Nonlinear electrodynamics nonminimally coupled to gravity: Symmetric-hyperbolicity and causal structure

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It is shown here that symmetric-hyperbolicity, which guarantees well-posedness, leads to a set of two inequalities for matrices whose elements are determined by a given theory. As a part of the calculation, carried out in a mostly covariant formalism, the general form for the symmetrizer, valid for a general Lagrangian theory, was obtained. When applied to nonlinear electromagnetism linearly coupled to curvature, the inequalities lead to strong constraints on the relevant quantities, which were illustrated with applications to particular cases. The examples show that nonlinearity leads to constraints on the field intensities, and nonminimal coupling imposes restrictions on quantities associated to curvature.

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I. INTRODUCTION

The well-posedness of the initial value problem stands out as a basic requirement to be satisfied by any relativistic field theory. Broadly speaking, it implies that solutions for a given problem exist, are unique, and depend continuously on the initial data. Hence, well-posedness is at the roots of physics, for it amounts to the predictability power of a given theory. While it is difficult to decide whether a given nonlinear theory has a well-posed initial value problem, a necessary and sufficient condition for well-posedness around a given solution is that all the linearized problems obtained by linearizing near such a solution are well-posed, see for instance [1,2].

In the linearized regime, well-posedness means that the equation of motion is hyperbolic. At least three notions of hyperbolicity can be distinguished [1,3]: (i) weak hyperbolicity, in which all the roots of the characteristic equation are real, (ii) strong hyperbolicity, which implies that there is an energy estimate that sets a bound for the energy of a solution at a given time in terms of the initial energy,[1] and (iii) symmetric hyperbolicity, which is a sufficient condition for well-posedness [4,5]. It follows that symmetric hyperbolicity implies strong hyperbolicity, which in turn implies weak hyperbolicity.

There are many examples in which the requirement of some kind of hyperbolicity imposes severe restrictions on the Lagrangian of a given theory. For instance, as shown in [2], the equations of motion of Lovelock's theory are always weakly hyperbolic for weak fields but not strongly hyperbolic in a generic weak-field background. The well-posedness of Horneski theory has been analyzed in [2,6]. As shown in the latter reference, the most general Horndeski theory that is strongly hyperbolic for weak fields in a generalized harmonic gauge is simply k-essence coupled to Einstein's gravity (see also [7]). The well-posedness of scalar-tensor effective field theory was studied in [8], where it was shown that the equations of motion are strongly hyperbolic at weak coupling.

We would like to explore here the constraints imposed by the requirement of well-posedness in the case of nonlinear electromagnetic theories coupled to gravity. Nonlinear electromagnetism has been widely studied in several contexts. A nonexhaustive list of applications and references includes black holes [9–13], astrophysics [14–16], and cosmology [17–21]. There are also several articles devoted to different aspects of the propagation of perturbations in nonlinear electromagnetic theories, such as [22–25]. The matter of (symmetric) hyperbolicity for nonlinear electromagnetism minimally coupled to gravity was analyzed in a flat spacetime background in [26], while the hyperbolicity of Maxwell's equations with a local (and possibly nonlinear) constitutive law in flat spacetime was considered in [27]. Our aim here is to determine the restrictions that follow from the requirement of symmetric hyperbolicity on nonlinear electromagnetism nonminimally coupled to gravity, several aspects of which have

[1]In fact, strong hyperbolicity is a necessary and sufficient condition for the initial-value problem to be well-posed.

been studied in detail in [28–31]. We shall restrict to couplings linear in the curvature, since higher-order couplings produce higher-than-second-order equations for the gravitational field [28].

As mentioned above, a sufficient condition for well-posedness to hold is that the system under study admits a symmetric-hyperbolic representation. The theory of first-order symmetric-hyperbolic systems, originally due to Friedrichs [32], has been extensively developed (see for instance [3] and references therein). To study the evolution of a system we shall adopt here the modern geometric approach to the subject outlined by Geroch in [33], in which covariance is kept during most of the calculation, instead of using a $3+1$ decomposition of spacetime (as for instance in [27]).[2] We shall see that such approach leads to a general form for the symmetrizer, and to conditions for symmetric hyperbolicity that are easier to evaluate than those for other types of hyperbolicity.

The structure of the paper is as follows. In Sec. II the basic notation used in the equations of motion is presented. Symmetric hyperbolicity, the related concept of symmetrizer, and the role of the constraints are analyzed in Sec. III. The equations defining the characteristic cones will be deduced in Sec. IV. Our main result, namely the explicit form of the matrices that are needed to investigate the symmetric hyperbolicity of any nonlinear electromagnetic theory nonminimally coupled to gravity can be found in Sec. V. In Sec. VI, some examples of the restrictions imposed by symmetric hyperbolicity are presented for different theories. Our closing remarks are presented in Sec. VII.

II. LAGRANGIANS AND EQUATIONS OF MOTION

To begin with, let $\mathcal{M}$ denote a smooth four-dimensional spacetime with a Lorentzian metric $\boldsymbol{g}$ of signature $(+,-,-,-)$. For the sake of concreteness we assume $\mathcal{M}$ to be also oriented and globally hyperbolic, i.e., $M \cong \mathbb{R}\times\Sigma$, with Σ a codimension-1 hypersurface. Sticking to the conventions $R_{ab} \equiv R^{c}{}_{acb}$ and $[ab] \equiv ab - ba$, the canonical decomposition of the Riemman tensor into its irreducible parts reads

$$R^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{1}{2}\delta^{[a}{}_{[c}S^{b]}{}_{d]} + \frac{R}{12}g^{ab}{}_{cd}, \tag{1}$$

where W_{abcd} is the Weyl conformal tensor, S_{ab} is the traceless part of the Ricci tensor, R is the scalar curvature, and $g_{abcd} = g_{a[c}g_{d]b}$ is the Kulkarni-Nomizu product of the metric with itself. Clearly, each factor in the decomposition has the same algebraic symmetries as the full Riemann tensor, that is

$$R_{abcd} = -R_{bacd} = -R_{abdc}, \qquad R_{abcd} = R_{cdab}, \tag{2}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{3}$$

An arbitrary rank four covariant tensor satisfying Eq. (2) has 21 independent components and is often referred to as a double symmetric (2,2) form (the skew pairs can be interchanged). If, in addition, the tensor satisfies Eq. (3), it is called an algebraic curvature tensor (see for instance [34]).

If we are given a double symmetric (2,2) form on $\mathcal{M}$, say χ, then we may obtain its left-dual $\star\chi$ and right-dual $\chi\star$ by

$$\star\chi_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{pq}\chi_{pqcd}, \qquad \chi_{abcd}\star = \frac{1}{2}\varepsilon^{pq}{}_{cd}\chi_{abpq}, \tag{4}$$

where ε_{abcd} is the Levi-Civita tensor with $\varepsilon_{0123} = \sqrt{-g}$. Notice that the place of the $\star$ indicates the pair of skew indices, which are Hodge dualized. A direct consequence is that

$$\star\star\chi_{abcd} = -\chi_{abcd}, \qquad \chi_{abcd}\star\star = -\chi_{abcd}. \tag{5}$$

We aim at constructing Lagrangians describing the coupling of gravity with the electromagnetic field using invariants that are at most linear in the curvature,[3] and respecting the $U(1)$ gauge invariance of electromagnetism. Recalling that every algebraic curvature tensor satisfy the Ruse-Lanczos identity, we may construct the following independent rank four tensors (see the Appendix for details)

$$^{(1)}\chi^{ab}{}_{cd} \equiv g^{ab}{}_{cd}, \qquad ^{(2)}\chi^{ab}{}_{cd} \equiv \varepsilon^{ab}{}_{cd}, \tag{6}$$

$$^{(3)}\chi^{ab}{}_{cd} \equiv R g^{ab}{}_{cd}, \qquad ^{(4)}\chi^{ab}{}_{cd} \equiv R\varepsilon^{ab}{}_{cd}, \tag{7}$$

$$^{(5)}\chi^{ab}{}_{cd} \equiv W^{ab}{}_{cd}, \qquad ^{(6)}\chi^{ab}{}_{cd} \equiv \star W^{ab}{}_{cd}, \tag{8}$$

$$^{(7)}\chi^{ab}{}_{cd} \equiv \delta^{[a}{}_{[c}S^{b]}{}_{d]}. \tag{9}$$

Notice that $^{(\Gamma)}\chi_{abcd}$ $(\Gamma = 1, 2, \ldots, 7)$ are double symmetric (2,2) forms by construction and it can be checked that they exhaust the interesting possibilities: the first pair, Eq. (6), involves only algebraic terms in the metric, while the remaining Eqs. (7)–(9) contain at most linear terms in the curvature.

Let $F_{ab} = \partial_{[a}A_{b]}$ denote a test electromagnetic (EM) field propagating on $\mathcal{M}$. We shall focus on the propagation properties of such a field on a fixed gravitational background. The EM field will be described by the gauge-invariant action functional of the form

[2] For a list of other approaches to the analysis of the evolution of minimally coupled electromagnetism, see [26].

[3] As discussed in [28], higher-order couplings lead to equations of motion with derivatives of the metric higher than 2.

$$S = \frac{1}{4}\int d^4x\sqrt{-g}\mathcal{L}(I^1, I^2, \ldots, I^7), \tag{10}$$

where the factor 1/4 is introduced for future convenience and the Lagrangian density is taken as an arbitrary smooth function of the following scalars

$$I^\Gamma \equiv {}^{(\Gamma)}H^{ab}F_{ab}, \quad \text{where } {}^{(\Gamma)}H^{ab} \equiv \frac{1}{2}\chi^{ab}{}_{cd}F^{cd}. \tag{11}$$

In analogy with electrodynamics in material media, we call ${}^{(\Gamma)}H^{ab}$ the Γth induction tensor and ${}^{(\Gamma)}\chi^{ab}{}_{cd}$ the Γth constitutive tensor. Particular instances described by Eq. (10) include Maxwell's theory, minimally-coupled nonlinear electrodynamics, and the three-parameter nonminimal Einstein-Maxwell model originating from QED vacuum polarization in a background gravitational field (see for instance [35]), among others. With the conventions presented above, the variation of the action with respect to the four potential yields a coupled system of first-order quasilinear partial differential equations (PDEs) for the fields, given by

$$\nabla_b H^{ab} = 0, \qquad \nabla_b \star F^{ab} = 0. \tag{12}$$

Here, ∇_a is the covariant derivative compatible with the metric and the full induction tensor is defined by

$$H^{ab} \equiv \sum \mathcal{L}_\Gamma {}^{(\Gamma)}H^{ab}, \tag{13}$$

with $\mathcal{L}_\Gamma \equiv \partial\mathcal{L}/\partial I^\Gamma$, for conciseness. Notice that the system defined by Eq. (12) is composed of eight equations for only six unknowns. In relevant physical situations, the equations must include six dynamical equations (defined with respect to some time coordinate, to be identified later) and two constraints, which are to be imposed on initial data.

III. SYMMETRIC-HYPERBOLICITY

In order to study whether the equations of motion admit a symmetric-hyperbolic representation, it is convenient to recast the system of equations Eq. (12) in a unified manner as

$$K_A{}^m{}_\beta(x,\Phi)\partial_m\Phi^\beta + J_A(x,\Phi) = 0, \tag{14}$$

where $x \in \mathcal{M}$, $K_A{}^m{}_\beta$ is the principal part of the PDE and $J_A(x,\Phi)$ stands for semilinear contributions (whose explicit form is unnecessary for our discussion). Here capital Latin indices ($A = 1, \ldots, 8$) stand for the space of multitensorial equations, lowercase Latin indices ($m = 0$, 1, 2, 3) stand for space-time indices, and Greek indices ($\beta = 1, \ldots, 6$) for tensorial unknowns. To start with, we introduce an ordering of the antisymmetric indices to obtain the six possible collective quantities

$$1 \to (01) \quad 2 \to (02) \quad 3 \to (03) \quad 4 \to (32)$$
$$5 \to (13) \quad 6 \to (21). \tag{15}$$

Making the identification $\Phi^\beta \to F^{bc}$ and performing simple manipulations in Eqs. (12), the principal part is then written as

$$K_A{}^m{}_\alpha = \frac{1}{2}(X_a{}^m{}_{bc}, \varepsilon_a{}^m{}_{bc}), \tag{16}$$

with

$$X_{abcd} \equiv \sum_\Gamma \mathcal{L}_\Gamma {}^{(\Gamma)}\chi_{abcd} + 4\sum_\Gamma\sum_\Lambda \mathcal{L}_{\Gamma\Lambda} {}^{(\Gamma)}H_{ab} {}^{(\Lambda)}H_{cd}. \tag{17}$$

X_{abcd} consists of a main term involving only first partial derivatives of the Lagrangian density and a nonlinear term including the Hessian matrix of the latter. Clearly, if the Lagrangian is a linear combination of the invariants I^Γ, the last term vanishes and linear equations of motion follow. More importantly, X_{abcd} is always a symmetric double (2,2)-form independently on the specific form of the Lagrangian. This is a direct consequence of the symmetries of the constitutive tensors together with the symmetry of the Hessian matrix, i.e., $\mathcal{L}_{\Gamma\Lambda} = \mathcal{L}_{\Lambda\Gamma}$.

In what follows we shall use the covariant approach for first-order symmetric-hyperbolic systems outlined in [33]: a symmetric hyperbolization of Eq. (14) means that there exist a smooth symmetrizer $h^A{}_\alpha$ and a covector field n_m, such that

(1) $\hat{K}^m_{\alpha\beta} \equiv h^A{}_\alpha K_A{}^m{}_\beta$ is symmetric in the indices α, β.

(ii) The matrix $\hat{K}_{\alpha\beta}(n) \equiv \hat{K}_\alpha{}^m{}_\beta n_m$ is positive definite.

Roughly speaking, the first statement means that it should be possible to construct from Eqs. (14) a new subsystem of first-order quasilinear PDEs given by

$$\hat{K}_\alpha{}^m{}_\beta\partial_m\Phi^\beta + \hat{J}_\alpha = 0, \tag{18}$$

with $\hat{J}_\alpha \equiv h^A{}_\alpha J_A$. Clearly, the new system contains only evolution equations and its dimension is equivalent to the number of unknown fields. The second statement means that the new system can be solved uniquely for any given set of initial conditions on a hypersurface Σ with normal covector n_m. What are the remaining equations the symmetrizer does not capture? For the system to be consistent they should not be of the evolution type. In other words, they must be satisfied automatically once they are satisfied initially, i.e., they should be what we normally call the constraints.[4]

The geometrical meaning of the covectors introduced in the previous paragraph is as follows. At a spacetime point $p \in \mathcal{M}$, the collection of all covectors satisfying condition (2) is denoted by S_p. This set defines a nonempty open,

[4]See Refs. [3,33,36] for additional details.

convex cone at p and the tangent vectors $p^a \in T_p\mathcal{M}$ such that $p^a n_a > 0$ for all $n_a \in S_p$ determine the cone of influence of the physical field, i.e., the maximal speed of propagation in any given direction. It turns out that the latter is also a nonempty open, convex cone at p.

In the next subsection, we start by showing that a family of symmetrizers parametrized by a vector field always exists for the system of first-order PDEs given by Eqs. (12). It is important to point out that our result is general in the sense that it does not depend on the specific form of the Lagrangian density. We then obtain the conditions that a theory must satisfy for positive definiteness [see condition (2) above] to hold. This is first obtained for a particular covector and henceforth generalized by inspecting the characteristic varieties (dispersion relations), which are necessarily well-behaved for symmetric hyperbolic systems.

A. Symmetrizer

In order to find a symmetrizer, it is convenient to work with projections. In other words, we seek a multitensorial field $h^A{}_\alpha$ such that the quantity $\delta\phi^\alpha(h^A{}_\alpha K_A{}^m{}_\beta)\delta\psi^\beta$ is symmetric in $\delta\phi$ and $\delta\psi$. Making the identifications

$$\delta\phi^\alpha \to A^{ab}, \qquad \delta\psi^\alpha \to B^{ab}, \tag{19}$$

where A and B are generic bivectors, we obtain from Eq. (16), the relation

$$K_A{}^m{}_\beta \delta\psi^\beta = \left(\frac{1}{2} X_a{}^m{}_{bc} B^{bc}, \star B_a{}^m\right). \tag{20}$$

Since there is no known practical procedure to obtain a symmetrizer for an arbitrary system of PDE's, hyperbolizations are found, for a sufficiently low number of dimensions, by solving explicitly the algebraic equations inherent to the system defined by Eq. (14) and, in higher dimensions, by guessing. Let us show that the symmetrizer is given by the projection

$$\delta\phi^\alpha h^A{}_\alpha = \left(A^a{}_q, \frac{1}{2}\star X^a{}_{qrs} A^{rs}\right) t^q, \tag{21}$$

where t^q is an auxiliary vector field and $\star X^a{}_{qrs}$ is the left Hodge dual as defined before. Indeed, multiplying (21) by (20) one obtains

$$\delta\phi^\alpha(h^A{}_\alpha K_A{}^m{}_\beta)\delta\psi^\beta = \frac{1}{2}(X_a{}^m{}_{bc} A^a{}_q B^{bc} + \star X^a{}_{qrs} A^{rs} \star B_a{}^m) t^q. \tag{22}$$

Now, defining

$$Y^a{}_q \equiv X^a{}_{qrs} A^{rs}, \qquad \star Y^a{}_q \equiv \star X^a{}_{qrs} A^{rs}, \tag{23}$$

and using the well-known identity valid for antisymmetric tensors

$$(\star Y^{aq})(\star B_{am}) = -\frac{1}{2}(Y_{ln} B^{ln})\delta_m{}^q + Y_{am} B^{aq}, \tag{24}$$

it follows that

$$\delta\phi^\alpha(h^A{}_\alpha K_A{}^m{}_\beta)\delta\psi^\beta = \frac{1}{2}\left[X_a{}^m{}_{bc}(A^a{}_q B^{bc} + A^{bc} B^a{}_q) - \frac{1}{2}(X_{abcd} B^{ab} A^{cd})\delta^m{}_q\right] t^q, \tag{25}$$

which is obviously symmetric in A and B since $X_{abcd} = X_{cdab}$. Therefore, the symmetrizer itself is given by the simple expression

$$h^A{}_\alpha = \frac{1}{2}(g^a{}_{qrz}, \star X^a{}_{qrz}) t^q, \tag{26}$$

It is remarkable that Eq. (26) is valid independently of the specific content of the tensor field X_{abcd}: in particular it does not matter whether the equations of motion contain quasilinear terms or not.

The application of Eq. (26) to (16) yields the object

$$\hat{K}_\alpha{}^m{}_\beta = \frac{1}{4}(g^a{}_{qrz} X_a{}^m{}_{bc} + \star X^a{}_{qrz} \varepsilon_a{}^m{}_{bc}) t^q, \tag{27}$$

which, after straightforward algebraic manipulations, becomes

$$\hat{K}_\alpha{}^m{}_\beta = -\frac{1}{4}(g_{q[a} X_{b]}{}^m{}_{cd} + g_{q[c} X_{d]}{}^m{}_{ab} + \delta^m{}_q X_{abcd}) t^q. \tag{28}$$

Notice that this equation is indeed symmetric in the exchange of antisymmetric indices $ab \Leftrightarrow cd$ and that the auxiliary vector field t^q remains (up to now) arbitrary, so we can use it at our disposal. This concludes the first step of our task.

B. Positive definiteness

Let us next investigate whether the above symmetrizer constitutes a true hyperbolization of the equations of motion. This will be achieved if we manage to find a covector field n_m such that the characteristic matrix

$$\hat{K}_{\alpha\beta}(t,n) = -\frac{1}{4}[t_{[a} X_{b]}{}^m{}_{cd} + t_{[c} X_{d]}{}^m{}_{ab} + t^m X_{abcd}] n_m \tag{29}$$

is positive definite, i.e.,

$$\delta\phi^\alpha \hat{K}_{\alpha\beta} \delta\phi^\beta > 0, \tag{30}$$

for all nonzero vectors $\delta\phi^\alpha \in \mathbb{R}^6$. A direct calculation gives the equivalent inequality

$$-\left[t_a X_b{}^m{}_{cd} A^{ab} A^{cd} + \frac{1}{4} t^m X_{abcd} A^{ab} A^{cd}\right] n_m > 0, \qquad (31)$$

where A^{ab} is an arbitrary but nonzero bivector. Notice that the latter is a fully covariant expression in the sense that it does not depend on any particular choice of coordinates. It will restrict, however, the choices of admissible covector and auxiliary vector fields.

In order to proceed, we shall decompose A^{ab} and X_{abcd} with respect to the auxiliary vector field t^q. To do so, assuming that t^q is timelike, future directed, and normalized, we write the bivector as

$$A_{ab} = \mathfrak{a}_{[a} t_{b]} + \varepsilon_{ab}{}^{cd} t_c \mathfrak{b}_d, \qquad (32)$$

with

$$\mathfrak{a}_a \equiv A_{ab} t^b, \quad \mathfrak{b}_a \equiv \star A_{ab} t^b, \quad \mathfrak{a}_a t^a = 0 \quad \mathfrak{b}_a t^a = 0. \qquad (33)$$

Similarly, for any double (2,2) form, we have[5]

$$X_{abcd} = \{g_{abpq}(g_{cdrs}\mathbb{P}^{pr} + \varepsilon_{cdrs}\mathbb{Q}^{pr}) + \varepsilon_{abpq}(g_{cdrs}\mathbb{R}^{pr} + \varepsilon_{cdrs}\mathbb{S}^{pr})\} t^q t^s, \qquad (34)$$

with the two-index tensors given by

$$\mathbb{P}_{ab} \equiv X_{acbd} t^c t^d, \qquad \mathbb{Q}_{ab} \equiv -X_{acbd}\star t^c t^d,$$
$$\mathbb{R}_{ab} \equiv -\star X_{acbd} t^c t^d, \qquad \mathbb{S}_{ab} \equiv \star X_{acbd}\star t^c t^d, \qquad (35)$$

and orthogonal to t^q by construction. In general, each of the latter has nine independent components since the number of independent components of X_{abcd} is 36. However, in the particular case in which $X_{abcd} = X_{cdab}$, the following simplified relations are valid:

$$\mathbb{P}_{ab} = \mathbb{P}_{ba}, \qquad \mathbb{S}_{ab} = \mathbb{S}_{ba}, \qquad \mathbb{Q}_{ab} = \mathbb{R}_{ba}, \qquad (36)$$

so that $\mathbb{P}_{ab}$ and $\mathbb{S}_{ab}$ have six independent components each, while $\mathbb{Q}_{ab}$ (or, equivalently, $\mathbb{R}_{ab}$) have the remaining nine components. Notice that, in the context of electrodynamics in material media, these tensors would be related to the permittivity, permeability, and magnetoelectric cross terms of the medium [38].

Using the above decompositions in Eq. (31), the new covariant inequality

$$(n_m t^m)[\mathbb{P}_{pr}\mathfrak{a}^p\mathfrak{a}^r - \mathbb{S}_{pr}\mathfrak{b}^p\mathfrak{b}^r] > 2(n_m t^n)\varepsilon^{mq}{}_{np}[\mathbb{R}_{qr}\mathfrak{a}^p\mathfrak{a}^r + \mathbb{S}_{qr}\mathfrak{a}^p\mathfrak{b}^r] \qquad (37)$$

follows, which is to be satisfied for any three vectors $\mathfrak{a}^p$ and $\mathfrak{b}^q$, not vanishing simultaneously. Furthermore, since our considerations are essentially algebraic we may restrict to an arbitrary point $p \in \mathcal{M}$. In order to complete our symmetric hyperbolization, it suffices to find a specific covector $n_a \in T_p^*\mathcal{M}$ satisfying inequality (37) since, if this is the case, there will be a unique connected, open and convex cone, S_p, containing the initial covector, and this cone will exhaust all possibilities. The natural choice here is the covector $n_a = g_{ab}t^b$, since the right-hand side of Eq. (37) will vanish. Now, since $n_m t^m > 0$ for the latter, we obtain the equivalent inequalities

$$\mathbb{P}_{pr}\mathfrak{a}^p\mathfrak{a}^r > 0, \qquad \mathbb{S}_{pr}\mathfrak{b}^p\mathfrak{b}^r < 0,$$

for all nonzero vectors $\mathfrak{a}^p$, $\mathfrak{b}^q$. In words, a symmetric hyperbolization is achieved for a covector n_a if the two-index tensors $\mathbb{P}_{ab}$ and $\mathbb{S}_{ab}$, obtained from X_{abcd} and $\star X_{abcd}\star$ via contractions with n^a, satisfy the above inequalities.

For the sake of completeness, let us a repeat the calculations in an adapted frame at p, such that

$$g_{ab}(p) = \eta_{ab}, \qquad t^q = \delta^q{}_0. \qquad (38)$$

Using Eq. (34), it can be shown that X_{abcd} has the following block matrix display:

$$X_{\alpha\beta} = \begin{pmatrix} \mathbb{P} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{S} \end{pmatrix}, \qquad (39)$$

where $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{S}$ denote the covariant 3×3 matrices constructed with the corresponding tensors in the obvious way. In order to compute Eq. (29) in matrix form it is convenient to define the auxiliary 3×3 matrices (a similar notation was used in [27])

$$\mathbf{A}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{A}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking into account that the transposition relation $(\mathbf{A}^k)^T = -\mathbf{A}^k$ holds, a direct calculation gives:

$$\hat{K}_{\alpha\beta}(n) = \begin{pmatrix} n_0\mathbb{P} - n_k(\mathbb{Q}\mathbf{A}^k - \mathbf{A}^k\mathbb{Q}^T) & n_k\mathbf{A}^k\mathbb{S} \\ -n_k\mathbb{S}\mathbf{A}^k & -n_0\mathbb{S} \end{pmatrix}. \qquad (40)$$

The above relation may be thought as a linear map from $T_x^*\mathcal{M}$ to the 21-dimensional space of symmetric 6×6 matrices Sym_6. The set of all symmetric positive definite matrices forms an open convex cone in Sym_6 with apex on the origin. It turns out that the image of the particular covector $n_m = \eta_{mn}t^n$ will lie inside this cone whenever

$$\mathbb{P} \succ 0, \qquad \mathbb{S} \prec 0, \qquad (41)$$

[5]See for instance [37] for similar decompositions.

since $n_m = (1, 0, 0, 0)$ in our frame. Therefore, if Eqs. (41) are satisfied point wisely, symmetric hyperbolicity is guaranteed for the corresponding covector field. In other words, if the auxiliary vector field t^a is vorticity free, then initial data given on a hypersurface Σ with normal covectors t_a are uniquely evolved away from the hypersurface. As will be shown in Sec. VI, the requirements set by Eq. (41) may yield severe constraints on the Lagrangian density, the intensity of the curvature tensor or the electromagnetic field.

C. Constraints

We now discuss the remaining equations, which are left aside by the symmetrizer. To do so, we recall that, according to Geroch's formalism, a constraint is a tensor c^{An} such that

$$c^{An} K_A{}^m{}_\alpha + c^{Am} K_A{}^n{}_\alpha = 0. \tag{42}$$

It is straightforward to show that, in our case, this tensor is given by

$$c^{An} = (x g^{an}, y g^{an}), \tag{43}$$

where $x, y \in \mathbb{R}$. Indeed, multiplying Eq. (43) by (16) one obtains

$$c^{An} K_A{}^m{}_\alpha = \frac{1}{2}(x X^{nm}{}_{bc} + y \epsilon^{nm}{}_{bc}) \tag{44}$$

which is obviously antisymmetric in n and m. Here, the emergence of two real numbers reveals that the vector space of constraints is actually two dimensional, as expected.

From the above calculation one concludes that the constraints are complete, in the sense that the number of constraint equations plus the number of evolution equations equals the number of initial equations. The constraints are integrable if the equation

$$c^{An} \nabla_n (K_A{}^m{}_\beta \nabla_m \Phi^\beta + J_A) = 0 \tag{45}$$

is identically satisfied solely due the algebraic structure of the principal symbol, independently of the Eq. (14) of the original system. We leave for the reader to verify that this is indeed the case. This means that the original system of equations is equivalent to a symmetric-hyperbolic one with two additional integrable constraints.

IV. CHARACTERISTIC CONES

Suppose that we manage to find a symmetric hyperbolization with constraints, as described in the previous sections. We have seen that this choice, however, is far from unique. Indeed, by the continuity of Eq. (40), any small deformation of t_a will be such that condition (2) of symmetric hyperbolization is satisfied. It turns out that the set of all admissible covectors S_p in $T^*_p\mathcal{M}$ is determined by the unique connected, open, convex, positive cone containing the initial covector [3]. Its existence is related to the hyperbolicity of the characteristic polynomial, defined by

$$p(n) \equiv \det(\hat{K}_{\alpha\beta}(n)), \tag{46}$$

which is a homogeneous multivariate polynomial of degree 6 in our case. Recall that, at a spacetime point, such a polynomial is called hyperbolic in a direction t_a if $p(t_a) > 0$ and the univariate polynomial $p(u_a + \lambda t_a)$ only has real roots for all covectors $u_a \neq t_a$. Now the vanishing set of the characteristic polynomial will define an algebraic variety: the cone of characteristic conormals (or characteristic cone, for brevity). In general, it will consist of different codimension 1 sheets that may be nested, intersect along lines, or even coincide. Geometrically, hyperbolicity in the direction of t_a is the requirement that every line parallel to t_a intersects this algebraic variety at exactly "6" points (counting multiplicities). Clearly, this condition severely constrains the topology of the characteristic cones, thus guaranteeing well-behaved propagation for small wavy excitations. In particular, it was shown by Garding [39] that the closure of the connected component of t_a in the set $\{n_a | p(n_a) \neq 0\}$ is necessarily convex: the hyperbolicity cone of the polynomial. That symmetric hyperbolicity implies the hyperbolicity of the characteristic polynomial is direct. To see this one simply observes that the equation

$$\det(\hat{K}_{\alpha\beta}(a + \lambda t)) = 0 \tag{47}$$

characterizes the eigenvalues of the quadratic form $\hat{K}_{\alpha\beta}(a_m)$ relative to the metric $\hat{K}_{\alpha\beta}(t_m)$—and these eigenvalues have to be real (see [3] for further details).

In order to compute the characteristic polynomial explicitly, we substitute Eq. (40) into (46) and use Schur's determinant identity to obtain the product of determinants

$$p(n) = -\det(\mathbb{S}) q(n), \tag{48}$$

where

$$q(n) \equiv \det(n_0^2 \mathbb{P} - n_0 n_k (\mathbb{Q}\mathbf{A}^k - \mathbf{A}^k \mathbb{Q}^T) - n_k n_l \mathbf{A}^k \mathbb{S} \mathbf{A}^l). \tag{49}$$

Notice that $q(t)$ is necessarily positive definite, since $\mathbb{P} > 0$ and $\mathbb{S} < 0$ for this particular covector. Interestingly, the determinant in Eq. (49) has been calculated several times in literature, see, e.g., [27,40]. In particular, it can be shown that it factorizes as

$$q(n) = (n_m t^m)^2 P(n), \tag{50}$$

where

$$P(n) = \frac{1}{24}\varepsilon_{a_1a_2a_3a_4}\varepsilon_{b_1b_2b_3b_4} \times X^{a_1a_2b_1c_1}X^{c_2a_3b_2c_3}X^{c_4a_4b_3b_4}n_{c_1}n_{c_2}n_{c_3}n_{c_4}. \quad (51)$$

In other words, the sixth-order polynomial always reduces to a product of a quadratic polynomial and a quartic polynomial. Since the vanishing set of the quadratic polynomial gives a noncompact variety (plane) inconsistent with the constraint equations, the characteristic cone is given by the covectors k_a that satisfy the fourth-order Fresnel equation

$$P(k) \sim (\star X_{pq}{}^{ar}X^{bpcs}X^{dq}{}_{rs}\star)k_ak_bk_ck_d = 0. \quad (52)$$

The fourth rank tensor defined by the terms between parentheses is called the Kummer tensor, whereas its totally symmetrized version is usually called the Tamm-Rubilar tensor. If symmetric hyperbolicity holds, then the above polynomial is necessarily hyperbolic in the direction of t_a, its vanishing set determining the causal structure of the theory up to a conformal factor. In specific situations, where X_{abcd} is sufficiently simple, Eq. (52) will reduce to the more familiar product of quadratic polynomials[6]

$$P(k) \sim (g_{(1)}^{ab}k_ak_b)(g_{(2)}^{ab}k_ck_d) = 0. \quad (53)$$

In these cases, symmetric hyperbolicity guarantees that the rank-2 contravariant tensors $g_{(1)}^{ab}$ and $g_{(2)}^{ab}$ are nondegenerate, necessarily of Lorentzian type and with the same signature. Furthermore, if these tensors coincide,[7] then only one of them must be considered in the dispersion relation (reduced polynomial), i.e.,

$$P_{\mathrm{red}}(k) \sim g_{(1)}^{ab}k_ak_b = 0. \quad (54)$$

It is important to point out that the above-mentioned result stating that symmetric hyperbolicity leads to a cone strongly constraints the possible shape of the Fresnel surfaces, defined in a convenient three space: they must be topologically equivalent to those obtained from the 4D hyperbolicity cone. Hence, open surfaces are not allowed by symmetric-hyperbolic propagation.[8]

V. GENERAL RESULT

We have seen that symmetric-hyperbolicity imposes restrictions on admissible physical theories. Importantly, they are always expressed in terms of matrix inequalities, which are much easier to guarantee than to check the hyperbolicity of the corresponding characteristic polynomial. In order to obtain explicit expressions for the matrix inequalities, it is convenient to introduce the projection tensor $h_{ab} = g_{ab} - t_at_b$, which projects arbitrary tensors onto the "rest spaces" orthogonal to t^q, i.e.,

$$h_{ab} = h_{ba}, \qquad h_{ac}h^c{}_b = h_{ab}, \qquad h_{ab}t^b = 0. \quad (55)$$

Similarly, we decompose the electromagnetic two form, the Weyl tensor and the traceless part of the Ricci tensor as

$$F_{ab} = E_{[a}t_{b]} + \varepsilon_{abcd}t^cB^d, \quad (56)$$

$$W_{abcd} = \{g_{abpq}(g_{cdrs}\mathcal{E}^{pr} - \varepsilon_{cdrs}\mathcal{B}^{pr}) - \varepsilon_{abpq}(g_{cdrs}\mathcal{B}^{pr} + \varepsilon_{cdrs}\mathcal{E}^{pr})\}t^qt^s, \quad (57)$$

$$S_{ab} = St_at_b + Q_{(a}t_{b)} + N_{ab}, \quad (58)$$

with the following definitions

$$E_a \equiv F_{ab}t^b, \qquad B_a \equiv \star F_{ab}t^b, \quad (59)$$

$$\mathcal{E}_{ab} \equiv W_{acbd}t^ct^d, \qquad \mathcal{B}_{ab} \equiv \star W_{acbd}t^ct^d, \quad (60)$$

$$S \equiv S_{ab}t^at^b, \quad Q_a \equiv h_a{}^bS_{bc}t^c, \quad N_{ab} \equiv h_a{}^ch_b{}^dS_{cd}. \quad (61)$$

According to the latter, the tensor fields $\{E_a, B_a, Q_a, \mathcal{E}_{ab}, \mathcal{B}_{ab}, N_{ab}\}$ are automatically orthogonal to the auxiliary vector field.

In order to compute $\mathbb{P}_{ab}$ and $\mathbb{S}_{ab}$ from Eq. (17), the following relations involving the constitutive tensors are useful

$${}^{(1)}\chi_{abcd}t^bt^d = h_{ac}, \qquad {}^{(2)}\chi_{abcd}t^bt^d = 0, \quad (62)$$

$${}^{(3)}\chi_{abcd}t^bt^d = Rh_{ac}, \qquad {}^{(4)}\chi_{abcd}t^bt^d = 0, \quad (63)$$

$${}^{(5)}\chi_{abcd}t^bt^d = \mathcal{E}_{ac}, \qquad {}^{(6)}\chi_{abcd}t^bt^d = \mathcal{B}_{ac}, \quad (64)$$

$${}^{(7)}\chi_{abcd}t^bt^d = N_{ac}. \quad (65)$$

Similarly, for the induction tensors, Eq. (11), one obtains

$${}^{(1)}H_{ab}t^b = E_a, \qquad {}^{(2)}H_{ab}t^b = B_a, \quad (66)$$

$${}^{(3)}H_{ab}t^b = RE_a, \qquad {}^{(4)}H_{ab}t^b = RB_a, \quad (67)$$

$${}^{(5)}H_{ab}t^b = \mathcal{E}_{ab}E^b - \mathcal{B}_{ab}B^b, \quad {}^{(6)}H_{ab}t^b = \mathcal{E}_{ab}B^b + \mathcal{B}_{ab}E^b, \quad (68)$$

$${}^{(7)}H_{ab}t^b = SE_a + N_{ab}E^b - \varepsilon_{abcd}t^bQ^cE^d, \quad (69)$$

Using the above relations and the Ruse-Lanczos identities (see the Appendix), it can be checked that the tensors $\mathbb{P}_{ab}$ and $\mathbb{S}_{ab}$ are given by

[6]A general ansatz leading to bimetricity in nonlinear electromagnetism was presented in [41].

[7]Covariant conditions on the Fresnel surface for birefringence to be absent were derived in [42].

[8]For examples of Fresnel surfaces in Maxwell's theory nonminimally coupled to gravity, see [38].

$$\mathbb{P}_{ab} = (\mathcal{L}_1 + \mathcal{L}_3 R)h_{ab} + \mathcal{L}_5\mathcal{E}_{ab} + \mathcal{L}_6\mathcal{B}_{ab} + \mathcal{L}_7 N_{ab} + 4E_aE_b(\mathcal{L}_{11} + 2R\mathcal{L}_{13}) + 4E_{(a}B_{b)}[\mathcal{L}_{12} + R(\mathcal{L}_{14} + \mathcal{L}_{23})]$$
$$+ 4B_aB_b(\mathcal{L}_{22} + 2R\mathcal{L}_{24}) + 4[\mathcal{L}_{15}E_{(a} + \mathcal{L}_{25}B_{(a}][\mathcal{E}_{b)c}E^c - \mathcal{B}_{b)c}B^c] + 4[\mathcal{L}_{16}E_{(a} + \mathcal{L}_{26}B_{(a}][\mathcal{E}_{b)c}B^c + \mathcal{B}_{b)c}E^c]$$
$$+ 4[\mathcal{L}_{17}E_{(a} + \mathcal{L}_{27}B_{(a}][SE_{b)} + N_{b)c}E^c - \varepsilon_{b)pqr}Q^p t^q E^r] + \cdots, \quad (70)$$

$$\mathbb{S}_{ab} = -(\mathcal{L}_1 + \mathcal{L}_3 R)h_{ab} - \mathcal{L}_5\mathcal{E}_{ab} - \mathcal{L}_6\mathcal{B}_{ab} + \mathcal{L}_7 N_{ab} + 4B_aB_b(\mathcal{L}_{11} + 2R\mathcal{L}_{13})$$
$$- 4E_{(a}B_{b)}[\mathcal{L}_{12} + R(\mathcal{L}_{14} + \mathcal{L}_{23})] + 4E_aE_b(\mathcal{L}_{22} + 2R\mathcal{L}_{24}) + 4[\mathcal{L}_{15}B_{(a} - \mathcal{L}_{25}E_{(a}][\mathcal{E}_{b)c}B^c + \mathcal{B}_{b)c}E^c]$$
$$- 4[\mathcal{L}_{16}B_{(a} - \mathcal{L}_{26}E_{(a}][\mathcal{E}_{b)c}E^c - \mathcal{B}_{b)c}B^c] - 4[\mathcal{L}_{17}B_{(a} - \mathcal{L}_{27}E_{(a}][SB_{b)} + N_{b)c}B^c - \varepsilon_{b)pqr}Q^p t^q B^r] + \cdots, \quad (71)$$

where the dots stand for possible nonlinear terms in the irreducible parts of the curvature tensor, which we shall not discuss in this work. The tensors $\mathbb{P}_{ab}$ and $\mathbb{S}_{ab}$ given above are the most general ones when dealing with symmetric hyperbolicity in models of nonlinear electromagnetism coupled linearly to curvature. The inequalities given in Eqs. (41), defined in terms of such tensors, lead to drastic restrictions on admissible theories: there must be a strong compromise between the electromagnetic quantities $\{E^a, B^a\}$, the spacetime curvature expressed via $\{\mathcal{E}_{ab}, \mathcal{B}_{ab}, Q_a, N_{ab}\}$ and the partial derivatives of the Lagrangian density.

VI. APPLICATIONS

Let us next present several examples that illustrate the restrictions obtained so far. For the sake of completeness, we display, in each case, the tensor X_{abcd}, the relevant inequalities, and the corresponding characteristic polynomial. A generic feature of the latter is nonfactorization: in general, the characteristic varieties are described by vanishing sets of fourth-order polynomials that do not split into products of second-order polynomials. Needless to say, this fact makes it considerably difficult to guarantee hyperbolicity straight from the characteristic polynomial. In contrast with this fact, symmetric-hyperbolicity automatically implies that the characteristic varieties are necessarily well behaved.

A. Maxwell electrodynamics $\mathcal{L} = -I^1$

Our first example is Maxwell's linear theory, which is the simplest possible case in our setting. Using Eq. (17), it follows that

$$X_{abcd} = -g_{abcd}. \quad (72)$$

Taking the auxiliary vector field t^q as timelike, future-directed, and normalized, Eqs. (70) and (71) give the two-index tensors as

$$\mathbb{P}_{ab} = -h_{ab}, \qquad \mathbb{S}_{ab} = h_{ab}. \quad (73)$$

In a frame such that $g_{ab} = \eta_{ab}$ and $t^q = \delta^q{}_0$, the corresponding 3×3 matrices are

$$\mathbb{P}_{ij} = \delta_{ij}, \qquad \mathbb{S}_{ij} = -\delta_{ij}, \qquad i, j = 1, 2, 3. \quad (74)$$

Since they satisfy Eq. (41) identically, symmetric-hyperbolicity is guaranteed for all timelike covector fields $n_m = \eta_{mn}t^n$. A direct calculation using Eq. (52) leads to

$$P(k) \sim (g^{ab}k_ak_b)^2.$$

Only one of the quadratic polynomials is needed to obtain the usual dispersion relation for linear electromagnetic waves in vacuum:

$$P_{\rm red}(k) \sim g^{ab}k_ak_b = 0. \quad (75)$$

Clearly, the cone of hyperbolicity at p, S_p, is the connected component of t_a, i.e., it coincides with the set of future-directed timelike covectors, as expected.

B. Minimally coupled nonlinear electrodynamics

1. Single invariant: $\mathcal{L} = \mathcal{L}(I^1)$

If the Lagrangian density is an arbitrary function of the invariant I^1, a direct inspection of Eq. (17) gives

$$X_{abcd} = \mathcal{L}_1 g_{abcd} + 4\mathcal{L}_{11}F_{ab}F_{cd}. \quad (76)$$

From Eqs. (70) and (71), we obtain the projections

$$\mathbb{P}_{ab} = \mathcal{L}_1 h_{ab} + 4\mathcal{L}_{11}E_aE_b, \quad \mathbb{S}_{ab} = -\mathcal{L}_1 h_{ab} + 4\mathcal{L}_{11}B_aB_b, \quad (77)$$

Simple manipulations using Eq. (41) then yield the matrix inequalities

$$\mathcal{L}_1\delta_{ij} - 4\mathcal{L}_{11}E_iE_j \prec 0, \quad (78)$$

$$\mathcal{L}_1\delta_{ij} + 4\mathcal{L}_{11}B_iB_j \prec 0. \quad (79)$$

In particular, they imply the condition $\mathcal{L}_1 < 0$, since the theory should be well behaved when either the electric or magnetic field vanish. However, notice that for field intensities violating the inequalities, the propagation of small wavy excitations about the background field will be, in general, ill posed.

A straightforward calculation using Eq. (52) gives the well-known bimetric dispersion relation

$$P(k) \sim (g^{ab}_{(1)} k_a k_b)(g^{ab}_{(2)} k_c k_d), \tag{80}$$

with the effective metrics given by

$$g^{ab}_{(1)} \equiv \mathcal{L}_1^2 g^{ab}, \qquad g^{ab}_{(2)} \equiv -(\mathcal{L}_1 g^{ab} + 4\mathcal{L}_{11}\tau^{ab}),$$
$$\tau^{ab} \equiv F^{ac}F^{b}{}_{c}, \tag{81}$$

as was shown for instance in [43]. Needless to say, if Eqs. (78) and (79) hold, the effective metrics are well behaved, and both possess the same Lorentzian signature. However, it should be clear from our discussion that symmetric-hyperbolicity requires more than the simple Lorentzian nature of the effective metrics.

2. Two invariants: $\mathcal{L}(I^1,I^2)$

The case of two invariants is naturally more involved. The symmetric double (2,2) form becomes

$$X_{abcd} = \mathcal{L}_1 g_{abcd} + 4\{\mathcal{L}_{11}F_{ab}F_{cd} + \mathcal{L}_{12}(F_{ab}\star F_{cd} + \star F_{ab}F_{cd}) + \mathcal{L}_{22}\star F_{ab}\star F_{cd}\}. \tag{82}$$

Apart from minor details on definitions, this tensor coincides with the so-called jump tensor obtained by Obukhov and Rubilar in [44]. Notice also that the term involving $\mathcal{L}_2$ in Eq. (17) was discarded, since it is proportional to the Bianchi identity. We then obtain the tensors [see Eqs. (70) and (71)]

$$\mathbb{P}_{ab} = \mathcal{L}_1 h_{ab} + 4\{\mathcal{L}_{11}E_aE_b + \mathcal{L}_{12}(E_aB_b + B_aE_b) + \mathcal{L}_{22}B_aB_b\}, \tag{83}$$

$$\mathbb{S}_{ab} = -\mathcal{L}_1 h_{ab} + 4\{\mathcal{L}_{11}B_aB_b - \mathcal{L}_{12}(E_aB_b + B_aE_b) + \mathcal{L}_{22}E_aE_b\}, \tag{84}$$

which, in the frame described above, lead to the inequalities

$$\mathcal{L}_1\delta_{ij} - 4\{\mathcal{L}_{11}E_iE_j + \mathcal{L}_{12}(E_iB_j + B_iE_j) + \mathcal{L}_{22}B_iB_j\} \prec 0, \tag{85}$$

$$\mathcal{L}_1\delta_{ij} + 4\{\mathcal{L}_{11}B_iB_j - \mathcal{L}_{12}(E_iB_j + B_iE_j) + \mathcal{L}_{22}E_iE_j\} \prec 0. \tag{86}$$

A tedious calculation using Eq. (52) yields the effective metrics

$$g^{ab}_{(1)} = \mathcal{X}g^{ab} + (\mathcal{Y} + \sqrt{\mathcal{Y}^2 - \mathcal{X}\mathcal{Z}})t^{ab}, \tag{87}$$

$$g^{ab}_{(2)} = \mathcal{X}g^{ab} + (\mathcal{Y} - \sqrt{\mathcal{Y}^2 - \mathcal{X}\mathcal{Z}})t^{ab}, \tag{88}$$

with

$$\mathcal{X} = \mathcal{L}_1^2 + 2\mathcal{L}_1(G - \mathcal{L}_{22}F) + (\mathcal{L}_{12}\mathcal{L}_{12} - \mathcal{L}_{11}\mathcal{L}_{22})G^2,$$
$$\mathcal{Y} = 2\mathcal{L}_1(\mathcal{L}_{11} + \mathcal{L}_{22}) + 4(\mathcal{L}_{12}\mathcal{L}_{12} - \mathcal{L}_{11}\mathcal{L}_{22})F,$$
$$\mathcal{Z} = (\mathcal{L}_{11}\mathcal{L}_{22} - \mathcal{L}_{12}\mathcal{L}_{12}),$$

and we have used the notation $I^1 \equiv F$, $I^2 \equiv G$, $t^{ab} \equiv F^{ac}F^{b}{}_{c}$ for conciseness. This result coincides with [44,45], with minor modifications of notation.

C. Nonminimally coupled nonlinear electrodynamics

Since the equations of motion that follow from Eq. (10) are very general, it is convenient to consider some particular cases. Hence, in what follows we assume that the Lagrangian density takes the form

$$\mathcal{L}(I^1, I^2) + \alpha I^3 + \beta I^5 + \gamma I^7, \tag{89}$$

where $\mathcal{L}(I^1, I^2)$ is an arbitrary function of the usual electromagnetic invariants, and α, β, and γ are phenomenological parameters to be determined in principle from experiments. Such a density is still sufficiently general to include most of the relevant important models present in literature.

Let us examine next some particular cases that follow from Eq. (89).

1. Linear nonminimal model

The corresponding Lagrangian density is given by

$$\mathcal{L} = -I^1 + \alpha I^3 + \beta I^5 + \gamma I^7, \tag{90}$$

since we expect to recover Maxwell's theory in the flat spacetime regime. This model describes several possible nonminimal couplings of linear electrodynamics with gravity, and it encompasses for instance the modifications of Maxwell's theory due to one-loop vacuum polarization contributions [35]. Clearly, the corresponding equations of motion are linear and, using Eq. (17), one obtains the double symmetric (2,2) form as

$$X_{abcd} = (\alpha R - 1)g_{abcd} + \beta W_{abcd} + \gamma(g_{ac}S_{bd} - g_{ad}S_{bc} + g_{bd}S_{ac} - g_{bc}S_{ad}). \tag{91}$$

Notice that the irreducible parts of the Riemann tensor contribute in different ways to the equations of motion. Using the above prescription in Eqs. (70) and (71), we obtain the following two-index tensors

$$\mathbb{P}_{ab} = (\alpha R - 1)h_{ab} + \beta\mathcal{E}_{ab} + \gamma N_{ab}, \tag{92}$$

$$\mathbb{S}_{ab} = (1 - \alpha R)h_{ab} - \beta \mathcal{E}_{ab} + \gamma N_{ab}. \tag{93}$$

Let us next consider several subcases.

Scalar curvature coupling ($\alpha \neq 0, \beta = 0, \gamma = 0$).—This is by far the simplest type of nonminimal coupling. Indeed, since the double symmetric (2,2) form in Eq. (91) reduces to

$$X_{abcd} = (\alpha R - 1)g_{abcd}, \tag{94}$$

the relevant two-index tensors read

$$\mathbb{P}_{ab} = (\alpha R - 1)h_{ab}, \qquad \mathbb{S}_{ab} = (1 - \alpha R)h_{ab}. \tag{95}$$

Hence, the simple matrix inequality

$$(1 - \alpha R)\delta_{ij} \succ 0 \tag{96}$$

follows. Clearly, symmetric-hyperbolicity requires that $\alpha < 1/R$, which may forbid good propagation for sufficiently high curvature, for a given α. Regarding the characteristic cone, Eq. (52) gives

$$P(k) \sim (g^{ab}k_a k_b)^2, \tag{97}$$

which shows that in this case the dispersion relation is governed by the background metric, i.e., the causal structure is not changed by the coupling.

Weyl coupling ($\alpha = 0, \beta \neq 0, \gamma = 0$).—It follows from Eqs. (92) and (93) that

$$\mathbb{P}_{ab} = -\mathbb{S}_{ab} = -h_{ab} + \beta \mathcal{E}_{ab}.$$

Although the ensuing inequalities need to be examined on a case-by-case basis, they will lead to limits on the components of the electric part of the Weyl tensor.[9]

In order to compute the dispersion relation explicitly, we first recall that the Weyl conformal tensor has only one independent Hodge dual, i.e., $\star W_{abcd} = W_{abcd}\star$. A direct calculation using Eq. (52) gives a quartic equation of the type

$$P(k) \sim G^{abcd}k_a k_b k_c k_d, \tag{98}$$

with the Kummer tensor given by

$$G^{abcd} \equiv g^{ab}g^{cd} - \frac{\beta^2}{3}\left(W^{apbq}W^{c}{}_{p}{}^{d}{}_{q} + \frac{1}{4}g^{ab}W^{cpqr}W^{d}{}_{pqr}\right) + \frac{\beta^3}{6}\star W_{pq}{}^{ar}W^{bpcs}W^{dq}{}_{rs}\star. \tag{99}$$

[9]Conversely, for a given geometry, the inequalities may furnish β-dependent limits on the region of space-time where the propagation is symmetric-hyperbolic.

Using the following identities due to Debever and Lanczos

$$W_{apcq}W_{b}{}^{p}{}_{d}{}^{q} - \star W_{apcq}\star W_{b}{}^{p}{}_{d}{}^{q} = A g_{ab}g_{cd}, \tag{100}$$

$$W_{apqr}W_{b}{}^{pqr} = 2A g_{ab}, \tag{101}$$

with $A \equiv \frac{1}{8}W_{pqrs}W^{pqrs}$, we obtain the simplified expression

$$G^{abcd} = \left(1 - \frac{\beta^2}{6}A\right)g^{ab}g^{cd} - \frac{\beta^2}{3}W^{apbq}W^{c}{}_{p}{}^{d}{}_{q} + \frac{\beta^3}{3}W_{pq}{}^{ar}W^{bpcs}W^{dq}{}_{rs}. \tag{102}$$

Traceless Ricci couplings ($\alpha = 0, \beta = 0, \gamma \neq 0$).—In this case, we have

$$X_{abcd} = -g_{abcd} + \gamma(g_{ac}S_{bd} - g_{ad}S_{bc} + g_{bd}S_{ac} - g_{bc}S_{ad}). \tag{103}$$

The relevant matrices take the form

$$\mathbb{P}_{ab} = -h_{ab} + \gamma N_{ab}, \tag{104}$$

$$\mathbb{S}_{ab} = h_{ab} + \gamma N_{ab}. \tag{105}$$

As in the previous case, the corresponding inequalities will relate the coupling constant γ to the curvature quantities described by N_{ab}. The Kummer tensor is given by

$$G^{abcd} = g^{ab}g^{cd} - \gamma g^{ab\,(7)}\chi^{cpd}{}_{p} + \frac{\gamma^2}{2}\left({}^{(7)}\chi^{apb}{}_{p}{}^{(7)}\chi^{cqd}{}_{q} - {}^{(7)}\chi^{apbq\,(7)}\chi^{c}{}_{q}{}^{d}{}_{q}\right) + \frac{\gamma^3}{6}{}^{(7)}\chi^{apbq\,(7)}\chi^{crds\,(7)}\chi^{prqs}.$$

Nonminimally coupled EM field in a cosmological background.—Another relevant example is that of the propagation in a cosmological background described by the flat Friedman-Lemâitre-Robertson-Walker metric, for which the Weyl tensor is null. In such a case, only the terms associated to $\mathcal{L}_1$, $\mathcal{L}_3$, and $\mathcal{L}_7$ survive in Eqs. (70) and (71). Using Einstein's equations, the traceless part of the Ricci tensor is given in terms of the matter by

$$S_{ab} = T_{ab} - \frac{T}{4}g_{ab},$$

where T_{ab} is the energy-momentum tensor and T, its trace. In a convenient tetrad basis, in which $T_{ab} = \mathrm{diag}(\rho, p, p, p)$, it follows that

$$\mathbb{P}_{ij} = -\delta_{ij}\left[1 + \beta(\rho - 3p) - \frac{1}{2}\gamma(\rho + p)\right], \tag{106}$$

$$\mathbb{S}_{ij} = \delta_{ij}\left[1 + \beta(\rho - 3p) + \frac{1}{2}\gamma(\rho + p)\right], \qquad (107)$$

Hence, the inequalities (41) lead to

$$1 + \beta(\rho - 3p) > \frac{1}{2}|\gamma|(\rho + p),$$

which is trivially satisfied in the case of a linear EM field. The propagation will be symmetric hyperbolic if this inequality is satisfied at all times for which the model is valid.

VII. CONCLUSIONS

Well-posedness is a basic requirement for any field theory, and it is guaranteed by symmetric-hyperbolicity. We have obtained a general form for the symmetrizer, given in Eq. (26), valid for a general Lagrangian theory. We have shown that symmetric-hyperbolicity leads to a set of two inequalities for the matrices $\mathbb{P}$ and $\mathbb{S}$, whose elements are determined by a given theory. Regarding the constraints, we have verified that they are integrable.

When applied to nonlinear electromagnetism linearly coupled to curvature, the matrices $\mathbb{P}$ and $\mathbb{S}$ are expressed in terms the fields, the Lagrangian, and its derivatives, and also of the different quantities associated to curvature. They lead to strong constraints on the relevant quantities, which were illustrated with applications to several particular cases. The examples show that while in the linear theory, no constraint arises from symmetric hyperbolic propagation, nonlinearity leads to constraints on the field intensities, and nonminimal coupling imposes restrictions on quantities associated to curvature. In the general case, symmetric hyperbolicity relates the electromagnetic quantities $\{E^a, B^a\}$, the spacetime curvature expressed via $\{\mathcal{E}_{ab}, \mathcal{B}_{ab}, Q_a, N_{ab}\}$ and the partial derivatives of the Lagrangian density.

The ideas presented here can be applied in other settings, such as electromagnetism in material media. We plan to return to this problem in a future publication.

APPENDIX: RUSE-LANCZOS IDENTITY

Let χ_{abcd} denote an arbitrary double symmetric (2,2) form at a point $p \in \mathcal{M}$. Its double Hodge dual is defined as

$$\star\chi_{abcd}\star = \frac{1}{4}\varepsilon_{ab}{}^{pq}\varepsilon_{cd}{}^{rs}\chi_{pqrs}. \qquad (A1)$$

Writing the traces of χ_{abcd} as

$$\chi_{ab} \equiv \chi^c{}_{acb}, \qquad \chi \equiv \chi^a{}_a, \qquad (A2)$$

we may construct the symmetric trace-free tensor

$$\psi_{ab} \equiv \chi_{ab} - \frac{1}{4}g_{ab}\chi. \qquad (A3)$$

Using elementary algebraic manipulations, it can be shown that

$$\star\chi_{abcd}\star + \chi_{abcd} = g_{a[c}\psi_{d]b} + g_{b[d}\psi_{c]a}, \qquad (A4)$$

which is called the Ruse-Lanczos identity [46,47]. Applying this identity to the constitutive tensors results in the following useful relations

$$\star{}^{(1)}\chi^{ab}{}_{cd}\star + {}^{(1)}\chi^{ab}{}_{cd} = 0,$$
$$\star{}^{(2)}\chi^{ab}{}_{cd}\star + {}^{(2)}\chi^{ab}{}_{cd} = 0,$$
$$\star{}^{(3)}\chi^{ab}{}_{cd}\star + {}^{(3)}\chi^{ab}{}_{cd} = 0,$$
$$\star{}^{(4)}\chi^{ab}{}_{cd}\star + {}^{(4)}\chi^{ab}{}_{cd} = 0,$$
$$\star{}^{(5)}\chi^{ab}{}_{cd}\star + {}^{(5)}\chi^{ab}{}_{cd} = 0,$$
$$\star{}^{(6)}\chi^{ab}{}_{cd}\star + {}^{(6)}\chi^{ab}{}_{cd} = 0,$$
$$\star{}^{(7)}\chi^{ab}{}_{cd}\star - {}^{(7)}\chi^{ab}{}_{cd} = 0.$$

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