Curious case of the Buchdahl-Land-Sultana-Wyman-Ibañez-Sanz spacetime

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We revisit Wyman's "other" scalar field solution of the Einstein equations and its Sultana generalization to positive cosmological constant, which has a finite 3-space and corresponds to a special case of a stiff fluid solution proposed by Buchdahl and Land and, later, by Ibañez and Sanz to model relativistic stars. However, there is a hidden cosmological constant and the peculiar geometry prevents the use of this spacetime to model relativistic stars.

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I. INTRODUCTION

An analytical solution of the Einstein field equations of general relativity (GR) that is static and spherically symmetric appears in two different contexts that are apparently unrelated. In the first context, it is a non-asymptotically flat solution of the Einstein equations with a free scalar field as a source and with zero cosmological constant Λ, and it was discovered by Wyman in 1981 [1]. This is sometimes called Wyman's "other" solution to distinguish it from the more well known solution found by Fisher [2] and rediscovered many times, which in the literature goes by the names Fisher-Bergmann-Leipnik-Janis-Newman-Winicour-Buchdahl-Wyman [1,3–5] (see also Ref. [6]) and is the general solution of the $\Lambda = 0$ Einstein equations which is static, spherically symmetric, asymptotically flat, and is sourced by a free scalar field [2–5], see [7] for a recent review.

In the second context, Wyman's "other" solution is a special case of geometries proposed to describe the interior of a relativistic star by Ibañez and Sanz [8] and corresponding to the stiff equation of state. Sultana generalized Wyman's other solution by including a positive cosmological constant [9], obtaining a special case of another class of perfect fluid solutions found by Ibañez and Sanz. More precisely, Wyman's other metric is a special case of a perfect fluid geometry found in 1982 by Ibañez and Sanz [8] and in 1968 by Buchdahl and Land [10], which is itself a special case of the Tolman IV class of GR solutions introduced in 1939 [11–13]. We summarize below the rather convoluted history of the GR solution that is the subject of this work and that we call Buchdahl-Land-Sultana-Wyman-Ibañez-Sanz (in short, BLSWIS) solution.

The BLSWIS metric is contained as a special limit in the Buchdahl and Land's [10] 1968 stiff fluid solution of the Einstein equations with vanishing cosmological constant but pressure

$$P = \rho - \rho_0 \tag{1.1}$$

where ρ is the fluid energy density and ρ_0 is a constant. This equation of state was meant [10] to generalize the Schwarzschild interior solution for an incompressible fluid [14] but, apparently unbeknownst to these authors, in practice it reintroduces Λ into the scenario. The general Buchdahl-Land solution is itself a special case of the 1939 Tolman IV class of solutions [11] describing the interior of a perfect fluid ball with Λ [12]. As most authors solving for relativistic stellar interiors, Buchdahl and Land [10] did not match the fluid solution to an exterior, the implicit assumption in this literature being that the interior is matched with a Schwarzschild exterior at the star boundary, where the pressure vanishes [12,13].

Wyman's other solution was found in 1981 [1] as a free scalar field solution of the $\Lambda = 0$ Einstein equations extending to infinite radius and nonasymptotically flat[1] (this work [1] by Wyman is better known because it

[1]The Wyman geometry, but with a different scalar field, is a special case of the spacetime reported as a solution of a scalar-tensor gravity with power-law potential in Ref. [15] without making the connection with [1,8,10]. However, the geometry and scalar field proposed in [15] fail to satisfy the corresponding field equations.

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rediscovered the different Fisher-Janis-Newman-Winicour-Buchdahl-Wyman solution and gave it in its most general form [7]). This is a very different context from stellar models. In 2015, Sultana [9] generalized Wyman's other scalar field solution [1] to the case in which a cosmological constant $\Lambda > 0$ appears in the Einstein equations. Sultana was well aware of the fact that the geometry thus obtained is a special case of the Ibañez and Sanz solution.[2] We will refer to the scalar field solution of [9] as the Sultana-Wyman solution (this is the same as the BLSWIS one, but the name "Sultana-Wyman" is a reminder of the fact that the spacetime is sourced by Λ and by a homogeneous scalar field).

The BLSWIS geometry is contained, as a special case, in the more general perfect fluid solution of the Einstein equations with equation of state $P = w\rho$, $w = \text{const}$, and $0 < w \leq 1$ found by Ibañez and Sanz in 1982 [8]. These authors remark that this special case had been previously found by Buchdahl and Land [10][3] but they were unaware of Wyman's (then recent) paper and they did not realize that, in their special case $w = 1$, they were introducing the cosmological constant even though their field equations are initially declared to have $\Lambda = 0$ [8].

As is common in the history of analytical solutions of the Einstein equations [13], the same spacetime has been discovered and reinterpreted more than once and it is time to introduce some order in the relevant literature spanning many decades. This is the purpose of the present work, where we revisit the BLSWIS spacetime and compare, as much as possible, the two different points of view, i.e., perfect fluid without scalar field versus scalar field solution with $\Lambda > 0$. In particular, the boundary conditions for the Einstein equations need to be discussed and make stellar models based on the BLSWIS geometry unappealing from the physical point of view, or even impossible.

We follow the notation of Ref. [14]: the metric signature is $-+++$ and we use units in which Newton's constant G and the speed of light c are unity, but we occasionally restore G to compare with previous literature. Λ denotes the cosmological constant and $\kappa \equiv 8\pi G$.

II. THE WYMAN AND SULTANA-WYMAN SCALAR FIELD SOLUTIONS OF THE EINSTEIN EQUATIONS

The Einstein equations sourced by a minimally coupled, free and massless scalar field ϕ are

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab}\mathcal{R} + \Lambda g_{ab} = \kappa\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2} g_{ab}\nabla^c\phi\nabla_c\phi\right), \tag{2.1}$$

$$\Box\phi = 0, \tag{2.2}$$

where $\mathcal{R}_{ab}$, $\mathcal{R}$, and g_{ab} are the Ricci tensor, Ricci scalar, and metric tensor, respectively, while ∇_a is the covariant derivative associated with g_{ab} and $\Box \equiv g^{ab}\nabla_a\nabla_b$ is the curved space d'Alembertian.

The general static, spherically symmetric, and asymptotically flat solution of these equations for $\Lambda = 0$ is the well known Fisher solution [1–6] (see the recent review [7] for a discussion of this and other spherical solutions). Under the assumption that the matter field ϕ depends only on the radial coordinate, the unique static, spherical, and asymptotically flat solution was found by Fisher [2] and later rediscovered, in other coordinates or in other forms, by Bergmann and Leipnik [3], Janis, Newman and Winicour [4], Buchdahl [5], and finally by Wyman [1], who wrote the most general form of this solution. Wyman proposed another family of solutions for $\Lambda = 0$ (generalized by Varela [18] to the case $\Lambda \neq 0$) corresponding to spherically symmetric and static geometry and with scalar field depending only on time, $\phi = \phi(t)$. In general, this class of solutions is expressed by power series and is not useful for practical calculations, but one of them (again, for $\Lambda = 0$) is particularly simple [1]:

$$ds^2 = -\kappa r^2 dt^2 + 2dr^2 + r^2 d\Omega_{(2)}^2, \tag{2.3}$$

$$\phi(t) = \phi_0 t, \tag{2.4}$$

where $d\Omega_{(2)}^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere and ϕ_0 is a dimensionless constant. We refer to this solution as Wyman's other solution. It is a special case (for $w = 1$) of the "scaling solution" published a year later by Ibañez and Sanz [8] for a perfect fluid with equation of state[4] $P = w\rho$

$$ds^2 = -r^{\frac{4w}{1+w}} dt^2 + \frac{w^2 + 6w + 1}{(w+1)^2} dr^2 + r^2 d\Omega_{(2)}^2. \tag{2.5}$$

In spite of the fact that the energy density [8]

$$\rho(r) = \frac{w}{2\pi(w^2 + 6w + 1)r^2} \tag{2.6}$$

and the pressure $P(r) = w\rho(r)$ are singular at $r = 0$, this solution is usually regarded as possessing regions that are realistic approximations to the bulk of a star on the verge of collapsing [13,19–21].

Sultana [9] has generalized Wyman's other solution to include a positive cosmological constant Λ. The scalar field remains linear in time as in Eq. (2.4) (see Appendix for a discussion), while the line element becomes

[2]Sultana's generalization was later used to generate an exact solution of Brans-Dicke theory [16] and of $f(\mathcal{R}) = \mathcal{R}^2$ gravity [17].

[3]Ibañez and Sanz [8] also do not match this interior solution to an exterior one.

[4]Ibañez and Sanz use units in which $\kappa = 1$. Therefore, their Eqs. (15) for the energy density and pressure differ from our Eqs. (2.11), (2.12) by a factor 8π in the denominator.

$$ds^2 = -\kappa r^2 dt^2 + \frac{2dr^2}{1-\frac{2\Lambda r^2}{3}} + r^2 d\Omega_{(2)}^2 \tag{2.7}$$

(we will refer to this, in conjunction with Eq. (2.4) for the scalar, as the "Sultana-Wyman solution"). The limit $\Lambda \to 0$ reproduces Wyman's other solution (2.3) and (2.4). Again, the Sultana-Wyman solution is a special case of a family found by Ibañez and Sanz [8] with the Heintzmann method [22], which generalizes the scaling solution (2.5):

$$ds^2 = -r^{\frac{4w}{1+w}} dt^2 + \frac{a}{1-Car^{2+b}} dr^2 + r^2 d\Omega_{(2)}^2, \tag{2.8}$$

where

$$a = \frac{w^2+6w+1}{(w+1)^2}, \tag{2.9}$$

$$b = \frac{4w(1-w)}{(w+1)(3w+1)}, \tag{2.10}$$

and where C is an arbitrary constant. The corresponding energy density and pressure are [8]

$$\rho_w(r) = \frac{1}{8\pi}\left[\frac{4w}{(1+w)^2 ar^2} + C(3+b)r^b\right], \tag{2.11}$$

$$P_w(r) = \frac{1}{8\pi}\left[\frac{4w^2}{(1+w)^2 ar^2} - \frac{C(1+5w)}{(1+w)} r^b\right], \tag{2.12}$$

which are singular at $r=0$ (Ibañez and Sanz consider the range of equation of state parameters $0 < w \leq 1$ and find no solutions for dust $w=0$, which would eliminate the divergence in ρ_w and P_w [8]). For $w=1$, which corresponds to the stiff equation of state of a free scalar field, it is $a=2$, $b=0$ and the energy density and pressure become

$$\rho_1(r) = \frac{1}{8\pi}\left[\frac{1}{2r^2} + 3C\right], \tag{2.13}$$

$$P_1(r) = \frac{1}{8\pi}\left[\frac{1}{2r^2} - 3C\right]. \tag{2.14}$$

It is interesting that Ibañez and Sanz [8] do not relate their constant C to the cosmological constant in the $w=1$, $b=0$ case, although it is clear that the last term in the right-hand side of Eq. (2.13) and of Eq. (2.14) can be regarded as the contribution of a cosmological constant $\Lambda = 3C$ to the total energy density and pressure, added to those of the free scalar field. Moreover, for $w=1$ the line element (2.8) generalizes the Wyman solution also to the case $\Lambda < 0$ (this solution is implicit in Sultana's paper [9]).

The physical nature of Wyman's other solution was studied in previous papers [8,10,17,21] and its Sultana generalization was used in [17] to generate a new solution of Brans-Dicke theory with a massive scalar by means of a conformal transformation to the Jordan frame (the same geometry is a solution of $f(\mathcal{R}) = \mathcal{R}^2$ gravity [17]). Using the same method, Ref. [9] generated new solutions of conformally coupled scalar field theory with a Higgs potential.

Let us analyze the physical properties of the Sultana-Wyman solution (2.7) and (2.4) for $\Lambda > 0$. The time and radial coordinates vary in the range

$$-\infty < t < +\infty, \qquad 0 \leq r < \sqrt{\frac{3}{2\Lambda}}. \tag{2.15}$$

A. Geometry and radial geodesics

The Sultana-Wyman geometry described by the line element (2.7) is static and spherically symmetric. By taking the limit $\Lambda \to 0$, one recovers Wyman's other solution (2.3), (2.4) extending to $0 \leq r < +\infty$.

In general, if (apparent) horizons are present in a spherically symmetric geometry, they are located by the roots of the equation

$$g^{ab}\nabla_a r \nabla_b r = g^{rr} = 0, \tag{2.16}$$

where r is the areal radius (e.g., [23]), which is always defined in the presence of spherical symmetry. (Furthermore, a single root denotes a black hole or white hole apparent horizon, while a double root denotes a wormhole horizon throat [23].) In the Sultana-Wyman case (2.7), this equation has the unique single root

$$r_* = \sqrt{\frac{3}{2\Lambda}} \tag{2.17}$$

which, however, does not correspond to a horizon. To understand this situation note that, in spite of the fact that the time direction $t^a = (\partial/\partial t)^a$ is a timelike Killing vector of the geometry (2.7), its norm

$$t_a t^a = -kr^2 \tag{2.18}$$

does not change sign anywhere and, unlike what happens for the Schwarzschild metric or the de Sitter metric, there is no Killing horizon here. The 3-dimensional space $t = \text{const}$ is finite and is covered by the range $0 < r \leq \sqrt{\frac{3}{2\Lambda}}$ of the radial coordinate. The scalar field and the cosmological constant satisfy the weak and null energy conditions and generic strong rigidity arguments lead one to exclude event horizons [24–28] given the absence of Killing horizons.

To confirm this property, consider the congruences of outgoing (+) and ingoing (−) radial null geodesics with tangents $l^c_{(\pm)}$ and components $l^\mu_{(\pm)} = (l^0, l^1, 0, 0)$. The normalization $l_a^{(\pm)} l^a_{(\pm)} = 0$ yields

$$l^1_{(\pm)} = \pm\sqrt{\frac{\kappa}{2}}r\sqrt{1-\frac{2\Lambda r^2}{3}}l^0_{(\pm)} \tag{2.19}$$

and, since a null vector can be rescaled by a function, we can choose $l^0 = 1$ obtaining

$$l^\mu_{(\pm)} = \left(1, \pm\sqrt{\frac{\kappa}{2}\left(1-\frac{2\Lambda r^2}{3}\right)}r, 0, 0\right). \tag{2.20}$$

The equation of radial null geodesics can be integrated remembering that $l^\mu \equiv dx^\mu(\lambda)/d\lambda$, where λ is an affine parameter along the null geodesics. Then we have

$$\frac{dt}{d\lambda} = 1, \tag{2.21}$$

$$\frac{1}{r\sqrt{1-2\Lambda r^2/3}}\frac{dr}{d\lambda} = \pm\sqrt{\frac{\kappa}{2}}, \tag{2.22}$$

and then

$$t(\lambda) = \lambda - \lambda_0, \tag{2.23}$$

$$-\text{arctanh}\left(\sqrt{1-\frac{2\Lambda r^2}{3}}\right) = \pm\sqrt{\frac{\kappa}{2}}(\lambda-\lambda_0). \tag{2.24}$$

Simple manipulations of Eq. (2.24) yield

$$r(\lambda) = \sqrt{\frac{3}{2\Lambda}}\frac{1}{\cosh\left[\sqrt{\frac{\kappa}{2}}(\lambda-\lambda_0)\right]}, \tag{2.25}$$

see Fig. 1, which shows that radial null geodesics can never reach radii larger than $\sqrt{\frac{3}{2\Lambda}}$ since $\cosh x \geq 1$. A photon at $r = 0$ is an infinite value of the affine parameter λ away from the turning point $r = \sqrt{\frac{3}{2\Lambda}}$. It takes an arbitrarily long time $t \sim \lambda$ for a photon arbitrarily close to $r = 0$ to arrive to the turning point $r = \sqrt{\frac{3}{2\Lambda}}$. Similarly, a photon traveling radially and starting at $r = \sqrt{\frac{3}{2\Lambda}}$ at $\lambda = \lambda_0$ (or at any finite radius) takes an infinite λ-time to reach the origin (Fig. 1).

This dynamics can be understood by rewriting the l^1 component of the four-tangent to radial null geodesics as

$$\frac{dr}{d\lambda} = \pm\sqrt{\frac{\kappa}{2}}r\sqrt{1-\frac{2\Lambda r^2}{3}}, \tag{2.26}$$

squaring, and dividing by 2, which yields

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = 0, \qquad V(r) = \frac{\kappa r^2}{2}\left(\frac{2\Lambda r^2}{3} - 1\right), \tag{2.27}$$

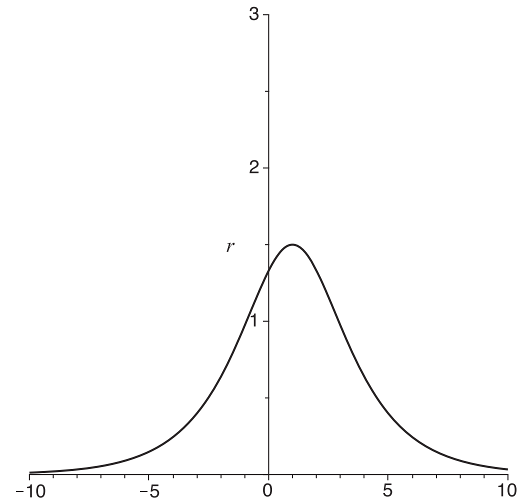

FIG. 1. The radial coordinate $r(\lambda)$ of a radial photon (vertical axis) versus the affine parameter λ (equivalently, the time t, on the horizontal axis) for the parameter values $\kappa = 1$, $\lambda_0 = 1$. The region $\lambda < \lambda_0$ to the left of the peak describes outgoing photons with $dr/d\lambda > 0$, while $\lambda > \lambda_0$ describes radial ingoing photons. A photon starting near $\lambda = -\infty$ and $r \simeq 0$ takes a very long time to reach the turning point [represented by the peak $r_{\text{max}} = \sqrt{\frac{3}{2\Lambda}}$ of $r(\lambda)$]. From there, the radial photon returns toward the origin $r = 0$ circling the finite 3-space, while approaching $r = 0$ in an infinite λ-time.

a formal energy conservation equation for a fictitious particle of unit mass and zero total energy in the effective potential $V(r)$. The latter intersects the r-axis at $r = 0$, $\sqrt{\frac{3}{2\Lambda}}$ and has a negative minimum $V_{\text{min}} = -\frac{3\kappa}{32\Lambda}$ at $r = \sqrt{\frac{3}{4\Lambda}}$ (Fig. 2). ($V(r)$ is an even function, but we are only interested in the region $r \geq 0$.)

Since the energy of the fictitious particle representing the radial photon is always zero, the motion is confined between $r = 0$ and the turning point $\sqrt{\frac{3}{2\Lambda}}$. If the radial photon starts near $r = 0$, it must do so with nearly zero kinetic energy and it takes an infinite amount of λ-time to reach the turning point $r = \sqrt{\frac{3}{2\Lambda}}$ at the end of 3-space. The point $r = 0$ is an unstable equilibrium point and a particle located there has zero energy and remains there. Once the radial photon is at the boundary $r = \sqrt{\frac{3}{2\Lambda}}$ of the finite space, it circles it toward the origin $r = 0$, but it takes an infinite λ-time to reach it as this photon slows down approaching it. The radial photon completes a single cycle of "oscillation" between $r = 0$ and $r = \sqrt{\frac{3}{2\Lambda}}$ in an infinite time, in a manner analogous to an overdamped oscillator.

It might appear that there is a stable circular photon orbit at $r = \sqrt{\frac{3}{4\Lambda}}$, where the potential is minimum, but photons cannot stay there because the total energy must be zero and,

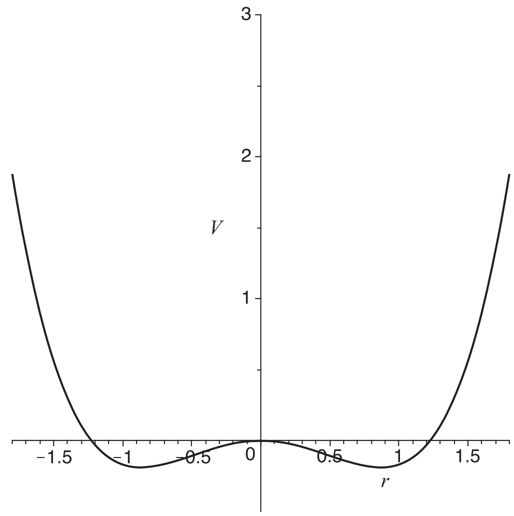

FIG. 2. The potential $V(r)$ (only the region $r \geq 0$ is physical and κ and Λ are set to unity for illustration). The motion is confined between $r = 0$ and the turning point $\sqrt{\frac{3}{2\Lambda}}$ because the total energy is zero. A radial outgoing photon ($dr/d\lambda > 0$) starting out arbitrarily close to $r = 0$ in the far past takes an arbitrarily long time (until λ_0) to reach the turning point $r = \sqrt{\frac{3}{2\Lambda}}$ and then heads again for $r = 0$, approaching in an infinite time and circling the finite 3-space. $dr/d\lambda$ vanishes as the photon approaches $r = 0$ or $r = \sqrt{\frac{3}{2\Lambda}}$. The photon cannot sit in the minimum of the potential $V_{\rm min} < 0$ because the total energy is forced to be zero.

since $V_{\rm min} < 0$, the positive kinetic energy $(dr/d\lambda)^2/2 = -V_{\rm min}$ moves it away from this radius.

We can calculate the expansions of the congruences of outgoing and ingoing radial null geodesics,

$$\begin{aligned}\theta_{(\pm)} &= \nabla_c l^c_{(\pm)} \\ &= \partial_\mu l^\mu_{(\pm)} + \Gamma^\mu_{\mu\alpha} l^\alpha_{(\pm)} \\ &= \partial_t l^0_{(\pm)} + \partial_r l^1_{(\pm)} + \Gamma^\mu_{\mu 0} l^0_{(\pm)} + \Gamma^\mu_{\mu 1} l^1_{(\pm)},\end{aligned} \tag{2.28}$$

where $\Gamma^\mu_{\alpha\beta}$ denote the Christoffel symbols. Using

$$\Gamma^\mu_{\mu 0} = 0, \qquad \Gamma^\mu_{\mu 1} = \frac{3 - 4\Lambda r^2/3}{r(1 - 2\Lambda r^2/3)}, \tag{2.29}$$

one obtains

$$\theta_{(\pm)} = \pm 2\sqrt{2\kappa}\sqrt{1 - \frac{2\Lambda r^2}{3}}. \tag{2.30}$$

In the limit $r \to \sqrt{\frac{3}{2\Lambda}}$ both expansions vanish. This anomalous behavior does not characterize a horizon (at which one of the expansions vanishes and the other does not), but signals the fact that 3-space ends at $r = \sqrt{\frac{3}{2\Lambda}}$ (more on this below).

Consider now outgoing/ingoing radial timelike geodesics with four-tangents

$$p^\mu_{(\pm)} = m u^\mu_{(\pm)} = m \left.\frac{dx^\mu}{d\tau}\right|_{(\pm)} = (p^0, p^1_{(\pm)}, 0, 0), \tag{2.31}$$

where m is the mass of a test particle of four-velocity $u^a_{(\pm)}$ and τ is the proper time along the timelike geodesic. The timelike Killing vector t^a guarantees conservation of energy along each geodesic:

$$p_a^{(\pm)} t^a = -E = \text{const}. \tag{2.32}$$

yielding

$$u^0 = \frac{\bar{E}}{\kappa r^2}, \tag{2.33}$$

where $\bar{E} \equiv E/m$ is the (constant) energy per unit mass. Then the normalization $u_a u^a = -1$ gives

$$u^\mu_{(\pm)} = \left(\frac{\bar{E}}{\kappa r^2}, \pm\sqrt{\frac{1}{2}\left(\frac{\bar{E}^2}{\kappa r^2} - 1\right)\left(1 - \frac{2\Lambda r^2}{3}\right)}, 0, 0\right). \tag{2.34}$$

The particle is at rest if either $r = \bar{E}/\sqrt{\kappa}$ (in which case $u^0 = 1$ and $u^1 = 0$), or if $r = \sqrt{\frac{3}{2\Lambda}}$ (in which case the particle is as far from the origin as possible).

The coordinate radial velocities of outgoing/ingoing massive test particles are

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{u^1_{(\pm)}}{u^0} = \pm\sqrt{\frac{(\bar{E}^2 - \kappa r^2)}{2\bar{E}}\left(1 - \frac{2\Lambda r^2}{3}\right)}, \tag{2.35}$$

which vanish in the limit $r \to \sqrt{\frac{3}{2\Lambda}}$ (while $u^0 \neq 0$, of course): at this radius, particles do not move either outward or inward, which would not happen at a horizon where only motion in one direction is forbidden (outward for a black hole horizon, inward for a cosmological or white hole horizon).

It is useful to compare the Sultana-Wyman geometry with the Einstein static universe, which has line element

$$ds^2 = -dt^2 + \frac{dr^2}{1 - K r^2} + r^2 d\Omega^2_{(2)}, \tag{2.36}$$

with constant curvature index $K > 0$ and finite 3-spaces of constant time and radial coordinate spanning the finite range $0 \leq r \leq 1/\sqrt{K}$. Naively, since this metric is spherically symmetric and r is the areal radius, a search for horizons with the equation

$$\nabla^c r \nabla_c r = g^{rr} = 1 - Kr^2 = 0 \tag{2.37}$$

would yield the unique positive single root $r = 1/\sqrt{K}$, but we know better. This is not a horizon, the norm of the timelike Killing vector $(\partial/\partial t)^a$ is always -1 and does not change sign anywhere, and we expect the expansions $\theta_{(\pm)}$ of radial null geodesics to exhibit pathological behavior at $r = 1/\sqrt{K}$. This is indeed the case. Let these geodesics have tangents $l^a_{(\pm)}$, then the normalization $l^a_{(\pm)} l_a^{(\pm)} = 0$ yields $l^1_{(\pm)} = \pm l^0 \sqrt{1 - Kr^2}$ and, choosing again $l^0 = 1$, one has

$$l^\mu_{(\pm)} = (1, \pm\sqrt{1 - Kr^2}, 0, 0). \tag{2.38}$$

The geodesic equations

$$\frac{dt}{d\lambda} = 1, \qquad \frac{dr}{d\lambda} = \pm\sqrt{1 - Kr^2}, \tag{2.39}$$

are easily integrated to $t(\lambda) = \lambda - \lambda_0$ (where λ_0 is an integration constant) and

$$\frac{\arcsin(\sqrt{K}r)}{\sqrt{K}} = \pm(\lambda - \lambda_0). \tag{2.40}$$

The last equation gives

$$r(\lambda) = \pm\frac{1}{\sqrt{K}} \sin[\sqrt{K}(\lambda - \lambda_0)], \tag{2.41}$$

where the sign of the right-hand side is chosen so that $r(\lambda)$ remains non-negative. The periodicity shows that a radial photon keeps circling the finite 3-space along the same spatial curve on the 3-sphere. We can rewrite the equation $l^1_{(\pm)} = dr/d\lambda = \pm\sqrt{1 - Kr^2}$ as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + W(r) = 0, \qquad W(r) = \frac{1}{2}(Kr^2 - 1). \tag{2.42}$$

The potential $W(r)$ is that of a simple harmonic oscillator with the origin of the energy shifted (Fig. 3), which intersects the r-axis at $r = 1/\sqrt{K}$, a turning point where the kinetic energy vanishes.

Since the total effective energy is zero, the motion is confined between $r = 0$ and the turning point $r = 1/\sqrt{K}$. Radial photons in this finite spacetime "oscillate" between $r = 0$ and $r = \sqrt{\frac{3}{2\Lambda}}$, which physically means that they keep going around the spherical 3-space, as described by the periodic solution (2.41). There are no stable or unstable circular orbits, except for the degenerate one at $r = 0$.

The expansions of the radial null geodesic congruences are

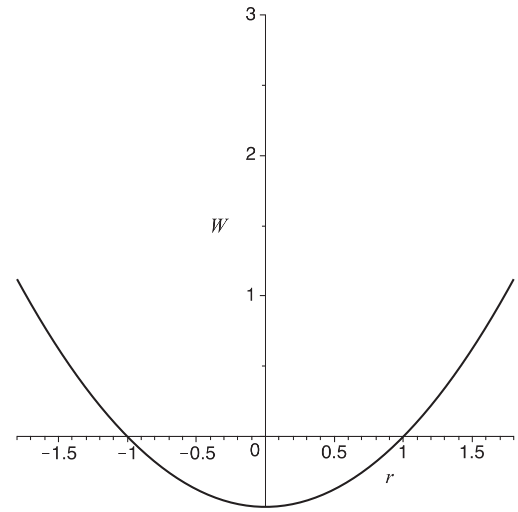

FIG. 3. The harmonic oscillator potential $W(r)$ (only the region $r \geq 0$ is physical and $K = 1$ for illustration). A radial photon oscillates between $r = 0$ and $r = 1/\sqrt{K}$, going around the finite hyperspherical 3-space again and again, each "oscillation" taking a finite λ-time.

$$\begin{aligned}\theta_{(\pm)} &= \nabla_a l^a_{(\pm)} = \partial_\mu l^\mu_{(\pm)} + \Gamma^\mu_{\mu\alpha} l^\alpha_{(\pm)} \\ &= \pm\partial_r\sqrt{1 - Kr^2} + \Gamma^\mu_{\mu 0} + \Gamma^\mu_{\mu 1}\sqrt{1 - Kr^2}.\end{aligned} \tag{2.43}$$

Using

$$\Gamma^\mu_{\mu 0} = 0, \qquad \Gamma^\mu_{\mu 1} = \frac{2}{r} - Kr(1 - Kr^2), \tag{2.44}$$

one obtains

$$\begin{aligned}\theta_{(\pm)} &= \pm\frac{1}{\sqrt{1 - Kr^2}} \\ &\quad \times \left\{-Kr + (1 - Kr^2)\left[\frac{2}{r} - Kr(1 - Kr^2)\right]\right\},\end{aligned} \tag{2.45}$$

which correctly reduce to $\pm 2/r$ in the degenerate Minkowski case $K = 0$. In the limit $r \to 1/\sqrt{K}$, we have $\theta_{(\pm)} \to \mp\infty$, signaling the fact that there is no horizon at this radius, but the 3-space is finite instead.

One can consider also radial timelike geodesics parametrized by the proper time τ. The timelike Killing vector $t^a = (\partial/\partial t)^a$ with unit norm gives energy conservation along each such geodesic: $p_c t^c = -E = \text{const}$ yields

$$u^0 = \bar{E} \equiv \frac{E}{m} \tag{2.46}$$

and the normalization $u_c u^c = -1$ then gives

$$u^\mu_{(\pm)} = \left(\bar{E}, \pm\sqrt{(\bar{E}^2 - 1)(1 - Kr^2)}, 0, 0\right). \tag{2.47}$$

Radial motion stops either if $\bar{E}=1$ (in which case $u^0=1$) or if $r=1/\sqrt{K}$, where the particle is as far away from the origin as possible in the Einstein static universe. It is easy to integrate the timelike geodesic equation, obtaining

$$t(\tau)=\bar{E}\tau+t_0, \tag{2.48}$$

$$r(\tau)=\pm\frac{1}{\sqrt{K}}\sin\left[\sqrt{K(\bar{E}^2-1)}(\tau-\tau_0)\right], \tag{2.49}$$

(where t_0 and τ_0 are integration constants) or, eliminating the parameter τ,

$$r(t)=\pm\frac{1}{\sqrt{K}}\sin\left[\sqrt{K\left(1-\frac{1}{\bar{E}^2}\right)}(t-t_0)\right]. \tag{2.50}$$

The radial position of the particle cannot exceed the maximum value $1/\sqrt{K}$.

To conclude, the 3-space of the Sultana-Wyman geometry is finite. Denoting with $g^{(3)}$ the determinant of the restriction $g^{(3)}_{ab}$ of the spacetime metric g_{ab} to this subspace, its volume is given by

$$\begin{aligned}V&=\int d^3\vec{x}\sqrt{g^{(3)}}=\int\frac{\sqrt{2}r^2\sin\vartheta}{\sqrt{1-2\Lambda r^2/3}}dr d\vartheta d\varphi\\&=4\pi\sqrt{2}\int_0^{\sqrt{\frac{3}{2\Lambda}}}dr\frac{r^2}{\sqrt{1-2\Lambda r^2/3}}\\&=\frac{4\pi\sqrt{2}}{8}\left[\frac{3\sqrt{6}}{\Lambda^{3/2}}\arcsin\left(\sqrt{\frac{2\Lambda}{3}}r\right)-2r\sqrt{1-\frac{2\Lambda r^2}{3}}\right]_0^{\sqrt{\frac{3}{2\Lambda}}}\\&=\frac{3\sqrt{3}\pi^2}{2\Lambda^{3/2}}\simeq 25.64\Lambda^{-3/2}.\end{aligned} \tag{2.51}$$

B. Central singularity

By computing the Ricci scalar from the Sultana-Wyman line element (2.7) one obtains $\mathcal{R}=4\Lambda-1/r^2$, while contracting the field equations (2.1) and using Eq. (2.4) yields

$$\mathcal{R}=4\Lambda+\kappa g^{ab}\nabla_a\phi\nabla_b\phi=4\Lambda-\frac{\phi_0^2}{r^2}, \tag{2.52}$$

which fixes the dimensionless integration constant to $\phi_0=\pm1$. The Ricci scalar diverges as $r\to0^+$ in both cases $\Lambda=0$ (Wyman's other solution) and $\Lambda>0$ (Sultana-Wyman solution). The total (i.e., including scalar field and cosmological constant) energy density and pressure obtained from Eqs. (2.9)–(2.12) for $w=1$,

$$\rho(r)=\frac{1}{8\pi}\left(\frac{1}{2r^2}+\Lambda\right), \tag{2.53}$$

$$P(r)=\frac{1}{8\pi}\left(\frac{1}{2r^2}-\Lambda\right), \tag{2.54}$$

are also singular but the spatially homogeneous scalar field is regular everywhere. The Sultana-Wyman solution is interpreted as a scalar field naked central singularity embedded in a "background" due to the cosmological constant.[5]

The equation of a sphere of constant radius r_0 is $f(r)=r-r_0=0$ and the normal to this surface has direction

$$N_\mu=\nabla_\mu f=\delta_{\mu1}; \tag{2.55}$$

its norm

$$N_cN^c=g^{\mu\nu}\delta_{\mu1}\delta_{\nu1}=g^{rr}=\frac{1}{2}\left(1-\frac{2\Lambda r^2}{3}\right) \tag{2.56}$$

is positive for any $r<r_*=\sqrt{\frac{3}{2\Lambda}}$. Taking the limit $r\to0^+$ in this equation, one obtains $N^cN_c|_{r=0}=1/2$, hence N^c is spacelike and the central singularity at $r=0$ is timelike.

In the $\Lambda\to0$ limit to Wyman's other solution, 3-spaces of constant time are infinite, the coordinate r extends to infinity, and the geometry describes a naked singularity embedded in a spacetime which is not asymptotically flat because the corresponding Ricci tensor

$$\mathcal{R}_{ab}=\kappa\nabla_a\phi\nabla_b\phi=\kappa\phi_0^2\delta_{a0}\delta_{b0} \tag{2.57}$$

does not vanish as $r\to+\infty$ and the energy density $\rho\sim1/r^2$ diverges when integrated between a finite radius and infinity (see below).

C. Quasilocal mass

In spherical symmetry, the Misner-Sharp-Hernandez mass M_{MSH} contained in a ball of radius r is defined by [29,30]

$$1-\frac{2GM_{\mathrm{MSH}}}{r}\equiv\nabla^c r\nabla_c r, \tag{2.58}$$

where r is the areal radius. The Hawking-Hayward quasilocal mass [31,32] reduces to the Misner-Sharp-Hernandez mass in spherical symmetry and is the Noether charge associated with the conservation of the

[5]The quotation marks are mandatory because, due to the nonlinearity of the Einstein equations, a metric cannot be split into a "background" plus a "deviation" from it in a covariant way, except for (generalized) Kerr-Schild metrics.

Kodama current [33]. For the Sultana-Wyman solution with $\Lambda > 0$, the Misner-Sharp-Hernandez mass is [17]

$$M_{\mathrm{MSH}}(r) = \frac{r}{4G}\left(1 + \frac{2\Lambda r^2}{3}\right). \tag{2.59}$$

By comparison, the Misner-Sharp-Hernandez mass of a ball in de Sitter space is $M_{\mathrm{dS}}(r) = \frac{\Lambda r^3}{6G}$, so the scalar field ϕ in the Sultana-Wyman geometry contributes an amount $r/(4G)$ added to the mass of de Sitter space. More precisely, the mass in Eq. (2.59) splits as

$$M_{\mathrm{MSH}}(r) = \frac{4\pi r^3}{3}\frac{\Lambda}{\kappa} + \frac{4\pi r^3}{3}\frac{1}{2\kappa r^2} = \frac{4\pi r^3}{3}(\rho_{(\phi)} + \rho_\Lambda), \tag{2.60}$$

where we used Eq. (2.13).

For $\Lambda = 0$ (in which case $0 \le r < +\infty$), the mass $M_{\mathrm{MSH}}(r)$ of the Wyman solution diverges linearly as the areal radius $r \to +\infty$, showing again that this geometry is not asymptotically flat (in which case M_{MSH} would be finite [34]).

For $\Lambda > 0$, the total Misner-Sharp-Hernandez mass contained in the finite Sultana-Wyman slices of constant time is

$$M_{\mathrm{MSH}}\left(r = \sqrt{\frac{3}{2\Lambda}}\right) = \sqrt{\frac{3}{8\Lambda G^2}}. \tag{2.61}$$

III. THE SULTANA-WYMAN GEOMETRY AS A FINITE FLUID BALL: BUCHDAHL-LAND AND IBAÑEZ-SANZ

It is well known that a minimally coupled scalar field is equivalent to a perfect fluid and that a free scalar corresponds to a stiff fluid with equation of state $P = \rho$. Therefore, the Sultana-Wyman solution can be interpreted as describing a spacetime filled with a stiff fluid and a cosmological constant. Indeed, it corresponds to a special case of a previous stiff fluid solution. This fact was apparently unknown to Wyman in the case $\Lambda = 0$, but Sultana identifies his generalization of Wyman's solution to $\Lambda > 0$ with a solution generated by Ibañez and Sanz [8] using the Heintzmann technique [22]. Ibañez and Sanz correctly identify it with the previous Buchdahl-Land solution [10] which is, in turn, a special case of the Tolman IV class of solutions of the Einstein equations with Λ [11]. The Buchdahl-Land solution was recently used by Jowsey and Visser [21] as an example in an unrelated context, the question of the existence of a maximum force in general relativity.

The Tolman IV solution of the Einstein equations with Λ and a perfect fluid is [11]

$$ds^2 = -\left(1 + \frac{r^2}{A^2}\right)dt^2 + \frac{1 + 2r^2/A^2}{(1 - \frac{r^2}{R^2})(1 + \frac{r^2}{A^2})}dr^2 + r^2 d\Omega_{(2)}^2, \tag{3.1}$$

where A and R are constants. By using the dimensionless time $\tau \equiv t/A$, this line element is rewritten as

$$ds^2 = -(A^2 + r^2)d\tau^2 + \frac{A^2 + 2r^2}{(1 - r^2/R^2)(A^2 + r^2)}dr^2 + r^2 d\Omega_{(2)}^2. \tag{3.2}$$

Taking the limit in which the parameter $A \to 0$ yields

$$ds^2 = -r^2 d\tau^2 + \frac{2}{1 - r^2/R^2}dr^2 + r^2 d\Omega_{(2)}^2. \tag{3.3}$$

Redefining the time coordinate as $\tau \equiv \sqrt{\kappa}\bar{t}$ and identifying $2\Lambda/3 \equiv 1/r_{\mathrm{H}}^2$ with $1/R^2$, one obtains the Buchdahl-Land line element [10]

$$ds^2 = -\kappa r^2 d\bar{t}^2 + \frac{2}{1 - 2\Lambda r^2/3}dr^2 + r^2 d\Omega_{(2)}^2, \tag{3.4}$$

which coincides with the Sultana-Wyman solution (2.7). In this notation, the latter has its boundary at radius $\sqrt{3/(2\Lambda)} = R$.

The scalar field is redefined according to

$$\phi = \phi_0 t = \phi_0 A\tau = \phi_0\sqrt{\kappa}\bar{t} \equiv \bar{\phi}_0\bar{t}. \tag{3.5}$$

Let us relate the "standard" view in the literature (the Buchdahl-Land/Ibañez-Sanz [8,10] geometry as a stiff fluid GR solution) and the Sultana-Wyman view of the same geometry as a scalar field solution with $\Lambda > 0$. Starting from the latter, the expression of the scalar field stress-energy tensor [14]

$$T_{ab}^{(\phi)} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi - Vg_{ab} \tag{3.6}$$

gives the well-known energy density and pressure

$$\rho_{(\phi)} = -\frac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi), \tag{3.7}$$

$$P_{(\phi)} = -\frac{1}{2}\nabla^c\phi\nabla_c\phi - V(\phi), \tag{3.8}$$

which make it clear that a free scalar field corresponds to a stiff fluid with equation of state $P_{(\phi)} = \rho_{(\phi)}$.

If one regards the Sultana-Wyman solution as a free scalar field solution of the Einstein equations with cosmological constant $\Lambda > 0$, then using $\phi = \bar{\phi}_0\bar{t}$, one has

$$\rho_{(\phi)} = P_{(\phi)} = \frac{\bar{\phi}_0^2}{2\kappa r^2}, \tag{3.9}$$

but the *total* effective energy density and pressure are obtained by viewing the Λ-term as an effective fluid with stress-energy tensor $T_{ab}^{(\Lambda)} = -\frac{\Lambda}{\kappa} g_{ab}$ in the right-hand side of the Einstein equations,

$$\rho_{\text{tot}} = \frac{\bar{\phi}_0^2}{2\kappa r^2} + \frac{\Lambda}{\kappa} = \frac{1}{16\pi}\left(\frac{\bar{\phi}_0^2}{r^2} + \frac{3}{R^2}\right), \tag{3.10}$$

$$P_{\text{tot}} = \frac{\bar{\phi}_0^2}{2\kappa r^2} - \frac{\Lambda}{\kappa} = \frac{1}{16\pi}\left(\frac{\bar{\phi}_0^2}{r^2} - \frac{3}{R^2}\right), \tag{3.11}$$

where we used the fact that $\Lambda = \frac{3}{2R^2}$. Alternatively, one can regard the Sultana-Wyman spacetime as a solution of the Einstein equations without cosmological constant but with a scalar field in the constant potential $V(\phi) = \Lambda/\kappa$, with the same result. If we set $\bar{\phi}_0 = 1$, Eqs. (3.10) and (3.11) match Eqs. (3.60) of Jowsey and Visser [21], who do not contemplate scalar fields and view the Buchdahl-Land solution as a stiff fluid ball.

The energy density and pressure are singular as $r \to 0^+$ and the pressure $P_{(\phi)}$ vanishes at the radius

$$\frac{|\bar{\phi}_0| R}{\sqrt{3}} = \frac{|\bar{\phi}_0| r_{\text{H}}}{\sqrt{3}} \tag{3.12}$$

the radius that is usually taken as the boundary of the star in the literature (this corresponds to $R_s \equiv R/\sqrt{3}$ in [21]).

Keeping $\bar{\phi}_0$ general, we encounter two possible situations:

(i) If $|\bar{\phi}_0| < \sqrt{3}$, the boundary r_* of the star is below the Sultana-Wyman maximum radius, $r_* < r_{\text{H}}$;
(ii) If $|\bar{\phi}_0| = \sqrt{3}$ the fluid configuration fills the entire Sultana-Wyman 3-space, i.e., it is not a star.

These scenarios are discussed in the following.

A. Star boundary below $\sqrt{\frac{3}{2\Lambda}}$

Assuming that the star extends from the origin $r = 0$ (where, however, there is spacetime singularity—see the discussion below) to a boundary r_0, one has to match the interior Sultana-Wyman scalar field solution with an exterior in order to build a stellar model. In spherical stellar models, the standard practice consists of matching an interior fluid solution with a Schwarzschild exterior (e.g., [7,12]). However, having established that the BLSWIS geometry solves the Einstein equations with $\Lambda > 0$, the interior must be matched with a Schwarzschild-de Sitter/Kottler exterior, which is the unique solution in this case, according to a straightforward generalization of the Birkhoff theorem [35,36].

There are two possibilities: either one regards the interior as a stiff fluid solution of the Einstein equations (with $\Lambda > 0$), or as a free scalar field solution of the Einstein equations (with $\Lambda > 0$). In the first case, since the pressure goes to zero at the star boundary $r_0 < \sqrt{\frac{3}{2\Lambda}}$, one would be tempted to match with a Schwarzschild exterior, as done for all fluid models of stars[6] (e.g., [12]), but Schwarzschild is not a solution of the Einstein equations in a $\Lambda > 0$ vacuum. Therefore, the interior should be matched smoothly with a Schwarzschild-de Sitter exterior, but this is impossible because the interior pressure P_{tot} given by Eq. (3.11) is always larger than the exterior pressure $P_\Lambda = -\Lambda/\kappa < 0$. One could allow for a discontinuity of matter on the star boundary, but this implies the presence of a layer of material on that surface, which is not a physical model of a star.

Let us consider the second possibility. Since the scalar field $\phi(t)$ does not depend on the spatial coordinates, it cannot be set to zero at the star boundary, or to a constant (with respect to time) in the star exterior, therefore the exterior solution must also be a scalar field solution of the Einstein equations with $\Lambda \neq 0$. The fluid ball is not surrounded by vacuum and its exterior geometry cannot be Schwarzschild-de Sitter/Kottler [35,36]. The homogeneous scalar field is linear in time, a feature that persists in the exterior by continuity. Therefore, the interior Sultana-Wyman solution does not match to the Fisher geometry [2] either, for which $\phi = \phi(r)$.

This situation is rather curious: in the two cases above, the field equations are different, and it happens that a certain geometry solves both.[7] In the present problem with the stiff fluid solution *of the* $\Lambda > 0$ *Einstein equations*, one wants to cut the solution at the specific value r_* of radius where P vanishes and join it smoothly with an exterior solution. However, in the other interpretation in which the geometry is a scalar field solution of the Einstein equations *with cosmological constant* $\Lambda > 0$, the field equations are different and one should not expect *a priori* that joining smoothly the same interior geometry with an exterior one is possible, or physically meaningful, or that it gives the same result. The interior solution solves two different sets of field equations, but continuing it smoothly to an exterior is an issue. The two different points of view contemplating different sources with a (hidden) Λ behave differently with respect to the continuation to an exterior. In one case, a discontinuous matching with a Schwarschild-de Sitter

[6] Buchdahl and Land [10], Ibañez and Sanz [8], and Jowsey and Visser [21] do not discuss this matching nor refer to it, but it is implicit in the large literature on stellar models that a stellar interior must be matched with an exterior Schwarschild [13].

[7] This situation is quite common: for example, any physically reasonable theory of gravity admits the Friedmann-Lemaître-Robertson-Walker solution. In these situations, although the field equations are very different [37,38], the same geometry solves both.

exterior is the only possibility, which entails a layer of material at the star surface. In the other situation, one must match the same interior geometry with an exterior that has homogeneous scalar field and $\Lambda > 0$ and is not Schwarzschild-de Sitter, or else one must impose the additional unphysical requirement that the scalar field is discontinuous. None of these two situations is interesting to build a physical model of a relativistic star (with or without Λ).

In any case, because of the central singularity, there is another potentially very serious issue in interpreting the BLSWIS solution as describing a fluid ball. In the literature, various authors seem to content themselves with assuming that only regions with $r > 0$ of this Buchdahl-Land solution describe realistic star geometries (see the comments by Buchdahl and Land [10], Ibañez and Sanz [8] and Jowsey and Visser [21] to this regard). In their monumental review of exact GR solutions, Stephani, Kramer, MacCallum, Hoenselaers and Herlt also suggest using solutions with a central singularity to model the outer layers of composite spheres [13], and using regions with different equations of state is common in the modeling of Newtonian stars, when their interiors are not well mixed. However, selecting certain limited spacetime regions as realistic solutions ultimately involves further matching with other nonsingular solutions extending down to $r = 0$. To the best of our knowledge, this possibility is not actively explored in the literature.

B. Star boundary at $r = \sqrt{\frac{3}{2\Lambda}}$

This potential possibility corresponds to a very strange situation. The pressure becomes negative in the region $\frac{|\bar{\phi}_0| R}{\sqrt{3}} < r < r_*$ (while $\rho_{(\phi)}$ remains positive), which is unphysical for a stellar interior. An exterior must necessarily have the same value of the cosmological constant $\Lambda > 0$ and the same scalar field $\phi = \phi_0 t$—that is, the solution is again Sultana-Wyman, but we know that its 3-spaces of constant time have finite extension, therefore one cannot consider an "exterior". Attempts to describe a stellar configuration with boundary at $r = \sqrt{\frac{3}{2\Lambda}}$ seem doomed.

IV. CONCLUSIONS

The history of the BLSWIS solution of the Einstein equations is a bit convoluted: it is derived either as a solution with a free homogeneous scalar field $\phi(t)$ and cosmological constant $\Lambda > 0$ [9], or as special limits of interior solutions for a relativistic star with a perfect fluid and, superficially, $\Lambda = 0$ [8,10,11]. However, when it is obtained through these special limits, there is a positive cosmological constant hidden in this solution that was not evident in the more general fluid solutions. The crucial difference between a real fluid and a cosmological constant, even when the latter is treated as an effective fluid, is that the former can be confined to a limited region of spacetime and vanish outside of it, but the latter permeates all of spacetime.

While the spacetime geometry is the same in the Sultana-Wyman and the perfect (stiff) fluid solutions, the boundary conditions at the surface of the would-be star differ in the two contexts. Matching smoothly the BLSWIS "interior" to an "exterior" in order to build a stellar model (necessarily, for $\Lambda > 0$ and with a homogeneous scalar field) does not make much sense physically, as discussed above. In particular, the constant time slices of the BLSWIS geometry are finite. In the Oppenheimer-Snyder model of gravitational collapse [39,40], a finite Friedmann-Lemaître-Robertson-Walker universe with positively curved spatial sections and filled with dust is matched smoothly with a Schwarschild exterior, and this case bears some resemblance to the BLSWIS case. However, in order to match to the Schwarzschild exterior, the fluid in the interior must necessarily have zero pressure. In the BLSWIS case, an exterior must necessarily have the same value of the cosmological constant $\Lambda > 0$ and the same scalar field $\phi = \phi_0 t$—that is, the solution is again Sultana-Wyman, but we know that its 3-spaces of constant time have finite extension, therefore one cannot consider an "exterior".

When examining the other boundary at the star's center $r = 0$, the central singularity of the BLSWIS geometry does not bode well for using the Wyman geometry or its Sultana generalization to describe the interior of stars. Ibañez and Sanz [8] and also Jowsey and Visser [21] join the existing literature [13] in regarding this geometry as capable of describing certain regions of relativistic stars. While this may be the case, extra caution must be exerted when applying this GR solution to realistic situations. Overall, the BLSWIS solution of the Einstein equations does not lend itself to model physically meaningful relativistic stars and joins the graveyard of exact solutions of the Einstein equations originally meant to model fluid balls which fail to do so for one reason or another [12]. The lesson learned from the BLSWIS case is that, in order to build realistic models of relativistic stars (or of regions of them), it is not sufficient to solve analytically the Einstein equations with a perfect fluid source, but attention must be paid to the exterior and to the star boundary, since a cosmological constant or a inhomogeneous matter source cannot be eliminated when passing from the interior to the exterior.

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APPENDIX: KLEIN-GORDON EQUATION FOR THE SULTANA-WYMAN SPACETIME

It is straightforward to show that $\phi = \phi_0 t$ solves the Klein-Gordon equation (2.2), which reads

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{A1}$$

for both the Wyman solution (for which $\Lambda = 0$) and for its Sultana generalization with $\Lambda > 0$.

Using $\sqrt{-g} = \sqrt{\frac{2\kappa}{1-2\Lambda r^2/3}}r^3 \sin\vartheta$ and $\partial_\mu\phi = \phi_0\delta_{\mu 0}$, this equation reduces to

$$\partial_t\left(\frac{\phi_0}{\sqrt{\kappa}}\frac{\sqrt{2}r\sin\vartheta}{\sqrt{1-2\kappa r^2/3}}\right) = 0, \tag{A2}$$

which is trivially satisfied since the argument of the round bracket depends only on r.

When the cosmological constant $\Lambda > 0$ is included by Sultana in the picture, it is equivalent to regard the total matter content of spacetime as (1) a free scalar field ϕ with Λ in the Einstein equations, or (2) as a scalar field with the constant potential $V = \Lambda/\kappa$ and no cosmological constant in the Einstein equations. In the first case, the Klein-Gordon equation does not change in form. By making the second choice, the Klein-Gordon equation for ϕ would be modified by the potential $V(\phi)$ according to

$$\Box\phi - \frac{dV}{d\phi} = 0, \tag{A3}$$

but since $dV/d\phi \equiv 0$ for $V = \Lambda/\kappa$, the form of this equation is unchanged (however, the cosmological constant Λ or, alternatively, the potential V changes the field equations (2.1) for the metric and, accordingly, the solution changes from the Wyman to the Sultana-Wyman one).

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