Planar black holes configurations and shear viscosity in arbitrary dimensions with shift and reflection symmetric scalar-tensor theories

Moisés Bravo-Gaete[*]

Facultad de Ciencias Básicas, Universidad Católica del Maule, Casilla 617, Talca, Chile

M. M. Stetsko[†]

Department for Theoretical Physics, Ivan Franko National University of Lviv, 12 Drahomanov Street, Lviv, UA-79005, Ukraine

In higher dimensions, we explore planar hairy black hole configurations for a special subclass of the Horndeski theory defined by two coupling functions depending on the kinetic term and enjoying shift symmetry and reflection symmetry. For this analysis, we derive a set of new solutions given by time-dependent as well as time-independent scalar field configurations. Additionally, we calculate their thermodynamic quantities by using Wald formalism satisfying the first law of thermodynamics as well as a Smarr relation. Together with the above, the Wald procedure allows us to compute the shear viscosity, showing that for a suitable choice of the coupling functions the Kovtun-Son-Starinets bound is violated.

DOI: 10.1103/PhysRevD.105.024038

I. INTRODUCTION

General relativity (GR) is without a doubt a very successful standard model of gravity. Nevertheless, astrophysical discoveries such as the accelerated expansion of the Universe [1,2] and the recent first detection of gravitational waves [3] have yielded the motivation to study theories of gravity beyond GR.

There are many ways to construct modifications of GR, one of them is to introduce new degrees of freedom given by scalar fields and denominated as scalar-tensor theories. In particular, Horndeski in the 1970s formulated a four-dimensional theory defined by the metric $g_{\mu\nu}$ and a scalar field ϕ, as well as their derivatives, where the equations of motion are at most of the second order [4]. This peculiarity makes the Horndeski theory a healthy one, because it does not have ghosts or instabilities caused by higher orders derivatives in the equations of motion.

An intuitive way to construct the Horndeski theory action follows from the Galileon theory [5,6], with the introduction of four functions dependent on the metric $g_{\mu\nu}$, the scalar field ϕ, and the kinetic term $X := -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ which reads [7]

$$
\begin{aligned}
S[g_{\mu\nu}, \phi] = \int d^4x \sqrt{-g} \Big[& G_2(X, \phi) - G_3(X, \phi) \Box\phi \\
& + G_{4X}((\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2) + R G_4(X, \phi) \\
& - \frac{1}{6} G_{5X}((\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3) \\
& + G_{\mu\nu} \nabla^\mu\nabla^\nu\phi G_{(5)}(X, \phi) \Big],
\end{aligned}
\tag{1}
$$

where we define

$$
G_{nX} := \frac{\partial G_n}{\partial X}, \qquad n = \{2, 3, 4, 5\},
$$
$$
(\nabla_\mu\nabla_\nu\phi)^2 := (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi),
$$
$$
(\nabla_\mu\nabla_\nu\phi)^3 := (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\rho\phi)(\nabla_\rho\nabla^\nu\phi),
$$

with R and $G_{\mu\nu}$ being the scalar curvature and the Einstein tensor respectively.

On the other hand, one of the peculiarities of planar black holes is their relation to an ideal fluid given by the gravity/gauge duality [8–10]. Within this scenario, it is possible to compute the well-known ratio between the shear viscosity η and the entropy density s, allowing us to check the conjecture about a universal bound known as the Kovtun-Son-Starinets (KSS) bound, which reads [11–14]

$$
\frac{\eta}{s} \geq \frac{1}{4\pi},
\tag{2}
$$

being demonstrated in a variety of gravity theories (see for example [15–18]), where the shear viscosity can be

[*]mbravo@ucm.cl
[†]mstetsko@gmail.com

obtained by effective coupling constants of the transverse graviton on the location of the event horizon, via the membrane paradigm [19], and corroborated by the Kubo formula [20,21]. Recently, constructing a Noether charge with a suitable choice of a spacelike Killing vector, and following the Wald formalism [22,23], the η/s ratio was calculated by using the infrared data on the black hole event horizon [24], greatly simplifying the steps in comparison with the previous procedure.

Nevertheless, in recent years it has been shown with specific examples that the bound can be violated. In fact, we can mention gravity theories such as the Einstein-Hilbert Gauss-Bonnet model [25,26] as well as a particular truncation of the Horndeski theory (1) [27,28].

In particular, in the present paper, we are interested in the study of a subclass of the action (1) based on the work developed in [29], where the theory in D dimensions takes the form

$$S[g_{\mu\nu}, \phi] = \int d^D x \sqrt{-g} \mathcal{L}, \qquad (3)$$

and the Lagrangian is expressed as

$$\mathcal{L} = G_2 + G_4 R + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \qquad (4)$$

where now G_2 and G_4 are arbitrary functions of the kinetic term X and, as before, $G_{4X} := \partial G_4/\partial X$. The corresponding equations of motion are of the following form:

$$\mathcal{E}_{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = 0, \qquad (5)$$

$$\mathcal{E}_\phi := \nabla_\mu J^\mu = 0, \qquad (6)$$

where

$$\begin{aligned} J^\mu := &-G_{2X}\nabla^\mu\phi + 2G_{4X}G^{\mu\nu}\nabla_\nu\phi \\ &- G_{4XX}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2]\nabla^\mu\phi \\ &- 2G_{4XX}(\Box\phi\nabla^\mu X - \nabla^\mu\nabla^\nu\phi\nabla_\nu X), \end{aligned} \qquad (7)$$

while the equations with respect to the metric $\mathcal{E}_{\mu\nu}$ are reported in the Appendix. Within this theory, we will focus on black holes in arbitrary dimensions with a planar base manifold for the event horizon, and the thermodynamics of these configurations will be also examined. In addition, we compute the shear viscosity η, and to perform this task in the present paper we will utilize the formalism developed in [24].

The rest of the paper is organized as follows. In Sec. II we explore planar black holes with a linear time-dependent scalar field ϕ, giving a general solution for some particular cases for the functions G_2 and G_4. In Sec. III the time-independent case is analyzed, whereas its thermodynamics

is studied in Sec. IV. In Sec. V, the shear viscosity is computed, where the η/s ratio is obtained, and a condition on the functions G_2 and G_4 is found where the KSS bound can be violated. Finally, Sec. VI is devoted to our conclusions and discussion.

II. DERIVATION OF THE SOLUTION WITH A LINEAR TIME-DEPENDENT SCALAR FIELD

One of the peculiarities of scalar fields in scalar-tensor theories of Horndeski type (3) and (4) is the existence of time-dependent configurations compatible with the gravitational sector. In particular, in the following studies, the ansatz for the metric will be

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \sum_{i=1}^{D-2} dx_i^2, \qquad (8)$$

and following the steps performed in [29,30], the scalar field reads

$$\phi(r,t) = \varphi(r) + qt, \qquad (9)$$

where q is a constant. It is worth pointing out that this structure for the scalar field has been successfully applied with other kinds of topologies for the event horizon (see for example [29–33]). Since the scalar field ϕ is time dependent, the kinetic term X takes the form

$$X = \frac{1}{2}\left(\frac{q^2}{h} - f(\varphi')^2\right). \qquad (10)$$

Imposing a condition on the radial part of the current $J^r = 0$, we obtain the following relation for the metric functions f and h:

$$f = -\frac{G_{2X}r^{D-2}h}{(D-2)[\mathcal{G}_X(r^{D-3}h)' + (D-3)q^2 r^{D-4}G_{4XX}]}, \qquad (11)$$

where $(')$ denotes the derivative with respect to the radial coordinate r and

$$\mathcal{G}(X) = (G_4 - 2XG_{4X}), \qquad \mathcal{G}_X := \partial\mathcal{G}/\partial X. \qquad (12)$$

It should be pointed out that for the time-dependent scalar field (9), in addition to the "diagonal" gravitational field equations (5) there is an "off-diagonal" term which corresponds to the (t, r) component, where the corresponding field equation takes the form

$$\mathcal{E}_{tr} := \left[-\frac{G_{2X}}{2} + G_{4X} \frac{(D-2)f}{2r^{D-2}h}(r^{D-3}h)' + G_{4XX} \right.$$
$$\left. \times \frac{(D-2)f}{2rh} \left(\frac{h'}{h} q^2 + \frac{(r^{D-3}h)'}{r^{D-3}} f(\varphi')^2 \right) \right] q\varphi' = 0.$$

(13)

This latter equation, together with (10), also gives rise to the relation (11), and it is worth noting that this fact reflects the consistency of our procedure. Taking into account the relation (11), we can recast the (r, r) component of the equations of motion (5) in the following form:

$$(r^{D-3}h)' \partial_X(G_2\mathcal{G}) + (D-3)q^2 r^{D-4} \partial_X(G_2 G_{4X}) = 0. \quad (14)$$

Having combined Eq. (14) and the relation for metric functions (11), we can also derive the following relation:

$$f = -\frac{r^2}{(D-2)(D-3)q^2} \frac{\partial_X(G_2\mathcal{G})}{\partial_X(G_4 G_{4X})} h, \quad (15)$$

where we note that it cannot be applied to a time-independent scalar field (this is $q = 0$).

Using Eqs. (11), (14), and (15), the (t, t) component of the Einstein equations (5) acquires the structure

$$-\frac{(D-2)h}{2r\mathcal{G}} \left(\frac{f}{h} \mathcal{G}^2 \right)' = 0, \quad (16)$$

and it follows immediately that

$$\frac{f}{h} \mathcal{G}^2 = C, \quad (17)$$

where C is an integration constant. Here it is important to note that we do not impose any asymptotic behavior at the beginning, but it follows from the solutions we will obtain below. Finally, combining the solution (17) together with the relation (15), we obtain

$$-\frac{r^2}{(D-2)(D-3)q^2} \frac{\partial_X(G_2\mathcal{G})}{\partial_X(G_4 G_{4X})} = \frac{C}{\mathcal{G}^2}. \quad (18)$$

This relation demonstrates clearly that kinetic term $X = X(r)$ can be derived algebraically for given functions G_2 and G_4, and it is in complete agreement with the result obtained previously for the four-dimensional case [29]. Having obtained the explicit form of the function $X(r)$, one can use Eq. (14) to find the metric function $h(r)$, and finally using the relation (15) we derive the metric function $f(r)$. We also point out that the constant C in the relation (17) should be positive ($C > 0$), because the metric functions $h(r)$ and $f(r)$ should be of the same sign in any point on their domains of variation. In the following lines, we set $C = 1$ without any loss of generality of conclusions.

Since no specific conditions have been imposed on the functions G_2 and G_4, we can choose them freely and try to find explicit structures for the metric functions $h(r)$ and $f(r)$, as well as the explicit expression for the kinetic term $X(r)$. Below we examine a few cases for the functions G_2 and G_4 for which it is possible to obtain explicit expressions of the metric functions, at least in a relatively simple form. Before we start considering some particular cases, we would like to stress that the only equation which has not been used explicitly when we derived the relations (15), (17), and (18) is the (x_i, x_i) component from the field equations (5). Therefore, we might expect that this equation gives rise to some conditions which should be imposed on the functions G_2 and G_4, being written as follows:

$$\mathcal{G} \left[\frac{1}{2} \sqrt{\frac{f}{h}} \left(\sqrt{\frac{f}{h}} h' \right)' + \frac{(D-3)}{2r^{D-3}h} (r^{D-4}fh)' \right]$$
$$- \frac{G_2}{2} + \mathcal{G}' \left(\frac{h'}{2h} + \frac{D-3}{r} \right) f + \frac{(D-3)q^2}{2r^{D-3}G_{4X}}$$
$$\times \left(\frac{r^{D-4}fG_{4X}^2}{h} \right)' = 0. \quad (19)$$

If we now use relation (17), the latter equation can be simplified to

$$\frac{(r^{D-3}h)''}{r^{D-3}\mathcal{G}} - G_2 + \frac{(D-3)q^2}{r^{D-3}G_{4X}} \left(\frac{r^{D-4}G_{4X}^2}{\mathcal{G}^2} \right)' = 0. \quad (20)$$

It is worth noting that the above-written equation can be applied to the time-independent field, namely in this case $q = 0$, and the equation can be cast as follows:

$$(r^{D-3}h)'' - r^{D-3}\mathcal{G}G_2 = 0. \quad (21)$$

In the following, we will use Eq. (20) to check whether some additional conditions for G_2 and G_4 appear, allowing us to obtain explicit expressions for $h(r)$ and $f(r)$.

A. Case A

As a first case, we assume that both G_2 and G_4 are linear functions of X:

$$G_2 = \alpha + \beta X, \qquad G_4 = \xi + \gamma X, \quad (22)$$

where α, β, ξ, and γ are constants. By using the relation (18) together with $\beta = -\gamma\alpha/\xi$, we obtain

$$X = \frac{1}{\gamma} \left[\xi - \left(\frac{\gamma\xi(D-2)(D-3)}{2\alpha} \right)^{1/3} \left(\frac{q}{r} \right)^{2/3} \right]. \quad (23)$$

Now, taking into account Eq. (14), one can obtain the explicit structure for the metric function $h(r)$, which takes the following form:

$$h(r) = -\frac{M}{r^{D-3}} - \left(\frac{2(D-3)^2\alpha\gamma^2 q^4}{\xi(D-2)}\right)^{1/3}\frac{3r^{2/3}}{(3D-7)}, \quad (24)$$

where M is a positive integration constant. In order to have a black hole solution the metric function $h(r)$ should be positive at least for sufficiently large r; consequently, it means that the second term in the relation (24) has to be positive and it takes place if $\alpha/\gamma < 0$ with $\xi > 0$. To derive the metric function $f(r)$, according to the relation (17), one also needs the explicit relation for the function $\mathcal{G}$ being obtained via the relations (12) and (23) given by

$$\mathcal{G}^2 = \left(\frac{\gamma\xi(D-2)(D-3)q^2}{2\alpha}\right)^{2/3}\frac{1}{r^{4/3}}. \quad (25)$$

Substituting the functions (24) and (25) into Eq. (20), we can check that this equation is satisfied as an identity without any conditions on the parameters of the functions G_2 and G_4. Here we would also like to emphasize that a similar conclusion can be made regarding all of the chosen forms of G_2, G_4, and $h(r)$ that we will consider below. Just for completeness, after a redefinition of the coupling constants γ, α, and ξ, as well as for the integration constant M, we recover the asymptotically Lifshitz black hole in arbitrary dimensions found in [34].

B. Case B

Now we suppose that the functions G_2 and G_4 are

$$G_2 = \alpha(\xi - \gamma X)^k, \qquad G_4 = \xi + \gamma X, \quad (26)$$

and using Eq. (18) we obtain

$$X = \frac{1}{\gamma}\left[\xi - \left(\frac{(D-2)(D-3)\gamma q^2}{(k+1)\alpha}\right)^{\frac{1}{k+2}}r^{-2/(k+2)}\right], \quad (27)$$

where $k \neq -1$ and $k \neq -2$, and Eq. (14) gives rise to the following form of the metric function $h(r)$:

$$h(r) = -\frac{M}{r^{D-3}} - \frac{k(k+2)(D-3)\gamma q^2}{(k+1)[(k+2)(D-3)+2]}\frac{r^{\frac{2}{k+2}}}{\bar{A}}, \quad (28)$$

where

$$\bar{A} = \left[\frac{(D-2)(D-3)\gamma q^2}{\alpha(k+1)}\right]^{1/(k+2)},$$

with M a positive integration constant. Finally, via Eq. (21) we obtain

$$\mathcal{G}^2 = \bar{A}^2 r^{-4/(k+2)}, \quad (29)$$

allowing us to obtain the expression for the metric function $f(r)$ by using the relation (17).

One can easily check that if $k = 1$, and after a redefinition of the constants, the relations (27)–(29) are reduced to the corresponding relations obtained for the previously examined situation (Case A) corresponding to asymptotically Lifshitz black holes in higher dimensions.

C. Case C

For this situation, the functions G_2 and G_4 are chosen as

$$G_2 = \alpha + \beta X^k, \qquad G_4 = \xi + \gamma\sqrt{X}, \quad (30)$$

and the relation (18) allows us to arrive at

$$X = \left[\frac{(D-2)(D-3)\gamma q^2}{4k\beta\xi^2}\right]^{\frac{2}{2k+1}}r^{-\frac{4}{2k+1}} \quad (31)$$

where $k \neq -\frac{1}{2}$. Using Eq. (14) we obtain the metric function $h(r)$, which reads

$$h(r) = -\frac{M}{r^{D-3}} + \frac{(D-3)\gamma q^2}{2k\xi\beta}\left[\frac{\alpha}{2(D-1)}\tilde{A}^{-\frac{2k+1}{2}}r^2 - \frac{(4k^2-1)\beta}{2[(D-3)(2k+1)+2]}\tilde{A}^{-\frac{1}{2}}r^{\frac{2k}{2k+1}}\right], \quad (32)$$

where, as before, M is a positive integration constant and we denoted for simplicity:

$$\tilde{A} = \left[\frac{\gamma}{4k\beta\xi^2}(D-2)(D-3)q^2\right]^{2/(2k+1)}.$$

It is easy to check that

$$\mathcal{G}^2 = \xi^2, \quad (33)$$

and as a consequence if we impose $\xi = 1$, we have that $f(r) = h(r)$. Here it is important to note that for this case we have an anti–de Sitter (AdS) or de Sitter configuration depending on the sign of

$$\frac{\alpha(D-3)\gamma q^2}{2(D-1)2k\xi\beta}\tilde{A}^{-\frac{2k+1}{2}},$$

together with $2k/(2k+1) < 0$. For the sake of completeness, starting with $k = 1/2$ from (30) we note that the third term in the relation (32) disappears and the metric functions take the very simple form

$$f(r) = h(r) = -\frac{M}{r^{D-3}} + \frac{\alpha}{(D-1)(D-2)}r^2. \quad (34)$$

Therefore, for this particular case, we completely recover the AdS or de Sitter situations of solutions known from the standard general relativity gravity. We also point out that

the solution (34) represents a black hole if we impose $\alpha > 0$.

D. Case D

Now, for an election of the functions given by

$$G_2 = \alpha X^2 (1 + \gamma X^2)^{3/4}, \qquad G_4 = \xi (1 + \gamma X^2)^{1/4}, \qquad (35)$$

Eq. (18) gives rise to

$$X = -\frac{\gamma}{4\alpha\xi} \frac{(D-2)(D-3)q^2}{r^2}, \qquad (36)$$

while the metric function $h(r)$ takes the form

$$h(r) = -\frac{M}{r^{D-3}} + \frac{3\gamma^2(D-2)(D-3)^2}{16\alpha\xi(D-5)} \frac{q^4}{r^2}, \qquad (37)$$

where from Eq. (21) we have

$$\mathcal{G}^2 = \xi^2 \left[1 + \frac{\gamma^3(D-2)^2(D-3)^2}{16\alpha^2\xi^2} \frac{q^4}{r^4} \right]^{-\frac{3}{2}}, \qquad (38)$$

while the metric function $f(r)$ is obtained via Eq. (17). Here it is important to note that, unlike the previous cases, this solution enjoys other asymptotic behavior, where to obtain a black hole, the presence of the constant q as well as a positive integration constant M are providential. It is worth noting that if $D = 5$ the second term in (37) instead of inverse quadratic dependence will have $\sim \ln(r)/r^2$ character, allowing us to study the number of the locations of the event horizon by using Lambert W functions [35]. It should be also pointed out that to have a black hole we have to impose $D \geq 5$, because if $D = 4$ the terms in (37) reverse their roles, and for the particular case $D = 3$ the function (37) keeps the first term only.

E. Case E

For this case, let us take the functions G_2 and G_4 in the following form:

$$G_2 = \alpha(1 + \gamma X^2)^{5/4}, \qquad G_4 = \xi(1 + \gamma X^2)^{1/4}, \qquad (39)$$

where from Eq. (18) we obtain

$$X = -\left[\frac{4\alpha^2\xi^2}{(D-2)^2(D-3)^2 q^4} r^4 - \gamma \right]^{-1/2}, \qquad (40)$$

and it follows immediately that the condition on the coordinate r that should be imposed in order to provide the function $X(r)$ to be real is $r^4 \geq \gamma(D-2)^2(D-3)^2 q^4/ (4\alpha^2\xi^2)$, but we assume that these parameters are chosen in such a way that this condition is fulfilled outside the event horizon of the black hole that we are going to find here.

Using (40), we are able to find that the metric function $h(r)$ is given by

$$h(r) = -\frac{M}{r^{D-3}} + \frac{(D-3)q^2}{2r^{D-3}}$$

$$\times \int dr\, r^{D-4} \left[\left(\frac{4\alpha^2\xi^2}{(D-2)^2(D-3)^2 q^4} r^4 - \gamma \right)^{\frac{1}{2}} \right.$$

$$\left. + 2\gamma \left(\frac{4\alpha^2\xi^2}{(D-2)^2(D-3)^2 q^4} r^4 - \gamma \right)^{-\frac{1}{2}} \right]. \qquad (41)$$

In general, the integrals in the above-written relations cannot be written in terms of elementary functions, but it can be represented in terms of hypergeometric functions as

$$h(r) = -\frac{M}{r^{D-3}} + \frac{\alpha\xi}{(D-1)(D-2)} r^2$$

$$\times {}_2F_1\left(-\frac{1}{2}, \frac{1-D}{4}; \frac{5-D}{4}; \frac{\gamma(D-2)^2(D-3)^2 q^4}{4\alpha^2\xi^2 r^4} \right)$$

$$+ \frac{\gamma(D-2)(D-3)^2 q^4}{2\alpha\xi(D-5)r^2}$$

$$\times {}_2F_1\left(\frac{1}{2}, \frac{5-D}{4}; \frac{9-D}{4}; \frac{\gamma(D-2)^2(D-3)^2 q^4}{4\alpha^2\xi^2 r^4} \right). \qquad (42)$$

The structure of the above-written solution might be a bit complicated; for instance, for large r ($r \to +\infty$) the asymptotic form of this function is as follows:

$$h(r) \simeq -\frac{M}{r^{D-3}} + \frac{\alpha\xi}{(D-1)(D-2)} r^2$$

$$+ \frac{\gamma(D-2)(D-3)^2 q^4}{2\alpha\xi(D-5)r^2}, \qquad (43)$$

because both hypergeometric functions tend to 1 if $r \to +\infty$. The asymptotic relation (43) shows that for large radius the leading term is of anti–de Sitter (or de Sitter) type: $\sim \frac{\alpha\xi}{(D-1)(D-2)} r^2$, depending on the sign of the parameters α and ξ. Here it is important to note that this leading term does not depend on parameter q, being completely defined by the parameters of the functions G_2 and G_4 given previously in (39).

It should also be stressed that the representation (42) is valid only for even D while for odd D there are some subtleties; it is easy to see that the first hypergeometric function in (42) has some peculiarity if $D = 5$ while the second one for $D = 9$. But from the integral form (41) it follows that for odd D the results of the integration might be written in terms of elementary functions. Namely, for $D = 5$ we arrive at the following explicit form for the metric function $h(r)$ (if $\gamma > 0$):

$$h(r) = -\frac{M}{r^2} + \frac{q^2}{4}\sqrt{\frac{\alpha^2\xi^2}{9q^4}r^4 - \gamma} + \frac{15\gamma q^4}{4\alpha\xi r^2}\text{arcosh}\left(\frac{\alpha\xi}{3q^2\sqrt{\gamma}}r^2\right),$$

$$(44)$$

whereas for $\gamma < 0$ instead of an arcosh function one should utilize arsinh. For both cases, for a large r we have that

$$h(r) \simeq \frac{|\alpha\xi|r^2}{12} + O\left(\frac{1}{r^2}\right),$$

and we can find AdS or de Sitter planar configurations depending on the sign of the constants α and ξ present in (39).

If $D = 7$ the metric function $h(r)$ takes an even simpler form; namely it might be represented as a combination of irrational functions:

$$h(r) = -\frac{M}{r^4} + \frac{\alpha\xi}{30r^4}\left(r^4 - \frac{100\gamma q^4}{\alpha^2\xi^2}\right)^{\frac{3}{2}} + \frac{20\gamma q^4}{\alpha\xi r^4}\left(r^4 - \frac{100\gamma q^4}{\alpha^2\xi^2}\right)^{\frac{1}{2}}.$$

$$(45)$$

It is straightforward to check that for large r, the given exact relation gives rise to an asymptotic relation of the form (43). Finally, from Eq. (21), the function $\mathcal{G}^2$ can be represented as

$$\mathcal{G}^2 = \frac{4\xi^2}{(1 + \gamma X^2)^{\frac{3}{2}}},$$

$$(46)$$

while the kinetic term X takes the form found previously in (40) for seven dimensions.

F. Case F

As a final case to analyze, we consider the following form for the functions G_2 and G_4:

$$G_2 = \alpha(1 + \gamma X^2)^{7/4}, \qquad G_4 = \xi(1 + \gamma X^2)^{1/4}. \quad (47)$$

Equation (18) gives rise to

$$X = -\frac{(D-2)(D-3)}{16\alpha\xi}\frac{q^2}{r^2},$$

$$(48)$$

and the metric function $h(r)$ takes the form

$$h(r) = -\frac{M}{r^{D-3}} + \frac{4\alpha\xi}{(D-1)(D-2)}r^2 + \frac{3\gamma(D-2)(D-3)^2}{16\alpha\xi(D-5)}\frac{q^4}{r^2}.$$

$$(49)$$

Here we would also like to note that if $D = 5$ the latter term is of the form $\sim \ln(r)/r^2$, where for this situation,

as Case D, the number of horizons can be analyzed in terms of Lambert W functions [35]. The obtained metric function (49) is very similar to the asymptotic relation (43). In particular, the first and the second terms in these relations coincide completely, whereas the third terms have the same dependence on r, dimension of spacetime D, and the parameter q.

Finally, we can write the explicit form for the function $\mathcal{G}^2$ as

$$\mathcal{G}^2 = \xi^2\left(1 + \frac{\gamma(D-2)^2(D-3)^2}{16\alpha^2\xi^2}\frac{q^4}{r^4}\right)^{-3/2}. \quad (50)$$

III. THE TIME-INDEPENDENT CASE

Now we will focus on a scalar field ϕ given by (9) with $q = 0$, where the kinetic term X is given by (10), being expressed as

$$X = -\frac{1}{2}f(\varphi')^2,$$

$$(51)$$

and the radial component $J^r = 0$ of the equations of motion with respect to the scalar field (7) is satisfied imposing the condition (11).

For the time-independent scalar field Eq. (14), the (r, r) component can be written in the following form:

$$(r^{D-3}h)'\partial_X(G_2\mathcal{G}) = 0.$$

$$(52)$$

According to [29], the above equation allows us to conclude that we can find the kinetic term X in an algebraic way, considering $X = $ constant such that $\partial_X(G_2\mathcal{G}) = 0$. Finally, the (t, t) component can be expressed as a differential equation with respect to the metric function f which reads

$$-h\left[\frac{(D-2)(\mathcal{G}^2 r^{D-3}f)'}{r^{D-2}\mathcal{G}} - G_2\right] = 0, \quad (53)$$

and the expressions for the metric functions f and h take the form

$$f(r) = h(r) = \frac{G_2}{(D-1)(D-2)\mathcal{G}}r^2 - \frac{M}{r^{D-3}}, \quad (54)$$

where M, as before, is an integration constant. For this situation, we can find a de Sitter or anti–de Sitter configuration, where the radius takes the form

$$\frac{G_2}{(D-1)(D-2)\mathcal{G}} := l^{-2}, \quad (55)$$

and the derivative of the scalar field as well as its explicit expression are given by

$$(\phi')^2 = -\frac{2X}{f},$$

$$\phi(r) = \pm\frac{2\sqrt{-2Xl^2}}{(D-1)}\ln\left[r^{\frac{D-3}{2}}\left(\frac{r}{l} + \sqrt{\frac{r^2}{l^2} - \frac{M}{r^{D-3}}}\right)\right], \quad (56)$$

together with the condition

$$2Xl^2 \leq 0,$$

while the remaining equations of motion with respect to the metric $\mathcal{E}_{x_ix_i} = 0$ from (21) are trivially satisfied.

Here, it is worth pointing out that a particular form of the functions G_2 and G_4, when both of them are linear functions of X, also belong to the class of functions for which $X = $ constant is satisfied, where this condition stems directly from the equations of motion (5) for the (r, r) component. If we consider the relation (22), it follows from Eq. (52) that

$$X = \frac{\beta\xi - \alpha\gamma}{2\beta\gamma}. \quad (57)$$

Taking into account Eqs. (11) (with $q = 0$) and (53), we obtain

$$f(r) = h(r) = -\frac{M}{r^{D-3}} + \frac{\beta}{(D-1)(D-2)\gamma}r^2. \quad (58)$$

On the other hand, if we come back to the relation (52) we can point out that this equation might be fulfilled if we do not impose the condition $X(r) = $ constant as it has been performed above. In fact, from the condition $\partial_X(G_2\mathcal{G}) = 0$, we have that

$$G_2\mathcal{G} = C_1, \quad (59)$$

where C_1 is an integration constant. It is worth emphasizing that here we do not impose any condition on the kinetic term $X(r)$. The important conclusion which stems immediately from the relation (59) is the fact that the functions G_2 and G_4 in the Lagrangian (4) cannot be chosen independently; they have to obey the relation (59).

Equation (53) together with the condition (59) leads to the following relation:

$$f\mathcal{G}^2 = -\frac{M}{r^{D-3}} + \frac{C_1}{(D-1)(D-2)}r^2, \quad (60)$$

and here again, M denotes an integration constant. We note that the right-hand side of the relation (60) is very similar to the metric functions f and h given by the relation (54), but in the left-hand side of the relation (60) in contrast with (54) we have product $f\mathcal{G}^2$. Surely, if the additional condition $X(r) = $ constant is imposed there will be a complete

coincidence of the results up to a redefinition of the constants.

To obtain the metric function h, we use the relation (11) with $q = 0$, once again taking into account the relations (59) and (60), and we arrive at the following expression:

$$h(r) = -\frac{M}{r^{D-3}} + \frac{C_1}{(D-1)(D-2)}r^2. \quad (61)$$

Equation (61) is completely the same form as (54). Therefore, the explicit form of the metric function $h(r)$ (61) is valid even if $X(r) = $ constant is not imposed, but for a less restrictive condition, it is given by (59). At the same time, the equality $f(r) = h(r)$ from (54) is violated in this a bit more general case. Here we would also like to stress that the form of the function $\mathcal{G}$ is completely defined by a chosen form of the functions G_2 or G_4. We might also expect that some constraints on the functions G_2 and G_4 might be imposed if one considers the field equation (5) for the (x_i, x_i) component, because it has not been used in the procedure described above. Nevertheless, a careful check shows that this equation does not impose any constraint. Therefore the functions G_2 and G_4 might be chosen freely with the only condition; namely, they should obey (59).

IV. THERMODYNAMICS FOR THE TIME-INDEPENDENT SOLUTION

Given the steps performed previously to obtain planar black holes solutions, in this section, we will explore their thermodynamic behavior by using the Wald formalism [22,23], where the main idea is the variation of the Hamiltonian from a conserved Noether current.

The variation of the action (3) and (4) reads

$$\delta S = \sqrt{-g}[\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_\phi\delta\phi] + \partial_\mu\mathcal{J}^\mu,$$

where, as before, $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$ are the equations of motion with respect to the metric and the scalar field given by (5) and (6) respectively, while $\mathcal{J}^\mu$ represents the surface term which reads

$$\begin{aligned}
\mathcal{J}^\mu = \sqrt{-g}\bigg[&2(P^{\mu(\alpha\beta)\gamma}\nabla_\gamma\delta g_{\alpha\beta} - \delta g_{\alpha\beta}\nabla_\gamma P^{\mu(\alpha\beta)\gamma}) \\
&+ \frac{\delta\mathcal{L}}{\delta(\phi_\mu)}\delta\phi - \nabla_\nu\left(\frac{\delta\mathcal{L}}{\delta(\phi_{\mu\nu})}\right)\delta\phi + \frac{\delta\mathcal{L}}{\delta(\phi_{\mu\nu})}\delta(\phi_\nu) \\
&- \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\mu\rho})}\phi^\sigma\delta g_{\sigma\rho} - \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\rho\mu})}\phi^\sigma\delta g_{\sigma\rho} \\
&+ \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\sigma\rho})}\phi^\mu\delta g_{\sigma\rho}\bigg],
\end{aligned} \quad (62)$$

where we have

$$P^{\mu\nu\lambda\rho} = \frac{\delta\mathcal{L}}{\delta R_{\mu\nu\lambda\rho}} = \frac{1}{2}G_4(g^{\mu\lambda}g^{\nu\rho} - g^{\mu\rho}g^{\nu\lambda}),$$

$$\frac{\delta\mathcal{L}}{\delta(\phi_\mu)} = J^\mu, \qquad \frac{\delta\mathcal{L}}{\delta(\phi_{\sigma\rho})} = 2G_{4X}(\Box\phi g^{\sigma\rho} - \nabla^\sigma\nabla^\rho\phi),$$

with J^μ given previously in (7).

In particular, for the metric (8) and a time-independent scalar field $\phi = \phi(r)$, the only nonzero component of the surface current $\mathcal{J}^\mu$ is the radial component $\mathcal{J}^r$ which takes the following form:

$$\mathcal{J}^r = \sqrt{\frac{h}{f}}r^{D-2}\left\{-G_4\left[\frac{f}{h}\left(\delta h' + \frac{h'}{2}\left(\frac{\delta f}{f} - \frac{\delta h}{h}\right)\right)\right.\right.$$
$$\left.+ \frac{D-2}{r}\delta f\right] + G_{4X}f(\varphi')^2\left[\left(\frac{h'}{2h} + \frac{D-2}{r}\right)\delta f\right.$$
$$\left.+ \frac{(D-2)f}{rh}\delta h\right] + 2G_{4X}f^2\left(\frac{h'}{2h} + \frac{D-2}{r}\right)\varphi'\delta\varphi'\right\}. \quad (63)$$

It is worth noting that the current J^r from (7) vanishes; therefore the term proportional to $\delta\phi$ does not appear in the relation (63).

To compute the entropy, by using the surface term given in (62) we define a 1-form $\mathcal{J}_{(1)} = \mathcal{J}_\mu dx^\mu$ and its Hodge dual $\Theta_{(D-1)} = (-1) * \mathcal{J}_{(1)}$. Then, after making use of the equations of motion (this is $\mathcal{E}_{\mu\nu} = 0$ and $\mathcal{E}_\phi = 0$), we have

$$\mathcal{J}_{(D-1)} = \Theta_{(D-1)} - i_\xi * \mathcal{L} = -d * \mathcal{J}_{(2)},$$

where i_ξ is a contraction of the vector field ξ^μ on the first index of $*\mathcal{L}$. The above relation allows us to define a $(D-2)$-form $Q_{(D-2)} = *\mathcal{J}_{(2)}$ such that $\mathcal{J}_{(D-1)} = dQ_{(D-2)}$, where

$$Q_{(D-2)} = Q_{\alpha_1\alpha_2\cdots\alpha_{D-2}} = \epsilon_{\alpha_1\alpha_2\cdots\alpha_{D-2}\mu\nu}Q^{\mu\nu}$$

with

$$Q^{\mu\nu} = \left[2P^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 4\xi_\sigma\nabla_\rho P^{\mu\nu\rho\sigma}\right.$$
$$\left.+ \frac{\delta\mathcal{L}}{\delta\phi_{\mu\sigma}}\phi^\nu\xi_\sigma - \frac{\delta\mathcal{L}}{\delta\phi_{\nu\sigma}}\phi^\mu\xi_\sigma\right]. \quad (64)$$

To obtain the first law of black hole thermodynamics, the vector field ξ^μ is supposed to be a time-translation vector, which is a Killing vector and it is null on the event horizon r_h. In particular, for a planar black hole metric, we obtain

$$Q^{tr} = \sqrt{\frac{h}{f}}r^{D-2}\left(-\frac{f}{h}G_4h' + \frac{2(D-2)}{r}G_{4X}f^2(\varphi')^2\right). \quad (65)$$

Finally, the variation of the Hamiltonian reads

$$\delta\mathcal{H} = \delta\int_\mathcal{C}\mathcal{J}_{(D-1)} - \int_\mathcal{C}d(i_\xi\Theta_{(D-1)})$$

$$= \int_{\Sigma^{(D-2)}}(\delta Q_{(D-2)} - i_\xi\Theta_{(D-1)})$$

$$= -\left\{(D-2)\sqrt{\frac{h}{f}}r^{D-3}\left[G_4\delta f\right.\right.$$

$$\left.+ 2G_{4X}f\left(\frac{(\varphi')^2}{2}\delta f - \delta X\right) - 4G_{4XX}fX\delta X\right]\Sigma_{D-2}\right\}$$

$$= -\frac{(D-2)}{\mathcal{G}}\sqrt{\frac{h}{f}}r^{D-3}\Sigma_{D-2}\delta(\mathcal{G}^2 f), \quad (66)$$

where $\mathcal{C}$ and $\Sigma^{(D-2)}$ are a Cauchy surface and its boundary respectively, Σ_{D-2} is the finite volume of the $(D-2)$-dimensional compact angular base manifold, $\mathcal{G}$ was given previously in (12), and δX denotes variation of the kinetic term for the scalar field X; namely, it equals $\delta X = -(\frac{(\varphi')^2}{2}\delta f + f\varphi'\delta\varphi')$. Here we note that (66) has two components, one of them located at infinity, and denoted as $\mathcal{H}_\infty$, and the other at the horizon, given by $\mathcal{H}_+$. Due to the fact that there are two types of black hole solutions for a time-independent scalar field, we split the following analysis into two branches, namely one of them is for the configuration (54)–(56), and we find

$$\delta\mathcal{H}_\infty = (D-2)\mathcal{G}\Sigma_{D-2}\delta M,$$

$$\delta\mathcal{H}_+ = \frac{(D-2)(D-1)\mathcal{G}\Sigma_{D-2}r_h^{D-2}}{l^2}\delta r_h$$

$$= T\delta(4\pi r_h^{D-2}\mathcal{G}\Sigma_{D-2}), \quad (67)$$

with the AdS or de Sitter radius l given previously in (55), and T is the Hawking temperature

$$T = \frac{\kappa}{2\pi}\bigg|_{r=r_h} = \frac{1}{4\pi}\left(h'(r)\sqrt{\frac{f(r)}{h(r)}}\right)\bigg|_{r=r_h} \quad (68)$$

constructed by the surface gravity κ which reads

$$\kappa = \sqrt{-\frac{1}{2}(\nabla_\mu\xi_\nu)(\nabla^\mu\xi^\nu)}, \quad (69)$$

and the timelike Killing vector $\partial_t = \xi^\mu\partial_\mu$ given by

$$T = \frac{(D-1)r_h}{4\pi l^2}. \quad (70)$$

According to the Wald procedure, the equality $\delta\mathcal{H}_\infty = \delta\mathcal{H}_+$ implies the first law of black holes thermodynamics

$$d\mathcal{M} = Td\mathcal{S}_W, \quad (71)$$

where from Eqs. (67)

$$\mathcal{M} = (D-2)\mathcal{G}\Sigma_{D-2}M = \frac{(D-2)\mathcal{G}r_h^{D-1}\Sigma_{D-2}}{l^2},$$

$$\mathcal{S}_W = 4\pi\mathcal{G}r_h^{D-2}\Sigma_{D-2}.$$

In order to have positive extensive thermodynamical quantities, we consider the case of AdS-planar black holes, where its radius l takes the form (55) with $G_2/\mathcal{G} > 0$. Just for completeness, a higher-dimensional Smarr relation [36]

$$\mathcal{M} = \left(\frac{D-2}{D-1}\right)T\mathcal{S}_W \qquad (72)$$

is satisfied.

On the other hand, and following the same steps as before, for the solution (59)–(61) we have

$$\delta\mathcal{H}_\infty = (D-2)\delta M\Sigma_{D-2} = \delta\mathcal{M},$$

$$\Rightarrow \mathcal{M} = (D-2)\Sigma_{D-2}M = \frac{C_1 r_h^{D-1}\Sigma_{D-2}}{(D-1)},$$

$$T = \frac{C_1 r_h}{2\pi(D-2)\mathcal{G}(X(r_h))},$$

where we suppose that $\mathcal{G}$ is a positive function for $r \geq r_h$, while C_1 and M are positive integration constants. Together with the above, to satisfy the first law (71)

$$dM = C_1 r_h^{D-2}\Sigma_{D-2}dr_h = Td\mathcal{S}_W$$

$$= \frac{C_1 r_h}{4\pi(D-2)\mathcal{G}(X(r_h))}d\mathcal{S}_W$$

$$\Rightarrow \mathcal{S}_W = 4\pi(D-2)\int dr_h\mathcal{G}(X(r_h))r_h^{D-3}\Sigma_{D-2}.$$

Obviously, if $\mathcal{G}(X(r_h))$ is a constant that does not depend on the location of the event horizon r_h, we have that

$$\mathcal{S}_W = 4\pi\mathcal{G}r_h^{D-2}\Sigma_{D-2},$$

fulfilling, in addition, the Smarr relation (72).

Here it is important to note that for some cases the Wald relation for the entropy (or more generally the Wald approach) might be ambiguous; this ambiguity appears in particular for Horndeski-type theories. To cure these ambiguities, the so-called solution phase-space method (SPSM) was established, which can be treated as a further elaboration of Wald's approach [37]. A recent study also shows that the definition of the black hole temperature should be modified for the Horndeski theory, and it can be explained due to different speeds of the propagations of photons and gravitons [38]. Therefore, the black holes's temperature now can be defined as follows:

$$T_{\text{bh}} = \mathcal{G}T, \qquad (73)$$

where T is the Hawking temperature (68) which is completely defined by the surface gravity κ (69). Now, from the relation at infinity $\delta\mathcal{H}_\infty$ for the time-independent scalar field solutions (54)–(56) and (59)–(61), we can infer that

$$\mathcal{M} = (D-2)\Sigma_{D-2}M,$$

where we point out that here $\mathcal{M}$ might be treated as a mass (thermodynamic) of the black hole.

On the other hand, considering the relation $\delta\mathcal{H}_+$ and taking into account the relation (73), we can write

$$\delta\mathcal{H}_+ = 4\pi(D-2)T_{\text{bh}}r_+^{D-3}\delta r_h\Sigma_{D-2} = T_{\text{bh}}\delta\bar{\mathcal{S}}_W,$$

where now $\bar{\mathcal{S}}_W$ is the entropy, which takes the following form:

$$\bar{\mathcal{S}}_W = 4\pi\Sigma_{D-2}r_h^{D-2}. \qquad (74)$$

Here we have an agreement with the corresponding relation in the framework of standard general relativity. Now we are able to write the first law of black hole thermodynamics, which now takes the form

$$dM = T_{\text{bh}}d\bar{\mathcal{S}}_W.$$

It can be easily shown that a Smarr relation

$$\mathcal{M} = \left(\frac{D-2}{D-1}\right)T_{\text{bh}}\bar{\mathcal{S}}_W$$

also holds.

V. EXPLORING THE VISCOSITY/ENTROPY DENSITY RATIO

As was mentioned in the Introduction, planar black hole configurations have a particularity which allows us to study the viscosity/entropy density (η/s) ratio, where in our case are the AdS solutions given by (54)–(56). As a first step, we perform a transverse and traceless perturbation of the metric (8) for $D > 3$ with $h = f$, which reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + 2r^2\Psi(t,r)dx_1 dx_2 + r^2\sum_{i=1}^{D-2}dx_i^2, \qquad (75)$$

where for the ansatz

$$\Psi(t,r) = \zeta t + h_{x_1 x_2}(r),$$

with ζ a constant identified as the gradient of the fluid velocity along the x_1 direction [24], yields the following linearized equation for $h_{x_1 x_2}$:

$$[\mathcal{G}r^{D-2}f(h_{x_1x_2})']' = 0. \qquad (76)$$

According to the Wald formalism [22,23] together with the method [24], the shear stress is associated with the current

$$\mathcal{J}^{x_2} = \sqrt{-g}Q^{rx_2} = \mathcal{G}r^{D-2}f(h_{x_1x_2})', \qquad (77)$$

where Q^{rx_2} is defined by (64) together with a spacelike Killing vector $\partial_{x_1} = \xi^\mu \partial_\mu$. The current (77) is conserved due to the linear equation (76). Imposing the ingoing horizon boundary condition

$$h_{x_1x_2} = \zeta\sqrt{\frac{G_4}{\mathcal{G}}}\frac{\log(r - r_h)}{4\pi T} + \cdots,$$

as well as a Taylor expansion in the near horizon region r_h, that is,

$$h = f = 4\pi T(r - r_h) + \cdots,$$

where T is the Hawking temperature (70), we have

$$\eta = \zeta\mathcal{G}\sqrt{\frac{G_4}{\mathcal{G}}}r_h^{D-2} = \frac{1}{4\pi}\sqrt{\frac{G_4}{\mathcal{G}}}\zeta s,$$

with the entropy density s given by

$$s = \frac{\mathcal{S}_W}{\Sigma_{D-2}} = 4\pi r_h^{D-2}\mathcal{G}.$$

Finally, the viscosity/entropy density ratio takes the form

$$\frac{\eta}{s} = \frac{1}{4\pi}\sqrt{\frac{G_4}{\mathcal{G}}} = \frac{1}{4\pi}\sqrt{\frac{G_4}{G_4 - 2XG_{4X}}}. \qquad (78)$$

Some comments about the above result can be made. First of all, similar to the cases [15–18], the (η/s) ratio (78) does not depend on the event horizon r_h. Nevertheless, together with the constant $1/(4\pi)$ a contribution depending on G_4 and $\mathcal{G}$ appears. In addition, the viscosity/entropy density ratio for the linear case (22), analyzed in [27], can be recovered. As the linear situation, the above result allows us to construct examples where

$$0 < \frac{G_4}{\mathcal{G}} = \frac{G_4}{G_4 - 2XG_{4X}} < 1,$$

with a suitable choice of parameters violating the KSS bound, for example, for $XG_{4X} < 0$ and $G_4 > 0$. It is worth pointing out that for the situation $G_4 = $ constant, we have the saturated situation [that is $\eta/s = 1/(4\pi)$]. Just for completeness, the viscosity/entropy density (η/s) ratio (78) also can be found following the steps described in [12,26].

VI. CONCLUSIONS AND DISCUSSION

In the present paper, we explore new hairy black hole solutions in arbitrary dimensions and a planar base manifold based on the work developed in [29]. For this case, the model is given by a special case of the Horndeski theory with shift symmetry and reflection symmetry (3) and (4) constructed by two coupling functions depending on the kinetic term X. For these configurations, we split our analysis based on the time dependence or time independence of the scalar field, obtaining a set of new configurations depending on the relation between the coupling functions G_2 and G_4. It is worth pointing out that depending on the form of the function $\mathcal{G}$ (12) constructed through the kinetic term X and the function G_4, together with Eq. (17), we can find planar black holes configurations with various asymptotic behaviors. For instance, there are asymptotically AdS or de Sitter solutions (given by the Cases C–F) for linear time-dependent scalar field configurations as well as for the time-independent case. On the other side, there are Lifshitz-type solutions and other generalizations (obtained in Cases A and B).

Together with the above, the thermodynamics is analyzed for the time-independent situation through the Wald formalism [22,23]. It is worth pointing out that the thermodynamical quantities are not unique due to the ambiguity present in the Wald approach, which becomes remarkable for this kind of scalar-tensor theory. To circumvent this inconvenience, we consider the SPSM, which can be treated as a further elaboration of Wald's method [37], redefining the black hole's temperature [38] and implying that the mass and the entropy resemble the standard general relativity extensive quantities. For both situations, the first law, as well as a higher-dimensional Smarr relation, are fulfilled.

In addition, the Wald procedure [22,23] together with the method [24] allows us to compute the shear viscosity/ entropy density (η/s) ratio, wherein our situation depends on the coupling functions G_2 and G_4, showing a new specific example where the KSS bound (2) can be violated.

Some natural extensions of this work would be for example to consider spherical or hyperbolical topologies for the event horizon, the inclusions of electromagnetic sources (see for example [35,39–42]), or even a recent extension denominated as the degenerate-higher-order-scalar-tensor theory, allowing us to add new degrees of freedom introducing a scalar field, and avoiding Ostrogradsky instability due to its degeneracy property [43–49].

APPENDIX: EQUATIONS OF MOTION WITH RESPECT TO THE METRIC

In the following section, we present the equations of motion (5) based on the computations performed in [7]:

$$\mathcal{G}^1_{\mu\nu} = -\frac{1}{2}G_{2X}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}G_2 g_{\mu\nu},$$

$$\begin{aligned}
\mathcal{G}^2_{\mu\nu} &= G_4 G_{\mu\nu} - \frac{1}{2}G_{4X}R\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}G_{4XX}[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2]\nabla_\mu\phi\nabla_\nu\phi \\
&\quad - G_{4X}\Box\phi\nabla_\mu\nabla_\nu\phi + G_{4X}\nabla_\lambda\nabla_\mu\phi\nabla^\lambda\nabla_\nu\phi + 2\nabla_\lambda G_{4X}\nabla^\lambda\nabla_{(\mu}\phi\nabla_{\nu)}\phi - \nabla_\lambda G_{4X}\nabla^\lambda\phi\nabla_\mu\nabla_\nu\phi \\
&\quad + g_{\mu\nu}\left\{ G_{4XX}\nabla_\alpha\nabla_\lambda\phi\nabla_\beta\nabla^\lambda\phi\nabla^\alpha\phi\nabla^\beta\phi + \frac{1}{2}G_{4X}[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] \right\} \\
&\quad + 2[G_{4X}R_{\lambda(\mu}\nabla_{\nu)}\phi\nabla^\lambda\phi - \nabla_{(\mu}G_{4X}\nabla_{\nu)}\phi\Box\phi] - g_{\mu\nu}[G_{4X}R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - \nabla_\lambda G_{4X}\nabla^\lambda\phi\Box\phi] \\
&\quad + G_{4X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi - G_{4XX}\nabla^\alpha\phi\nabla_\alpha\nabla_\mu\phi\nabla^\beta\phi\nabla_\beta\nabla_\nu\phi,
\end{aligned}$$

where the equations are given by

$$\mathcal{E}_{\mu\nu} = \sum_{i=1}^{2}\mathcal{G}^i_{\mu\nu} = 0.$$

[1] A. G. Riess *et al.* (Supernova Search Team), Astron. J. **116**, 1009 (1998).
[2] S. Perlmutter *et al.* (Supernova Cosmology Project), Astrophys. J. **517**, 565 (1999).
[3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. Lett. **116**, 061102 (2016).
[4] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[5] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D **80**, 064015 (2009).
[6] C. Charmousis, Lect. Notes Phys. **892**, 25 (2015).
[7] T. Kobayashi, M. Yamaguchi, and J. 'i. Yokoyama, Prog. Theor. Phys. **126**, 511 (2011).
[8] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[9] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[10] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[11] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).
[12] D. T. Son and A. O. Starinets, J. High Energy Phys. 09 (2002) 042.
[13] P. Kovtun, D. T. Son, and A. O. Starinets, J. High Energy Phys. 10 (2003) 064.
[14] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
[15] A. Buchel and J. T. Liu, Phys. Rev. Lett. **93**, 090602 (2004).
[16] A. Buchel, Phys. Lett. B **609**, 392 (2005).
[17] P. Benincasa, A. Buchel, and R. Naryshkin, Phys. Lett. B **645**, 309 (2007).
[18] K. Landsteiner and J. Mas, J. High Energy Phys. 07 (2007) 088.
[19] N. Iqbal and H. Liu, Phys. Rev. D **79**, 025023 (2009).
[20] R. G. Cai, Z. Y. Nie, and Y. W. Sun, Phys. Rev. D **78**, 126007 (2008).
[21] R. G. Cai, Z. Y. Nie, N. Ohta, and Y. W. Sun, Phys. Rev. D **79**, 066004 (2009).
[22] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).
[23] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[24] Z. Y. Fan, Phys. Rev. D **97**, 066013 (2018).
[25] Y. Kats and P. Petrov, J. High Energy Phys. 01 (2009) 044.
[26] M. Brigante, H. Liu, R. C. Myers, S. Shenker, and S. Yaida, Phys. Rev. D **77**, 126006 (2008).
[27] X. H. Feng, H. S. Liu, H. Lü, and C. N. Pope, J. High Energy Phys. 11 (2015) 176.
[28] F. A. Brito and F. F. Santos, Europhys. Lett. **129**, 50003 (2020).
[29] T. Kobayashi and N. Tanahashi, Prog. Theor. Exp. Phys. **2014**, 73E02 (2014).
[30] E. Babichev and C. Charmousis, J. High Energy Phys. 08 (2014) 106.
[31] A. Anabalon, A. Cisterna, and J. Oliva, Phys. Rev. D **89**, 084050 (2014).
[32] M. Rinaldi, Phys. Rev. D **86**, 084048 (2012).
[33] M. Bravo-Gaete and M. Hassaine, Phys. Rev. D **90**, 024008 (2014).
[34] M. Bravo-Gaete and M. Hassaine, Phys. Rev. D **89**, 104028 (2014).
[35] M. Bravo Gaete, S. Gomez, and M. Hassaine, Eur. Phys. J. C **79**, 200 (2019).
[36] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973); **30**, 521(E) (1973).
[37] K. Hajian and M. M. Sheikh-Jabbari, Phys. Rev. D **93**, 044074 (2016).

[38] K. Hajian, S. Liberati, M. M. Sheikh-Jabbari, and M. H. Vahidinia, Phys. Lett. B **812**, 136002 (2021).

[39] J. F. Plebánski, *Lectures on Non-Linear Electrodynamics*, Monograph of the Niels Bohr Institute (Nordita, Copenhagen, 1968).

[40] A. Alvarez, E. Ayón-Beato, H. A. González, and M. Hassaïne, J. High Energy Phys. 06 (2014) 041.

[41] M. M. Stetsko, arXiv:2012.14902.

[42] M. M. Stetsko, Gen. Relativ. Gravit. **53**, 2 (2021).

[43] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, J. High Energy Phys. 12 (2016) 100.

[44] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, J. Cosmol. Astropart. Phys. 07 (2016) 033.

[45] E. Babichev, C. Charmousis, A. Cisterna, and M. Hassaine, J. Cosmol. Astropart. Phys. 06 (2020) 049.

[46] O. Baake, M. F. Bravo Gaete, and M. Hassaine, Phys. Rev. D **102**, 024088 (2020).

[47] O. Baake and M. Hassaine, Eur. Phys. J. C **81**, 642 (2021).

[48] O. Baake, C. Charmousis, M. Hassaine, and M. San Juan, J. Cosmol. Astropart. Phys. 06 (2021) 021.

[49] F. F. Santos, arXiv:2010.10942.