

Temperature and free energy of multi-black-hole systems

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In the present paper, we compute the Euclidean action of a generic system consisting of N arbitrary Kerr-Newman black holes located on the symmetry axis and separated from each other by massless struts. This allows us to introduce for the first time the notion of “Hawking average temperature” \hat{T} of the multi-black-hole system via the condition of vanishing the entire set of terms involving the singular horizons due to periodic time, and the resulting formula for this temperature contains solely the surface gravities κ_i and horizon areas A_i^H of the black hole constituents. We also show that the corresponding expression for the free energy of the system defined by \hat{T} is consistent with the first law of thermodynamics and Smarr mass relations.

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I. INTRODUCTION

The Euclidean action approach to black hole physics was introduced by Gibbons and Hawking in their seminal paper [1], where they confirmed in an elegant new way the previously found analytic relations of the Hawking temperature to the surface gravity [2] and of the black hole’s entropy to the area of the event horizon [3]. Though this approach was originally applied to the study of the thermodynamic behavior of a single black hole, the subsequent interest in the binary black-hole configurations, stimulated by the development of modern solution generating techniques and construction of various multi-black-hole metrics [4,5], later led to the appearance of a number of papers in which the Euclidean formalism was employed for the analysis of some specific double-black-hole systems with *equal* surface gravities of the constituents. Thus, in the paper [6], Costa and Perry evaluated the free energy of two equal Schwarzschild black holes and, compared to the paper [1], the action integral of the binary system had an additional term arising from the massless strut [7] separating the constituents. The Euclidean action of a pair of identical (up to a sign of charges) Reissner-Nordström non-extreme black holes was calculated by Emparan and Teo [8], while the thermodynamical properties of equal Kerr black holes, with a brief analysis of the action integrals, were studied numerically by Herdeiro *et al.* [9]. In our recent paper [10], we have computed the Euclidean action for two different binary systems of equal counterrotating Kerr-Newman (KN) black holes and obtained the corresponding expressions for the free energy of each system. Note that in the above papers [6,8–10] all the binary systems were in thermal equilibrium because the black holes in them had the same surface gravities and hence the same Hawking temperature T_H .

In the paper [10], we have observed that it is quite simple to guess the explicit expression of the free energy, which is closely related to the Euclidean action, in the general case of the double-Schwarzschild solution [11] from the respective formula obtained in the special case of equal Schwarzschild black holes. This naturally raises the question of whether the Euclidean action method can be extended to the systems of arbitrary black holes with *nonequal* surface gravities, similar to thermodynamics, which goes far beyond the black-hole configurations in thermal equilibrium [12]. The reason of why such an extension of the method has not yet been attempted up to now is, in fact, known and looks quite meaningful at first sight—a Wick rotation of the multi-black-hole spacetime introduces conical singularities at the horizons due to the periodic Euclidean time τ ; whereas one conical singularity, say, on the first horizon can be eliminated by choosing appropriately the period of τ , the other horizons will remain singular unless they have the same surface gravities as the first horizon. However, as will be shown in the present paper, the way out of this seemingly unresolvable situation consists in calculating the entire Euclidean action, including the terms arising from all the singular horizons; then the condition of vanishing the combined contribution of the latter “singular horizon” terms will fix the value of the period of τ , which will allow us to introduce for the first time the notion of the Hawking average temperature of the system as the inverse of the latter period. It is remarkable that the temperature introduced in this way turns out to be determined by a very concise formula involving exclusively the horizons’ areas and surface gravities; moreover, the corresponding free energy of the system becomes consistent with the first law of thermodynamics and the well-known Smarr relations [13].

This paper is organized as follows. In the next section, we will reexamine the derivation of the Euclidean action in the case of a single KN black hole [14], evaluating explicitly the contribution of the conical singularity due to Euclidean time. In Sec. III, we calculate the Euclidean action for a system of N collinear arbitrary KN black holes described by the extended N -soliton electrovac solution [15]. Here two different types of the conical singularity contributions will appear—the one coming from the singular horizons and the other arising from the struts. The Hawking average temperature \hat{T} is introduced in Sec. IV, where we find its form in terms of the surface gravities and horizons' areas of black holes; we also give the expression of the free energy W defined by \hat{T} and show its consistency with both the first law of thermodynamics and Smarr's mass formula. Section V contains concluding remarks.

Throughout the paper, units are used in which $c = G = \hbar = k_B = 1$.

II. EUCLIDEAN ACTION OF A SINGLE KERR-NEWMAN BLACK HOLE

The covariant Euclidean action we are interested in can be represented in the form [1]

$$I_E = I_{EH} + I_{GH} + I_{em}, \quad (1)$$

where the first term I_{EH} is the well-known Einstein-Hilbert action in Euclidean signature

$$I_{EH} = -\frac{1}{16\pi} \int_{\mathcal{M}} R\sqrt{g}, \quad (2)$$

the second term I_{GH} is the Gibbons-Hawking boundary term

$$I_{GH} = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} [K]\sqrt{h}, \quad (3)$$

with $[K] := K - K_0$, K being the extrinsic curvature of $\partial\mathcal{M}$, and K_0 being the extrinsic curvature of the same surface $\partial\mathcal{M}$ in flat space. The last term I_{em} is the standard electromagnetic action in the absence of currents

$$I_{em} = \frac{1}{8\pi} \int_{\mathcal{M}} F \wedge \star F. \quad (4)$$

The free energy of the system is related to I_E by the simple formula

$$W = I_E/\beta, \quad \beta = 1/T, \quad (5)$$

where β is the period of the Euclidean time $\tau = it$, and T denotes the temperature.¹

In the presence of a conical singularity, the action integral I_{EH} can be evaluated with the aid of the formula [6,17,18]

$$\frac{1}{2} \int_{\mathcal{M}} R\sqrt{g} = Area \cdot \delta, \quad (6)$$

where $Area$ is the area of the surface spanned by the conical singularity, and δ is the angle deficit.

As for the Gibbons-Hawking boundary term I_{GH} that must be computed at spatial infinity, it can be seen that it consists of two parts. The first one, namely,

$$\int_{\partial\mathcal{M}} K\sqrt{h}, \quad (7)$$

is intrinsically divergent and requires a counterterm for regularization; the latter term is given by the second part of I_{GH} ,

$$\int_{\partial\mathcal{M}} K_0\sqrt{h}. \quad (8)$$

One possible way to compute (7) is by using formula [1,19]

$$\int_{\partial\mathcal{M}} K\sqrt{h} = n(\text{Vol}\partial\mathcal{M}), \quad (9)$$

where n is a unit normal vector field over $\partial\mathcal{M}$, while $\text{Vol}\partial\mathcal{M}$ stands for the volume of $\partial\mathcal{M}$.

We find it instructive, before treating the general case of N arbitrary KN black holes, first to illustrate the use of the above formulas by the example of a single KN black hole. A Wick rotated KN metric, written in Boyer-Lindquist coordinates, has the form

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr - q^2}{\Sigma}\right) d\tau^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{C}{\Sigma} \sin^2\theta d\varphi^2 + \frac{2ia(2mr - q^2)}{\Sigma} \sin^2\theta d\tau d\varphi, \\ \Sigma &= r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 - 2mr + a^2 + q^2, \\ C &= (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta, \end{aligned} \quad (10)$$

¹Euclidean time can also be introduced in a coordinate-invariant way [16].

the parameters m , a , and q representing, respectively, the mass, angular momentum per unit mass, and electric charge of the KN black hole.

We start the calculation of the Euclidean action of the KN solution with the evaluation of the term I_{EH} via formula (6). The area spanned by the conical singularity at the horizon is precisely the horizon's area A^H , while for the conical deficit δ^r in terms of the surface gravity κ we obtain

$$\delta^r = 2\pi - \int_0^\beta \kappa d\tau = 2\pi \left(1 - \frac{\kappa}{2\pi} \beta\right), \quad (11)$$

whence it follows that

$$I_{\text{EH}} = -\frac{1}{8\pi} A^H \delta^r, \quad (12)$$

the concrete well-known values of A^H and κ of the KN black hole being

$$A^H = 4\pi(r_+^2 + a^2), \quad \kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)},$$

$$r_\pm = m \pm \sqrt{m^2 - a^2 - q^2}. \quad (13)$$

Taking into account the positivity of A^H , it is clear that the only possibility to get rid of the conical singularity contribution and achieve $I_{\text{EH}} = 0$ would be demanding $\delta^r = 0$, which implies

$$\beta = \frac{2\pi}{\kappa} \Rightarrow T = \frac{\kappa}{2\pi}, \quad (14)$$

so that the removal of the conical singularity due Euclidean time automatically fixes the temperature T of the KN black hole in the Hawking form (14).

Turning now to the evaluation of the Gibbons-Hawking term I_{GH} , we must bear in mind that the boundary lies at spatial infinity where each of the integrals (7) and (8) diverges. Therefore, on the one hand, we must first compute the integrals for some finite r and then take the limit $r \rightarrow \infty$ in the final expression for I_{GH} ; on the other hand, we can use the large- r approximation to slightly simplify the computational process.

Let \mathcal{N} be a hypersurface defined by $r = \text{const}$. Then the determinant h of the induced metric over \mathcal{N} for large r takes the form

$$h \simeq \left(1 - \frac{2m}{r}\right) r^4 \sin^2 \theta, \quad (15)$$

and we get for the volume of \mathcal{N} ,

$$\text{Vol}\mathcal{N} = \int_0^\pi \int_0^{2\pi} \int_0^\beta \sqrt{h} d\theta d\varphi d\tau \simeq 4\pi\beta r^2 \left(1 - \frac{m}{r}\right). \quad (16)$$

If we consider now a normalized vector field n over \mathcal{N} defined as $g_{rr}^{-1/2} \partial_r$, then in our approximation

$$n \simeq \left(1 - \frac{m}{r}\right) \partial_r, \quad (17)$$

so that

$$n(\text{Vol}\mathcal{N}) \simeq 4\pi\beta(2r - 3m), \quad (18)$$

and

$$-\frac{1}{8\pi} \int_{\mathcal{N}} K \sqrt{h} \simeq -\frac{1}{2} \beta(2r - 3m). \quad (19)$$

On the other hand, the extrinsic curvature of \mathcal{N} in flat space is just $K_0 = 2/r$, so we get

$$-\frac{1}{8\pi} \int_{\mathcal{N}} K_0 \sqrt{h} \simeq -\frac{1}{2} \beta \left(1 - \frac{m}{r}\right) 2r, \quad (20)$$

and the combination of (19) and (20) yields

$$-\frac{1}{8\pi} \int_{\mathcal{N}} [K] \sqrt{h} = \frac{1}{2} \beta m + O(r^{-1}). \quad (21)$$

Taking the limit $r \rightarrow \infty$ in (21), we finally arrive at the Gibbons-Hawking term I_{GH} of the KN black hole,

$$I_{\text{GH}} = \frac{1}{2} \beta m. \quad (22)$$

To find the remaining part of the Euclidean action related to the electromagnetic field, we shall employ the approach earlier used in the papers [8,19] to avoid manipulations with the electric potential on the horizon made in the original paper [1]. In fact, this calculational procedure is similar to the one just employed for the evaluation of the Gibbons-Hawking term I_{GH} —we shall perform the computations for some generic finite value of r , but at the end, instead of the limit $r \rightarrow \infty$, we shall take the limit $r \rightarrow r_+$ corresponding to the horizon of the KN black hole. This, in particular, excludes the use of any approximation tricks during the calculations.

In the Euclidean signature, the electromagnetic potential of the KN solution has the form

$$A = A_\tau(r, \theta) d\tau + A_\varphi(r, \theta) d\varphi = \frac{qr}{\Sigma} (i d\tau + a \sin^2 \theta d\varphi), \quad (23)$$

and these components of A must be substituted into the integral (4), which can be readily rewritten as a boundary term

$$\int_{\mathcal{M}} F^{\mu\nu} F_{\mu\nu} \sqrt{g} d^4x = 2 \int_{\partial\mathcal{M}} F^{\nu\mu} A_\mu n_\nu \sqrt{h} d^3x. \quad (24)$$

As before, the integral on the right-hand side (rhs) of (24) will be computed over a hypersurface \mathcal{N} defined by $r = \text{const}$, with the same normal unit vector n , and the integration is straightforward,

$$\begin{aligned} \int_{\mathcal{N}} F^{\nu\mu} A_\mu n_\nu \sqrt{h} d^3x &= 2\pi\beta \int_0^\pi \frac{d\theta}{\sin\theta} [(g_{\varphi\tau} A_\varphi - g_{\varphi\varphi} A_\tau) \partial_r A_\tau + (g_{\tau\varphi} A_\tau - g_{\tau\tau} A_\varphi) \partial_r A_\varphi] \\ &= 2\pi\beta q^2 r \int_0^\pi \frac{\Sigma - 2r^2}{\Sigma^2} \sin\theta d\theta \\ &= 2\pi\beta q^2 r \partial_r \int_0^\pi \frac{r}{\Sigma} \sin\theta d\theta \\ &= 4\pi\beta q^2 r \partial_r \arctan(a/r) \\ &= -4\pi\beta \frac{q^2 r}{r^2 + a^2}. \end{aligned} \quad (25)$$

Taking now the limit $r \rightarrow r_+$, we immediately obtain

$$\lim_{r \rightarrow r_+} \int_{\mathcal{N}} F^{\nu\mu} A_\mu n_\nu \sqrt{h} d^3x = -4\pi\beta \frac{q^2 r_+}{r_+^2 + a^2} = -4\pi\beta q \Phi, \quad (26)$$

where the value of the electric potential Φ on the horizon is determined by the formula

$$\Phi = \frac{q r_+}{r_+^2 + a^2}. \quad (27)$$

Therefore, for the electromagnetic action integral I_{em} , we finally get

$$I_{\text{em}} = -\frac{1}{2} \beta q \Phi, \quad (28)$$

so that the entire Euclidean action I_E takes the form

$$I_E = \frac{1}{2} \beta (m - q\Phi) - \frac{1}{8\pi} A^H \delta^\tau, \quad (29)$$

with A^H and δ^τ defined by (13).

As it has already been mentioned, the choice of β in the form (14) causes vanishing of the last term in (29), which is required for the regularity of the horizon. Thus, when $\beta = 1/T = 2\pi/\kappa$, the free energy of the KN black hole, as it follows from (5), is given by

$$W = \frac{1}{2} (m - q\Phi). \quad (30)$$

We now turn to the discussion of the general case of N black holes.

III. EUCLIDEAN ACTION OF N KERR-NEWMAN BLACK HOLES

The configuration of N collinear arbitrary KN black holes is described by a subfamily of the extended N -soliton electrovac solution [15] constructed with the aid of Sibgatullin's integral method [20]. It is curious that a year after the publication of the paper [15] with a concise explicit form of all the metrical fields, an article appeared in a mathematical journal [21] in which solely the proof of the existence and uniqueness of the Ruiz *et al.* N -soliton solution was attempted. We recall that the Ernst potentials [22] of the solution [15] are defined by the expressions

$$\mathcal{E} = E_+ / E_-, \quad \Phi = F / E_-,$$

$$E_\pm = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \pm 1 & \frac{r_1}{\alpha_1 - \beta_1} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_1} \\ \vdots & \vdots & \ddots & \vdots \\ \pm 1 & \frac{r_1}{\alpha_1 - \beta_N} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_N} \\ 0 & \frac{h_1(\alpha_1)}{\alpha_1 - \beta_1^*} & \dots & \frac{h_1(\alpha_{2N})}{\alpha_{2N} - \beta_1^*} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{h_N(\alpha_1)}{\alpha_1 - \beta_N^*} & \dots & \frac{h_N(\alpha_{2N})}{\alpha_{2N} - \beta_N^*} \end{vmatrix}, \quad F = \begin{vmatrix} 0 & f(\alpha_1) & \dots & f(\alpha_{2N}) \\ -1 & \frac{r_1}{\alpha_1 - \beta_1} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_1} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \frac{r_1}{\alpha_1 - \beta_N} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_N} \\ 0 & \frac{h_1(\alpha_1)}{\alpha_1 - \beta_1^*} & \dots & \frac{h_1(\alpha_{2N})}{\alpha_{2N} - \beta_1^*} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{h_N(\alpha_1)}{\alpha_1 - \beta_N^*} & \dots & \frac{h_N(\alpha_{2N})}{\alpha_{2N} - \beta_N^*} \end{vmatrix}, \quad (31)$$

where the coordinates ρ and z enter the $(2N+1) \times (2N+1)$ determinants E_\pm and F only through the functions $r_n = \sqrt{\rho^2 + (z - \alpha_n)^2}$, and the constant objects $h_l(\alpha_n)$ and $f(\alpha_n)$ are defined as follows:

$$\begin{aligned}
h_l(\alpha_n) &= e_l^* + 2f_l^* f(\alpha_n), & f(\alpha_n) &= \sum_{l=1}^N \frac{f_l}{\alpha_n - \beta_l}, \\
e_l &= \frac{2 \prod_{n=1}^{2N} (\beta_l - \alpha_n)}{\prod_{k \neq l}^N (\beta_l - \beta_k) \prod_{k=1}^N (\beta_l - \beta_k^*)} - 2 \sum_{k=1}^N \frac{f_l f_k^*}{\beta_l - \beta_k^*},
\end{aligned} \tag{32}$$

the asterisk meaning complex conjugation. The set of arbitrary parameters involved in formulas (31) and (32) consists of N complex constants β_l , N complex constants f_l , and $2N$ real parameters supplied by the α_n 's, which can be real valued or occur in complex conjugate pairs.

Potentials \mathcal{E} and Φ determine the functions f , γ , and ω in the stationary axisymmetric line element

$$ds^2 = f^{-1} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f (dt - \omega d\varphi)^2, \tag{33}$$

and these are given by the formulas

$$\begin{aligned}
f &= \frac{D}{2E_+ E_-^*}, & e^{2\gamma} &= \frac{D}{2K_0 K_0^* \prod_{n=1}^{2N} r_n}, & \omega &= -\frac{2\text{Im}[E_-^* (G + H) + FI^*]}{D}, \\
D &= E_+ E_-^* + E_+^* E_- + 2FF^*,
\end{aligned} \tag{34}$$

with the following explicit form of the determinants I , G , H , and K_0 [15]:

$$\begin{aligned}
G &= \begin{vmatrix} 0 & r_1 + \alpha_1 - z & \dots & r_{2N} + \alpha_{2N} - z \\ -1 & \frac{r_1}{\alpha_1 - \beta_1} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_1} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \frac{r_1}{\alpha_1 - \beta_N} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_N} \\ 0 & \frac{h_1(\alpha_1)}{\alpha_1 - \beta_1^*} & \dots & \frac{h_1(\alpha_{2N})}{\alpha_{2N} - \beta_1^*} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{h_N(\alpha_1)}{\alpha_1 - \beta_N^*} & \dots & \frac{h_N(\alpha_{2N})}{\alpha_{2N} - \beta_N^*} \end{vmatrix}, & H &= \begin{vmatrix} z & 1 & \dots & 1 \\ -\beta_1 & \frac{r_1}{\alpha_1 - \beta_1} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_1} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_N & \frac{r_1}{\alpha_1 - \beta_N} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_N} \\ e_1^* & \frac{h_1(\alpha_1)}{\alpha_1 - \beta_1^*} & \dots & \frac{h_1(\alpha_{2N})}{\alpha_{2N} - \beta_1^*} \\ \vdots & \vdots & \ddots & \vdots \\ e_N^* & \frac{h_N(\alpha_1)}{\alpha_1 - \beta_N^*} & \dots & \frac{h_N(\alpha_{2N})}{\alpha_{2N} - \beta_N^*} \end{vmatrix}, \\
I &= \begin{vmatrix} \sum_{l=1}^N f_l & 0 & f(\alpha_1) & \dots & f(\alpha_{2N}) \\ z & 1 & 1 & \dots & 1 \\ -\beta_1 & -1 & \frac{r_1}{\alpha_1 - \beta_1} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_N & -1 & \frac{r_1}{\alpha_1 - \beta_N} & \dots & \frac{r_{2N}}{\alpha_{2N} - \beta_N} \\ e_1^* & 0 & \frac{h_1(\alpha_1)}{\alpha_1 - \beta_1^*} & \dots & \frac{h_1(\alpha_{2N})}{\alpha_{2N} - \beta_1^*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_N^* & 0 & \frac{h_N(\alpha_1)}{\alpha_1 - \beta_N^*} & \dots & \frac{h_N(\alpha_{2N})}{\alpha_{2N} - \beta_N^*} \end{vmatrix}, & K_0 &= \begin{vmatrix} \frac{1}{\alpha_1 - \beta_1} & \dots & \frac{1}{\alpha_{2N} - \beta_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\alpha_1 - \beta_N} & \dots & \frac{1}{\alpha_{2N} - \beta_N} \\ \frac{h_1(\alpha_1)}{\alpha_1 - \beta_1^*} & \dots & \frac{h_1(\alpha_{2N})}{\alpha_{2N} - \beta_1^*} \\ \vdots & \ddots & \vdots \\ \frac{h_N(\alpha_1)}{\alpha_1 - \beta_N^*} & \dots & \frac{h_N(\alpha_{2N})}{\alpha_{2N} - \beta_N^*} \end{vmatrix}.
\end{aligned} \tag{35}$$

The subfamily describing N arbitrary KN black holes located on the symmetry axis and separated from each other by massless struts is contained in the general formulas as a special asymptotically flat case characterized by $4N - 1$ real parameters representing the individual N masses, N angular momenta, N electric charges, and $N - 1$ relative

distances between the black holes. Moreover, since we restrict ourselves to the black-hole sector of the solution (31) only, then all $2N$ parameters α_n entering the functions r_n and determining the positions of black holes on the symmetry axis must be real valued (see Fig. 1). At the same time, the fact that the general N -soliton solution has $6N$

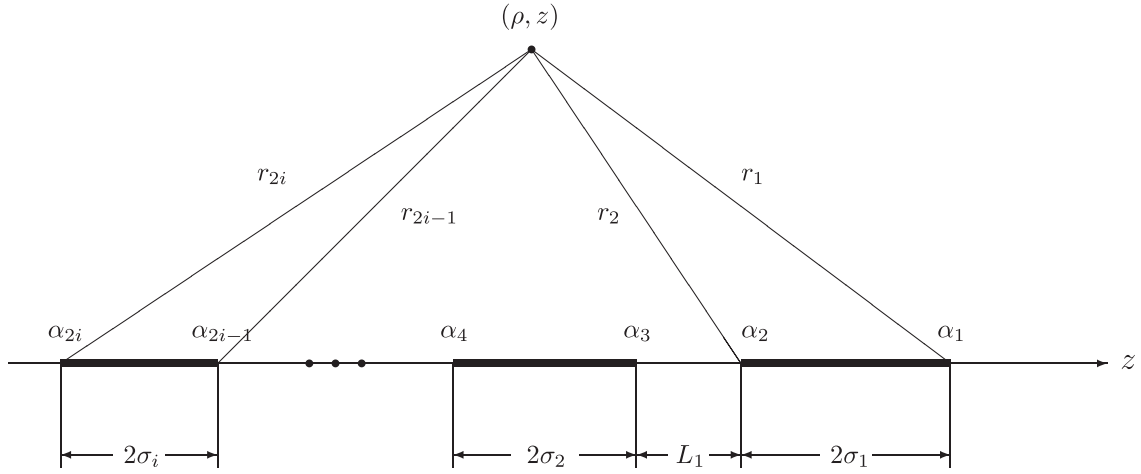


FIG. 1. Location of the KN black holes on the symmetry axis.

arbitrary real parameters means that its multi-black-hole subfamily we are interested in arises by imposing $2N + 1$ restrictions/conditions on the parameters of the general solution. Such restrictions are quite simple: the first one is just the condition of asymptotic flatness, or absence of the Newman-Unti-Tamburino parameter of the system (one restriction); furthermore, the separation of black holes means that the metric function ω must verify the so-called “axis condition” on the parts of the symmetry axis separating the black-hole horizons ($N - 1$ restrictions). Of course, since we consider the conventional KN black holes endowed with electric charges, we must also exclude the individual magnetic charges of all the black holes (N more restrictions). To these $2N$ restrictions, it is necessary to add the last one related to the possibility of a translation along the z axis, and this liberty will be abolished by fixing the position of the origin of coordinates. In mathematical terms, the above restrictions can be formulated as

$$\text{Im}\left(\sum_{l=1}^N e_l\right) = 0 \quad (36)$$

(the condition of asymptotic flatness),

$$\omega(\rho = 0, \alpha_{2k+1} \leq z \leq \alpha_{2k}) = 0, \quad k = 1, 2, \dots, N - 1 \quad (37)$$

(the axis conditions), where one must bear in mind that ω takes constant values on the symmetry axis, and

$$\begin{aligned} \text{Re}[\Phi(\rho = 0, z = \alpha_{2k-1}) - \Phi(\rho = 0, z = \alpha_{2k})] &= 0, \\ k &= 1, 2, \dots, N \end{aligned} \quad (38)$$

(absence of magnetic charges). Last, the position of the origin of coordinates can be fixed, say, by subjecting α_n 's to the constraint

$$\sum_{n=1}^{2N} \alpha_n = 0, \quad (39)$$

which turns out to be particularly useful in the case of the equatorially symmetric configurations.

To calculate the Euclidean action for the system of N collinear KN black holes, we must perform a Wick rotation in the metric (33), yielding

$$d\bar{s}^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(id\tau + \omega d\varphi)^2, \quad (40)$$

and then follow the calculation procedures outlined in the previous section. We shall compute the terms I_{EH} , I_{GH} , and I_{em} in the same order as this was done in the case of a single KN black hole.

A. The Einstein-Hilbert term

In the general case of N black holes, two groups of conical singularities are present: the first group is composed of N singularities on the horizons of black holes as a consequence of the Wick rotation, and the second group accounts for $N - 1$ conical singularities due to the usual massless struts that prevent the black holes from falling onto each other. Although these types of singularities look different, their contribution to the action integral I_{EH} must be evaluated by means of the same formula (6). Below we will start with the conical singularities arising from the struts.

To compute the Area A_i^S associated with $\mathcal{L}_i \times S^1$, where \mathcal{L}_i is the i th strut joining the points α_{2i} and α_{2i+1} of the symmetry axis and S^1 refers to the periodic time, we can use the formula

$$A_i^S = \int_{\mathcal{L}_i \times S^1} \sqrt{h}, \quad (41)$$

where h is the determinant of the induced metric over the hypersurface $\mathcal{L}_i \times S^1$. From (40) it follows that $h = \exp(2\gamma)$, and we obtain

$$A_i^S = \int_{\alpha_{2i+1}}^{\alpha_{2i}} \int_0^\beta e^\gamma|_{\mathcal{L}_i} dz d\tau = \beta e^{\gamma_i} L_i, \quad (42)$$

where $L_i = \alpha_{2i} - \alpha_{2i+1}$ is the coordinate length of the i th strut and γ_i is the (constant) value of the metric function γ on the i th strut. Introducing further the ‘‘thermodynamic length’’ [23] of the i th strut by the formula $l_i = \exp(\gamma_i)L_i$, [9,24] we finally get

$$A_i^S = \beta l_i. \quad (43)$$

On the other hand, the corresponding deficit angle related to the coordinate φ can be written as

$$\delta_i^\varphi = 2\pi - \int_0^{2\pi} e^{-\gamma_i} d\varphi = 2\pi(1 - e^{-\gamma_i}) = -8\pi\mathcal{F}_i, \quad (44)$$

where \mathcal{F}_i represents the interaction force between the i th and $(i+1)$ th black holes [7,25]

$$\mathcal{F}_i = \frac{1}{4}(e^{-\gamma_i} - 1), \quad i = 1, 2, \dots, N-1. \quad (45)$$

As for the N conical singularities generated by the Euclidean time τ on the black-hole horizons, it was already mentioned in the previous section that their *Area*’s entering formula (6) are just areas of black holes A_i^H , and technically these can be computed by means of the formula

$$A_i^H = \int_{H_i} \sqrt{h}, \quad (46)$$

where $h = -e^{2\gamma}\omega^2$, as it follows from (40); then

$$A_i^H = \int_{\alpha_{2i}}^{\alpha_{2i-1}} \int_0^{2\pi} \sqrt{-e^{2\gamma_i}\omega_i^2} dz d\varphi = 4\pi\sigma_i \sqrt{-e^{2\gamma_i}\omega_i^2}. \quad (47)$$

Here γ_i and ω_i are the constant values of the metric functions γ and ω on the i th horizon, and $\sigma_i = (\alpha_{2i-1} - \alpha_{2i})/2$ is the half length of the i th horizon. Formula (47) can also be rewritten in the form

$$A_i^H = 4\pi\sigma_i\kappa_i^{-1}, \quad (48)$$

where $\kappa_i = (-e^{2\gamma_i}\omega_i^2)^{-1/2}$ is the surface gravity of the i th black-hole horizon [26,27].

For the corresponding deficit angle δ_i^τ related to the i th horizon and associated with the periodic time τ , we have

$$\delta_i^\tau = 2\pi - \int_0^\beta \kappa_i d\tau = 2\pi \left(1 - \frac{\kappa_i}{2\pi}\beta\right), \quad (49)$$

and therefore the expression of the action term I_{EH} takes the form

$$\begin{aligned} I_{\text{EH}} &= -\frac{1}{8\pi} \sum_{i=1}^{N-1} A_i^S \delta_i^\varphi - \frac{1}{8\pi} \sum_{i=1}^N A_i^H \delta_i^\tau \\ &= \beta \sum_{i=1}^{N-1} l_i \mathcal{F}_i - \frac{1}{4} \sum_{i=1}^N A_i^H \left(1 - \frac{\kappa_i}{2\pi}\beta\right). \end{aligned} \quad (50)$$

B. The Gibbons-Hawking term

In order to calculate the Gibbons-Hawking term (3) of the Euclidean action for our multi-black-hole system, it is convenient to make use of spherical coordinates (ζ, θ) related to the Weyl-Papapetrou cylindrical coordinates (ρ, z) by the formulas

$$\zeta = \sqrt{\rho^2 + z^2}, \quad \cos \theta = z/\sqrt{\rho^2 + z^2}. \quad (51)$$

Then, by analogy with the case of a single KN black hole, we can consider a hypersurface \mathcal{N} defined by $\zeta = \text{const}$. Clearly, as $\zeta \rightarrow \infty$, the asymptotic behavior of the functions r_n of the solution (31) is $r_n \rightarrow \zeta$.

Since the spacetime under consideration is asymptotically flat, then the component $g_{\tau\varphi}$ of the metric tensor is of the order $O(\zeta^{-1})$, and the induced metric over \mathcal{N} can be written in the form

$$\begin{aligned} d\sigma^2 &= f_{\mathcal{N}} d\tau^2 + f_{\mathcal{N}}^{-1} \zeta^2 (e^{2\gamma_{\mathcal{N}}} d\theta^2 + \sin^2 \theta d\varphi^2) \\ &\quad + O(\zeta^{-1}) d\tau d\varphi, \end{aligned} \quad (52)$$

with the determinant

$$\begin{aligned} h &= \begin{vmatrix} f_{\mathcal{N}} & 0 & 0 \\ 0 & f_{\mathcal{N}}^{-1} \zeta^2 \sin^2 \theta & O(\zeta^{-1}) \\ 0 & O(\zeta^{-1}) & f_{\mathcal{N}}^{-1} \zeta^2 e^{2\gamma_{\mathcal{N}}} \end{vmatrix} \\ &= f_{\mathcal{N}}^{-1} e^{2\gamma_{\mathcal{N}}} \zeta^4 \sin^2 \theta + O(\zeta^{-2}), \end{aligned} \quad (53)$$

where the subscript \mathcal{N} indicates that the functions are restricted to $\mathcal{N}(\zeta)$.

We note that I_{GH} is defined at spatial infinity, so we are free to use a large- ζ approximation for our purposes, within which we can neglect the terms of order $O(\zeta^{-2})$. In particular, we will drop for that reason the last term in (53) involving imaginary quantities. Computing now the volume of \mathcal{N} for large ζ , we obtain

$$\text{Vol}\mathcal{N} = \int_{\mathcal{N}} \sqrt{h} d\tau d\varphi d\theta \simeq 4\pi\beta\zeta^2 f_{\mathcal{N}}^{-1/2} e^{\gamma_{\mathcal{N}}}, \quad (54)$$

where it has been supposed that $f_{\mathcal{N}}$ as well as $\gamma_{\mathcal{N}}$ are constant on $\mathcal{N}(\zeta)$.

The asymptotic behavior of the function $f_{\mathcal{N}}$ of the solution (31) is well known, and it is determined by the expression

$$f_{\mathcal{N}} \simeq 1 - \frac{2M}{\zeta}, \quad M = -\frac{1}{2} \operatorname{Re} \left(\sum_{l=1}^N e_l \right), \quad (55)$$

where M is the total mass of the system. In the particular case of the black-hole spacetime, the struts give zero contribution to the total mass, the latter then representing a sum of individual Komar masses [28] of the black-hole constituents

$$M = \sum_{i=1}^N M_i. \quad (56)$$

Of course, M here can also be viewed as the Arnowitt-Deser-Misner mass of the multi-black-hole system [29].

Now, taking into account (55), we get

$$\begin{aligned} g_{\zeta\zeta}^{-1/2} \partial_{\zeta} \operatorname{Vol} \mathcal{N} &\simeq 4\pi\beta \left(1 + \frac{2M}{\zeta} \right)^{-1/2} \partial_{\zeta} \left[\left(1 + \frac{2M}{\zeta} \right)^{1/2} \zeta^2 \right] \\ &= 4\pi\beta(2\zeta - M) + O(\zeta^{-1}), \end{aligned} \quad (57)$$

whence it follows that

$$\int_{\mathcal{N}} K \sqrt{h} = g_{\zeta\zeta}^{-1/2} \partial_{\zeta} \operatorname{Vol} \mathcal{N} = 4\pi\beta(2\zeta - M) + O(\zeta^{-1}). \quad (58)$$

In addition,

$$\int_{\mathcal{N}} K_0 \sqrt{h} = 4\pi\beta(2\zeta) + O(\zeta^{-1}), \quad (59)$$

and hence

$$\int_{\mathcal{N}} [K] \sqrt{h} = -4\pi\beta M + O(\zeta^{-1}). \quad (60)$$

Therefore, after taking the limit $\zeta \rightarrow \infty$ in (60), we finally arrive at

$$I_{\text{GH}} = \frac{1}{2} \beta M. \quad (61)$$

Of course, this result is an expected one, and in the case of asymptotically flat spacetimes it may even look trivial because all it says is that the total mass of the system is a combined contribution of all the constituents of the system. However, the calculation of the total mass can become a nontrivial exercise if a spacetime is not asymptotically flat globally [30].

C. The electromagnetic term

Like in the case of a single KN black hole, the electromagnetic potential A of the generic configuration of N black-hole constituents has two nonzero components and is defined by the formula

$$A = A_{\tau} d\tau + A_{\varphi} d\varphi, \quad (62)$$

where the explicit form of A_{τ} and A_{φ} is given in the paper [15].

To compute the electromagnetic term I_{em} of the Euclidean action, it is advantageous first to rewrite (4) as a boundary integral

$$I_{\text{em}} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} A \wedge \star dA, \quad (63)$$

also noting that since the field A vanishes at infinity we have $\partial\mathcal{M} = \sum_i H_i \times S^1$, H_i denoting as usual the horizon of i th KN black hole.

To perform the calculation, we introduce a family of N hypersurfaces $N_i = C_i \times S^1$, where C_i is a cylinder with radius $\rho = \text{const}$ and height $\alpha_{2i-1} - \alpha_{2i}$. In this way, we will have the behavior $N_i \rightarrow H_i \times S^1$ as $\rho \rightarrow 0$. Since the integrals over the bases of the cylinders do not contribute in the final result, the computation for each N_i yields

$$A \wedge \star dA|_{N_i} = \rho^{-1} [A_{\tau} (g_{\varphi\tau} \partial_{\rho} A_{\varphi} - g_{\varphi\varphi} \partial_{\rho} A_{\tau}) + A_{\varphi} (g_{\tau\varphi} \partial_{\rho} A_{\tau} - g_{\tau\tau} \partial_{\rho} A_{\varphi})] d\tau \wedge dz \wedge d\varphi|_{N_i}. \quad (64)$$

Following Carter [26], we now define $\lambda_i := -i\partial_{\tau} + \Omega_i^H \partial_{\varphi}$ and $\Omega_i^H := -ig_{\tau\varphi}/g_{\varphi\varphi}|_{H_i}$. Then we can rewrite the first A_{τ} in (64) as $A_{\tau} - i\Omega_i^H A_{\varphi} + i\Omega_i^H A_{\varphi}$ and rearrange the terms, thus obtaining

$$\begin{aligned} \int_{N_i} A \wedge \star dA &= \beta \int_{N_i - S^1} \rho^{-1} (A_{\tau} - i\Omega_i^H A_{\varphi}) (g_{\varphi\tau} \partial_{\rho} A_{\varphi} - g_{\varphi\varphi} \partial_{\rho} A_{\tau}) dz d\varphi \\ &\quad + 2\pi\beta \int_{\alpha_{2i}}^{\alpha_{2i-1}} \rho^{-1} A_{\varphi} [(g_{\tau\varphi} - i\Omega_i^H g_{\varphi\varphi}) \partial_{\rho} A_{\tau} - (g_{\tau\tau} - i\Omega_i^H g_{\varphi\tau}) \partial_{\rho} A_{\varphi}] dz. \end{aligned} \quad (65)$$

Observing further that

$$g_{\tau\varphi} - i\Omega_i^H g_{\varphi\varphi} \approx 0, \quad g_{\tau\tau} - i\Omega_i^H g_{\varphi\tau} \approx \rho^2 g_{\varphi\varphi}^{-1} \quad (66)$$

in the vicinity of the i th horizon, and also that $\rho \rightarrow 0$ as $N_i \rightarrow H_i \times S^1$, with which the second integral on the rhs of (65) vanishes on the horizon, we get

$$\begin{aligned} \int_{H_i \times S^1} A \wedge \star dA &= \lim_{\rho \rightarrow 0} \int_{N_i} A \wedge \star dA \\ &= \beta i \Phi_i \int_{H_i} \star F + 2\pi\beta \lim_{\rho \rightarrow 0} \int_{\alpha_{2i}}^{\alpha_{2i-1}} \rho^{-1} A_\varphi [(g_{\tau\varphi} - i\Omega_i^H g_{\varphi\varphi}) \partial_\rho A_\tau \\ &\quad - (g_{\tau\tau} - i\Omega_i^H g_{\varphi\tau}) \partial_\rho A_\varphi] dz, \\ &= \beta \Phi_i \left(i \int_{H_i} \star F \right), \end{aligned} \quad (67)$$

where we have exploited the fact that the potential

$$\Phi_i := A(\lambda_i)|_{H_i} = iA_\tau + \Omega_i^H A_\varphi|_{H_i} \quad (68)$$

takes constant value on the i th horizon [26] and also made use of the relation

$$\star F|_{H_i} = \star dA|_{H_i} = -\rho^{-1} (g_{\varphi\tau} \partial_\rho A_\varphi - g_{\varphi\varphi} \partial_\rho A_\tau) dz \wedge d\varphi|_{H_i}. \quad (69)$$

Last, using formula [26]

$$i \int_{H_i} \star F = -4\pi Q_i, \quad (70)$$

where Q_i is the electric charge of the i th KN black hole, we arrive at the desired result for the electromagnetic action term,

$$I_{\text{em}} = -\frac{1}{2} \beta \sum_{i=1}^N Q_i \Phi_i. \quad (71)$$

Putting now the expressions obtained for I_{EH} , I_{GH} , and I_{em} together, we can write down the final formula for the Euclidean action I_{E} of N arbitrary KN black holes,

$$I_{\text{E}} = \frac{\beta}{2} \left(M - \sum_{i=1}^N Q_i \Phi_i \right) + \beta \sum_{i=1}^{N-1} l_i \mathcal{F}_i - \frac{1}{4} \sum_{i=1}^N A_i^H \left(1 - \frac{\kappa_i}{2\pi} \beta \right). \quad (72)$$

It is clear that the last term in (72) involving A_i^H and κ_i is likely to be removed from the action for the regularity reason.

IV. HAWKING AVERAGE TEMPERATURE AND FREE ENERGY OF N KN BLACK HOLES

We have seen that in the case of a single KN black hole the conical singularity formed on the horizon by the periodic time can be eliminated by an appropriate choice of β , which is equivalent to introducing the Hawking temperature of a black hole. However, in the configurations consisting of various black holes that have different surface gravities, the conical singularity can be removed (by fixing β) only on one horizon, as it follows from the structure of the last term in (72), whereas the horizons of the other black holes will remain singular. This, in particular, explains why the Euclidean action method has been applied so far exclusively to the binary systems of equal black holes.

At the same time, the advantage of evaluating explicitly the singular terms in the Euclidean action is quite clear—this opens a good opportunity to analyze these terms in detail and helps one understand better the whole problem with the conical singularities due to Euclidean time and possible approaches to tackle it. Thus, an immediate proposal for regularizing the Euclidean action would be demanding that the entire combined contribution of the singular horizon terms in (72) must vanish. As can be easily seen, this fixes *uniquely* the choice of the parameter β in the action; indeed,

$$\begin{aligned} \sum_{i=1}^N A_i^H \left(1 - \frac{\kappa_i}{2\pi} \beta \right) &= 0 \\ \Rightarrow \beta &\equiv \hat{\beta} = \left(\sum_{i=1}^N A_i^H \right) \left(\sum_{i=1}^N \frac{\kappa_i}{2\pi} A_i^H \right)^{-1}, \end{aligned} \quad (73)$$

whence we get the expression for the temperature that generalizes the notion of the Hawking temperature to the case of multiple black holes,

$$T \equiv \hat{T} = \frac{1}{2\pi} \left(\sum_{i=1}^N \kappa_i A_i^H \right) \left(\sum_{i=1}^N A_i^H \right)^{-1}. \quad (74)$$

From now on we will call \hat{T} the Hawking average temperature (HAT) of a multi-black-hole system.

Note that, in the particular cases of one black hole ($\kappa_1 = \kappa$, $\kappa_i = 0$, $i = 2, \dots, N$), or a system of black holes possessing the same surface gravity ($\kappa_i = \kappa$, $i = 1, \dots, N$), one recovers from (74) the well-known conventional formula for the Hawking temperature,

$$T_H = \frac{\kappa}{2\pi}. \quad (75)$$

Therefore, for the regularized Euclidean action in the presence of massless struts, we obtain the formula

$$I_E = \frac{\hat{\beta}}{2} \left(M - \sum_{i=1}^N Q_i \Phi_i \right) + \hat{\beta} \sum_{i=1}^{N-1} l_i \mathcal{F}_i, \quad (76)$$

which in turn supplies us, via (5), with the expression for the free energy of the system,

$$W = \frac{1}{2} \left(M - \sum_{i=1}^N Q_i \Phi_i \right) + \sum_{i=1}^{N-1} l_i \mathcal{F}_i. \quad (77)$$

We would like to emphasize that the introduction of HAT \hat{T} does not actually deny anyhow the existence of individual Hawking temperatures T_i of the black-hole constituents. Precisely, the possibility of removing the conical singularity on any particular horizon by means of the choice $\beta_i = 1/T_i = 2\pi/\kappa_i$ in (72) shows, on the one hand, that the individual black-hole temperatures do exist and are of Hawking's type and, on the other hand, that they may not coincide with the average temperature \hat{T} of the system. In this respect, it should be pointed out that we assume as usual that each black hole satisfies the Smarr mass relation [13,26,31], and the sum of such relations can be written as

$$M = \sum_{i=1}^N M_i = \sum_{i=1}^N (2T_i S_i + 2\Omega_i^H J_i + \Phi_i Q_i), \quad (78)$$

where J_i is the angular momentum and S_i is the entropy of the i th black hole; while Ω_i^H introduced in the previous section represents the angular velocity of the i th horizon and is equal to the inverse of the metric function ω evaluated on that horizon. In fact, it is not difficult to find a representation of \hat{T} in terms of the individual Hawking temperatures T_i . Indeed, performing in (74) the substitutions $\kappa_i = T_i/2\pi$, $A_i^H = 4S_i$, and introducing the total entropy S of the system by the formula

$$S := \sum_{i=1}^N S_i = \sum_{i=1}^N \frac{A_i^H}{4}, \quad (79)$$

we get for \hat{T} the following concise expression:

$$\hat{T} = \frac{1}{S} \sum_{i=1}^N T_i S_i, \quad (80)$$

with which, for instance, the Smarr relation (78) rewrites as

$$M = 2\hat{T}S + \sum_{i=1}^N (2\Omega_i^H J_i + \Phi_i Q_i). \quad (81)$$

A useful corollary of (81) is yet another form of the free energy, namely,

$$W = M - \hat{T}S - \sum_{i=1}^N (\Omega_i^H J_i + \Phi_i Q_i) + \sum_{i=1}^{N-1} l_i \mathcal{F}_i, \quad (82)$$

which generalizes the respective formula for W known for the case of two equal black-hole constituents [6,10].

As was shown in [10], the variation of W is given by the formula

$$dW = -Sd\hat{T} - \sum_{i=1}^N (J_i d\Omega_i^H + Q_i d\Phi_i) + \sum_{i=1}^{N-1} \mathcal{F}_i dl_i, \quad (83)$$

so by taking the differential of (82) and making use of (83), we obtain the first law of thermodynamics for the system of N KN black holes,

$$dM = \hat{T}dS + \sum_{i=1}^N (\Omega_i^H dJ_i + \Phi_i dQ_i) - \sum_{i=1}^{N-1} l_i d\mathcal{F}_i. \quad (84)$$

Returning now to the temperature \hat{T} , let us consider its representation involving the quantities σ_i that are defined by the parameters α_n of the N -soliton solution and represent the half lengths of the respective horizons. As it follows from formula (48), we have $\kappa_i A_i^H = 4\pi\sigma_i$, so that, after changing the sum of the horizon areas to the sum of the entropies, we can rewrite (74) in the form

$$\hat{T} = \frac{1}{2S} \sum_{i=1}^N \sigma_i. \quad (85)$$

The above representation of \hat{T} is of interest for two reasons. First, it demonstrates in a simple way that in the limiting case of N extreme KN black holes (when all $\sigma_i = 0$) the HAT \hat{T} takes zero value. Second, formula (85) simplifies even further for a static configuration of N Schwarzschild black holes [32], the thermodynamics of which has recently been studied in papers [30,33,34], because in this case the quantities σ_i are equal to the

masses of black holes, $\sigma_i = M_i$; as a consequence, (85) takes a remarkably simple form

$$\hat{T} = \frac{M}{2S}, \quad (86)$$

where M is the total mass of the Schwarzschild black holes.

In the simplest case of two nonequal Schwarzschild black holes [11], the explicit form of \hat{T} can be worked out with the aid of the formulas of paper [35], yielding

$$\hat{T} = \frac{M[R^2 - (M_1 - M_2)^2]}{8\pi(R + M)[R(M_1^2 + M_2^2) - M(M_1 - M_2)^2]},$$

$$M = M_1 + M_2, \quad (87)$$

where R is the coordinate distance between the centers of black holes. If $M_1 > M_2$, then \hat{T} turns out to be greater than the Hawking temperature of the black hole with larger mass M_1 and less than the temperature of the black hole with smaller mass M_2 ,

$$\frac{R + M_1 - M_2}{8\pi M_1(R + M)} < \hat{T} < \frac{R - M_1 + M_2}{8\pi M_2(R + M)}, \quad (88)$$

which lends support to our interpretation of \hat{T} as an average temperature of the system. Note that \hat{T} is an increasing function of R , taking its maximum value

$$\hat{T}_{\max} = \frac{M}{8\pi(M_1^2 + M_2^2)}, \quad (89)$$

in the limit $R \rightarrow \infty$, and its minimum value $\hat{T}_{\min} = (8\pi M)^{-1}$ at $R = M$, when the two black holes merge and form one Schwarzschild black hole with mass M . The gravitational interaction of black holes therefore decreases the temperature of the system.

V. CONCLUDING REMARKS

Therefore, we have computed the Euclidean action describing the system of N arbitrary KN black holes and shown how the problem of singular horizon terms due to periodic time can be obviated by introducing the notion of average temperature of a multi-black-hole configuration. In this way, we have extended the applicability of the path-integral method to the systems of nonequal black holes with different surface gravities, which, in particular, permitted us to obtain the expression for the free energy of the system in the generic case and find a natural generalization of the Hawking temperature to the case of multiple black holes in the form of HAT \hat{T} determined concisely by the surface

gravities and horizon areas of black holes. It is worth noting in this respect that the Euclidean action method seems to be ideally suited for treating the black-hole systems, as it permits a far-reaching analysis just on the basis of the well-known general properties and characteristics of black holes, without the need to use the explicit form of the multi-black-hole solution.

While the average temperature \hat{T} introduced in the present paper offers a possibility to approximate a system of black holes by treating it as one body with mass M (the total mass of the system) and temperature \hat{T} , one may also ask whether the system could have some other averaged physical characteristics describing it as an integral unit. Interestingly, looking at the Smarr relation in the form (81), it is tempting to speculate that the second and third terms on the rhs of (81) could be rewritten similar to the first term involving the total entropy. Indeed, by introducing the total angular momentum J and total charge Q of the system by the usual formulas

$$J = \sum_{i=1}^N J_i, \quad Q = \sum_{i=1}^N Q_i, \quad (90)$$

we can define, in analogy with the HAT \hat{T} , the average angular velocity $\hat{\Omega}$ and average electric potential $\hat{\Phi}$ as

$$\hat{\Omega} = \frac{1}{J} \sum_{i=1}^N \Omega_i^H J_i, \quad \hat{\Phi} = \frac{1}{Q} \sum_{i=1}^N \Phi_i Q_i, \quad (91)$$

with which the Smarr relation (81) takes the form

$$M = 2\hat{T}S + 2\hat{\Omega}J + \hat{\Phi}Q. \quad (92)$$

The only reason why this formula has not received attention in the literature so far is, of course, the necessity to justify \hat{T} as the average temperature of the system, something that slightly surpasses the limits of the classical black-hole thermodynamics. We hope that our paper, providing such a justification for \hat{T} by means of the Euclidean action approach, makes formula (92) fully plausible now.

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