the JNW interior becomes the Schwarzschild interior.

In Sec. II A, gravity is deparametrized by the scalar field which provides a material reference frame of time. However, for the case of $\widehat{p}_\varphi=0$, the reference frame of time will disappear. The physical Hamiltonian becomes the Hamiltonian constraint. However, it is still necessary to understand the dynamics of the system as the relational evolution with respect to certain gravitational degree of freedom. The information of the dynamics is encoded in the solutions to $\widehat{h}\psi=0$. Thus we need to solve this equation.

A. Dynamics for $\widehat{p}_\varphi \neq 0$

By (3.12), the Hilbert space $L^2(\sigma_\mathfrak{c}, \mathrm{d}\mu_\mathfrak{c}; \mathcal{H}_\mathfrak{b}(\cdot))$ of the model consists of functions $\psi: m \mapsto \psi(m) \in \mathcal{H}_\mathfrak{b}(m)$. As shown in Sec. III B, $\mathcal{H}_\mathfrak{b}(m)$ can be chosen as the one defined by (3.21) with $\varepsilon_b=0$ and some m-dependent δ_b, denoted by $\delta_b^{(m)}$ whose explicit expression depends on the schemes to quantize $\mathfrak{h}$. With this convention, a state $\psi \in L^2(\sigma_\mathfrak{c}, \mathrm{d}\mu_\mathfrak{c}; \mathcal{H}_\mathfrak{b}(\cdot))$ can be represented by a family of functions

$$\psi(m,\cdot): \eta \to \psi(m, 2\delta_b^{(m)}\eta)$$

with $\eta \in \mathbb{Z}$. Given a state ψ, according to (2.9) and (2.11), an associated dynamical state reads

$$\psi(\varphi, m) = e^{\mp i \frac{\sqrt{4\pi G}}{L_0 \gamma \ell_\mathrm{p}^2}\varphi \sqrt{\widehat{\mathfrak{h}}^{(m)}}} \widehat{P}_{[0,\infty)}\psi(m). \tag{5.1}$$

Let $|\omega(m)\rangle \in \mathcal{H}_\mathfrak{b}(m)$ be the normalized eigenvector of $\widehat{\mathfrak{h}}^{(m)}$ with respect to the eigenvalue $\omega(m)$. Then (5.1) is simplified as

$$\psi(\varphi, m) = \sum_{\omega(m)\geq 0} e^{\mp i \frac{\sqrt{4\pi G\omega(m)}}{L_0\gamma\ell_\mathrm{p}^2}\varphi} \langle \omega(m)|\psi(m)\rangle |\omega(m)\rangle. \tag{5.2}$$

We choose $\psi(m)$ as

$$\psi(m,\eta) = e^{-\frac{(m-m_0)^2}{2\sigma_m^2}+i\lambda m} e^{-\frac{(\eta-\eta_0)^2}{2\sigma_\eta^2}-i\beta\eta}, \tag{5.3}$$

which carries some semiclassical features. According to (5.1), the initial state evolved by the Hamiltonian is $\widehat{P}_{[0,\infty)}\psi(m)$. Then, it is possible that $\widehat{P}_{[0,\infty)}\psi(m)$ is no longer a semiclassical wave packet even though $\psi(m)$ is. To see how to avoid this possibility, we introduce the expectation value of $\widehat{\mathfrak{h}}^{(m)}$:

$$\omega_o(m) \coloneqq \langle \psi(m)|\widehat{\mathfrak{h}}^{(m)}|\psi(m)\rangle$$

and its uncertainty

$$\Delta\omega(m) \coloneqq \sqrt{\langle \psi(m)|(\widehat{\mathfrak{h}}^{(m)})^2|\psi(m)\rangle - \langle \psi(m)|\widehat{\mathfrak{h}}^{(m)}|\psi(m)\rangle^2}.$$

For each m, we think of $\psi(m)$ as a wave function of the eigenvalues ω of $\widehat{\mathfrak{h}}^{(m)}$, so it is some wave packet peaked at $\omega_0(m)$ with fluctuation $\Delta\omega(m)$. The projection $\widehat{P}_{[0,\infty)}$ cuts off $\psi(m)$ at $m=0$ and vanishes it for all $\omega(m)\leq 0$. Therefore, $\widehat{P}_{[0,\infty)}\psi(m)$ can keep the wave-packet feature of $\psi(m)$ only if $|\omega_o(m)| \gg \Delta\omega(m)$. This condition is the criterion to choose the parameters in (5.3).

For a properly chosen $\psi(m)$, its evolution reads

$$\psi(m,\eta,\varphi) = e^{-\frac{(m-m_0)^2}{2\sigma_m^2}+i\lambda m} \sum_{\omega(m)\geq 0} \langle \mu_\eta^{(m)}|\omega(m)\rangle e^{i\frac{\sqrt{4\pi G\omega(m)}}{L_0\gamma\ell_\mathrm{p}^2}\varphi} \sum_{\eta'} e^{-\frac{(\eta'-\eta_0)^2}{2\sigma_\eta^2}-i\beta\eta'} \langle \omega(m)|\mu_{\eta'}^{(m)}\rangle. \tag{5.4}$$

To check the consistence between the quantum dynamics and effective dynamics, we calculate the expectation value of the operator $\widehat{p}_b$ as

$$\begin{aligned}\frac{\langle \widehat{p}_b\rangle}{\gamma\delta_b\ell_\mathrm{p}^2} &= \mathcal{N}^2 \int_{-\infty}^{\infty} \mathrm{d}m e^{-\frac{(m-m_0)^2}{\sigma_m^2}} \sum_\eta \eta \left| \sum\nolimits_{\omega(m)\geq 0} \langle \mu_\eta^{(m)}|\omega(m)\rangle e^{i\frac{\sqrt{4\pi G\omega(m)}}{\ell_\mathrm{p}^2\gamma L_0}\varphi} \sum\nolimits_{\eta'} e^{-\frac{(\eta'-\eta_0)^2}{2\sigma_\eta^2}-i\beta\eta'} \langle \omega(m)|\mu_{\eta'}^{(m)}\rangle \right|^2 \\ &\eqqcolon \frac{1}{\sqrt{\pi}\sigma_m} \sum_\eta \eta \int_{-\infty}^{\infty} \mathrm{d}m e^{-\frac{(m-m_0)^2}{\sigma_m^2}} \mathcal{P}(\eta, m, \varphi),\end{aligned} \tag{5.5}$$

where the normalization factor $\mathcal{N}$ is given by

$$\mathcal{N}^{-2} = \int_{-\infty}^{\infty} \mathrm{d}m e^{-\frac{(m-m_0)^2}{\sigma_m^2}} \sum_{\omega(m)\geq 0} \left| \sum_\eta e^{-\frac{(\eta-\eta_0)^2}{2\sigma_\eta^2} - i\beta\eta} \langle \omega(m) | \mu_\eta^{(m)} \rangle \right|^2 =: \int_{-\infty}^{\infty} \mathrm{d}m e^{-\frac{(m-m_0)^2}{\sigma_m^2}} \mathfrak{n}(m). \tag{5.6}$$

The integral in (5.5) can be calculated by using the saddle point approximation as

$$\frac{\langle \widehat{p}_b \rangle}{\gamma \delta_b \ell_{\mathrm{p}}^2} = \sum_\eta \eta \mathcal{P}(\eta, m_0, \varphi) \left(1 + \frac{\sigma_{m_0}^2}{4} \frac{\mathcal{P}''(\eta, m_0, \varphi)}{\mathcal{P}(\eta, m_0, \varphi)} + \cdots \right). \tag{5.7}$$

Therefore, as far as the leading order of the evolution is concerned, we only need to compute $\mathcal{P}(\eta, m_0, \varphi)$. The result of (5.6) reads

$$\mathcal{N}^{-2} = \sqrt{\pi} \sigma_{m_0} \mathfrak{n}(m_0) \left(1 + \frac{\sigma_m^2}{4} \frac{\mathfrak{n}''(m_0)}{\mathfrak{n}(m_0)} + \cdots \right). \tag{5.8}$$

Hence one has

$$\mathcal{P}(\eta, m_0, \varphi) \cong \frac{\left| \sum_{\omega(m_0)\geq 0} \langle \mu_\eta^{(m_0)} | \omega(m_0) \rangle e^{i \frac{\sqrt{4\pi G \omega(m_0)}}{\ell_{\mathrm{p}}^2 \gamma L_0} \varphi} \sum_{\eta'} e^{-\frac{(\eta'-\eta_0)^2}{2\sigma_\eta^2} - i\beta\eta'} \langle \omega(m_0) | \mu_{\eta'}^{(m_0)} \rangle \right|^2}{\sum_{\omega(m_0)\geq 0} \left| \sum_\eta e^{-\frac{(\eta-\eta_0)^2}{2\sigma_\eta^2} - i\beta\eta} \langle \omega(m_0) | \mu_\eta^{(m_0)} \rangle \right|^2}. \tag{5.9}$$

By this formula, $\mathcal{P}(\eta, m_0, \varphi)$ can be computed numerically easily. The numerical results of $\mathcal{P}(\eta, m_0, \varphi)^{1/2}$ and $\langle \widehat{p}_b \rangle / (\gamma \delta_b \ell_{\mathrm{p}}^2) \cong \sum_\eta \eta \mathcal{P}(\eta, m_0, \varphi)$ are shown in Fig. 1, where we choose $\delta_b = \sqrt{\Delta}$ with Δ being the area gap in LQG. Moreover, one can also calculate the evolution of p_b with respect to the effective Hamiltonian

$$H = \frac{4\pi}{L_0 G \gamma} \sqrt{\frac{2}{\delta_b \delta_c} p_b \sin(\delta_b b) p_c \sin(\delta_c c) + \frac{1}{\delta_b^2} p_b^2 \sin(\delta_b b)^2 + \gamma^2 p_b^2}. \tag{5.10}$$

As a comparison, the results of the effective dynamics and the classical dynamics are also plotted in Fig. 1.

As shown in Fig. 1, both of the classical singularities could be resolved by the effective dynamics where $|\eta|$ is prevented from reaching 0 by the bounces at the local minimums. Then the classical spacetime is extended periodically. The quantum evolution matches very well with the effective dynamics for several periods around $\varphi = 0$ when the semiclassical feature of the coherent state is well kept. The effective evolution and thus the quantum evolution match well with the classical dynamics in the classical regime. Thus the current quantum theory has a correct semiclassical limit and its semiclassical features can be responded properly by the effective dynamics. However, the coherent property of the state cannot be kept along the whole evolution, since the width of the wave packet grows as the time φ runs far away from the initial value $\varphi = 0$. This leads to a significant difference between the quantum and effective dynamics in late time.

B. Dynamics for $\widehat{p}_\varphi = 0$

In the case of $\widehat{p}_\varphi = 0$, the dynamics is encoded in the equation

$$\widehat{\mathfrak{h}} \psi = 0. \tag{5.11}$$

Alternatively, one could also consider the Hamiltonian constraint operator corresponding to the vacuum Hamiltonian constraint multiplied by volume as a lapse function [50]. To solve (5.11), it is convenient to represent $\psi(m)$ for each m by $\psi(m, \cdot) : \omega(m) \mapsto \psi(m, \omega(m)) \in \mathbb{C}$ with $\omega(m) \in \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})$. Then the action of $\overline{\widehat{\mathfrak{h}}}$ on ψ reads

$$(\widehat{\mathfrak{h}} \psi)(m, \omega(m)) = \omega(m) \psi(m, \omega(m)). \tag{5.12}$$

By (3.13) the inner product of two states ψ_1 and ψ_2 reads

$$(\psi_1, \psi_2) = \int_{\sigma_{\mathfrak{c}}} \mathrm{d}\mu_{\mathfrak{c}} \sum_{\omega(m) \in \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})} \psi_1(m, \omega(m))^* \psi_2(m, \omega(m)), \tag{5.13}$$

where $*$ means the complex conjugate.

Given a solution ψ to (5.11), Eq. (5.12) implies

$$\psi(m, \omega(m)) = 0, \quad \forall \ \omega(m) \neq 0. \tag{5.14}$$

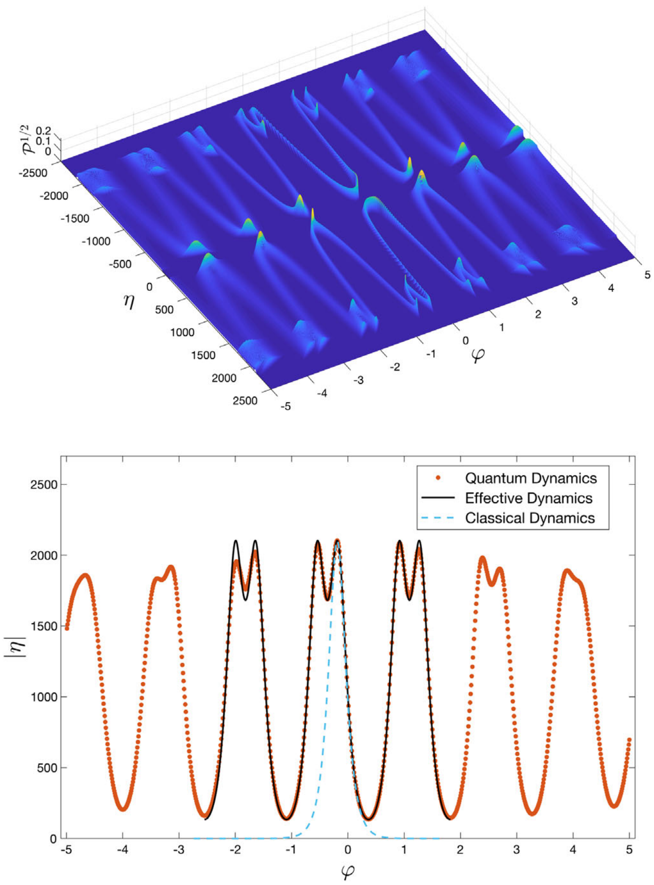

FIG. 1. Plots of the evolution of the wave packet (top panel) and the evolution of $|\eta| = p_b/(\gamma\delta_b\ell_{\mathrm{p}}^2)$ derived by the quantum, the effective dynamics and the classical dynamics (bottom panel). According to the results, the quantum dynamics as well as the effective dynamics matches well with the classical dynamics in the classical regime. Thus the quantum model admits a correct classical limit. Moreover, the quantum evolution of p_b matches very well with the effective dynamics in the domain where the wave packet is sharply peaked. However, the width of the wave packet grows as φ goes beyond the domain. Then the effective dynamics is no longer valid.

Thus the support of the solution ψ is contained in the set

$$\mathcal{S}_0 = \{(m, \omega(m)) | m \in \sigma_{\mathfrak{c}}, \omega(m) \in \sigma(\widehat{\mathfrak{h}}^{(m)}), \omega(m) = 0\}, \tag{5.15}$$

which is identified naturally to the set

$$\sigma_\xi \coloneqq \{m \in \sigma_{\mathfrak{c}}, 0 \in \sigma(\widehat{\mathfrak{h}}^{(m)})\}. \tag{5.16}$$

Given two functions ψ_i with $i = 1, 2$ on σ_ξ, (5.13) can be expressed by

$$(\psi_1, \psi_2) = \int_{\sigma_\xi} \mathrm{d}\mu_{\mathfrak{c}} \psi_1(m)^* \psi_2(m). \tag{5.17}$$

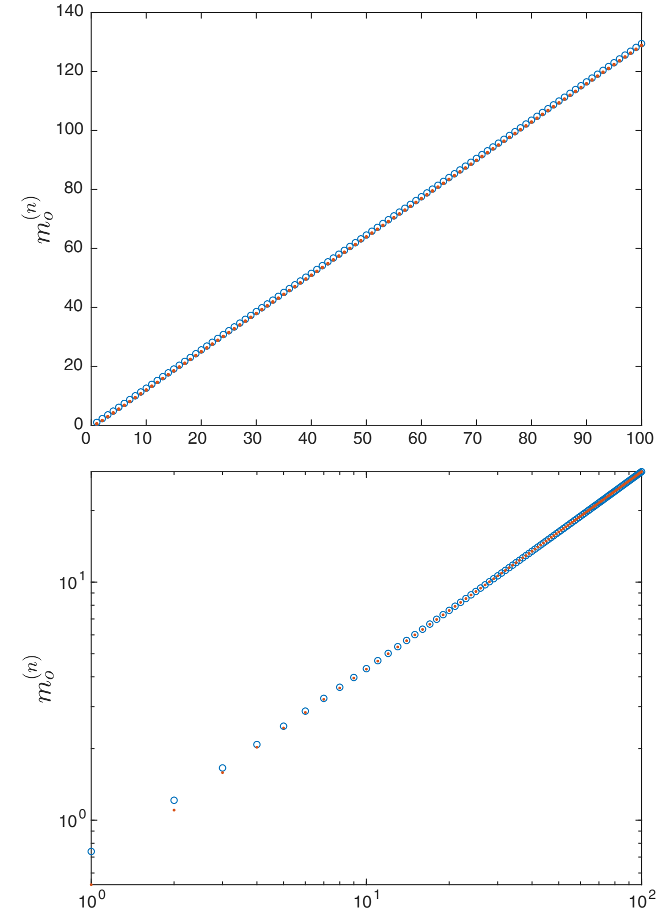

FIG. 2. Plots of σ_ξ for the μ_o scheme with $\delta_b = \sqrt{\Delta} = \delta_c$ (top panel) and the scheme with $\delta_b = \sqrt{\Delta}/(2|m|)$ and $\delta_c = \sqrt{\Delta}$ (bottom panel). As shown in the figure, nearby each value of $m_o^{(n)}$ there exists an adjoint value of $m_o^{(n')}$. 0 does not belong to σ_ξ. The parameters are chosen as $\gamma = 0.2374$, $\Delta = 4\sqrt{3}\pi\gamma\ell_{\mathrm{p}}^2$ and $\ell_{\mathrm{p}} = 1$.

Thus, one could naively deem that the functions on σ_ξ with the inner product (5.17) would constitute the physical Hilbert space. However, depending on the explicit expression of $\mu_{\mathfrak{c}}$, it could occur that the right-hand side of (5.17) vanishes for regular functions ψ_1 and ψ_2. To see how this happens, let us assume $\sigma_{\mathfrak{c}} = \mathbb{R}$ at first. Because the eigenvalues of $\widehat{\mathfrak{h}}^{(m)}$ can be expanded at some m_o closed to m by a power series of $m - m_o$ and $\sigma(m_o)$ is discrete, in general they are not 0. Hence it is reasonable to expect that there are only countably many $m \in \sigma_{\mathfrak{c}}$ such that $0 \in \sigma(\widehat{\mathfrak{h}}^{(m)})$. Then, both σ_ξ and $\mathcal{S}_0$ are countable sets. This speculation is confirmed by our numerical computation in the μ_o scheme as we as the scheme with $\delta_c = \sqrt{\Delta}$ and $\delta_b = \sqrt{\Delta}/(2|m|)$, as shown in Fig. 2. For the case of $\sigma_{\mathfrak{c}} \neq \mathbb{R}$, one has $\sigma_{\mathfrak{c}} \subset \mathbb{R}$ because $p_c \widehat{\sin(\delta_c c)}/\delta_c$ was assumed to be self-adjoint. Then the resulting σ_ξ is just a subset of that for the case of $\sigma_\xi = \mathbb{R}$. Therefore, σ_ξ is always countable. Since σ_ξ is countable, it could occur that

$\sigma_\xi \subset \sigma_{\mathfrak{c}}$ is of vanishing measure, i.e., $\mu_{\mathfrak{c}}(\sigma_\xi) = 0$. For this case, the right-hand side of (5.17) will vanish and thus it cannot define an inner product for the physical Hilbert space. Hence we introduce the following procedure to define the inner product in the solution space, which is valid not only for the case of $\mu_{\mathfrak{c}}(\sigma_\xi) = 0$ but also for the case of $\mu_{\mathfrak{c}}(\sigma_\xi) \neq 0$. Let $\tilde{\delta}(m_o, \cdot)$ for each $m_o \in \sigma_{\mathfrak{c}}$ be the function on $\sigma_{\mathfrak{c}}$ such that (i) $\tilde{\delta}(m_o, m) = 0$ for all $m \neq m_o$ and (ii) $\int_{\sigma_{\mathfrak{c}}} \mathrm{d}\mu_{\mathfrak{c}} \tilde{\delta}(m_o, m) = 1$. Thus $\tilde{\delta}(m_o, \cdot)$ is the Dirac-δ distribution for $\mu_{\mathfrak{c}}(\sigma_\xi) = 0$ and proportional to the Kronecker δ function otherwise. Given a regular function $\psi(m, \omega(m))$, a solution to (5.11) can be generated as

$$\Psi(m) = \sum_n \tilde{\delta}(m_o^{(n)}, m)\delta_{0,\omega_m}\psi(m, \omega(m)), \tag{5.18}$$

where $m_o^{(n)} \in \sigma_\xi \subset \sigma_{\mathfrak{c}}$. By choosing an appropriate dense subspace $\mathcal{S} \subset \mathcal{H}$, these Ψ of (5.18) are indeed antilinear functionals on $\mathcal{S}$ as

$$\begin{aligned} \Psi: \phi \mapsto \Psi[\phi] &:= \int_{\sigma_{\mathfrak{c}}} \mathrm{d}\mu_{\mathfrak{c}} \sum_{\omega_m} \Psi(m)^* \phi(m, \omega_m) \\ &= \sum_{m_o^{(n)}} \psi(m_o^{(n)}, 0)^* \phi(m_o^{(n)}, 0) \end{aligned} \tag{5.19}$$

for all $\phi \in \mathcal{S}$. Hence Ψ is the solution to (5.11) in the sense that $\Psi[\widehat{\mathfrak{h}}\phi] = 0$ for all $\phi \in \mathcal{S}$. Thus (5.18) defines a rigging map $\mathbb{P}: \psi \mapsto \mathbb{P}\psi$ on $\mathcal{S}$. Therefore, by the refined algebraic quantization procedure [6], the physical inner product of two solutions $\Psi_i = \mathbb{P}\psi_i$ with $i = 1, 2$ reads

$$(\Psi_1|\Psi_2) = \Psi_1[\psi_2] = \sum_{n \in \mathbb{Z}} \psi_1(m_o^{(n)})^* \psi_2(m_o^{(n)}), \tag{5.20}$$

which coincides with (5.17) if $\mu_{\mathfrak{c}}(\sigma_\xi) \neq 0$. Hence the physical Hilbert space of the solutions is given by

$$\mathcal{H}_{\mathrm{BH}} := \overline{\{f: \sigma_\xi \to \mathbb{C}, \sum_n |f(m_o^{(n)})|^2 < \infty\}}. \tag{5.21}$$

The Dirac observable $\widehat{\xi}_{\delta_c} := p_c \widehat{\sin(\delta_c c)}/\delta_c$ in $\mathcal{H}$ can be promoted to an operator $\widehat{\xi}'_{\delta_c}$ in $\mathcal{H}_{\mathrm{BH}}$ by the dual action such that

$$(\widehat{\xi}'_{\delta_c} f)(m_o^{(n)}) = L_0 \gamma \delta_c m_o^{(n)} f(m_o^{(n)}), \quad \forall\ m_o^{(n)}. \tag{5.22}$$

Equation (5.22) implies that each $m_o^{(n)}$ is an eigenvalue of $\widehat{\xi}'_{\delta_c}$, and $\widehat{\xi}'_{\delta_c}$ is self-adjoint in $\mathcal{H}_{\mathrm{BH}}$ with the spectrum $\overline{\sigma_\xi}$ as the closure of σ_ξ.

Given $\psi \in \mathcal{H}_{\mathfrak{b}}(m)$, Eq. (4.33) indicates

$$(\overline{\widehat{\mathfrak{h}}^{(-m)}}\psi)(\eta)^* = (\overline{\widehat{\mathfrak{h}}^{(m)}}\psi^*)(\eta), \tag{5.23}$$

where $\psi^* \in \mathcal{H}_{\mathfrak{b}}(m)$ is defined by $\psi^*(\eta) = \psi(\eta)^*$. For a given $m_o^{(n)} \in \sigma_\xi$, let ψ_0 be the eigenvector of $\overline{\widehat{\mathfrak{h}}^{(m_o^{(n)})}}$ with the eigenvalue 0. Then (5.23) ensures that

$$\overline{\widehat{\mathfrak{h}}^{(-m_o^{(n)})}}\psi_0^* = \overline{\widehat{\mathfrak{h}}^{(m_o^{(n)})}}\psi_0 = 0. \tag{5.24}$$

Thus, ψ_0^* is an eigenvector of $\widehat{\mathfrak{h}}^{(-m_o^{(n)})}$ with the eigenvalue 0. Therefore, one has $-m_o^{(n)} \in \overline{\sigma_\xi}$ provided $m_o^{(n)} \in \overline{\sigma_\xi}$.

For $m = 0$, one has the operator

$$\widehat{\mathfrak{h}}^{(0)} = \frac{1}{\delta_b^2}\widehat{\beta}_{\delta_b}^2 + \gamma^2 \widehat{p}_b^2.$$

Assume that there is an eigenvector ϕ of $\overline{\widehat{\mathfrak{h}}^{(0)}}$ with the eigenvalue 0. Because of $\widehat{\beta}_{\delta_b}^2 \geq 0$ and $\widehat{p}_b^2 \geq 0$, ϕ would satisfy

$$\widehat{\beta}_{\delta_b}^2 \phi = 0 = \widehat{p}_b^2 \phi. \tag{5.25}$$

By the definitions of $\widehat{\beta}_{\delta_b}$ and $\widehat{p}_b^2$, one can easily check that there is no nontrivial ϕ satisfying (5.25). Hence, 0 is not an eigenvalue of $\overline{\widehat{\mathfrak{h}}^{(0)}}$. Taking account of Theorem IV.1, one gets the conclusion that $0 \notin \sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})$ for sufficiently small $|m|$. Therefore, there exists a gap between the spectrum $\overline{\sigma_\xi}$ and 0, i.e., $0 \notin \overline{\sigma_\xi}$. Note that $\overline{\sigma_\xi}$ is the spectrum of the operator $\widehat{\xi}'_{\delta_c}$ in the physical Hilbert space, whose classical limit of $p_c \widehat{\sin(\delta_c c)}/\delta_c$ is proportional to the mass of the Schwarzschild BH. Thus the above analysis shows the discreteness of this mass spectrum. If a certain mechanism of BH evaporation could be introduced into our quantum model consistently such that the BH evaporates from one eigenstate of $\widehat{\xi}'_{\delta_c}$ to another, in such BH evaporation models, the operator $\widehat{\xi}'_{\delta_c}$ should refer to the quasilocal mass of the BH itself (e.g., the mass of isolated horizon defined in [65]) which does not includes the mass of the radiation, because $\widehat{\xi}'_{\delta_c}$ is comprised of the symmetry-reduced variables inside the BH. Then the BH would evaporate its mass discretely, and the evaporation would eventually halt at the stable ground state with a nonvanishing minimal eigenvalue of $\widehat{\xi}'_{\delta_c}$. We call this ground state the BH remnant. Note that the above discussion, extrapolating from a quantum description of the BH interior, assumes that the exterior quantum description of the BH and the inclusion of Hawking radiation will not change the mass spectrum.

The above analysis is compatible with the numerical results in Fig. 2, which shows that the values of $m_o^{(n)}$ are discrete and have the following characters. First, for each $m_o^{(n)}$, there exists an adjoint $m_o^{(n')}$ nearby it. This property comes from the symmetric property (4.41) of the eigenvectors of $\widehat{\mathfrak{h}}^{(m)}$. Second, the lowest value of $|m_o^{(n)}|$ in σ_ξ can be obtained as $|m_o^{(\mathrm{lwt})}| = 0.5499$ for the μ_o scheme and $|m_o^{(\mathrm{lwt})}| = 0.5362$ for the other scheme, where the parameters are chosen as $\gamma = 0.2374$, $\Delta = 4\sqrt{3}\pi\gamma\ell_{\mathrm{p}}^2$ and $\ell_{\mathrm{p}} = 1$.

VI. CONCLUDING REMARK

The loop quantization of the model of a Schwarzschild interior coupled to a massless scalar field has been studied in the previous sections. By applying the deparametrization procedure, we get the physical Hamiltonian $\mathfrak{h}$ of this model with respect to the scalar field. Since $p_c c$ is a Dirac observable in the classical theory, $\mathfrak{h}$ is promoted to an operator $\widehat{\mathfrak{h}}$ commutating with the operator $\widehat{p_c \sin(\delta_c c)/\delta_c}$ which corresponds to $p_c c$. Replacing $\widehat{p_c \sin(\delta_c c)/\delta_c}$ in $\widehat{\mathfrak{h}}$ with its spectrum $L_o \gamma m \in \sigma_{\mathfrak{c}}$, we obtain a family of operators $\widehat{\mathfrak{h}}^{(m)}$. It is shown that both $\widehat{\mathfrak{h}}^{(m)}$ and $\widehat{\mathfrak{h}}$ are self-adjoint. The spectrum of $\widehat{\mathfrak{h}}^{(m)}$ and its analyticity with respect to m are studied. Moreover, we develop a numerical method to diagonalize $\widehat{\mathfrak{h}}^{(m)}$. Based on these results, the dynamics for the cases of $\widehat{p}_\varphi \neq 0$ and $\widehat{p}_\varphi = 0$ are studied, respectively.

For the case of $\widehat{p}_\varphi \neq 0$, the evolution of a wave packet is considered and the results are compared with the effective dynamics governed by the effective Hamiltonian. It turns out that the quantum evolution matches well with the effective dynamics in the domain where the wave packet is sharply peaked. However, the width of the wave packet would increase as the relational time ϕ evolves. Thus an inconsistence between the quantum dynamic and the effective dynamics would occur at late time. The numerical codes to compute the evolution can be found in [66].

For the case of $\widehat{p}_\varphi = 0$, the constraint $\widehat{\mathfrak{h}} = 0$ is imposed to get the physical states of the loop quantum Schwarzschild interior model. Its physical Hilbert space $\mathcal{H}_{\mathrm{BH}}$ is obtained. The spectrum $\overline{\sigma_\xi}$ of the Dirac observable $\widehat{\xi}'_{\delta_c}$, i.e., the dual of $\widehat{p_c \sin(\delta_c c)/\delta_c}$ in $\mathcal{H}_{\mathrm{BH}}$, is analyzed by both analytical and numerical methods. It turns out that the $\overline{\sigma_\xi}$ is discrete and it does not contain 0, provided that the parameters $\delta_b^{(m)}$ and $\varepsilon_b(m)$ satisfy the suitable analyticity. Thus, there exists a gap between $\overline{\sigma_\xi}$ and 0. Since the classical limit of $\widehat{\xi}'_{\delta_c}$ is proportional to the mass of the Schwarzschild BH, $\overline{\sigma_\xi}$ is referred to as its mass spectrum. Moreover, by the numerical method to diagonalize $\widehat{\mathfrak{h}}$, we can also compute $\overline{\sigma_\xi}$ numerically [66]. Note that the observable $\widehat{\xi}'_{\delta_c}$ in our models refers to the quasilocal mass of the BH. For instance, it agrees with the quasilocal mass M_Δ defined with respect to a naturally chosen pair of null vector fields (ℓ^a, n^a) on the horizon Δ [65], where the null normal field ℓ^a to Δ and future directed null field n^a transverse to Δ satisfy $\ell^a n_a = -1$. Some interesting results would be obtained if certain mechanism of BH evaporation could be introduced into our quantum model such that the evaporation can be regarded in a quasistatic process from the Schwarzschild BH of a given mass to the Schwarzschild BH with another mass. Then, it is reasonable to expect that the evaporation occurs in a quantum-jump manner, since the mass of BH can take only discrete values. Moreover, because of $0 \notin \overline{\sigma_\xi}$, the evaporation of the BH would eventually halt at the remnant. It should be noted that this discussion extrapolates from a quantum description of the BH interior. It is assumed that the exterior quantum description of the BH and the inclusion of Hawking radiation will not change the mass spectrum. We leave the detailed investigation of these issues for our future work.

It should be noted that the analyses in the current paper are valid for a quite general class of schemes such that (i) a separable Hilbert subspace $\mathcal{H}_{\mathfrak{c}} \subset \tilde{\mathcal{H}}_{\mathfrak{c}}$ can be chosen to define an operator $\widehat{p_c \sin(\delta_c c)/\delta_c}$ corresponding to $p_c \sin(\delta_c c)/\delta_c$, which is self-adjoint and commutates with the physical Hamiltonian $\widehat{h}$; and (ii) the quantum parameter δ_b is a constant or any function of $p_c \sin(\delta_c c)/\delta_c$. With the numerical method developed in this paper, it becomes possible to further study the Hawking radiation with the matter backreaction and the distortion of the Hawking spectrum in detail. We leave this open issue for our future works. Thus the discrete mass spectrum predicted by our LQG model provides a solid starting point to study the possibilities of considering the BH remnants as dark matter candidates, as well as solving the puzzle of information loss in BH evaporation.

ACKNOWLEDGMENTS

This work is supported by National Science Foundation of China with Grants No. 11961131013, No. 11875006, and No. 11775082. C. Z. acknowledges the support by the Polish Narodowe Centrum Nauki, Grant No. 2018/30/Q/ST2/00811.

APPENDIX A: ASYMPTOTIC BEHAVIOR OF $\widehat{\beta}_{\delta_b}$

By definition, the eigenequation of $\widehat{\beta}_{\delta_b}$ implies that its eigenfunctions satisfy

$$\psi(\mu + 2\delta_b) = \frac{4i\omega}{\gamma\ell_{\mathrm{p}}^2} \frac{1}{\mu + 2\delta_b} \psi(\mu) + \frac{\mu}{\mu + 2\delta_b} \psi(\mu - 2\delta_b), \tag{A1}$$

which can be rewritten as

$$\vec{\psi}(\mu+2\delta_b)=A(\mu)\vec{\psi}(\mu) \tag{A2}$$

with

$$\vec{\psi}(\mu+2\delta_b)\coloneqq\begin{pmatrix}\psi(\mu+2\delta_b)\\ \psi(\mu)\end{pmatrix}\quad\text{and}$$

$$A(\mu)\coloneqq\begin{pmatrix}\frac{4i\omega}{\gamma\ell_{\rm p}^2}\frac{1}{\mu+2\delta_b} & \frac{\mu}{\mu+2\delta_b}\\ 1 & 0\end{pmatrix}.$$

Define

$$B(\mu)\eqqcolon\begin{pmatrix}\psi_0^+(\mu+2\delta_b) & \psi_0^-(\mu+2\delta_b)\\ \psi_0^+(\mu) & -\psi_0^-(\mu)\end{pmatrix},$$

where

$$\psi_0^\pm(\mu)=\frac{1}{\sqrt{|\mu|}}e^{\pm ik\ln(|\mu|)}.$$

We obtain that the vector-valued function

$$\vec{\chi}(\mu)=\begin{pmatrix}\chi_1(\mu)\\ \chi_2(\mu)\end{pmatrix},$$

defined by

$$\vec{\psi}(\mu+2\delta_b)\eqqcolon B(\mu)\vec{\chi}(\mu+2\delta_b), \tag{A3}$$

satisfies

$$\vec{\chi}(\mu+2\delta_b)=B(\mu)^{-1}A(\mu)B(\mu-2\delta_b)\vec{\chi}(\mu)\coloneqq M(\mu)\vec{\chi}(\mu), \tag{A4}$$

which is obtained by applying (A2). Our purpose is to derive the condition with which the vector-valued function $\vec{\chi}(\mu)$ in right-hand side of (A3) can be approximated by some constant up to some $O(\mu^{-1})$ term. This can be achieved if the matrix $M(\mu)$ satisfies

$$M(\mu)=M+O(|\mu|^{-2}), \tag{A5}$$

with some constant M. Substituting (A5) into (A4), we obtain

$$k=\frac{\omega}{\gamma\ell_{\rm p}^2\delta_b} \tag{A6}$$

with

$$M(\mu)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+O(|\mu|^{-2}). \tag{A7}$$

Therefore, one gets

$$\psi(\mu)=\begin{cases}\chi_1\psi_0^+(\mu)+\chi_2\psi_0^-(\mu), & \mu=\varepsilon_b+4n\delta_n,\\ \chi_1\psi_0^+(\mu)-\chi_2\psi_0^-(\mu), & \mu=\varepsilon_b+4n\delta_n+2\delta_b\end{cases} \tag{A8}$$

with $n\in\mathbb{Z}$.

APPENDIX B: THE SELF-ADJOINTNESS OF $\widehat{\mathfrak{h}}^{(m)}$ AND $\widehat{\mathfrak{h}}$

We now prove that $\widehat{\mathfrak{h}}^{(m)}$ and $\widehat{\mathfrak{h}}$ are essentially self-adjoint. Define an operator $\widehat{A}$ as

$$\begin{aligned}\widehat{A}|\mu\rangle&=\left(\frac{1}{\delta_b^2}\frac{\gamma^2\ell_{\rm p}^4}{16}((\mu+2\delta_b)^2+\mu^2)+\frac{1}{4}\gamma^4\ell_{\rm p}^4(\mu-2\mathfrak{b})^2\right)|\mu\rangle\\ &\eqqcolon A(\mu)|\mu\rangle,\end{aligned} \tag{B1}$$

whose domain reads

$$D(\widehat{A})=\left\{|\psi\rangle,\sum_\mu|A(\mu)\psi(\mu)|^2<\infty\right\}. \tag{B2}$$

Then $\widehat{A}$ is self-adjoint because $A(\mu)$ is real. Let $\widehat{B}$ be the operator defined on the domain $D_{\mathfrak{b}}$ as

$$\widehat{B}=\frac{2mL_0\gamma}{\delta_b}\widehat{\beta}_{\delta_b}+\frac{1}{\delta_b^2}\left(\widehat{\beta}_{\delta_b}^2-\frac{1}{4}(\widehat{p}_b^2+(\widehat{p}_b+\delta_b\gamma\ell_{\rm p}^2)^2)\right). \tag{B3}$$

Then $\widehat{B}$ can be expressed as

$$\widehat{B}=\frac{2mL_0\gamma}{\delta_b}\frac{1}{2i}(\alpha^\dagger-\alpha)-\frac{1}{4\delta_b^2}((\alpha^\dagger)^2+\alpha^2), \tag{B4}$$

where

$$\begin{aligned}\alpha^\dagger|\mu\rangle&=\frac{\gamma\ell_{\rm p}^2}{2}(\mu+2\delta_b)|\mu+2\delta_b\rangle,\\ \alpha|\mu\rangle&=\frac{\gamma\ell_{\rm p}^2}{2}\mu|\mu+2\delta_b\rangle.\end{aligned} \tag{B5}$$

Given $\psi\in D_{\mathfrak{b}}$, we have

$$\begin{aligned}\|\widehat{B}\psi\|\le&\left\|\left(\frac{2mL_0\gamma}{\delta_b}\frac{1}{2i}\alpha^\dagger-\frac{1}{4\delta_b^2}(\alpha^\dagger)^2\right)\psi\right\|\\ &+\left\|\left(\frac{2mL_0\gamma}{\delta_b}\frac{1}{2i}\alpha+\frac{1}{4\delta_b^2}\alpha^2\right)\psi\right\|.\end{aligned} \tag{B6}$$

Moreover, a straightforward calculation gives

$$\left\|\left(\frac{2mL_0\gamma}{\delta_b}\frac{1}{2i}\alpha^\dagger-\frac{1}{4\delta_b^2}(\alpha^\dagger)^2\right)\psi\right\|^2 \le \sum_\mu\left(\frac{4m^2L_0^2\gamma^4\ell_{\rm p}^4}{16\delta_b^2}\mu^2+\frac{\gamma^4\ell_{\rm p}^8}{256\delta_b^4}\mu^2(\mu-2\delta_b)^2 + \frac{\gamma^2\ell_{\rm p}^4}{16\delta_b^2}\frac{2mL_0\gamma}{\delta_b}\frac{\gamma\ell_{\rm p}^2}{4}(\mu^2(\mu+2\delta_b)+(\mu-2\delta_b)^2\mu)\right)|\psi(\mu)|^2 =: \sum_\mu B^+(\mu)|\psi(\mu)|^2, \tag{B7}$$

and

$$\left\|\left(\frac{2mL_0\gamma}{\delta_b}\frac{1}{2i}\alpha-\frac{1}{4\delta_b^2}\alpha^2\right)\psi\right\|^2 \le \sum_\mu\left(\frac{4m^2L_0^2\gamma^4\ell_{\rm p}^4}{16\delta_b^2}(\mu+2\delta_b)^2+\frac{\gamma^4\ell_{\rm p}^8}{256\delta_b^4}(\mu+4\delta_b)^2(\mu+2\delta_b)^2 + \frac{1}{64\delta_b^2}\frac{2mL_0\gamma^4\ell_{\rm p}^6}{\delta_b}((\mu+2\delta_b)^2\mu+(\mu+4\delta_b)^2(\mu+2\delta_b))\right)|\psi(\mu)|^2 =: \sum_\mu B^-(\mu)|\psi(\mu)|^2. \tag{B8}$$

By Eqs. (B6)–(B8) we obtain

$$\|\widehat{B}\psi\| \le \sqrt{\sum_\mu B^+(\mu)|\psi(\mu)|^2}+\sqrt{\sum_\mu B^-(\mu)|\psi(\mu)|^2} \le \sqrt{\sum_\mu 4B^-(\mu)|\psi(\mu)|^2}. \tag{B9}$$

Moreover, one has

$$\|\widehat{A}\psi\| = \sqrt{\sum_\mu A(\mu)^2|\psi(\mu)|^2}. \tag{B10}$$

By the expressions of $B^-(\mu)$ and $A(\mu)$, there exists some real number $b\ge 0$ such that

$$4B^-(\mu) \le \frac{1}{1+2\gamma^4\delta_b^4}A(\mu)^4+b, \tag{B11}$$

which implies

$$\|\widehat{B}\psi\|^2 \le \frac{1}{1+2\gamma^4\delta_b^4}\|\widehat{A}\psi\|^2+b\|\psi\|^2. \tag{B12}$$

Therefore, $\widehat{\mathfrak{h}}^{(m)}=\widehat{B}+\widehat{A}$ is self-adjoint on $D(A)$ and thus essentially self-adjoint on $D_{\mathfrak{b}}$ according to the Kato-Rellich theorem (see Theorem X.12 in [59]). Thanks to the self-adjointness of $\widehat{\mathfrak{h}}^{(m)}$, the self-adjointness of $\widehat{\mathfrak{h}}$ can be proven as follows. Let $g^\pm$ be elements in the orthogonal complement of the range of $\widehat{\mathfrak{h}}\pm i$, respectively, i.e., $g^\pm\in\mathrm{Ran}(\widehat{\mathfrak{h}}\pm i)^\perp$. Then we have

$$0=\langle(\widehat{A}\pm i)\psi,g^\pm\rangle = \int_{\sigma_{\mathfrak{c}}}\mathrm{d}\mu_{\mathfrak{c}}\langle(\widehat{\mathfrak{h}}^{(m)}\pm i)\psi(m),g^\pm(m)\rangle,\quad \forall\ \psi\in D(\widehat{\mathfrak{h}}). \tag{B13}$$

Let f be a square-integrable function f on $\sigma_{\mathfrak{c}}$ with respect to $\mu_{\mathfrak{c}}$, i.e., $f\in L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}})$. Given $\psi\in D(\widehat{\mathfrak{h}})$, one has

$$\left|\int_{\sigma_{\mathfrak{c}}}\mathrm{d}\mu_{\mathfrak{c}}\langle f(m)\widehat{\mathfrak{h}}^{(m)}\psi(m),f(m)\widehat{\mathfrak{h}}^{(m)}\psi(m)\rangle\right|^2 \le \int_{\sigma_{\mathfrak{c}}}\mathrm{d}\mu_c|f(m)|^2\int_{\sigma_{\mathfrak{c}}}\|\widehat{\mathfrak{h}}^{(m)}\psi(m)\|^2<\infty, \tag{B14}$$

which implies that $f\psi: m\mapsto f(m)\psi(m)$ is also in $D(\widehat{\mathfrak{h}})$. Thus substituting $f\psi$ into (B13), we conclude

$$\int_{\sigma_{\mathfrak{c}}}\mathrm{d}\mu_{\mathfrak{c}}\overline{f(m)}\langle(\widehat{\mathfrak{h}}^{(m)}\pm i)\psi(m),g^\pm(m)\rangle=0,\ \ \forall\ f\in L^2(\sigma_{\mathfrak{c}},\mathrm{d}\mu_{\mathfrak{c}}). \tag{B15}$$

Therefore, it holds almost everywhere for m that

$$\langle(\widehat{\mathfrak{h}}^{(m)}\pm i)\psi(m),g^\pm(m)\rangle=0. \tag{B16}$$

Since this conclusion is true for all $\psi(m)\in D_{\mathfrak{b}}(m)$, one has that $g^\pm(m)\in\mathrm{Ran}(\widehat{\mathfrak{h}}^{(m)}\pm i)^\perp$ almost everywhere for m. However, $\widehat{\mathfrak{h}}^{(m)}$ is essentially self-adjoint for all m. Hence one gets

$$\mathrm{Ran}(\widehat{\mathfrak{h}}^{(m)}\pm i)^\perp=\{0\},\quad \forall\ m. \tag{B17}$$

This ensures that $g^\pm(m)=0$ for all m. The self-adjointness of $\widehat{A}$ is thus obtained because of the basic criterion for self-adjointness (see, e.g., Theorem VIII.3 in [57]).

APPENDIX C: THE ANALYTICITY OF $\overline{T_m}$

Theorem C.1.—$\overline{T_m}$ is an analytic family of forms of type (a) in the sense of Kato.

By definition, this theorem can be obtained directly from the following lemmas.

Lemma C.1.—The norm $\|\cdot\|_m$ defined on $D_{\mathfrak{b}}(m_o)$ for each m is equivalent to the norm

$$\|\psi\|_+ = \sqrt{\langle\psi|\widehat{p}_b^2|\psi\rangle + \langle\psi|\psi\rangle}; \tag{C1}$$

i.e., there exist constants $c, C > 0$ such that

$$c\|\psi\|_+ \leq \|\psi\|_m \leq C\|\psi\|_+, \quad \forall\ \psi \in D_{\mathfrak{b}}(m_o). \tag{C2}$$

Proof.—Given $|\psi\rangle = \sum_\mu \psi_\mu|\mu\rangle \in D_{\mathfrak{b}}(m_o)$, by (4.2) one gets

$$\begin{aligned}
&T_m(\psi,\psi) + (L_0^2m^2\gamma^2+1)\langle\psi|\psi\rangle \\
&\quad\geq \gamma^2\langle\psi|(\widehat{p}_b - \gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2|\psi\rangle + \langle\psi|\psi\rangle \\
&\quad\geq \gamma^2 c\langle\psi|\widehat{p}_b^2|\psi\rangle + \langle\psi|\psi\rangle,
\end{aligned} \tag{C3}$$

where

$$c = 1 + \frac{(\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2}{2} - \frac{\gamma\ell_{\mathrm{p}}^2\mathfrak{b}\sqrt{4+(\gamma\ell_{\mathrm{p}}^2\mathfrak{b})^2}}{2} > 0. \tag{C4}$$

Moreover, consider the operator $(1+\widehat{p}_b^2)^{-\frac{1}{2}}(\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1} + L_0^2\gamma^2m^2+1)(1+\widehat{p}_b^2)^{-\frac{1}{2}}$ on $D_{\mathfrak{b}}(m_o)$. It can be verified straightforwardly that

$$|\langle\mu|(1+\widehat{p}_b^2)^{-\frac{1}{2}}(\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1}+L_0^2\gamma^2m^2+1)(1+\widehat{p}_b^2)^{-\frac{1}{2}}|\mu'\rangle| < \infty. \tag{C5}$$

Thus the operator $(1+\widehat{p}_b^2)^{-\frac{1}{2}}(\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1} + L_0^2\gamma^2m^2+1)\times(1+\widehat{p}_b^2)^{-\frac{1}{2}}$ is bounded; i.e., there exist $C>0$ such that

$$|\langle\psi|(1+\widehat{p}_b^2)^{-\frac{1}{2}}(\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1}+L_0^2\gamma^2m^2+1)(1+\widehat{p}_b^2)^{-\frac{1}{2}}|\psi\rangle| \leq C\langle\psi|\psi\rangle, \quad \forall\ \psi\in D_{\mathfrak{b}}(m_o). \tag{C6}$$

By definition, $(1+\widehat{p}_b^2)^{-\frac{1}{2}}: D_{\mathfrak{b}}(m_o)\to D_{\mathfrak{b}}(m_o)$ is surjective. Hence (C6) implies

$$|\langle\psi|(\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1}+L_0^2\gamma^2m^2+1)|\psi\rangle| \leq C\langle\psi|(1+\widehat{p}_b^2)|\psi\rangle, \quad \forall\ \psi\in D_{\mathfrak{b}}(m_o). \tag{C7}$$

Thus

$$T_m(\psi,\psi) + (L_0^2m^2\gamma^2+1)\langle\psi|\psi\rangle \leq C\langle\psi|(1+\widehat{p}_b^2)|\psi\rangle, \quad \forall\ \psi\in D_{\mathfrak{b}}(m_o). \tag{C8}$$

Then (C2) is proven by (C3) and (C8). ∎

By Lemma C.1, for each m, $Q(m)$ is indeed the closure of $D_{\mathfrak{b}}(m_o)$ with respect to the norm $\|\cdot\|_+$ given in (C1). Thus one gets $Q(m) = Q(m_o)$ for all m.

*Lemma C.2.*Suppose $\delta_b^{(m)}$ and $\varepsilon_b(m)$ to be analytic functions of m at m_o and $\delta_b^{(m_o)}\neq 0$. Then $\overline{T_m}(\psi,\psi)$, for each $\psi\in Q(m_o)$, is an analytic function of m at m_o.

Proof.—The action of $\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1}$ on $|\mu_n\rangle$ with $\mu_n = \varepsilon_b^o + 2n\delta_b^o$ reads

$$\begin{aligned}
\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1}|\mu_n\rangle = &-\frac{1}{4(\delta_b^o)^2}(\mathfrak{f}(m)+\widehat{p}_b)(\mathfrak{f}(m)+\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\delta_b^o)|\mu_{n+2}\rangle + \frac{2L_0m\gamma}{2\delta_b^o i}(\mathfrak{f}(m)+\widehat{p}_b)|\mu_{n+1}\rangle \\
&+\left(\frac{1}{4(\delta_b^o)^2}((\mathfrak{f}(m)+\widehat{p}_b+\gamma\ell_{\mathrm{p}}^2\delta_b^o)^2 + (\mathfrak{f}(m)+\widehat{p}_b)^2) + \gamma^2(\delta_b^{(m)})^2(\mathfrak{f}(m)+\widehat{p}_b-\gamma\ell_{\mathrm{p}}^2\mathfrak{b}/\delta_b^{(m)})^2\right)|\mu_n\rangle \\
&-\frac{2L_0m\gamma}{2\delta_b^o i}(\mathfrak{f}(m)+\widehat{p}_b+\gamma\ell_{\mathrm{p}}^2\delta_b^o)|\mu_{n-1}\rangle - \frac{1}{4(\delta_b^o)^2}(\mathfrak{f}(m)+\widehat{p}_b+2\gamma\ell_{\mathrm{p}}^2\delta_b^o)(\mathfrak{f}(m)+\widehat{p}_b+\gamma\ell_{\mathrm{p}}^2\delta_b^o)|\mu_{n-2}\rangle,
\end{aligned} \tag{C9}$$

where $\mathfrak{f}(m) \coloneqq \varepsilon_b(m)\delta_b^o/\delta_b^{(m)} - \varepsilon_b^o$ is analytic at m_o. Given $|\psi\rangle\in Q(m_o)$, Eq. (C9) implies that $\langle\psi|\mathfrak{i}_m\widehat{\mathfrak{H}}^{(m)}\mathfrak{i}_m^{-1}|\psi\rangle$ is a finite linear combination of $\langle\psi|\widehat{p}_b^2|\psi\rangle$, $\langle\psi|\widehat{p}_b|\psi\rangle$ and $\langle\psi|\psi\rangle$ with coefficients depending on m analytically. Since $Q(m)$ is the closure of $D_{\mathfrak{b}}(m_o)$ with respect to the norm $\|\cdot\|_+$, both $\langle\psi|\widehat{p}_b^2|\psi\rangle$ and $\langle\psi|\widehat{p}_b|\psi\rangle$ are well defined for all $\psi\in Q(m)$. Then the analyticity of $\overline{T_m}(\psi,\psi)$ is proven. ∎

APPENDIX D: PROOF OF EQS. (4.12) AND (4.13)

We notice that

$$\begin{aligned}\lambda_i^{(k)} &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)}\psi\rangle \\ &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \end{aligned} \tag{D1}$$

because of $\widehat{P}^{(k)}\overline{\widehat{\mathfrak{h}}^{(m)}}\widehat{P}^{(k)} = \widehat{P}^{(k)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(k)}$, and $\widehat{P}^{(k)}\psi = \psi$ for all $\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}$.

Given $k' > k$, one has $\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)} \subset \mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')}$. Let $\tilde{\phi}_1,\tilde{\phi}_2,\ldots\tilde{\phi}_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')}$ be some vectors such that $\tilde{\phi}_{i-n},\tilde{\phi}_{i-n+1},\ldots,\tilde{\phi}_{i-1}\notin\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}$. Then we have

$$\begin{aligned}&\inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\tilde{\phi}_l\rangle=0,\ \forall\ l=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \\ &\quad= \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\tilde{\phi}_l\rangle=0,\ \forall\ l=1,2,\ldots,i-n-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \le \lambda_{i-n}^{(k)} \le \lambda_i^{(k)}.\end{aligned} \tag{D2}$$

Therefore, one obtains

$$\begin{aligned}\lambda_i^{(k)} &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \\ &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle. \end{aligned} \tag{D3}$$

Furthermore, for given $\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')}$, we have

$$\begin{aligned}&\{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)} | \|\psi\|=1; \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1\}\\ &\subset\{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')} | \|\psi\|=1; \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1\}.\end{aligned} \tag{D4}$$

Thus, one has

$$\inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \ge \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle, \tag{D5}$$

which implies

$$\begin{aligned}\lambda_i^{(k)} &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \\ &\ge \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k')};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle = \lambda_i^{(k')}. \end{aligned} \tag{D6}$$

Similarly, we have

$$\begin{aligned}\lambda_i^{(k)} &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \\ &= \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}} \inf_{\substack{\psi\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b,k)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle \\ &\ge \sup_{\phi_1,\phi_2,\ldots,\phi_{i-1}\in\mathcal{H}_{\mathfrak{b}}^{(\varepsilon_b)}} \inf_{\substack{\psi\in\overline{D_{\mathfrak{b}}(m)};\|\psi\|=1\\ \langle\psi,\phi_n\rangle=0,\ \forall\ n=1,2,\ldots,i-1}} \langle\psi, \widehat{\mathfrak{h}}^{(m)}\psi\rangle = \omega_i. \end{aligned} \tag{D7}$$

Thus the proof of (4.12) is completed, and the existence of the limit $\lim_{k\to\infty}\lambda_i^{(k)}$ can be obtained directly from (4.12).

APPENDIX E: THE FINITE CUTOFF APPROXIMATION

Theorem E.1.—Each λ_i given in (4.13) is an eigenvalue of $\overline{\widehat{\mathfrak{h}}^{(m)}}$. The space Λ_i is an eigenspace corresponding to the eigenvalue λ_i.

Proof.—By definition, one has

$$\|\widehat{P}^{(k)}(\widehat{\mathfrak{h}}^{(m)}-\lambda_i)\psi_i^{(k)}\| = \|(\lambda_i^{(k)}-\lambda_i)\psi_i^{(k)}\| \le |\lambda_i^{(k)}-\lambda_i|, \tag{E1}$$

which implies

$$\lim_{k\to\infty}\|\widehat{P}^{(k)}(\widehat{\mathfrak{h}}^{(m)}-\lambda_i)\psi_i^{(k)}\| = 0. \tag{E2}$$

Given an arbitrary $\varphi\in D_{\mathfrak{b}}(m)$, by the definition (3.39) of $D_{\mathfrak{b}}(m)$, there exists an integer N_φ such that $\varphi\in\mathcal{H}_{\mathfrak{b}}{}^{(\varepsilon_b,k)}$, $\forall\ k\ge N_\varphi$. Thus, for each $\varphi\in D_{\mathfrak{b}}(m)$, one gets

$$\langle\varphi|\widehat{P}^{(k)}(\widehat{\mathfrak{h}}^{(m)}-\lambda_i)|\psi_i^{(k)}\rangle = \langle\varphi|(\widehat{\mathfrak{h}}^{(m)}-\lambda_i)|\psi_i^{(k)}\rangle, \quad \forall\ k\ge N_\varphi, \tag{E3}$$

where we used $\widehat{P}^{(k)}\varphi=\varphi$ for all $k\ge N_\varphi$ and that $\widehat{\mathfrak{h}}^{(m)}$ is symmetric. Hence, we have

$$\begin{aligned}0 &= \lim_{k\to\infty}\langle\varphi|\widehat{P}^{(k)}(\widehat{\mathfrak{h}}^{(m)}-\lambda_i)|\psi_i^{(k)}\rangle \\ &= \langle\varphi|(\widehat{\mathfrak{h}}^{(m)}-\lambda_i)|\psi_i\rangle, \quad \forall\ \varphi\in D_{\mathfrak{b}}(m).\end{aligned} \tag{E4}$$

By this equation, ψ_i is in the domain of the adjoint of $\widehat{\mathfrak{h}}^{(m)}$. Since $\widehat{\mathfrak{h}}^{(m)}$ is essentially self-adjoint, its adjoint is equal to its closure $\overline{\widehat{\mathfrak{h}}^{(m)}}$. Thus ψ_i is in $\overline{D_{\mathfrak{b}}(m)}$. Moreover, (E4) also implies

$$(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\psi_i = 0, \tag{E5}$$

which ensures either $\psi_i = 0$ or that ψ_i is an eigenvector of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ with the eigenvalue λ_i. We now show that $\psi_i \neq 0$ and hence ψ_i can only be an eigenvector of $\overline{\widehat{\mathfrak{h}}^{(m)}}$. Given an eigenvalue ω of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, let $|\omega,\alpha\rangle$ be an orthonormal basis of the eigenspace with respect to ω. Define a projection $\widehat{P}$ as

$$\widehat{P}|\psi\rangle \coloneqq \sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|\omega,\alpha\rangle\langle\omega,\alpha|\psi\rangle. \tag{E6}$$

Because $\overline{\widehat{\mathfrak{h}}^{(m)}}$ is bounded from below and each eigenvalue ω is of finite multiplicity, the summation in the rhs consists of only finite terms. Then for each vectors $\psi_i^{(n_l)}$ in (4.14), we have

$$\begin{aligned}&\left\|(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\widehat{P}\psi_i^{(n_l)}-\sum_{\omega\in(-\infty,\lambda_k]}\sum_{\alpha}|\omega,\alpha\rangle\langle\omega,\alpha|\psi_i\rangle(\omega-\lambda_i)\right\|\\&\quad\leq\sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|(\langle\omega,\alpha|\psi_i^{(n_l)}\rangle-\langle\omega,\alpha|\psi_i\rangle)||\omega-\lambda_i|.\end{aligned} \tag{E7}$$

Taking account of (E7), (4.14) and the fact that the summation contains finitely many terms, we obtain

$$\begin{aligned}\lim_{l\to\infty}&\left\|(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\widehat{P}\psi_i^{(n_l)}\right.\\&\left.-\sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|\omega,\alpha\rangle\langle\omega,\alpha|\psi_i\rangle(\omega-\lambda_i)\right\|=0,\end{aligned} \tag{E8}$$

i.e.,

$$\begin{aligned}&\lim_{l\to\infty}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\widehat{P}\psi_i^{(n_l)}\\&\quad=\sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|\omega,\alpha\rangle\langle\omega,\alpha|\psi_i\rangle(\omega-\lambda_i).\end{aligned} \tag{E9}$$

Moreover, because of (E5) and

$$\begin{aligned}&\sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|\omega,\alpha\rangle\langle\omega,\alpha|\psi_i\rangle(\omega-\lambda_i)\\&\quad=\sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|\omega,\alpha\rangle\langle\omega,\alpha|(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\psi_i\rangle,\end{aligned} \tag{E10}$$

we finally have

$$\lim_{l\to\infty}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\widehat{P}\psi_i^{(n_l)}=0. \tag{E11}$$

Furthermore, by using

$$\|\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\psi_i^{(n_l)}\|=\|(\lambda_i^{(n_l)}-\lambda_i)\psi_i^{(n_l)}\|\leq|\lambda_i^{(n_l)}-\lambda_i|, \tag{E12}$$

one has

$$\lim_{l\to\infty}\|\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\psi_i^{(n_l)}\|=0. \tag{E13}$$

Combining (E13) with the inequality

$$\begin{aligned}&\|\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)(1-\widehat{P})\psi_i^{(n_l)}\|\\&\quad\leq\|\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\psi_i^{(n_l)}\|+\|(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\widehat{P}\psi_i^{(n_l)}\|,\end{aligned} \tag{E14}$$

we finally obtain

$$\begin{aligned}&\lim_{l\to\infty}\|\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)(1-\widehat{P})\psi_i^{(n_l)}\|\\&\quad\leq\lim_{l\to\infty}\|(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)\widehat{P}\psi_i^{(n_l)}\|=0,\end{aligned} \tag{E15}$$

which implies

$$\lim_{l\to\infty}\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)(1-\widehat{P})\psi_i^{(n_l)}=0. \tag{E16}$$

Defining $\tilde{\omega}\coloneqq\inf\{\omega\in\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}}),\omega>\lambda_i\}$, we have

$$\begin{aligned}&\langle\widehat{P}^{(n_l)}(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)(1-\widehat{P})\psi_i^{(n_l)}|\psi_i^{(n_l)}\rangle\\&\quad=\langle(1-\widehat{P})(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_i)(1-\widehat{P})\psi_i^{(n_l)}|\psi_i^{(n_l)}\rangle\\&\quad\geq(\tilde{\omega}-\lambda_i)\|(1-\widehat{P})\psi_i^{(n_l)}\|,\end{aligned} \tag{E17}$$

where the last inequality is resulted from $\langle(\overline{\widehat{\mathfrak{h}}^{(m)}}-\lambda_k)(1-\widehat{P})\varphi,\varphi\rangle\geq(\tilde{\omega}-\lambda_i)\langle(1-\widehat{P})\varphi,\varphi\rangle$ for all $\varphi\in\overline{D_{\mathfrak{h}}(m)}$. The combination of (E17) and (E16) leads to

$$\lim_{l\to\infty}\|(1-\widehat{P})\psi_i^{(n_l)}\|=0. \tag{E18}$$

Furthermore, because of

$$\begin{aligned}\|\psi_i^{(n_l)}-\psi_i\|&\leq\|\widehat{P}\psi_i^{(n_l)}-\widehat{P}\psi_i\|+\|(1-\widehat{P})\psi_i^{(n_l)}\|\\&\quad+\|(1-\widehat{P})\psi_i\|,\end{aligned} \tag{E19}$$

we obtain

$$\lim_{l\to\infty}\|\psi_i^{(n_l)}-\psi_i\|\leq\lim_{l\to\infty}\|\widehat{P}\psi_i^{(n_l)}-\widehat{P}\psi_i\|+\|(1-\widehat{P})\psi_i\|. \tag{E20}$$

The first term in the rhs of (E20) satisfies

$$\|\widehat{P}\psi_i^{(n_l)}-\widehat{P}\psi_i\|\leq\sum_{\omega\in(-\infty,\lambda_i]}\sum_{\alpha}|(\langle\omega,\alpha|\psi_k^{(n_l)}\rangle-\langle\omega,\alpha|\psi_k\rangle)|. \tag{E21}$$

Taking account of (4.14) and the fact that the summation in the rhs contains finite terms, Eq. (E21) implies

$$\lim_{l\to\infty}\|\widehat{P}\psi_i^{(n_l)}-\widehat{P}\psi_i\|=0. \tag{E22}$$

Therefore, we have

$$\lim_{l\to\infty}\|\psi_i^{(n_l)}-\psi_i\|\le\|(1-\widehat{P})\psi_i\|\le\|\psi_i\|, \tag{E23}$$

which implies that $|\psi_i\rangle\neq 0$. Otherwise, one would get

$$0=\lim_{l\to\infty}\|\psi_i^{(n_l)}-\psi_i\|=\lim_{l\to\infty}\|\psi_i^{(n_l)}\|, \tag{E24}$$

which is contradictory to $\|\psi_i^{(n)}\|=1$. Therefore, according to (E5), λ_i is an eigenvalue of $\overline{\widehat{\mathfrak{h}}^{(m)}}$ and $|\psi_i\rangle$ is a corresponding eigenvector. ■

The above proof is inspired by the works [67,68] and Sec. VIII.7 in [57].

Theorem E.2.—Given λ_i and λ_{i+1} as defined in (4.13), if $\lambda_i\neq\lambda_{i+1}$, i.e., $\lambda_i<\lambda_{i+1}$, one has

$$\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})\cap(\lambda_i,\lambda_{i+1})=\varnothing. \tag{E25}$$

Proof.—Consider an interval (a,b) with $\lambda_i<a<b<\lambda_{i+1}$. By definition of λ_i and (4.12), there exists an integer $\tilde{N}$ such that

$$\lambda_i^{(k)}\le a,\quad \forall\ k\ge\tilde{N}. \tag{E26}$$

Then for an eigenvalue $\lambda_{i'}^{(k)}$ of $\widehat{P}^{(k)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(k)}$ with $k\ge\tilde{N}$,

(i) if $i'\le i$, one has

$$\lambda_{i'}^{(k)}\le\lambda_i^{(k)}\le a; \tag{E27}$$

(ii) if $i'>i$, or equivalently $i'\ge i+1$, one has

$$\lambda_{i'}^{(k)}\ge\lambda_{i+1}^{(k)}\ge\lambda_i\ge b. \tag{E28}$$

The above analysis indicates

$$\sigma(\widehat{P}^{(k)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(k)})\cap(a,b)=\varnothing,\quad\forall\ k\ge\tilde{N}, \tag{E29}$$

where $\sigma(\widehat{P}^{(k)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(k)})$ denote, as usual, the set of eigenvalues of $\widehat{P}^{(k)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(k)}$. Let z be the complex number

$$z=\frac{a+b}{2}+i\frac{a-b}{2}. \tag{E30}$$

Given $|\varphi\rangle\in D_{\mathfrak{h}}(m)$, one has $|\psi\rangle\coloneqq(\widehat{\mathfrak{h}}^{(m)}-z)|\varphi\rangle\in D_{\mathfrak{h}}(m)$ by the expression of $\widehat{\mathfrak{h}}^{(m)}$. Hence there exists a large integer $n\ge\tilde{N}$ such that

$$\widehat{P}^{(n)}|\varphi\rangle=|\varphi\rangle,\quad\widehat{P}^{(n)}|\psi\rangle=|\psi\rangle,\quad\widehat{P}^{(n)}\widehat{\mathfrak{h}}^{(m)}|\varphi\rangle=\widehat{\mathfrak{h}}^{(m)}|\varphi\rangle. \tag{E31}$$

Then one has

$$|\psi\rangle=(\widehat{\mathfrak{h}}^{(m)}-z)|\varphi\rangle=(\widehat{P}^{(n)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(n)}-z)|\varphi\rangle, \tag{E32}$$

which leads to

$$(\widehat{\mathfrak{h}}^{(m)}-z)^{-1}|\psi\rangle=(\widehat{P}^{(n)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(n)}-z)^{-1}|\psi\rangle. \tag{E33}$$

Because of $\widehat{P}^{(n)}\psi=\psi$, i.e., $\psi\mathcal{H}_{\mathfrak{h}}^{(n)}$, we have

$$\|(\widehat{P}^{(n)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(n)}-z)^{-1}\psi\|^2=\sum_{i'=1}^{2n+1}|(\lambda_{i'}^{(n)}-z)^{-1}|^2|\langle\psi_{i'}^{(n)}|\psi\rangle|^2, \tag{E34}$$

where $\psi_{i'}^{(n)}$ is the normalized eigenvector of $\widehat{P}^{(n)}\widehat{\mathfrak{h}}^{(m)}\widehat{P}^{(n)}$ corresponding to the eigenvalue $\lambda_{i'}^{(n)}$. According to the inequality $\lambda_1^{(n)}\le\lambda_2^{(n)}\le\cdots\le\lambda_k^{(n)}\le a\le b\le\lambda_{k+1}^{(n)}\le\cdots\le\lambda_n^{(n)}$, we have that

$$|(\lambda_{i'}^{(n)}-z)^{-1}|^2\le\frac{2}{(a-b)^2}, \tag{E35}$$

which implies

$$\begin{aligned}\|(\widehat{\mathfrak{h}}^{(m)}-z)^{-1}\psi\|^2&\le\frac{2}{(a-b)^2}\sum_{i'=1}^{2n+1}|\langle\psi_{i'}^{(n)}|\psi\rangle|^2\\&=\frac{2}{(a-b)^2}\|\psi\|^2,\ \forall\ |\psi\rangle\in D_{\mathfrak{h}}(m).\end{aligned} \tag{E36}$$

Thus, one has

$$\|(\overline{\widehat{\mathfrak{h}}^{(m)}}-z)^{-1}\psi\|^2\le\frac{2}{(a-b)^2}\|\psi\|^2,\quad\forall\ |\psi\rangle\in\overline{D_{\mathfrak{h}}(m)}. \tag{E37}$$

As a consequence,

$$\rho((\overline{\widehat{\mathfrak{h}}^{(m)}}-z)^{-1})\le\frac{\sqrt{2}}{b-a}, \tag{E38}$$

where $\rho((\overline{\widehat{\mathfrak{h}}^{(m)}}-z)^{-1})$ is the spectral radius of $(\overline{\widehat{\mathfrak{h}}^{(m)}}-z)^{-1}$. Because of the self-adjointness of $\overline{\widehat{\mathfrak{h}}^{(m)}}$, the spectrum of $(\overline{\widehat{\mathfrak{h}}^{(m)}}-z)^{-1}$ is

$$\sigma((\overline{\widehat{\mathfrak{h}}^{(m)}}-z)^{-1})=\{(\lambda-z)^{-1},\lambda\in\sigma(\overline{\widehat{\mathfrak{h}}^{(m)}})\}. \tag{E39}$$

Therefore, (E38) leads to

$$(a,b)\cap\sigma(\widehat{\mathfrak{h}}^{(m)})=\varnothing, \tag{E40}$$

which is true for any $(a,b)\subset(\lambda_i,\lambda_{i+1})$. Hence, one obtains

$$(\lambda_i,\lambda_{i+1})\cap\sigma(\widehat{\mathfrak{h}}^{(m)})=\varnothing. \tag{E41}$$

■

APPENDIX F: THE EFFECTIVE DYNAMICS

The effective Hamiltonian constraint reads

$$\begin{aligned}H&=p_\varphi^2-\frac{4\pi}{GL_0^2\gamma^2}\left(\frac{2}{\delta_b\delta_c}p_b\sin(\delta_b b)p_c\sin(\delta_c c)\right.\\&\quad\left.+\frac{1}{\delta_b^2}p_b^2\sin^2(\delta_b b)+\gamma^2p_b^2\right)\\&=:p_\varphi^2-\frac{4\pi}{GL_0^2\gamma^2}\mathfrak{h}.\end{aligned} \tag{F1}$$

As $p_c\sin(\delta_c c)=\gamma m L_0\delta_c$ is a constant of motion, it is sufficient to consider the following Hamiltonian constraint for the evolution of p_b:

$$\begin{aligned}H^{(m)}&=p_\varphi^2-\frac{4\pi}{GL_0^2\gamma^2}\left(\frac{2\gamma m L_0}{\delta_b}p_b\sin(\delta_b b)\right.\\&\quad\left.+\frac{1}{\delta_b^2}p_b^2\sin^2(\delta_b b)+\gamma^2p_b^2\right)\\&=:p_\varphi^2-\frac{4\pi}{GL_0^2\gamma^2}\mathfrak{h}^{(m)}.\end{aligned} \tag{F2}$$

The evolution of $y\coloneqq p_b\sin(\delta_b b)$ with respect to φ is given by

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\varphi}&=\frac{\sqrt{4\pi}}{\sqrt{G}L_0\gamma}\left\{y,\sqrt{\mathfrak{h}^{(m)}}\right\}\\&=\frac{4\pi}{GL_0^2\gamma^2}\frac{1}{p_\varphi}p_b^2\gamma^3\delta_b\cos(\delta_b b)\\&=\pm\frac{4\pi\delta_b\gamma}{GL_0^2}\frac{1}{p_\varphi}\sqrt{p_b^4-p_b^2y^2}.\end{aligned} \tag{F3}$$

The Hamiltonian constraint $H^{(m)}=0$ can also be written as

$$p_b^2=-\frac{y^2}{\gamma^2\delta_b^2}-\frac{2L_0my}{\gamma\delta_b}+\frac{GL_0^2p_\varphi^2}{4\pi}, \tag{F4}$$

which, together with (F3), gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\varphi}&=\pm\frac{4\pi\sqrt{1+\gamma^2\delta_b^2}}{GL_0^2\gamma\delta_b}\frac{1}{p_\varphi}\\&\quad\times\sqrt{(y-y_-)(y-y_-')(y-y_+')(y-y_+)}\end{aligned} \tag{F5}$$

with

$$\begin{aligned}y_\pm&=L_0\gamma\delta_b m\left(-1\pm\sqrt{1+\frac{Gp_\varphi^2}{4\pi m^2}}\right),\\y_\pm'&=\frac{L_0\gamma\delta_b m}{1+\gamma^2\delta_b^2}\left(-1\pm\sqrt{1+\frac{Gp_\varphi^2(1+\gamma^2\delta_b^2)}{4\pi m^2}}\right).\end{aligned} \tag{F6}$$

Moreover, according to (F4), the maximal value of p_b along its dynamical trajectory is

$$(p_b^{\max})^2=L_0^2m^2\left(1+\frac{Gp_\varphi^2}{4\pi m^2}\right) \tag{F7}$$

with which $y_\pm$ and $y_\pm'$ can be rewritten as

$$\begin{aligned}y_\pm&=L_0\gamma\delta_b m\left(-1\pm\frac{p_b^{\max}}{L_0m}\right),\\y_\pm'&=\frac{L_0\gamma\delta_b m}{1+\gamma^2\delta_b^2}\left(-1\pm\sqrt{(1+\gamma^2\delta_b^2)\frac{(p_b^{\max})^2}{L_0^2m^2}-\gamma^2\delta_b^2}\right).\end{aligned} \tag{F8}$$

Equation (F8) can be used to fix the dynamical solution by the initial data of $p_b^{\max}$.

Solution to (F5) is

$$\begin{aligned}\varphi&=\pm\frac{GL_0^2\gamma\delta_b}{4\pi\sqrt{1+\gamma^2\delta_b^2}}p_\varphi\\&\quad\times\int_{y_a}^{y_b}\frac{\mathrm{d}y}{\sqrt{(y-y_-)(y-y_-')(y-y_+')(y-y_+)}}.\end{aligned} \tag{F9}$$

As $p_b^2=-(y-y_-)(y-y_+)\geq0$ and $y_-\leq y_-'\leq y_+'\leq y_+$, we have

$$y_a\in[y_-',y_+'],\qquad y_b\in[y_-',y_+']. \tag{F10}$$

Moreover, because (F4) can be written as

$$p_b^2=-\frac{1}{\gamma^2\delta_b^2}(y-y_-)(y-y_+), \tag{F11}$$

(F10) implies that p_b cannot reach 0 and will bounce at

$$p_b^{(\pm)}=-(y_\pm'-y_-)(y_\pm'-y_+). \tag{F12}$$

In order to calculate the integral (F9), we define

$$t(y) = \sqrt{\frac{y'_+ - y_-}{y'_+ - y'_-}}\sqrt{\frac{y - y'_-}{y - y_-}}. \tag{F13}$$

Then (F9) becomes

$$\varphi = \pm \frac{GL_0^2\gamma\delta_b}{2\pi\sqrt{1+\gamma^2\delta_b^2}} p_\varphi \sqrt{\frac{1}{(y'_+ - y_-)(y_+ - y'_-)}} \times \int_{t_a}^{t_b} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{F14}$$

By choosing the initial data $\varphi_0 = \varphi(y'_-)$, we finally have

$$\varphi(y) - \varphi_0 = \pm \frac{GL_0^2\gamma\delta_b}{2\pi\sqrt{1+\gamma^2\delta_b^2}} \times p_\varphi \sqrt{\frac{1}{(y'_+ - y_-)(y_+ - y'_-)}} F(\arcsin(t(y))|k), \tag{F15}$$

where $F(x|k)$ is the elliptic integral of the first type.

[1] K. Akiyama *et al.*, First M87 event horizon telescope results. I. The shadow of the supermassive black hole, Astrophys. J. **875**, L1 (2019).

[2] A. Ashtekar and R. S. Tate, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991), Vol. 6.

[3] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004).

[4] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Classical Quantum Gravity **21**, R53 (2004).

[5] M. Han, Y. Ma, and W. Huang, Fundamental structure of loop quantum gravity, Int. J. Mod. Phys. D **16**, 1397 (2007).

[6] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2008).

[7] A. Abhay and P. Jorge, *Loop Quantum Gravity: The First 30 Years* (World Scientific, Singapore, 2017), Vol. 4.

[8] C. Rovelli and L. Smolin, Discreteness of area and volume in quantum gravity, Nucl. Phys. **B442**, 593 (1995).

[9] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann, Quantization of diffeomorphism invariant theories of connections with local degrees of freedom, J. Math. Phys. (N.Y.) **36**, 6456 (1995).

[10] A. Ashtekar and J. Lewandowski, Quantum theory of geometry: I. Area operators, Classical Quantum Gravity **14**, A55 (1997).

[11] A. Ashtekar and J. Lewandowski, Quantum theory of geometry II: Volume operators, Adv. Theor. Math. Phys. **1**, 388 (1997).

[12] T. Thiemann, Quantum spin dynamics (QSD), Classical Quantum Gravity **15**, 839 (1998).

[13] T. Thiemann, Quantum spin dynamics (QSD): II. The kernel of the Wheeler-Dewitt constraint operator, Classical Quantum Gravity **15**, 875 (1998).

[14] Y. Ma and Y. Ling, $\widehat{\mathrm{Q}}$ operator for canonical quantum gravity, Phys. Rev. D **62**, 104021 (2000).

[15] J. Lewandowski, A. Okolow, H. Sahlmann, and T. Thiemann, Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, Commun. Math. Phys. **267**, 703 (2006).

[16] J. Yang and Y. Ma, New volume and inverse volume operators for loop quantum gravity, Phys. Rev. D **94**, 044003 (2016).

[17] E. Alesci, F. Cianfrani, and C. Rovelli, Quantum-reduced loop gravity: Relation with the full theory, Phys. Rev. D **88**, 104001 (2013).

[18] A. Ashtekar and M. Bojowald, Quantum geometry and the Schwarzschild singularity, Classical Quantum Gravity **23**, 391 (2006).

[19] L. Modesto, Loop quantum black hole, Classical Quantum Gravity **23**, 5587 (2006).

[20] C. G. Boehmer and K. Vandersloot, Loop quantum dynamics of the Schwarzschild interior, Phys. Rev. D **76**, 104030 (2007).

[21] M. Campiglia, R. Gambini, and J. Pullin, Loop quantization of spherically symmetric midi-superspaces: The interior problem, AIP Conf. Proc. **977**, 52 (2008).

[22] D.-W. Chiou, Phenomenological loop quantum geometry of the Schwarzschild black hole,Phys. Rev. D **78**, 064040 (2008).

[23] R. Gambini and J. Pullin, Loop Quantization of the Schwarzschild Black Hole, Phys. Rev. Lett. **110**, 211301 (2013).

[24] A. Corichi and P. Singh, Loop quantization of the Schwarzschild interior revisited, Classical Quantum Gravity **33**, 055006 (2016).

[25] M. Bojowald, S. Brahma, and D.-H. Yeom, Effective line elements and black-hole models in canonical loop quantum gravity, Phys. Rev. D **98**, 046015 (2018).

[26] A. Ashtekar, J. Olmedo, and P. Singh, Quantum Transfiguration of Kruskal Black Holes, Phys. Rev. Lett. **121**, 241301 (2018).

[27] A. Ashtekar, J. Olmedo, and P. Singh, Quantum extension of the Kruskal spacetime, Phys. Rev. D **98**, 126003 (2018).

[28] C. Rovelli and F. Vidotto, Small black/white hole stability and dark matter, Universe **4**, 127 (2018).

[29] E. Bianchi, M. Christodoulou, F. D'Ambrosio, H. M. Haggard, and C. Rovelli, White holes as remnants: A surprising scenario for the end of a black hole, Classical Quantum Gravity **35**, 225003 (2018).

[30] J. B. Achour, F. Lamy, H. Liu, and K. Noui, Polymer Schwarzschild black hole: An effective metric, Europhys. Lett. **123**, 20006 (2018).

[31] E. Alesci, S. Bahrami, and D. Pranzetti, Quantum evolution of black hole initial data sets: Foundations, Phys. Rev. D **98**, 046014 (2018).

[32] E. Alesci, S. Bahrami, and D. Pranzetti, Quantum gravity predictions for black hole interior geometry, Phys. Lett. B **797**, 134908 (2019).

[33] N. Bodendorfer, F. M. Mele, and J. Münch, Mass and horizon dirac observables in effective models of quantum black-to-white hole transition, Classical Quantum Gravity **38**, 095002 (2021).

[34] N. Bodendorfer, F. M. Mele, and J. Münch, Effective quantum extended spacetime of polymer Schwarzschild black hole, Classical Quantum Gravity **36**, 195015 (2019).

[35] J. G. Kelly, R. Santacruz, and E. Wilson-Ewing, Black hole collapse and bounce in effective loop quantum gravity, Classical Quantum Gravity **38**, 04LT01 (2021).

[36] M. Han and H. Liu, Improved effective dynamics of loop-quantum-gravity black hole and nariai limit, arXiv:2012.05729.

[37] E. Alesci, S. Bahrami, and D. Pranzetti, Asymptotically de Sitter universe inside a Schwarzschild black hole, Phys. Rev. D **102**, 066010 (2020).

[38] C. Zhang, Reduced phase space quantization of black holes: Path integrals and effective dynamics, Phys. Rev. D **104**, 126003 (2021).

[39] S. P. Song, H. D. Li, Y. G. Ma, and C. Zhang, Entropy of black holes with arbitrary shapes in loop quantum gravity, Sci. China Phys. Mech. Astron. **64**, 120411 (2021).

[40] S. W. Hawking, Black hole explosions?, Nature (London) **248**, 30 (1974).

[41] J. D. Barrow, E. J. Copeland, and A. R. Liddle, The cosmology of black hole relics, Phys. Rev. D **46**, 645 (1992).

[42] B. J. Carr, J. H. Gilbert, and J. E. Lidsey, Black hole relics and inflation: Limits on blue perturbation spectra, Phys. Rev. D **50**, 4853 (1994).

[43] I. Dalianis and G. Tringas, Primordial black hole remnants as dark matter produced in thermal, matter, and runaway-quintessence postinflationary scenarios, Phys. Rev. D **100**, 083512 (2019).

[44] J. Preskill, Do black holes destroy information, in *Proceedings of the International Symposium on Black Holes, Membranes, Wormholes and Superstrings*, edited by S. Kalara and D. V Nanopoulos (World Scientific, Singapore, 1993), pp. 22–39.

[45] P. Chen, Y. C. Ong, and D.-H. Yeom, Black hole remnants and the information loss paradox, Phys. Rep. **603**, 1 (2015).

[46] J. D. Bekenstein, Quantum black holes as atoms, arXiv:gr-qc/9710076.

[47] D. Pranzetti, Radiation from Quantum Weakly Dynamical Horizons in Loop Quantum Gravity, Phys. Rev. Lett. **109**, 011301 (2012).

[48] A. Barrau, X. Cao, K. Noui, and A. Perez, Black hole spectroscopy from loop quantum gravity models, Phys. Rev. D **92**, 124046 (2015).

[49] K. Lochan and S. Chakraborty, Discrete quantum spectrum of black holes, Phys. Lett. B **755**, 37 (2016).

[50] C. Zhang, Y. Ma, S. Song, and X. Zhang, Loop quantum Schwarzschild interior and black hole remnant, Phys. Rev. D **102**, 041502 (2020).

[51] E. Alesci, M. Assanioussi, J. Lewandowski, and I. Mäkinen, Hamiltonian operator for loop quantum gravity coupled to a scalar field, Phys. Rev. D **91**, 124067 (2015).

[52] A. Dapor, W. Kamiński, J. Lewandowski, and J. M. K. Świeżewski, Relational evolution of observables for Hamiltonian-constrained systems, Phys. Rev. D **88**, 084007 (2013).

[53] C. Zhang and X. Zhang, Quantum geometry and effective dynamics of Janis-Newman-Winicour singularities, Phys. Rev. D **101**, 086002 (2020).

[54] M. Domagała, K. Giesel, W. Kamiński, and J. Lewandowski, Gravity quantized: Loop quantum gravity with a scalar field, Phys. Rev. D **82**, 104038 (2010).

[55] A. Ashtekar, M. Bojowald, and J. Lewandowski, Mathematical structure of loop quantum cosmology, Adv. Theor. Math. Phys. **7**, 233 (2003).

[56] A. Ashtekar, T. Pawlowski, and P. Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D **74**, 084003 (2006).

[57] M. Reed, *Methods of Modern Mathematical Physics: Functional Analysis* (Elsevier, New York, 2003).

[58] C. Zhang, J. Lewandowski, and Y. Ma, Towards the self-adjointness of a hamiltonian operator in loop quantum gravity, Phys. Rev. D **98**, 086014 (2018).

[59] M. Reed, B. Simon, and S. Reed, *Methods of Modern Mathematical Physics: Fourier Analysis, Self-adjointness* (Elsevier, New York, 2003).

[60] M. Reed, B. Simon, and S. Reed, *Methods of Modern Mathematical Physics: Analysis of Operators* (Elsevier, New York, 2003).

[61] T. Kato, *Perturbation Theory for Linear Operators* (Springer Science & Business Media, New York, 2013), Vol. 132.

[62] T. Kato, On the upper, and lower bounds of eigenvalues, J. Phys. Soc. Jpn. **4**, 334 (1949).

[63] E. M. Harrell, Generalizations of Temple's inequality, Proc. Am. Math. Soc. **69**, 271 (1978).

[64] A. I. Janis, E. T. Newman, and J. Winicour, Reality of the Schwarzschild Singularity, Phys. Rev. Lett. **20**, 878 (1968).

[65] A. Ashtekar, B. Christopher, and F. Stephen, Mechanics of isolated horizons, Classical Quantum Gravity **17**, 253 (2000).

[66] https://github.com/czhangUW/LoopSchwarzschildBH.

[67] M. Levitin and E. Shargorodsky, Spectral pollution and second-order relative spectra for self-adjoint operators, IMA J. Numer. Anal. **24**, 393 (2004).

[68] M. Lewin and É. Séré, Spectral pollution and how to avoid it, Proc. Lond. Math. Soc. **100**, 864 (2010).