

Equatorially symmetric configurations of two Kerr-Newman black holes

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(Received 9 November 2020; accepted 2 January 2022; published 12 January 2022)

In this paper, we employ the general equatorially symmetric two-soliton solution of the Einstein-Maxwell equations for elaborating two physically meaningful configurations describing a pair of equal Kerr-Newman corotating black holes separated by a massless strut. The first configuration is characterized by opposite magnetic charges of its constituents, while in the second configuration the black holes carry equal electric and opposite magnetic charges, thus providing a nontrivial example of a binary dyonic black-hole system. The thermodynamic properties of these binary configurations are studied and the first law of thermodynamics taking correctly into account the magnetic field contribution is formulated for each case.

DOI: [10.1103/PhysRevD.105.024036](https://doi.org/10.1103/PhysRevD.105.024036)

I. INTRODUCTION

In the paper [1], the general six-parameter two-soliton solution of the Einstein-Maxwell equations possessing equatorial symmetry was constructed with the aid of Sibgatullin's integral method [2]. It is able to describe the exterior gravitational and electromagnetic fields of compact objects, as well as of the binary systems of identical black holes or hyperextreme sources. While the former application of that solution is better known in the literature (see, e.g., Refs. [3,4]), the latter possibility of the solution's usage for the analysis of the black hole binary systems has been scantily exploited only recently in its pure vacuum sector, and therefore it would be certainly of interest to make use of the solution [1] (henceforth referred to as the MMR solution) for obtaining its physically interesting generic electrovacuum subfamilies representing two equal (up to the sign of the charges) Kerr-Newman (KN) black holes [5] separated by a massless strut [6]. The main objective of the present paper will be the derivation and analysis of a nontrivial binary configuration of dyonic KN black holes carrying equal electric and opposite magnetic charges and formulation for it of the first law of thermodynamics.

Though our main results which will be discussed in the present paper were obtained more than a year ago, their publication was postponed due to the paper [7] criticizing the extension of the well-known Smarr mass formula [8] to the case of dyonic black holes [9], and the criticism has been refuted only recently [10] by demonstrating that the model of the dyonic KN solution worked out in [7] was frankly unphysical. However, lately an effort has been made [11] to use the binary configurations of dyonic KN black holes for rehabilitating the approach of the paper [7]

to the Smarr formula, so we find it necessary and instructive to briefly comment in the discussion section of our paper on the contradictions of the preprint [11].

The plan of the present paper is as follows. In the next section we will write down the MMR solution in a form simpler than the original one thanks to some technical improvements in the construction procedure that have been found over the years. This representation is fundamental for the subsequent working out the particular and generic cases of our interest. In Sec. III we consider a binary configuration of corotating KN black holes endowed with opposite magnetic charges. This particular binary system will permit us to present the corresponding magnetic version of the Smarr formula and show that it practically does not differ from the usual mass relation in the case of opposite electric charges. Here we also derive the first law of thermodynamics for that binary system and find the corresponding expression of the thermodynamic length. In Sec. IV we show how the binary configuration of corotating KN black holes with equal electric and opposite magnetic charges is contained in the general MMR solution and analyze its thermodynamical properties, including the corresponding first law of thermodynamics and correct account for the magnetic contribution in it. Discussion of the results obtained and concluding remarks can be found in Sec. V, where in particular we touch an interesting question of the nonuniqueness of the binary systems of KN sources with the same masses, angular momenta and charges.

II. ENHANCED FORM OF THE MMR SOLUTION

The MMR solution was constructed from the expressions of the Ernst complex potentials [12] on the upper part of the symmetry axis (the axis data) of the form

$$e(z) = \frac{(z - m - ia)(z + ib) + k}{(z + m - ia)(z + ib) + k}, \quad f(z) = \frac{qz + ic}{(z + m - ia)(z + ib) + k}, \quad (1)$$

where six arbitrary real parameters m, a, b, k, q and c are related to the first two nonzero mass, angular momentum, electric and magnetic multipoles [13–15] by the formulas

$$\begin{aligned} M_0 &= m, & M_2 &= -m(k + a^2), & J_1 &= ma, & J_3 &= -m[k(2a - b) + a^3], \\ Q_0 &= q, & Q_2 &= -q(k + b^2) - (a - b)(c + aq), & B_1 &= c + q(a - b), \\ B_3 &= -c(k + b^2) - (a - b)[q(a^2 + b^2 + 2k) + ac]. \end{aligned} \quad (2)$$

The position of the sources on the symmetry axis is defined by four roots α_i of the algebraic equation

$$e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0 \quad (3)$$

(a bar over a symbol means complex conjugation), and so α_i have the form

$$\begin{aligned} \alpha_1 &= -\alpha_4 = \frac{1}{2}(\kappa_+ + \kappa_-), & \alpha_2 &= -\alpha_3 = \frac{1}{2}(\kappa_+ - \kappa_-), \\ \kappa_{\pm} &= \sqrt{m^2 - a^2 - b^2 - q^2 - 2k \pm 2d}, & d &= \sqrt{(k + ab)^2 - m^2b^2 + c^2}. \end{aligned} \quad (4)$$

The form of the Ernst potentials \mathcal{E} and Φ in the whole (ρ, z) space obtainable from the axis data (1) is given by the expressions

$$\begin{aligned} \mathcal{E} &= (A - B)/(A + B), & \Phi &= C/(A + B), \\ A &= \kappa_+^2 \{ [(d - ab - k)\kappa_-^2 + k(m^2 - q^2) - (aq + c)(bq - c)](R_+r_- + R_-r_+) \\ &\quad + i\kappa_- [(a - b)(ab + k - d) - m^2b + qc](R_+r_- - R_-r_+) \} \\ &\quad + \kappa_-^2 \{ [(d + ab + k)\kappa_+^2 - k(m^2 - q^2) + (aq + c)(bq - c)](R_+r_+ + R_-r_-) \\ &\quad - i\kappa_+ [(a - b)(ab + k + d) - m^2b + qc](R_+r_+ - R_-r_-) \} \\ &\quad - 4d[k(m^2 - q^2) - (aq + c)(bq - c)](R_+R_- + r_+r_-), \\ B &= m\kappa_+\kappa_- \{ d[\kappa_+\kappa_-(R_+ + R_- + r_+ + r_-) - (m^2 - a^2 + b^2 - q^2)(R_+ + R_- - r_+ - r_-)] \\ &\quad + ibd[(\kappa_+ + \kappa_-)(R_+ - R_-) + (\kappa_+ - \kappa_-)(r_- - r_+)] \\ &\quad + i[b(m^2 - a^2) - ak - qc][(\kappa_+ + \kappa_-)(r_+ - r_-) + (\kappa_+ - \kappa_-)(R_- - R_+)] \}, \\ C &= qB/m + \kappa_+\kappa_-(bq - c)[2d(b - a)(R_+ + R_- - r_+ - r_-) \\ &\quad - i\kappa_+(d + ab + k)(R_+ - R_- - r_+ + r_-) - i\kappa_-(d - ab - k)(R_+ - R_- + r_+ - r_-)], \\ R_{\pm} &= \sqrt{\rho^2 + \left[z \pm \frac{1}{2}(\kappa_+ + \kappa_-) \right]^2}, & r_{\pm} &= \sqrt{\rho^2 + \left[z \pm \frac{1}{2}(\kappa_+ - \kappa_-) \right]^2}, \end{aligned} \quad (5)$$

and these formulas are presented in a simpler form than in the original paper [1]. The corresponding metric functions f, γ and ω from the Weyl-Papapetrou stationary axisymmetric line element [16]

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2 \quad (6)$$

have the following form:

$$\begin{aligned}
 f &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A} + \bar{B})}, & e^{2\gamma} &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16d^2\kappa_+^4\kappa_-^4 R_+ R_- r_+ r_-}, & \omega &= -\frac{\text{Im}[G(\bar{A} + \bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}}, \\
 G &= -2(z - ia)B - qC + \kappa_+\kappa_-^2[d(2m^2 - q^2) - 2m^2b^2 + c^2](R_-r_- - R_+r_+) \\
 &\quad + \kappa_+^2\kappa_- \{ [d(2m^2 - q^2) + 2m^2b^2 - c^2](R_-r_+ - R_+r_-) - i\kappa_- (2m^2b - qc)(R_+ - R_-) \\
 &\quad \times (r_+ - r_-) \} + i \{ (a-b)[k(2m^2 - q^2) - (aq+c)(bq-c)] - m^2q(bq-c) \} \\
 &\quad \times [\kappa_-^2(R_+r_+ + R_-r_-) - \kappa_+^2(R_+r_- + R_-r_+) + 4d(R_+R_- + r_+r_-)] + m\kappa_+\kappa_- \\
 &\quad \times \{ 2k[\kappa_-(d - ab - k)(R_- + r_- - R_+ - r_+) + \kappa_+(d + ab + k)(R_- - r_- - R_+ + r_+)] \\
 &\quad + c(c - bq)[(\kappa_+ - \kappa_-)(R_- - R_+) - (\kappa_+ + \kappa_-)(r_- - r_+)] \\
 &\quad + 2id[2k(a-b) + q(c - bq)](R_- - r_- + R_+ - r_+) \}, \\
 I &= q(A+B) + icB/m - [z - i(a-b)]C + m\kappa_+\kappa_-[\kappa_-(dq - bc)(R_-r_- - R_+r_+) \\
 &\quad + \kappa_+(dq + bc)(R_-r_+ - R_+r_-) - 2id(bq - c)(R_+R_- - r_+r_- + 2\kappa_+\kappa_-)] \\
 &\quad - imd(bq + c)[\kappa_-^2(R_+r_+ + R_-r_-) + \kappa_+^2(R_+r_- + R_-r_+)] + im[bq(m^2 - a^2) \\
 &\quad - c(b^2 + q^2) - k(aq + c)][\kappa_-^2(R_+r_+ + R_-r_-) - \kappa_+^2(R_+r_- + R_-r_+)] \\
 &\quad - 2imd[(bq - c)(m^2 - a^2 + b^2 + q^2) - 2kq(a-b)](R_+R_- + r_+r_-) - \kappa_+\kappa_- \\
 &\quad \times \{ \kappa_+[(d + ab + k)(ac - abq - kq) + 2m^2b(bq - c)](R_- - r_- - R_+ + r_+) \\
 &\quad + \kappa_-[(d - ab - k)(ac - abq - kq) - 2m^2b(bq - c)](R_- + r_- - R_+ - r_+) \\
 &\quad + 2id[(a-b)(ac - abq - kq) + 2m^2(bq - c)](R_- - r_- + R_+ - r_+) \}, \tag{7}
 \end{aligned}$$

while the nonzero components of the electromagnetic four-potential are defined as

$$A_t = -\text{Re}\left(\frac{C}{A+B}\right), \quad A_\varphi = \text{Im}\left(\frac{I}{A+B}\right). \tag{8}$$

It may be noted that the expression of the metric function ω is determined by only two additional potentials G and I , in contradistinction to the three such potentials in the original paper [1], which obviously improves the presentation of the MMR solution.

Due to its multipole structure (2) involving physically important multipole moments, the MMR metric is able to describe the exterior field of compact massive objects endowed with electric charge and magnetic dipole moment, and in this relation its most recent application was considered in the paper [4]. At the same time, the above formulas also contain, as special subfamilies, the solutions for two equal corotating KN sources, black holes or hyperextreme objects, and we now turn to the consideration of these binary configurations, mainly concentrating on the black-hole systems.

III. TWO EQUAL COROTATING KN BLACK HOLES WITH OPPOSITE CHARGES

After the publication of our work on two corotating identical Kerr sources [17] it was of course logical for us to turn our attention to searching the analogous binary equatorially symmetric configurations of KN sub- and

hyperextreme constituents. It appears that the MMR solution provides the simplest way to identify and describe the latter configurations because these arise from the formulas of the previous section by just imposing the condition $\omega = 0$ on the intermediate part of the symmetry axis (the axis condition). While treating the problem of two KN sources separated by a massless strut, it is advantageous to reparametrize the quantities α_i and the axis data (1) in the form

$$\alpha_1 = -\alpha_4 = \frac{1}{2}R + \sigma, \quad \alpha_2 = -\alpha_3 = \frac{1}{2}R - \sigma, \tag{9}$$

and

$$\begin{aligned}
 e(z) &= \frac{z^2 - 2(m+ia)z - \frac{1}{4}R^2 + 2(m^2 - a^2 - q^2) - \sigma^2 + i\delta}{z^2 + 2(m-ia)z - \frac{1}{4}R^2 + 2(m^2 - a^2 - q^2) - \sigma^2 - i\delta}, \\
 f(z) &= \frac{2qz + ib}{z^2 + 2(m-ia)z - \frac{1}{4}R^2 + 2(m^2 - a^2 - q^2) - \sigma^2 - i\delta}, \tag{10}
 \end{aligned}$$

with

$$\begin{aligned}
 \delta &= \epsilon\sqrt{(\sigma^2 - m^2 + a^2 + q^2)[R^2 - 4(m^2 - a^2 - q^2)] + b^2}, \\
 \epsilon &= \pm 1, \tag{11}
 \end{aligned}$$

where the set of six arbitrary parameters is now comprised of m , a , R , σ , q , and b . Note that the idea of the reparametrization consists in introducing the roots of Eq. (3) explicitly

into the axis data, and one can see that formulas (10)–(11) reduce to the axis data for two equal corotating Kerr sources considered in [17] in the vacuum limit ($q = b = 0$). The substitution that casts the axis data (1) into the form (10) is the following:

$$\begin{aligned} m &\rightarrow 2m, & a &\rightarrow \frac{4ma - \delta}{2m}, & b &\rightarrow -\frac{\delta}{2m}, \\ k &\rightarrow \frac{4m(ms + a\delta) - \delta^2}{4m^2}, & q &\rightarrow 2q, & c &\rightarrow b, \\ s &\equiv 2(m^2 - a^2 - q^2) - \frac{1}{4}R^2 - \sigma^2. \end{aligned} \quad (12)$$

We also notice that for some calculations it may be advantageous to use δ as an arbitrary parameter, in which case the expression of σ in terms of δ following from (11) has the form

$$\sigma = \sqrt{m^2 - a^2 - q^2 + \frac{\delta^2 - b^2}{R^2 - 4(m^2 - a^2 - q^2)}}. \quad (13)$$

The subfamily of the MMR spacetime representing two equal KN sources separated by a massless strut is segregated from the general case by the condition

$$\omega\left(\rho = 0, |z| \leq \frac{1}{2}R - \text{Re}(\sigma)\right) = 0, \quad (14)$$

which ensures that the constituents do not overlap. The quickest way to get the explicit form of (14) is to use the formulas for ω from the previous section and in the axis expression of ω calculated for $\rho = 0, |z| \leq \alpha_2$ to perform the parameter change (12) supplemented with the substitutions

$$\kappa_+ \rightarrow R, \quad \kappa_- \rightarrow 2\sigma, \quad d \rightarrow (R^2 - 4\sigma^2)/4. \quad (15)$$

Unlike in the vacuum case of corotating Kerr sources [17] where the condition (14) results in a quadratic equation for the quantity σ , in the case of the reparametrized MMR solution the axis condition leads to a biquadratic equation for σ that can be readily solved yielding

$$\begin{aligned} \sigma^2 &= \frac{1}{32a^2(R^2 + 2mR + 4a^2 + 2q^2)^2} \left(-D + 2(R^2 + 2mR + 4a^2 + 2q^2)\{(R^2 - 4m^2 + 4a^2 + 4q^2) \right. \\ &\quad \times [(R^2 + 2mR + 4q^2)^2 + 4a^2(R^2 + 4m^2 - 4a^2 + 4q^2) + 24aqb] + 8a[(mR + 2m^2 - q^2)(R^2a + 4qb) + ab^2]\} \\ &\quad \left. \pm [(R + 2m)(R^2 + 2mR + 4a^2 + 4q^2) + 8ma^2]\sqrt{D(R^2 + 4mR + 4m^2 + 4a^2)} \right), \\ D &= (R^2 - 4m^2 + 4a^2 + 4q^2)^2[(R^2 + 2mR + 4q^2)^2 + 4a^2(R^2 + 8q^2)] \\ &\quad + 32ab(R^2 + 2mR + 4a^2 + 2q^2)[q(R^2 - 4m^2 + 4a^2 + 4q^2) + ab], \end{aligned} \quad (16)$$

and this determines the subfamily of equatorially symmetric configurations of two KN sources, black holes or naked singularities, kept apart from each other by a massless strut.

With the reparametrization made, the moments $M_0, J_1, Q_0,$ and B_1 defining, respectively, the total mass, total angular momentum, total charge and magnetic dipole moment of the binary system take the form

$$\begin{aligned} M_0 &\equiv M_T = 2m, & J_1 &\equiv J_T = 4ma - \delta, \\ Q_0 &\equiv Q_T = 2q, & B_1 &\equiv \mu = 4aq + b, \end{aligned} \quad (17)$$

so that we can further precise the interpretation of the KN sources in the subfamily (16) as carrying equal electric and opposite magnetic charges.

The particular case for which one might expect simplification of the expression of σ in (16) is the absence of electric charges ($q = 0$), when only two opposite magnetic charges are present. In what follows we shall elaborate on this case in more detail, restricting our analysis exclusively to the

black-hole configurations corresponding to real valued σ . The case of nonzero q and vanishing magnetic charges constitutes a specialization of the general subfamily of binary systems which will be considered later, and one may recall in this respect that the electric charge in principle can be readily introduced into a binary system of Kerr black holes via the well-known Ernst-Harrison transformation [12,18].

After setting $q = 0$, the axis data (1) take the form

$$\begin{aligned} e(z) &= \frac{z^2 - 2(m + ia)z - \frac{1}{4}R^2 + 2(m^2 - a^2) - \sigma^2 + i\delta}{z^2 + 2(m - ia)z - \frac{1}{4}R^2 + 2(m^2 - a^2) - \sigma^2 - i\delta}, \\ f(z) &= \frac{ib}{z^2 + 2(m - ia)z - \frac{1}{4}R^2 + 2(m^2 - a^2) - \sigma^2 - i\delta}, \end{aligned} \quad (18)$$

and for our purposes we must write down the corresponding Ernst potentials and metric functions using the formulas of the previous section together with the substitutions (12) and (15). In this way we obtain for the Ernst potentials the expressions

$$\begin{aligned}
 \mathcal{E} &= (A - B)/(A + B), \quad \Phi = C/(A + B), \\
 A &= R^2 \{ [4m^2(m^2 - 2\sigma^2) - (R^2 + 4a^2)(a^2 - \sigma^2) + 4ma\delta](R_+r_- + R_-r_+) - 2i\sigma[a(R^2 - 4m^2 + 4a^2) - 2m\delta](R_+r_- - R_-r_+) \} \\
 &\quad + 4\sigma^2 \{ [2m^2(R^2 - 2m^2) - (R^2 - 4a^2)(a^2 + \sigma^2) - 4ma\delta](R_+r_+ + R_-r_-) - 2iR[2a(m^2 - a^2 - \sigma^2) + m\delta](R_+r_+ - R_-r_-) \} \\
 &\quad + (R^2 - 4\sigma^2)[(R^2 + 4a^2)(a^2 + \sigma^2) - 4m^4 - 4ma\delta](R_+R_- + r_+r_-), \\
 B &= 2R\sigma \{ (R^2 - 4\sigma^2)[mR\sigma(R_+ + R_- + r_+ + r_-) - (2m^3 - 2ma^2 + a\delta)(R_+ + R_- - r_+ - r_-)] \\
 &\quad + i[ma(R^2 + 4\sigma^2 - 8m^2 + 8a^2) - 2(m^2 + a^2)\delta][(R - 2\sigma)(R_- - R_+) + (R + 2\sigma)(r_+ - r_-)] \\
 &\quad + iR\sigma\delta[(R - 2\sigma)(R_- - R_+) - (R + 2\sigma)(r_+ - r_-)] \}, \\
 C &= 2Rb\sigma[a(R^2 - 4\sigma^2)(R_+ + R_- - r_+ - r_-) + 2iR(m^2 - a^2 - \sigma^2)(R_+ - R_- - r_+ + r_-) \\
 &\quad + i\sigma(R^2 - 4m^2 + 4a^2)(R_+ - R_- + r_+ - r_-)], \\
 R_{\pm} &= \sqrt{\rho^2 + \left[z \pm \left(\frac{1}{2}R + \sigma \right) \right]^2}, \quad r_{\pm} = \sqrt{\rho^2 + \left[z \pm \left(\frac{1}{2}R - \sigma \right) \right]^2}, \tag{19}
 \end{aligned}$$

while for the metric functions we get

$$\begin{aligned}
 f &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16R^4\sigma^4(R^2 - 4\sigma^2)^2R_+R_-r_+r_-}, \quad \omega = -\frac{\text{Im}[G(\bar{A} + \bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}}, \\
 G &= 4R\sigma^2[2(R^2 + 4a^2)(2m^2 - a^2 - \sigma^2) - 8m^4 - b^2](R_-r_- - R_+r_+) \\
 &\quad + 2R^2\sigma \{ [2(R^2 - 8m^2 + 4a^2)(a^2 + \sigma^2) + 8m^4 + b^2](R_-r_+ - R_+r_-) + 8im\delta\sigma(R_+ - R_-)(r_+ - r_-) \} \\
 &\quad + 2ia[2(R^2 + 4a^2)(a^2 + \sigma^2) - 8m^4 + b^2 - 8ma\delta][R^2(R_+ - r_+)(r_- - R_-) - 4\sigma^2(R_+ - r_-)(r_+ - R_-)] \\
 &\quad - 2R(\sigma/m) \{ [(R^2 + 4a^2)(a^2 + \sigma^2) - 4m^4 + b^2 - 4ma\delta][2R(m^2 - a^2 - \sigma^2)(R_- - R_+ - r_- + r_+) \\
 &\quad + \sigma(R^2 - 4m^2 + 4a^2)(R_- - R_+ + r_- - r_+) + ia(R^2 - 4\sigma^2)(R_- + R_+ - r_- - r_+)] \\
 &\quad - 2m^2b^2[(R - 2\sigma)(R_- - R_+) - (R + 2\sigma)(r_- - r_+)] \} - 2zB + i(4a - \delta/m)B, \\
 I &= 2Rb\sigma[R\delta(R_+r_- - R_-r_+) - 2\delta\sigma(R_+r_+ - R_-r_-) + im(R^2 - 4\sigma^2)(R_+R_- - r_+r_- + 4R\sigma)] \\
 &\quad - (i/2)mB_0(R^2 - 4\sigma^2)[R^2(R_+r_- + R_-r_+) + 4\sigma^2(R_+r_+ + R_-r_-)] + (i/2)b[m(R^2 - 8m^2 + 8a^2 + 4\sigma^2) - 4a\delta] \\
 &\quad \times [R^2(R_+r_- + R_-r_+) - 4\sigma^2(R_+r_+ + R_-r_-)] + 2ib(R^2 - 4\sigma^2)[2m(m^2 - a^2) + a\delta](R_+R_- + r_+r_-) \\
 &\quad - Rb(\sigma/m) \{ 2R[(4ma - \delta)(m^2 - a^2 - \sigma^2) + 4m^2\delta](R_- - R_+ - r_- + r_+) + \sigma[(4ma - \delta)(R^2 - 4m^2 + 4a^2) \\
 &\quad - 16m^2\delta](R_- - R_+ + r_- - r_+) - i(R^2 - 4\sigma^2)(8m^3 - 4ma^2 + a\delta)(R_- + R_+ - r_- - r_+) \} + ibB/(2m) - (z - 2ia)C. \tag{20}
 \end{aligned}$$

Formulas (8) for the electromagnetic potentials A_t and A_ϕ do not change.

The key point in the simplification of σ in (16) is finding the form of the parameter b in terms of the individual magnetic charge β of one of the black-hole constituents determined by the formula

$$\beta = \frac{1}{2} \int_H \omega A_{t,z} dz, \tag{21}$$

where both functions ω and A_t in (21) must be evaluated on the horizon. Of course, in view of the equatorial symmetry of our binary configuration it is sufficient to calculate the physical characteristics of only one of the black holes. The tedious but

straightforward calculations eventually lead to the following rather simple relation

$$b = -\frac{\beta R(R^2 - 4m^2 + 4a^2)[(R + 2m)^2 + 4a^2]}{(R^2 + 2mR + 4a^2)[(R + 2m)^2 + 4a^2] - 8a^2\beta^2}, \tag{22}$$

where β is the magnetic charge of the lower black hole whose horizon is the rod $-\frac{1}{2}R - \sigma \leq z \leq -\frac{1}{2}R + \sigma$ located on the symmetry axis (see Fig. 1), the magnetic charge of the upper constituent being $-\beta$.

The substitution of (22) into (16) converts the radicand in the latter formula into a perfect square, so that choosing in (16) the minus sign we arrive at the final expression for σ in the form

$$\sigma = \sqrt{m^2 - a^2 + \frac{(R^2 - 4m^2 + 4a^2)\{4a^2[m - \beta_0(R + 2m)]^2 - R^2\beta^2\}}{[R^2 + 2mR + 4a^2(1 - 2\beta_0)]^2}}, \quad (23)$$

while for δ , taking into account (22) and (23), we get from (11)

$$\delta = \frac{2a(R^2 - 4m^2 + 4a^2)[m - \beta_0(R + 2m)]}{R^2 + 2mR + 4a^2(1 - 2\beta_0)},$$

$$\beta_0 \equiv \frac{\beta^2}{(R + 2m)^2 + 4a^2}, \quad (24)$$

where we have introduced a dimensionless parameter β_0 for writing down the results in a more concise form.

Since the magnetically charged KN black holes are equal and corotating, their individual Komar [19] masses and angular momenta are just halves of the respective total quantities M_T and J_T , so that m is the mass of each black hole, and for the individual angular momenta J we get from (17) and (24)

$$J = \frac{a[(R + 2m)^2 + 4a^2][m + \beta_0(R - 2m)]}{R^2 + 2mR + 4a^2(1 - 2\beta_0)}. \quad (25)$$

The other physical characteristics that might be of interest to us are the horizon's area \mathcal{A} , the surface gravity κ , horizon's angular velocity Ω and the magnetic potential Φ_m ,

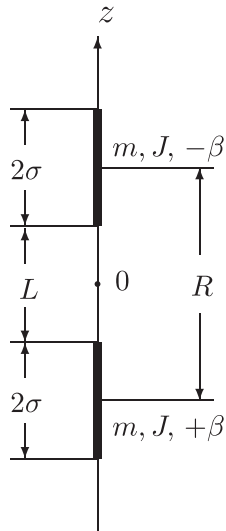


FIG. 1. Location of two equal corotating KN black holes with opposite magnetic charges on the symmetry axis. $L = R - 2\sigma$ is the coordinate length of the strut.

which all can be calculated by means of the formulas of the paper [20], taking into account the relation of Φ_m to the electric potential Φ_e of the associated problem [9]. Assuming the validity of the Bekenstein-Hawking formula $S = \mathcal{A}/4$ between the entropy S and horizon's area \mathcal{A} [21,22], and also recalling that the Hawking temperature T is related to the surface gravity as $T = \kappa/(2\pi)$, we give below the formulas for S , T , Φ_m and Ω calculated for the lower black hole of our particular binary configuration:

$$S = \frac{\sigma}{2T} = \frac{\pi[(R + 2m)^2 + 4a^2]\lambda_0}{(R + 2\sigma)[R^2 + 2mR + 4a^2(1 - 2\beta_0)]},$$

$$\Omega = \frac{a\nu_0}{[(R + 2m)^2 + 4a^2]\lambda_0},$$

$$\Phi_m = \frac{\beta(R^2 - 4m^2 + 4a^2)[(R + 2m)(m + \sigma) - 2a^2]}{[(R + 2m)^2 + 4a^2]\lambda_0}, \quad (26)$$

where

$$\lambda_0 = 2m[(R + 2m)(m + \sigma) - 2a^2]$$

$$- \beta_0[(R + 2m)(R^2 - 4m^2) + 8a^2(R + m + \sigma)],$$

$$\nu_0 = [R^2 + 2\sigma(R + 2\sigma) - 4m^2 + 4a^2]$$

$$\times [R^2 + 2mR + 4a^2(1 - 2\beta_0)]$$

$$- 4m(R^2 - 4m^2 + 4a^2)[m - (R + 2m)\beta_0], \quad (27)$$

and these thermodynamical quantities verify the Smarr mass formula [8]

$$m = 2TS + 2\Omega J + \Phi_m \beta, \quad (28)$$

which also holds for the upper black hole whose magnetic potential is $-\Phi_m$ and magnetic charge $-\beta$.

To the above thermodynamic characteristics we must add the expressions of the interaction force \mathcal{F} [6] and thermodynamic length ℓ [23] which were shown to enter explicitly into the first law of thermodynamics in the static and stationary vacuum [23–26] and electrovacuum cases [27,28]. It is remarkable that both \mathcal{F} and ℓ are defined in terms of the value γ_0 of the metric function γ on the strut, and whereas the formula for \mathcal{F} is well known, the analogous formula for ℓ , namely, $\ell = L \exp(\gamma_0)$, where L is the coordinate length of the strut, has been discovered only recently [27]. The form of \mathcal{F} and ℓ in our case has been found to be

$$\mathcal{F} = \frac{[(R+2m)^2 - 4a^2](m^2 - 4a^2\beta_0^2) + \beta_0[R^2(R+2m)^2 + 16a^2(m^2 - a^2)]}{(R^2 - 4m^2 + 4a^2)[(R+2m)^2 + 4a^2]},$$

$$\ell = \frac{(R^2 - 4m^2 + 4a^2)^2[(R+2m)^2 + 4a^2]}{(R+2\sigma)[R^2 + 2mR + 4a^2(1 - 2\beta_0)]^2}, \quad (29)$$

so that the corresponding first law of thermodynamics can be written, following the procedure described in [26,27], in the form

$$dM_T = 2TdS + 2\Omega dJ + 2\Phi_m d\beta - \ell d\mathcal{F}, \quad M_T = 2m. \quad (30)$$

It is worth noting that the case of corotating KN black holes with opposite electric charges is trivially obtainable from the above configuration of magnetically charged KN black holes by formally changing in (19) the electromagnetic Ernst potential Φ to $i\Phi$, in which case b becomes an electric dipole parameter, while β becomes the electric charge. Moreover, the transformation $b \rightarrow b - ip$, $b^2 \rightarrow b^2 + p^2$ in the formulas (19), (43), and (22) leads, after the analogous complex extension of the magnetic charge parameter $\beta \rightarrow \beta - iq$, $\beta^2 \rightarrow \beta^2 + q^2$ to the case of two dyonic KN black holes endowed with opposite electric and magnetic charges, and then the Smarr mass relation takes the form discussed in [9]. In the paper [29] it was clarified that in order to treat correctly the solutions involving both electric and magnetic charges it is best to identify first the particular case in which only the electric charges are present and then apply the extension parameter procedure.

Our purely magnetic solution considered in this section illustrates well that the solution with solely magnetic charges is equally suitable as a starting point for consistently treating the more general cases.

IV. TWO COROTATING DYONIC KN BLACK HOLES WITH EQUAL ELECTRIC AND OPPOSITE MAGNETIC CHARGES

We now turn to the general 5-parameter subfamily of the MMR spacetime representing a pair of KN black holes with a separating strut, and our objective is to add a nonzero net charge $2q$ to the solution considered in the previous section and get the general expression for σ in (16) in terms of q and β . Note that the case of two KN black holes with equal electric and opposite magnetic arbitrary charges has not been considered before and it represents a physically and mathematically nontrivial example of a binary dyonic configuration.

To fulfill our goal, we must first reparametrize the entire MMR solution using the transformation formulas (12) and (15). The expressions of the Ernst potentials \mathcal{E} and Φ thus obtained are the following:

$$\mathcal{E} = (A - B)/(A + B), \quad \Phi = C/(A + B),$$

$$A = R^2 \{ [(R^2 + 4a^2)(\sigma^2 - a^2) - 4(m^2 - q^2)(2\sigma^2 - m^2 + q^2) + 4a(qb + m\delta)] \times (R_+ r_- + R_- r_+) - 2i\sigma[a(R^2 - 4\Delta) - 2(qb + m\delta)](R_+ r_- - R_- r_+) \} + 4\sigma^2 \{ [2(m^2 - q^2)(R^2 - 2m^2 + 2q^2) - (R^2 - 4a^2)(\sigma^2 + a^2) - 4a(qb + m\delta)] \times (R_+ r_+ + R_- r_-) + 2iR[2a(\sigma^2 - \Delta) - qb - m\delta](R_+ r_+ - R_- r_-) \} + (R^2 - 4\sigma^2)[(R^2 + 4a^2)(\sigma^2 + a^2) - 4(m^2 - q^2)^2 - 4a(qb + m\delta)](R_+ R_- + r_+ r_-),$$

$$B = 2R\sigma \{ (R^2 - 4\sigma^2)[mR\sigma(R_+ + R_- + r_+ + r_-) - (2m\Delta + a\delta)(R_+ + R_- - r_+ - r_-)] + i[ma(R^2 + 4\sigma^2 - 8\Delta) - 4mqb - 2\delta(2m^2 - \Delta)][(R - 2\sigma)(R_- - R_+) + (R + 2\sigma)(r_+ - r_-)] + iR\sigma\delta[(R - 2\sigma)(R_- - R_+) - (R + 2\sigma)(r_+ - r_-)] \},$$

$$C = (q/m)B + 2R(\sigma/m)(mb + q\delta)[a(R^2 - 4\sigma^2)(R_+ + R_- - r_+ - r_-) - 2iR(\sigma^2 - \Delta) \times (R_+ - R_- - r_+ + r_-) + i\sigma(R^2 - 4\Delta)(R_+ - R_- + r_+ - r_-)],$$

$$R_{\pm} = \sqrt{\rho^2 + \left[z \pm \left(\frac{1}{2}R + \sigma \right) \right]^2}, \quad r_{\pm} = \sqrt{\rho^2 + \left[z \pm \left(\frac{1}{2}R - \sigma \right) \right]^2}, \quad \Delta \equiv m^2 - a^2 - q^2, \quad (31)$$

and formulas (7) for the metric functions take the form

$$\begin{aligned}
f &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, & e^{2\gamma} &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16R^4\sigma^4(R^2 - 4\sigma^2)^2 R_+ R_- r_+ r_-}, & \omega &= -\frac{\text{Im}[G(\bar{A} + \bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}}, \\
G &= 4R\sigma^2[(R^2 - 4\sigma^2)(2m^2 - q^2) + b^2 - 2\delta^2](R_- r_- - R_+ r_+) + 2R^2\sigma\{[(R^2 - 4\sigma^2) \\
&\times (2m^2 - q^2) - b^2 + 2\delta^2](R_- r_+ - R_+ r_-) + 8i\sigma(bq + 2m\delta)(R_+ - R_-)(r_+ - r_-)\} \\
&- 2i[a(2m^2 - q^2)(R^2 + 4\sigma^2 - 8\Delta) - 4a^2(qb + 2m\delta) - 4mq(mb + q\delta) - a(b^2 - 2\delta^2)] \\
&\times [R^2(R_+ - r_+)(R_- - r_-) - 4\sigma^2(R_+ - r_-)(R_- - r_+)] - 2R(\sigma/m) \\
&\times \{[m^2(R^2 + 4\sigma^2 - 8\Delta) - \delta(4ma - \delta)][\sigma(R^2 - 4\Delta)(R_- - R_+ + r_- - r_+) \\
&- 2R(\sigma^2 - \Delta)(R_- - R_+ - r_- + r_+) + ia(R^2 - 4\sigma^2)(R_+ + R_- - r_+ - r_-)] \\
&- 2mb(mb + q\delta)[(R - 2\sigma)(R_- - R_+) - (R + 2\sigma)(r_- - r_+)] - 2imq(R^2 - 4\sigma^2) \\
&\times (mb + q\delta)(R_- + R_+ - r_- - r_+)\} - 2zB + i(4ma - \delta)B/m - 2qC, \\
I &= 2R\sigma\{R[mq(R^2 - 4\sigma^2) - b\delta](R_- r_+ - R_+ r_-) - 2\sigma[mq(R^2 - 4\sigma^2) + b\delta](R_+ r_+ - R_- r_-) \\
&+ i(R^2 - 4\sigma^2)(mb + q\delta)(R_+ R_- - r_+ r_- + 4R\sigma)\} - (i/2)(R^2 - 4\sigma^2)(mb - q\delta) \\
&\times [R^2(R_+ r_- + R_- r_+) + 4\sigma^2(R_+ r_+ + R_- r_-)] + (i/2)[(R^2 + 4\sigma^2 - 8\Delta) - 4ab\delta \\
&- 16mq(qb + m\delta)][R^2(R_+ r_- + R_- r_+) - 4\sigma^2(R_+ r_+ + R_- r_-)] - 2i(R^2 - 4\sigma^2) \\
&\times [maq(R^2 + 4\sigma^2 - 8\Delta) - 2q\delta(2m^2 - \Delta) - 2mb(m^2 - a^2 + q^2) - ab\delta](R_+ R_- + r_+ r_-) \\
&- R(\sigma/m)\{2R[(mq(R^2 + 4\sigma^2 - 8\Delta) + b(4ma - \delta))(\Delta - \sigma^2) + 4m\delta(mb + q\delta)] \\
&\times (R_- - R_+ - r_- + r_+) + \sigma[(mq(R^2 + 4\sigma^2 - 8\Delta) + b(4ma - \delta))(R^2 - 4\Delta) \\
&- 16m\delta(mb + q\delta)](R_- - R_+ + r_- - r_+) - i(R^2 - 4\sigma^2)[8m^2(mb + q\delta) - ab(4ma - \delta) \\
&- maq(R^2 + 4\sigma^2 - 8\Delta)](R_- + R_+ - r_- - r_+)\} + 2q(A + B) + ibB/(2m) - (z - 2ia)C. \tag{32}
\end{aligned}$$

As before, the electromagnetic potentials A_t and A_ϕ are determined by formulas (8).

In the presence of the strut, which means that σ is not arbitrary but verifies (16), the parameter q is the electric charge of each KN black hole, while the magnetic charge β must be introduced by means of the relation of the magnetic dipole parameter b to the charges q and β . Such a relation turns out to be slightly more complicated than in the pure magnetic case considered in the previous section, and it can be written as

$$b = -\frac{(R^2 - 4\Delta)(2aq + R\beta + 4q\mu)}{R^2 + 2mR + 4a^2 + 8a\mu}, \quad \mu \equiv \frac{a(q^2 - \beta^2) + q\beta(R + 2m)}{(R + 2m)^2 + 4a^2}. \tag{33}$$

Then after the substitution of (33) into (16) and choosing the minus sign we get the desired final formula for σ , namely,

$$\sigma = \sqrt{m^2 - a^2 - q^2 + \frac{(R^2 - 4\Delta)\{4[ma + (R + 2m)\mu]^2 - (2aq + R\beta + 4q\mu)^2\}}{(R^2 + 2mR + 4a^2 + 8a\mu)^2}}, \tag{34}$$

while the expression for δ obtainable from (11), (33), and (34) has the form

$$\delta = \frac{2(R^2 - 4\Delta)[ma + (R + 2m)\mu]}{R^2 + 2mR + 4a^2 + 8a\mu}. \tag{35}$$

The angular momentum of each black hole is now defined by the expression

$$J = \frac{[(R + 2m)^2 + 4a^2][ma - (R - 2m)\mu] - 4q^2[ma + (R + 2m)\mu]}{R^2 + 2mR + 4a^2 + 8a\mu}, \tag{36}$$

so that the two black holes have the same mass m , angular momentum J and electric charge q , but they differ in their magnetic charges: β of the lower and $-\beta$ of the upper black hole (see Fig. 2). Therefore, we have a nontrivial binary

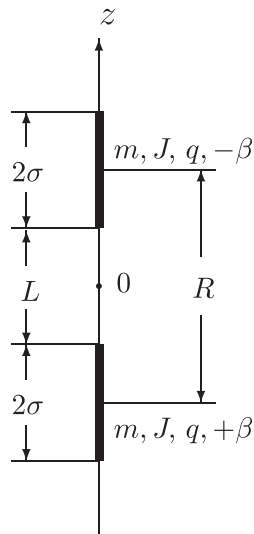


FIG. 2. Location of two equal corotating KN black holes with equal electric and opposite magnetic charges on the symmetry axis. $L = R - 2\sigma$ is the coordinate length of the strut.

system of dyonic KN black holes in which the magnetic charges are not introduced via the duality rotation of the potential Φ , in contrast to all the dyonic solutions studied for example in the paper [9]. In what follows we shall see that thermodynamics of the black holes in our system is subject to the generalized Smarr mass formula which takes into account the contribution of the magnetic field.

The calculations performed for the lower black hole with the aid of the standard Tomimatsu's formulas [20] give for the entropy, Hawking temperature, horizon's angular velocity and the electric potential the following expressions:

$$\begin{aligned} S &= \frac{\sigma}{2T} = \frac{\pi \{ [mR + (R + 2m)\sigma + 2\Delta]^2 + [a(R + 2\sigma) + \delta]^2 \}}{R(R + 2\sigma)}, \\ \Omega &= \frac{a[R^2 - 4\Delta + 2\sigma(R + 2\sigma)] - 2(qb + m\delta)}{[mR + (R + 2m)\sigma + 2\Delta]^2 + [a(R + 2\sigma) + \delta]^2}, \\ \Phi_e &= \frac{q(R + 2\sigma)[mR + (R + 2m)\sigma + 2\Delta] - b[a(R + 2\sigma) + \delta]}{[mR + (R + 2m)\sigma + 2\Delta]^2 + [a(R + 2\sigma) + \delta]^2}, \end{aligned} \quad (37)$$

and we have used the same way of writing these quantities as in the paper [30].¹

The above formulas must be supplemented with the expression of the magnetic potential Φ_m which, according to the papers [9,20], is defined by the equation

$$\beta\Phi_m = -\frac{1}{2} \int_H (A_\phi A'_\phi)_{,z} dz, \quad (38)$$

where $A'_\phi = \text{Im}(\Phi)$. Remarkably, the magnetic potential Φ_m can be written in a concise form

$$\Phi_m = -\frac{[2q\delta + b(R + 2m)]\{q[a(R + 2\sigma) + \delta] + \beta[mR + (R + 2m)\sigma + 2\Delta]\}}{[2aq + \beta(R + 2m)]\{[mR + (R + 2m)\sigma + 2\Delta]^2 + [a(R + 2\sigma) + \delta]^2\}}, \quad (39)$$

and it is not difficult to check that in the absence of electric charge ($q = 0$) formula (39) reduces to the expression of Φ_m in (26).

The thermodynamical variables obtained verify the generalized mass relation

$$m = 2TS + 2\Omega J + \Phi_e q + \Phi_m \beta, \quad (40)$$

and it should be remarked that the same relation holds for the upper black hole because the integral on

the right-hand side of (38) gives the same result as for the lower black hole, which must be interpreted as changing the sign of Φ_m (and β) on the upper horizon, while all other thermodynamical quantities remain unchanged.

To write out the corresponding first law of thermodynamics for our binary system, we still need the expressions of the interaction force and thermodynamic length. The calculations give for the former quantity the expression

$$\mathcal{F} = \frac{(R^2 + 4mR + 4\Delta)(m^2 - q^2 - 4\mu^2) + 4q[q\Delta + R\beta(a + 2\mu)] + R^2\beta^2 + 4a\mu(R^2 - 4\Delta)}{(R^2 - 4\Delta)[(R + 2m)^2 + 4a^2]}, \quad (41)$$

¹Note that formulas (37) are valid in the case of the general MMR solution, independently of the existence of a strut, and hence in principle need further processing to introduce explicitly the magnetic charge parameter β . However, this way of writing the thermodynamical quantities permits one to see a little bit better the mathematical structure of the potential Φ_m which we introduce later on and its relation to other thermodynamic characteristics.

while the latter quantity was found to have the form

$$\ell = \frac{(R^2 - 4\Delta)^2[(R + 2m)^2 + 4a^2]}{(R + 2\sigma)(R^2 + 2mR + 4a^2 + 8a\mu)^2}. \quad (42)$$

Then the first law reads as follows:

$$\begin{aligned} dM_T &= 2TdS + 2\Omega dJ + 2\Phi_e dq + 2\Phi_m d\beta - \ell d\mathcal{F}, \\ M_T &= 2m, \end{aligned} \quad (43)$$

and here both the electric and magnetic contributions are taken into account consistently. At the same time, while the potentials Φ_e and Φ_m are symmetric with respect to the change $q \rightarrow \beta$, $\beta \rightarrow q$ in the solutions where the magnetic charges are introduced by means of the duality rotation of the Ernst potential Φ [9], in our nontrivial dyonic configuration these Φ_e and Φ_m are defined by different, non-symmetric expressions. As a consequence, the generalized Smarr formula (40) in our case cannot be cast into a more elegant form (by introducing a complex charge $q + i\beta$) like this was done in the paper [9].

V. DISCUSSION

Since the MMR solution is the general equatorially symmetric 2-soliton solution of the stationary axisymmetric electrovac problem (of course up to an arbitrary duality rotation of the electromagnetic potential Φ [31]) then its 5-parameter subfamily considered in the previous section can be viewed as describing the general configuration of two identical corotating black holes with a massless strut in between. This in turn means that any known exact solution for a binary system with equatorial symmetry must be a particular specialization of the latter subfamily or obtainable from it via the constant phase transformation $\exp(i\alpha)$ of the potential Φ . In this respect, the recent solutions for corotating KN black holes with identical or opposite electric charges considered in [30] belong to our 5-parameter subfamily because the first solution is just its $\beta = 0$ particular case, while the second one follows immediately from its $q = 0$ specialization by applying the constant phase transformation with $\alpha = \pi/2$. The dyonic generalizations of the solutions [30] performed in [11] are also trivially obtainable from the $\beta = 0$ and $q = 0$ specializations of our subfamily. Note that the main physical difference between our nontrivial dyonic solution and those presented in [11] is that the latter solutions become static in the absence of the rotation parameter a , while the former solution at $a = 0$ still remains stationary due to the well-known frame-dragging effect by a charged magnetic dipole [32,33]. It can be also trivially seen that the transformation $\Phi \rightarrow i\Phi$ in our dyonic configuration leads to a binary system of black holes with equal magnetic and opposite electric charges, in which case the charges q , β and the potentials Φ_e , Φ_m interchange their roles: q and Φ_e then

become, respectively, the magnetic charge and magnetic potential, whereas β and Φ_m will play the role of the electric charge and electric potential, respectively, thus not changing the form of the generalized Smarr formula (40). However, such a configuration would be unphysical due to the presence of a nonzero net magnetic charge, in contradistinction to our dyonic system in which the opposite magnetic charges always form a magnetic dipole whose zero value is achieved by vanishing of one of these charges.

The 5-parameter dyonic configuration defined by formulas (33)–(36) has proved to be a good example of a binary system whose thermodynamics is subject to the generalized Smarr formula which takes into account the contribution of magnetic charges. It may be recalled in this regard that the recent paper [7] has argued that the magnetic potential Φ_m should not arise in the Smarr mass relation, which would mean in particular that the latter relation, say, for the magnetically charged KN black hole must look like in the case of an uncharged Kerr black hole. Though the constructions of the paper [7] were already shown to be frankly unphysical and inconsistent [10], a recent preprint [11] still makes an effort to rehabilitate the results of the paper [7] through the analysis of a specific binary dyonic configuration of KN black holes. The main contradiction of the author of [11] is that he starts with the mass relation without the magnetic potential Φ_m (like in the paper [7]) but eventually, after some manipulations, arrives at the Smarr formula of the paper [9] in which the potential Φ_m is already present, thus fully ignoring that precisely his final result was the subject of criticism in the paper [7]. We hope that our analysis of the first law of thermodynamics carried out in the previous section confirms convincingly the correctness of the original Tomimatsu's vision of the Smarr mass formula.

It is remarkable that formula (41) for the interaction force permits us to shed additional light on the equilibrium binary configurations of extremal KN black holes discovered and analyzed two decades ago in the paper [34]. Thus, in Table I from [34] the magnetic dipole moment of the equilibrium configurations was presumably attributed to rotating electric charges, without mentioning any contribution of the individual magnetic charges. In our solution (31) this would correspond to vanishing parameter b , and hence by setting $m = 2$, $R = 3.93076$, $q = 2.09509$, which closely approximates the characteristics of the first configuration from Table I of [34], we can find from Eqs. (33) and (41), after putting $b = 0$, $\mathcal{F} = 0$, that $a \simeq 5.12164$, $\beta \simeq 22.5323$, thus clearly demonstrating that large magnetic charges of opposite signs are involved in the binary equilibrium configurations from Table I of [34]. Moreover, with the above choice of the parameters, we also have $\sigma \simeq 0.00525$, so that the constituents in that particular binary system are nonextreme black holes. Surprisingly, despite of all our efforts, we have been unable to find equilibrium

configurations of corotating KN black holes with equal electric charges but zero magnetic charges ($\beta = 0$). We therefore conjecture that such equilibrium states are impossible. At the same time, balance turns out possible when the nonextreme black-hole constituents have no electric charges but are endowed with opposite magnetic charges, and a possible example is the following: $m = 2$, $R = 4.4$, $q = 0$, $a \simeq 5.11855$, $\beta \simeq 19.7747$, leading to $\sigma \simeq 0.05866$. Apparently, by virtue of the duality between electric and magnetic charges, this implies that analogous balance of corotating KN black holes with opposite electric charges and zero magnetic charges is also achievable.

An intriguing aspect of the binary charged black hole configurations which is of special interest to us and which we would like to briefly comment here is the following. In our papers on the binary systems of identical Kerr sources [17,35] we have shown that the uniqueness of the binary configurations with fixed masses and angular momenta can be broken for some particular values of the parameters, so that up to three different configurations with the same masses and angular momenta may exist due to relation of the rotation parameter a to the individual angular momentum J via the cubic equation. Since the analogous relation of the parameter a to J in the formulas (25) and (36) is determined, as can be easily seen, by a quintic equation, a natural question arises of whether the electromagnetic field of KN black holes is able to increase the nonuniqueness in the binary systems of charged black holes up to five different

configurations with the same masses, angular momenta, and charges? Our first numerical examination of Eqs. (25) and (36) has not yet been able to detect the parameter sets at which these equations would get five real roots for a . In the majority of cases these equations have one real root and two pairs of complex conjugate roots, they also may have three real and two complex roots. In the latter case a situation is possible when in the initial parameter sets ensuring three real roots of Eqs. (25) and (36) the subsequent increase of the values of q and β (keeping m and J unchanged) leads to the disappearance of two real roots, thus getting unique configurations from nonunique ones. Anyway, should any particular parameter sets at which the above quintic equations admit five real roots exist, they must belong to a highly restricted sector of the parameter space which yet has to be identified in the future.

ACKNOWLEDGMENTS

We are thankful to the referee for valuable comments. Many interesting and helpful discussions with Hugo García-Compeán and Carlos Ramírez-Valdez on thermodynamics of charged black holes are also gratefully acknowledged. This work was partially supported by CONACyT of Mexico, by Project No. SA096P20 from Junta de Castilla y León of Spain, and by Project No. PGC2018-096038-B-100 funded by the Spanish “Ministerio de Ciencia e Innovación” and FEDER “A way of making Europe.”

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