

Slowly rotating gravastars

Philip Beltracchi^{1,*}, Paolo Gondolo^{1,2,3,†} and Emil Mottola^{4,5,‡}

¹*Department of Physics and Astronomy, University of Utah, Salt Lake City, Utah 84112, USA*

²*Department of Physics, Tokyo Institute of Technology,
2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan*

³*Kavli Institute for the Physics and Mathematics of the Universe, The University of Tokyo,
Kashiwa, Chiba 277-8583, Japan*

⁴*Theoretical Division, T-2, MS B283, Los Alamos National Laboratory,
Los Alamos, New Mexico 87545, USA*

⁵*Department of Physics and Astronomy, University of New Mexico,
Albuquerque, New Mexico 87131, USA*

 (Received 11 July 2021; accepted 31 October 2021; published 3 January 2022)

We solve Einstein's equations for slowly rotating gravitational condensate stars (gravastars) up to second order in the rotation by expanding about the spherically symmetric gravastar with de Sitter interior and Schwarzschild exterior matched at their common horizon. Requiring that the perturbations are finite on the null surface reduces the exterior geometry to that of a Kerr black hole, implying that a slowly rotating gravastar cannot be distinguished from a Kerr black hole by any measurement or observation restricted to the macroscopic spacetime exterior to the horizon. We determine the interior solution, the surface stress tensor, and the Komar mass and angular momentum localized on the slowly rotating horizon surface. With the interior equation of state fixed at $p = -\rho$, finite junction conditions on the null horizon surface necessarily lead to an interior solution with a singular core, where the perturbative expansion breaks down. Comparison to other models and implications for more rapidly rotating gravastars are briefly discussed.

DOI: [10.1103/PhysRevD.105.024002](https://doi.org/10.1103/PhysRevD.105.024002)

I. INTRODUCTION

Extension of the exterior geometry of a black hole (BH) through the horizon and into the interior involves the physical assumption that the vacuum Einstein equations apply at and inside the horizon. Motivation for considering alternative BH interiors arises first from the curvature singularities in the geometries of mathematical BHs, which according to the classical singularity theorems are a general feature once some set of conditions involving trapped surfaces and energy conditions applies [1,2]. In addition to curvature singularities, the BH interiors may show acausal characteristics, such as the closed timelike curves of the Kerr solution [3]. Such acausal behavior is widely held to be unphysical, especially since it arises on macroscopic scales comparable to the scale of the BH horizon.

A BH horizon is a marginally trapped surface from which matter and information cannot exit classically. This horizon boundary is the root of severe difficulties BHs pose for quantum theory, most notably the apparent nonconservation of probability and enormous BH entropy implied by the Hawking effect [4]. The resulting “BH information

paradox” [5] has been the source of perplexity and speculation for more than four decades, with a wide range of views on its possible resolution [6], some of them quite radical, in which no classical spacetime interior may survive at all [7].

Another noteworthy property of horizons is the potentially large, unbounded semiclassical stress tensor from vacuum polarization effects that can occur on them [8–10]. If by any mechanism a nonvanishing surface stress tensor is present on the horizon, a globally vacuum solution is inappropriate and the interior may not be singular or have any unphysical or acausal features at all. Consistency with quantum unitary evolution can then be maintained in this case. Physical surface stresses on a BH horizon, whatever their origin, are perfectly consistent with the equivalence principle as any physical surface is, and can be described within classical general relativity in terms of a sharply peaked anisotropic stress tensor—in the infinitely thin surface limit, a Dirac δ -function distribution—localized on the horizon.

An explicit example of surface stresses and a quite different interior solution is provided by the gravitational condensate star (gravastar) in [11], where the Schwarzschild BH exterior is matched to a non-singular de Sitter interior with a positive vacuum energy but negative pressure,

*philipbeltracchi@gmail.com

†paolo.gondolo@utah.edu

‡mottola.emil@gmail.com, emottola@unm.edu

$p = -\rho$, at a surface located at their respective horizons. A similar proposal based on an analogy to quantum phase transitions in condensed matter was suggested in Ref. [12]. An abrupt change in ground state vacuum energy at the horizon is characteristic of a quantum phase transition, and would lead to a dark energy de Sitter interior if $\rho = -p > 0$ [10]. Independently of the microscopic origin of the phase transition boundary layer, the interior static de Sitter and exterior Schwarzschild geometries are “glued” together at their respective horizons at $R_S = 2GM/c^2 = c/H$, where M is the Schwarzschild mass and H is related to the de Sitter energy density by $\rho = 3H^2/(8\pi G)$. The discontinuity $[\kappa]$ in the surface gravities at the horizon results in a positive surface tension $\tau_s = [\kappa]/8\pi G = c^4(8\pi GR_S)^{-1}$ [11], and a surface stress tensor δ -function localized on the gravastar physical surface that replaces the mathematical BH horizon.

A variety of other models for “nonsingular BHs” or “BH mimickers” have been proposed over the years [13–17]. In addition to their different interiors, several of these models have also been called “gravastars” by their authors, some with thin shells [18–21], some with shells of quite large macroscopic thickness consisting of stiff matter [22,23] or anisotropic matter [22,24,25]. It is important to recognize that these alternate models have a *timelike* outer boundary at radius $R > 2GM/c^2$, *not* at the null horizon as in [11]. This distinction is important in whether or not the usual Israel junction conditions [26] can be used, since these are unsuitable for null hypersurfaces [27].

More importantly, the distinction between timelike and horizon surfaces manifests itself in physical properties and observational signatures that can distinguish such ultra-compact objects from mathematical BHs, for example in their tidal deformability. In the era of gravitational wave and multimessenger astronomy it becomes of paramount importance to distinguish these various alternative models, with different consequences for observations.

At the present time there is no direct observational evidence of the existence of a BH event horizon. Indirect arguments, based upon accretion models [28,29], are limited by the assumptions of the models [30]. The LIGO/LSC gravitational wave data [31], and even the spectacular images of M87 obtained by the Event Horizon Telescope [32], are sensitive to the light rings at the innermost stable circular orbits, probing the geometry well outside the BH horizon [33,34]. As a result, while some thick shell variants are already ruled out [23], the gravastar of Ref. [11], defined by an extremely thin shell localized on the horizon, remains consistent with observations [34]. The physics of a surface at or near a horizon, and possible interior structure of ultracompact thin-shell models, can be probed by the after-merger ringdown and/or new phenomena such as “echoes” of the original merger event signal [34,35], for which at present, there are suggestions but no conclusive evidence [36,37].

In this paper we shall reserve the term “gravastar” for the universal gravitational condensate star solution of Ref. [11], where the lightlike null horizon plays a privileged role as the locus of joining of interior and exterior classical geometries, and the surface layer there is of negligible thickness, so that its stress tensor is well approximated by a Dirac δ -function. Its generalization to the case of slow rotation is the subject of this paper.

Given the spherically symmetric gravastar, the method of Hartle-Thorne [38,39] can be used to find slowly rotating solutions by expanding in powers of a small rotation parameter, under the assumption that the equation of state of the interior is unchanged from the nonrotating case. In Ref. [40] this method was applied to a slowly rotating constant-density Schwarzschild star of mass M and its radius $R > 2GM/c^2$ was allowed to approach $2GM/c^2$ numerically. In this paper we show that analytic solution of the perturbation equations to second order in the rotation and matching directly on the horizon at radius $2GM/c^2$ is possible by the newly developed methods of [41], hereafter referred to as Paper I. As we shall see, matching at the horizon leads to different conclusions from that of [40], or that of other models matched at a radius different from $2GM/c^2$ [19–21].

The structure of the paper is as follows. In the next section the nonrotating spherical gravastar of Ref. [11] is briefly reviewed. In Sec. III we derive the Einstein equations for the slowly rotating gravastar. In Secs. IV and V we give the general analytic solution of these equations for the exterior and interior rotating gravastar metric functions respectively, showing that finiteness of the perturbations on the horizon requires the exterior geometry to be identical to that of a Kerr BH. In Sec. VI, we discuss the joining of the interior and exterior at the mutual horizon boundary, relying on the analytic method of Paper I [41]. In Sec. VII we give the δ -distributional stress energy tensor on the null horizon and surface gravity discontinuity resulting from this gluing of interior to exterior. Section VIII contains the Komar mass, angular momentum and moment of inertia of the slowly rotating gravastar solution, showing that some integration constants can be fixed by eliminating sources at the origin. Section IX contains a Summary and Discussion of our results, and comparison to other models, while Sec. X contains our Conclusions and consequences for observations.

There are three Appendices. Appendix A contains the Einstein tensor component for the rotating metric (3.1) expanded up to second order in the small rotation parameter. Appendix B contains the Weyl tensor of this metric, and Appendix C contains the conformal diagram that results from gluing of the nonrotating gravastar interior to the exterior at the horizon surface. Hereafter we generally use geometric units where $G = c = 1$ to simplify the notation, restoring them only when useful for clarity, and MTW metric and curvature conventions [42].

II. THE SPHERICAL GRAVASTAR AND HORIZON SURFACE TENSION

In order to establish notation and conventions we first briefly review the spherically symmetric nonrotating gravastar of Ref. [11]. As shown in [11], this gravastar solution can be obtained from Schwarzschild's constant density interior solution [43,44] by a limiting process in which the star radius $R \rightarrow R_S \equiv 2GM/c^2$, the Schwarzschild horizon radius. The gravastar of [11] may be viewed as a universal limit of the gravitational condensate star model first proposed in [45,46], where a thin layer of ultrarelativistic $p = \rho$ material was interposed, straddling the mutual Schwarzschild and de Sitter horizons. It is universal in the sense that when the thickness of the intervening surface layer localized on the horizon is taken to zero, the resulting solution is independent of any assumptions about the intervening layer equation of state, and the surface stress tensor is instead completely determined by the matching of the interior and exterior spacetimes on their respective horizons. This universal (nonrotating) gravastar has maximal compactness $GM/R = 1/2$ (up to possible Planck scale corrections), and evades the Buchdahl bound $R \geq (9/4)GM$, applicable to isotropic fluid spheres [47], by having an anisotropic stress at its surface. Like the arbitrarily thin shell model of [45,46], the gravastar of [11] is a low entropy, cold condensed solution, in no conflict with unitary quantum evolution or statistical mechanics, possesses no enormous entropy, and hence no information paradox [45,46].

The general static, spherically symmetric line element can be expressed as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.1)$$

with $f(r)$, $h(r)$ two arbitrary functions of r . Alternately one may write

$$f(r) = e^{2\nu_0(r)} = \frac{h(r)}{[j(r)]^2}, \quad h(r) = 1 - \frac{2m(r)}{r} \quad (2.2)$$

in terms of the gravitational potential $\nu_0(r)$, the mass function $m(r)$, and Hartle's function $j(r)$, the latter defined to be positive.

The general spherically symmetric solution of Einstein's equations requires the three stress-energy tensor components $T^t_t = -\rho$, $T^r_r = p_r$, and $T^\theta_\theta = T^\phi_\phi = p_\perp$, which are functions of r only. If $p_\perp = p_r \equiv p$, the pressure is isotropic. If in addition $\rho + p = 0$, as in the exterior and interior of a gravastar, Einstein's equations imply that $j(r)$ is constant and $m(r)$ is either a Schwarzschild term or a de Sitter term (or a sum of the two).

The spherical nonrotating gravastar solution is given by the following piecewise continuous functions $f(r)$ and $h(r)$ in the condensate interior and vacuum exterior respectively [11],

$$f(r) = \begin{cases} \frac{1}{4}(1 - \frac{r^2}{R_S^2}), & 0 \leq r \leq R_S \\ 1 - \frac{R_S}{r}, & r \geq R_S \end{cases} \quad (2.3a)$$

$$h(r) = \begin{cases} 1 - \frac{r^2}{R_S^2}, & 0 \leq r \leq R_S \\ 1 - \frac{R_S}{r}, & r \geq R_S. \end{cases} \quad (2.3b)$$

The line element (2.1) with (2.3) describes an interior static patch of de Sitter space with

$$\rho = -p = \frac{3H^2}{8\pi} = \frac{3}{8\pi R_S^2}, \quad r < R_S \quad (2.4)$$

joined to a Schwarzschild exterior with

$$\rho = p = 0, \quad r > R_S \quad (2.5)$$

at their mutual horizons

$$r = R_S = 2M = H^{-1}. \quad (2.6)$$

The function $j(r)$

$$j(r) = \sqrt{\frac{h}{f}} = \begin{cases} 2, & 0 \leq r < R_S \\ 1, & r > R_S \end{cases} \quad (2.7)$$

is discontinuous at the horizon surface, and the mass function is given by

$$m(r) = \begin{cases} \frac{1}{2}H^2 r^3, & 0 \leq r < R_S \\ M, & r > R_S. \end{cases} \quad (2.8)$$

Note that just as a Schwarzschild BH can have any mass, so too the spherically symmetric gravastar can have any mass M , provided only that the de Sitter vacuum dark energy density is fixed by (2.4) and (2.6).

The discontinuity $[\kappa] = \kappa_+ - \kappa_-$ in the surface gravity

$$\kappa(r) = \frac{1}{2} \sqrt{\frac{h}{f}} \frac{df}{dr} = \begin{cases} -\frac{1}{2R_S}, & r \rightarrow R_S^- \\ +\frac{1}{2R_S}, & r \rightarrow R_S^+ \end{cases} \quad (2.9)$$

as $r \rightarrow R_S^\pm$, gives rise to a δ -distributional anisotropic stress tensor whose nonzero components are

$$(\Sigma)T^A_B \sqrt{\frac{f}{h}} = S^A_B \delta(r - R_S), \quad S^A_B = \frac{[\kappa]}{8\pi G} \delta^A_B \quad (2.10)$$

for $A, B = \theta, \phi$. This stress tensor corresponds to a surface tension [11,41]

$$\tau_s = \frac{[\kappa]}{8\pi G} = \frac{c^4}{8\pi G R_S} = \frac{c^6}{16\pi G^2 M}. \quad (2.11)$$

The surface stress tensor $(\Sigma)T^A_B$ in (2.10) is a well-defined distribution when integrated against the standard coordinate invariant volume measure $d^4x\sqrt{-g} = dt dr r^2 \sqrt{f/h} d\theta d\phi \sin\theta$, since $r^2 \sin\theta$ is continuous on the horizon. The result (2.10)–(2.11) may be derived directly from the Einstein tensor density for the piecewise continuous metric (2.3) [41].

The surface stress S^A_B differs from that obtained from the usual Israel junction conditions by [11,41]

$$(S^A_B)_{\text{here}} = \sqrt{f(R)}(S^A_B)_{\text{Israel}}. \quad (2.12)$$

Whereas the junction conditions as originally formulated in [26] do not apply to the null horizon hypersurface where $f(R_H) = 0$, junction conditions modified by the $\sqrt{f} = e^{\nu_0}$ redshift factor and with one contravariant and one covariant index as in (2.12) give the well-defined finite result (2.10)–(2.11) [41]. The relation (2.12) is important for any comparison of the gravastar with its horizon surface tensor (2.10) to other models with matching at some radius away from the horizon by Israel’s method, in either the cases of zero or nonzero angular momentum.

III. LINEARIZED EINSTEIN EQUATIONS FOR SLOW ROTATION

We are interested in small stationary and axisymmetric perturbations due to rotation about the static spherically symmetric gravastar solution summarized in the previous section. There are several forms for the general

axisymmetric and stationary metric line element in the literature which may be used for this purpose [48–54]. The condition of stationarity and axial symmetry is invariantly defined by the geometry through two independent Killing vectors corresponding to time translations and rotations around the fixed rotation axis. Coordinates t and ϕ adapted to these symmetries can be introduced such that these Killing vectors are $\partial/\partial t$ and $\partial/\partial\phi$, respectively. The two remaining coordinates r and θ can be chosen so that the stationary axisymmetric line element assumes the form

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\phi - \omega dt)^2 + e^{2\alpha} dr^2 + e^{2\beta} d\theta^2 \quad (3.1)$$

where the five functions $\nu, \psi, \alpha, \beta, \omega$ are functions of r and θ . There remains the freedom to define a coordinate condition to reduce these five functions to the minimal four functions necessary to specify the general such metric. Hartle’s condition [51]

$$e^\psi = e^\beta \sin\theta \quad (3.2)$$

fixes this local coordinate freedom, leaving reparametrizations of r still allowed.

Consider now a stationary, axisymmetric, slowly rotating, nearly spherically symmetric solution, i.e., small perturbations about the static solution that are invariant under the transformations $(t, \phi) \rightarrow (-t, -\phi)$ and $\theta \rightarrow \pi - \theta$. Hartle and Chandrasekhar-Miller expanded the line element (3.1) to second order in the angular momentum as [39,51]

$$ds^2 = -e^{2\nu_0(r)} [1 + 2h_0(r) + 2h_2(r)P_2(\cos\theta)] dt^2 + \frac{r}{r - 2m(r)} \left\{ 1 + \frac{2}{r - 2m(r)} [m_0(r) + m_2(r)P_2(\cos\theta)] \right\} dr^2 + r^2 [1 + 2k_2(r)P_2(\cos\theta)] [d\theta^2 + \sin^2\theta (d\phi - \omega(r)dt)^2]. \quad (3.3)$$

Here $P_l(\cos\theta)$ is the Legendre polynomial of order l , $m(r)$ and $\nu_0(r)$ are the metric functions of the nonrotating solution, and $h_l(r)$, $m_l(r)$, $k_l(r)$ are the monopole ($l=0$) and quadrupole ($l=2$) contributions of second order in rotation respectively. The choice $k_0(r) = 0$ is part of Hartle’s choice of gauge. The function $\omega(r)$ is the first-order contribution that gives rise to inertial frame dragging.

The metric line element (3.3) is an expansion of (3.1) with the identifications

$$e^{2\nu} = e^{2\nu_0(r)} [1 + 2h_0(r) + 2h_2(r)P_2(\cos\theta)] \quad (3.4a)$$

$$e^{2\psi} = r^2 \sin^2\theta [1 + 2k_2(r)P_2(\cos\theta)] \quad (3.4b)$$

$$e^{2\alpha} = \frac{r}{r - 2m(r)} \left[1 + 2 \frac{m_0(r) + m_2(r)P_2(\cos\theta)}{r - 2m(r)} \right] \quad (3.4c)$$

$$e^{2\beta} = r^2 [1 + 2k_2(r)P_2(\cos\theta)] \quad (3.4d)$$

which will be used both in the interior and exterior of the slowly rotating gravastar, matching at the gravastar null horizon surface.

The perturbed stress-energy tensor is taken to be of the same form as in the nonrotating background gravastar solution, namely

$$T^\mu_\nu = -\mathcal{E} \delta^\mu_\nu \quad (3.5)$$

where \mathcal{E} can be expanded up to second order in the rotation as

$$\mathcal{E} = \rho(r) + \mathcal{E}_0(r) + \mathcal{E}_2(r)P_2(\cos\theta). \quad (3.6)$$

Here ρ is the background energy density, constant in the interior and exterior of the gravastar, given by (2.4)–(2.5), and

\mathcal{E}_0 , \mathcal{E}_2 are its monopole and quadrupole perturbations. The covariant conservation equation $\nabla_\mu T^\mu_\nu = 0$ for the T^μ_ν in Eq. (3.5) becomes $\partial_\mu \mathcal{E} = 0$, and in particular its $\nu = r, \theta$ components and $\rho = \text{const}$ require

$$\mathcal{E}_0 = \text{const}, \quad \mathcal{E}_2 = 0 \quad (3.7)$$

in the interior and exterior regions, with $\mathcal{E}_0 = 0$ in the exterior.

The forms of the perturbed Einstein equations in Refs. [39,51] cannot be applied directly to the case of a rotating gravastar, because they were derived assuming a perfect fluid under uniform rotation Ω , and use definitions of pressure and energy density perturbations that become undefined when $\rho + p = 0$, which is the case of interest here. We have therefore derived the perturbed Einstein equations for the cases of a slowly rotating gravastar anew. The components of the Einstein tensor for the metric (3.3) are given in Appendix A.

Because j is constant in both the interior and exterior of the gravastar, and thus its derivatives vanish there, and because of the specific constant form of the energy-momentum tensor (3.5), having the same $p = -\rho$ equation of state as the nonrotating case, the resulting linear perturbation equations are somewhat simpler than the perfect fluid case considered in [39,51]. In particular, the uniform angular velocity Ω and the auxiliary function $\varpi = \Omega - \omega$ of [39,51] do not appear.

The first order perturbed Einstein equation

$$\frac{d}{dr} \left(r^4 \frac{d\omega}{dr} \right) = 0 \quad (3.8a)$$

comes from $G^t_\phi = 0$ and gives the frame dragging function $\omega(r)$. Making use of (3.8a), the second order monopole terms in the Einstein equations $G^t_t = 8\pi T^t_t$ and $G^r_r = 8\pi T^r_r$ are

$$\frac{dm_0}{dr} = 4\pi r^2 \mathcal{E}_0 + \frac{j^2 r^4}{12} \left(\frac{d\omega}{dr} \right)^2 \quad (3.8b)$$

$$\frac{dh_0}{dr} = \frac{1 - 2m'}{(r - 2m)^2} m_0 - \frac{1}{r - 2m} \frac{dm_0}{dr} \quad (3.8c)$$

Here $m' \equiv dm/dr = 4\pi r^2 \rho$. These equations can be combined to give

$$h_0 = -\frac{m_0}{r - 2m} + C \quad (3.8d)$$

where C is a constant. For the quadrupole functions, with $v_2 = h_2 + k_2$ from [39], we find

$$m_2 = -(r - 2m) \left[h_2 - \frac{1}{6} j^2 r^4 \left(\frac{d\omega}{dr} \right)^2 \right] \quad (3.8e)$$

$$\frac{dv_2}{dr} = -2\nu'_0 h_2 + (1 + r\nu'_0) \frac{j^2 r^3}{6} \left(\frac{d\omega}{dr} \right)^2 \quad (3.8f)$$

$$\begin{aligned} \frac{dh_2}{dr} = & -\frac{2}{r(r - 2m)\nu'_0} v_2 - 2 \left(\nu'_0 + \frac{m}{r^2(r - 2m)\nu'_0} \right) h_2 \\ & + \left(\nu'_0 - \frac{1}{2r(r - 2m)\nu'_0} \right) \frac{j^2 r^4}{6} \left(\frac{d\omega}{dr} \right)^2 \end{aligned} \quad (3.8g)$$

where $\nu'_0 \equiv d\nu_0/dr = (m - 4\pi r^3 \rho)/[r(r - 2m)]$. The algebraic equation for m_2 comes from the quadrupole perturbations of $G^\theta_\theta - G^\phi_\phi = 0$. The equation for v_2 comes from the equation $G^r_\theta = 0$ with the replacement of m_2 from Eq. (3.8e). Finally, the differential equation (3.8g) for h_2 comes from the quadrupole perturbation term in $G^r_r = 8\pi T^r_r$. Equation (3.8e) can also be written in the form

$$h_2 = -\frac{m_2}{r - 2m} + \frac{1}{6} j^2 r^4 \left(\frac{d\omega}{dr} \right)^2 \quad (3.9)$$

which is analogous to (3.8d). The function k_2 is then given by $k_2 = v_2 - h_2$.

IV. EXTERIOR SOLUTION

Outside the gravastar ($r > 2M$), the spacetime is empty, and from (2.3)–(2.4), we have

$$m(r) = M, \quad h = f = e^{2\nu_0} = 1 - \frac{2M}{r}, \quad j = 1. \quad (4.1)$$

The solution of (3.8a) for the frame dragging that goes to zero as $r \rightarrow \infty$ is

$$\omega = \frac{2J}{r^3} \quad (4.2)$$

where J is an integration constant that can be identified as the total angular momentum of the solution. Using (4.2) in the Eq. (3.8b) for m_0 , and then using (3.8d) for h_0 , gives

$$m_0 = \delta M - \frac{J^2}{r^3} \quad (4.3a)$$

$$h_0 = -\frac{m_0}{r - 2M} + C_E \quad (4.3b)$$

in terms of two integration constants δM and C_E in the exterior.

For the quadrupole functions m_2 , k_2 , h_2 in the exterior, the two coupled linear first order equations (3.8f) and (3.8g) for v_2 and h_2 can be converted to a single second order equation for h_2 , which with the change of variable to $z = r/M - 1$ is recognized as the associated Legendre differential equation of degree $l = 2$ and order $m = 2$, with a particular inhomogeneous term in ω'^2 . The general solution for h_2 is thus the sum of a particular solution of the

inhomogeneous equation and two independent solutions of the associated Legendre equation, which can be taken to be the associated Legendre functions of the first and second kind, namely $P_2^2(z) = 3(z^2 - 1)$ and

$$Q_2^2(z) = \frac{3(z^2 - 1)}{2} \ln\left(\frac{z+1}{z-1}\right) + \frac{z(5 - 3z^2)}{z^2 - 1}. \quad (4.4)$$

The branch cut in this function is chosen along the real axis from $-\infty$ to $+1$, since in the exterior of the gravastar, $r > 2M$ and thus $z = r/M - 1 > 1$. The function v_2 then follows from h_2 by use of Eq. (3.8f), and contains associated Legendre functions $P_2^1(z) = 3z(z^2 - 1)^{1/2}$ and $Q_2^1(z)$, again with a branch along the real axis from $-\infty$ to $+1$, where

$$\frac{1}{\sqrt{z^2 - 1}} Q_2^1(z) = \frac{3z}{2} \ln\left(\frac{z+1}{z-1}\right) + \frac{2 - 3z^2}{z^2 - 1}. \quad (4.5)$$

The general solutions for the quadrupole perturbations are thus found to be

$$h_2 = \frac{J^2(r+M)}{Mr^4} + A_E \frac{r(r-2M)}{2M^2} + B_E Q_2^2\left(\frac{r}{M} - 1\right) \quad (4.6a)$$

$$v_2 = h_2 + k_2 = -\frac{J^2}{r^4} - A_E \frac{r-M}{M} - \frac{2B_E M}{\sqrt{r(r-2M)}} Q_2^1\left(\frac{r}{M} - 1\right) \quad (4.6b)$$

$$m_2 = -(r-2M) \left(h_2 - \frac{6J^2}{r^4} \right) \quad (4.6c)$$

with A_E and B_E two additional constants of integration besides J , δM , and C_E .

If we require the solution to be asymptotically flat, then $A_E = 0$, and moreover a nonzero C_E in (4.3b) is simply a rescaling of the time coordinate $g_{tt}(A_E = 0, r \rightarrow \infty) = -1 - 2C_E$. If the time coordinate t and Killing vector $\partial/\partial t$ are fixed by $g_{tt} \rightarrow -1$ asymptotically, then $C_E = 0$ is also fixed. The asymptotically flat exterior Hartle-Thorne metric (3.3) depends then on the remaining constants J , B_E , and δM . As $r \rightarrow 2M$, the quadrupole function k_2 behaves as

$$k_2 \rightarrow B_E \left(8 + 3 \ln \frac{r-2M}{2M} \right) - \frac{J^2}{4M^4} \quad (4.7)$$

where we retain the subdominant terms for later reference. Not only is k_2 supposed to be a small perturbation, but when it diverges the equatorial circumference also diverges. Finiteness of k_2 at $r = 2M$ requires

$$B_E = 0. \quad (4.8)$$

Note that $B_E = 0$ also eliminates the $B_E/(r-2M)$ behavior in the function h_2 as $r \rightarrow 2M$. With (4.8), the exterior solution simplifies to

$$m_0 = \delta M - \frac{J^2}{r^3} \quad (4.9a)$$

$$h_0 = -\frac{m_0}{r-2M} \quad (4.9b)$$

$$h_2 = \frac{J^2}{r^3} \left(\frac{1}{r} + \frac{1}{M} \right) \quad (4.9c)$$

$$m_2 = \frac{J^2(r-2M)(5M-r)}{Mr^4} \quad (4.9d)$$

$$k_2 = -\frac{J^2(r+2M)}{Mr^4} \quad (4.9e)$$

$$\omega = \frac{2J}{r^3}. \quad (4.9f)$$

With (4.9), we see that the combined term $e^{2\nu_0} h_0 = -m_0/r$ in the metric function $e^{2\nu}$ in (3.4a) remains finite at all $r \geq 2M$, while the function h_0 itself generally diverges

$$h_0 \rightarrow -\frac{1}{r-2M} \left(\delta M - \frac{J^2}{8M^3} \right) \quad (4.10)$$

as $r \rightarrow 2M$ unless

$$\delta M = \frac{J^2}{R_S^3} = \frac{J^2}{8M^3}. \quad (4.11)$$

As shown originally in Ref. [38], the Hartle-Thorne metric with $A_E = B_E = 0$ is equivalent to the Kerr metric for a BH of total mass $\mathcal{M} = M + \delta M$ and angular momentum $J = a\mathcal{M} \simeq aM$ to second order in rotation. In fact, the Kerr metric to second order in $a = J/\mathcal{M}$ in Boyer-Lindquist coordinates $(r_{\text{BL}}, \theta_{\text{BL}})$ results from the Hartle-Thorne metric in (3.3) with (4.9) through the coordinate transformation

$$r_{\text{BL}} = r \left\{ 1 - \frac{a^2}{2r^2} \left[\left(1 + \frac{2\mathcal{M}}{r} \right) \left(1 - \frac{\mathcal{M}}{r} \right) - \cos^2 \theta \left(1 - \frac{2\mathcal{M}}{r} \right) \left(1 + \frac{3\mathcal{M}}{r} \right) \right] \right\}, \quad (4.12a)$$

$$\theta_{\text{BL}} = \theta - \frac{a^2}{2r^2} \cos \theta \sin \theta \left(1 + \frac{2\mathcal{M}}{r} \right) \quad (4.12b)$$

where we have corrected the sign of the $\cos^2 \theta$ term in [38].

Thus the requirement that the perturbations of the spherically symmetric gravastar solution due to rotation remain finite on the gravastar horizon surface, together with asymptotic flatness, forces the exterior geometry to be *identical* to the second order expansion of a slowly rotating Kerr BH. A slowly rotating gravastar has compactness

$M/R_S = 1/2$ (up to possible Planck scale corrections), and cannot be distinguished from a Kerr BH by any measurement or observation sensitive only to the external spacetime metric. Note that this conclusion would not follow if we did not require finiteness of the perturbations on the horizon, since (4.8) would not be required in that case.

V. INTERIOR SOLUTION

The background metric of the gravastar interior is a region of de Sitter space with a rescaled time coordinate, with

$$m(r) = \frac{1}{2}H^2r^3, \quad h = 1 - H^2r^2, \quad j = 2. \quad (5.1)$$

We temporarily leave j unspecified to allow for easier comparison to other work [19–21,55], where typically $j = 1$ is used for Hartle perturbations on de Sitter space. The frame dragging and monopole perturbations are again found by integrating (3.8a)–(3.8d),

$$\omega = W_1 + \frac{W_2}{r^3} \quad (5.2a)$$

$$m_0 = \delta M_I + \frac{4\pi\mathcal{E}_0}{3}r^3 - \frac{j^2W_2^2}{4r^3} \quad (5.2b)$$

$$h_0 = -\frac{m_0}{r(1-H^2r^2)} + C \quad (5.2c)$$

where $W_1, W_2, \delta M_I, C$ are integration constants. The stress energy tensor for the interior is given by (3.5)–(3.7) with (2.4) and nonzero \mathcal{E}_0 in general.

The quadrupole perturbations are a linear combination of two independent homogeneous solutions and a particular solution of Eqs. (3.8f)–(3.8g). Upon making the change of variable to $z = 1/(Hr)$, the homogeneous equations may again be combined into an associated Legendre differential equation of degree $l = 2$ and order $m = 2$, this time for rh_2 , whose solutions are the associated Legendre functions $P_2^2(z)$ and $Q_2^2(z)$ of (4.4). The general solution for the quadrupole perturbations in the interior is then

$$v_2 = h_2 + k_2 = -\frac{j^2W_2^2(3 - 10H^2r^2 + 5H^4r^4)}{8r^4} + \frac{2A_I}{r} - \frac{B_I Hr}{\sqrt{1-H^2r^2}} Q_2^1(1/Hr) \quad (5.3a)$$

$$m_2 = \frac{j^2W_2^2(1-H^2r^2)(8-5H^2r^2)}{4r^3} + A_I \frac{(1-H^2r^2)^2}{H^2r^2} - \frac{B_I(1-H^2r^2)}{2H} Q_2^2(1/Hr) \quad (5.3b)$$

$$h_2 = -\frac{m_2}{r(1-H^2r^2)} + \frac{3j^2W_2^2}{2r^4} \quad (5.3c)$$

which together with (5.2) depends upon the six integration constants $W_1, W_2, \delta M_I, A_I, B_I$, and $C_I \equiv C$.

As $r \rightarrow 1/H$, the perturbation functions m_0, m_2 are finite while k_2 behaves as

$$k_2 \simeq 2HA_I - \frac{1}{2}j^2W_2^2H^4 + \frac{B_I}{2} \left(7 + 3 \ln \frac{1-Hr}{2} \right) \quad (5.4)$$

where we retain the subdominant terms for later reference. Finiteness of k_2 at the de Sitter horizon $r = 1/H$ requires

$$B_I = 0. \quad (5.5)$$

The (5.5) finiteness condition on the perturbation k_2 also removes the $B_I/(1-Hr)$ behavior from h_2 . From (5.2c), the combined term $e^{2\nu_0}h_0 = -m_0/r$ in $e^{2\nu}$ in (3.4a) remains finite as $r \rightarrow 1/H$, while the function h_0 itself generally diverges

$$h_0 \rightarrow -\frac{H}{2(1-Hr)} \left(\delta M_I + \frac{4\pi}{3H^3}\mathcal{E}_0 - \frac{j^2H^3W_2^2}{4} \right) \quad (5.6)$$

as $r \rightarrow 1/H$, unless

$$\mathcal{E}_0 = \frac{3}{4\pi R_S^3} \left(\frac{j^2W_2^2}{4R_S^3} - \delta M_I \right). \quad (5.7)$$

Finally, as $r \rightarrow 0$, the functions ω, m_0, m_2 , and k_2 diverge unless $W_2 = A_I = 0$. We do not impose these conditions since as we show in the following Sec. VI it implies $J = 0$ in the exterior and so no rotation at all of the gravastar, which is clearly too strong a condition.

VI. JOINING EXTERIOR TO INTERIOR ON THE NULL HORIZON SURFACE

The two Killing vectors $\partial/\partial t, \partial/\partial\phi$ determine the location of the null hypersurface $r = R_H$ where the gluing of the interior and exterior solutions occurs by the geometrically invariant condition that the norm of the vector $\ell = \partial/\partial t + \omega\partial/\partial\phi$ vanishes there, i.e.,

$$\ell \cdot \ell = e^{2\nu} = f(r)\{1 + 2h_0(r) + 2h_2(r)P_2(\cos\theta)\} = 0. \quad (6.1)$$

If we impose that h_0 and h_2 are finite at $r = 2M$ in the exterior and at $r = 1/H$ in the interior, resulting in Eqs. (4.11) and (5.7), and in addition use Eq. (2.6) from the nonrotating gravastar, then the solution of (6.1) is $r = R_H = R_S$.

In the coordinates (t, θ, ϕ) adapted to the Killing vectors, the induced metric on the horizon hypersurface at $r = R_H$ is

$$ds_H^2 = \Sigma(\theta)[d\theta^2 + \sin^2\theta(d\phi - \omega_H dt)^2] \quad (6.2)$$

where

$$\omega_H = \omega(R_H) = \frac{2J}{R_H^3} \quad (6.3)$$

and using (4.11), (5.7),

$$\Sigma(\theta) = e^{2\beta(R_H, \theta)} = R_H^2[1 + 2k_2(R_H)P_2(\cos\theta)]. \quad (6.4)$$

The induced metric (6.2) is regular on the null hypersurface, and independent of the perturbation functions h_2, m_2 . The first junction condition is that the induced metric (6.2) is the same when approached from both sides, which requires that $R_H, k_2(R_H)$, and $\omega(R_H)$ have the same value on both sides. These conditions also ensure continuity of the inner products of the Killing vectors, i.e., $\partial/\partial t \cdot \partial/\partial t$, $\partial/\partial t \cdot \partial/\partial\phi$, and $\partial/\partial\phi \cdot \partial/\partial\phi$.

The equality of the perturbation function $k_2(r)$ as $r \rightarrow R_H^\pm$ from both sides of the horizon implies that the interior limit

$$\lim_{r \rightarrow R_H^-} k_2(r) = \frac{2A_I}{R_H} - \frac{j^2 W_2^2}{2R_H^4} \quad (6.5)$$

from Eq. (5.4) with $B_I = 0$ and $H = R_H^{-1}$, and the exterior limit

$$\lim_{r \rightarrow R_H^+} k_2(r) = -\frac{4J^2}{R_H^4} \quad (6.6)$$

from Eq. (4.7) with $B_E = 0$ and $M = R_H/2$, coincide. Equating Eqs. (6.5) and (6.6) gives

$$A_I = \frac{j^2 W_2^2 - 8J^2}{4R_H^3} \quad (6.7)$$

fixing the interior integration constant A_I in terms of W_2 and J . Finally, the equality of $\omega(r)$ when the horizon is approached from both sides imposes

$$W_1 = \frac{2J - W_2}{R_H^3} \quad (6.8)$$

and fixes the interior integration constant W_1 .

With the finiteness conditions $B_E = B_I = 0$, and the conditions (4.11), (5.7), (6.7), (6.8) on the horizon, the perturbation functions in the interior of the gravastar $r < R_H = H^{-1} = 2M = R_S$, where $j = 2$, become

$$\omega = \frac{2J - W_2}{R_H^3} + \frac{W_2}{r^3} \quad (6.9a)$$

$$m_0 = \delta M_I \left(1 - \frac{r^3}{R_H^3}\right) + W_2^2 \left(\frac{r^6 - R_H^6}{r^3 R_H^6}\right) \quad (6.9b)$$

$$h_0 = C + \frac{R_H^2 m_0}{r(r^2 - R_H^2)} \quad (6.9c)$$

$$m_2 = \frac{W_2^2(r^2 - R_H^2)}{r^3 R_H^5} (r^3 + 5r^2 R_H - r R_H^2 - 8R_H^3) - \frac{2J^2(r^2 - R_H^2)^2}{r^2 R_H^5} \quad (6.9d)$$

$$h_2 = \frac{6W_2^2}{r^4} + \frac{R_H^2 m_2}{r(r^2 - R_H^2)} \quad (6.9e)$$

$$k_2 = -\frac{W_2^2}{2r^4 R_H^4} (r - R_H)(R_H^3 + 3r R_H^2 + 3r^2 R_H + 5r^3) - \frac{2J^2(r^2 + R_H^2)}{r^3 R_H^3} \quad (6.9f)$$

The interior solution (6.9) still depends on the three interior integration constants: $W_2, \delta M_I$, and C . The first two of these can be fixed by the requirement of the absence of δ -functions at the origin, as we shall show in Sec. VIII. The Weyl curvature of the interior solution is given in Appendix B.

We note that (6.7) precludes both W_2 and A_I from vanishing simultaneously for any $J \neq 0$, which would be necessary to have a regular solution at the origin as can be seen from Eqs. (5.2a) and (5.3c). Conversely, if we had demanded that the perturbation functions were regular at the origin (requiring $W_2 = A_I = 0$), then Eq. (6.7) forces $J = 0$, which is no rotation at all.

The reason for this is that the $\ell = 2$ quadrupole perturbations for h_2 (exterior) or rh_2 (interior) satisfy an associated Legendre differential equation with argument $z = r/M - 1$ or $z = 1/Hr$ respectively, with a particular inhomogeneous term depending upon ω'^2 . The singular points $z = (1, \infty)$ of the Legendre equation correspond to $r = 2M = R_S$ and $r = \infty$ for the exterior solution, and to $r = 1/H = R_H$ and $r = 0$ for the interior solution. The homogeneous solutions $Q_2^2(z)$ or $P_2^2(z)$ are singular at $z = 1$ or $z = \infty$, respectively, cf. (4.4). Requiring that the quadrupole perturbations remain finite at these singular points for both the exterior and interior solutions would set all four integration constants $A_E = B_E = A_I = B_I = 0$ in (4.6a)–(4.6b) and (5.3a)–(5.3b). But then requiring the junction condition of continuity of the induced metric (6.2)–(6.4) at $r = R_H = R_S$ requires (6.5) and (6.6) be equal and thus $W_2^2 = 2J^2$, which if nonzero gives a diverging interior solution for ω, h_2, k_2, m_2 in (5.2a) and (5.3). Hence the only way to eliminate all divergences

in the quadrupole functions is to set $W_2 = J = 0$, i.e., no rotation at all.

Thus it is impossible to obtain an interior solution for a slowly rotating gravastar with finite values at the origin and the horizon, matched there to an asymptotically flat finite solution at infinity in the Hartle-Thorne framework—if the equation of state of the interior is unchanged from $p = -\rho$ of the nonrotating case. We discuss the implications of this and how the divergences might be eliminated in a more complete treatment in the Discussion of Sec. IX and Conclusions of Sec. X.

VII. SURFACE GRAVITY AND STRESS TENSOR OF THE NULL HORIZON SURFACE

For timelike or spacelike hypersurfaces, one may relate the surface stress-energy to the discontinuity of the extrinsic curvature defined through the surface normal vector \mathbf{n} [26]. For a null surface where $\mathbf{n} \cdot \mathbf{n} = 0$, the extrinsic curvature vanishes, and the second junction conditions cannot be applied as formulated in [26]. In the Barrabès-Israel method [27], the surface stress-energy for a null surface is defined in terms of the discontinuity of an “oblique” curvature defined in terms of a second transverse vector \mathbf{N} . As discussed in Paper I [41], this method also does not apply to the case at hand here where $j = \sqrt{h/f}$ and $\sqrt{-g}$ are discontinuous. Instead Paper I [41] contains a derivation of the second junction conditions for a null surface directly from the Einstein tensor density $\sqrt{-g}G^\mu_\nu$ for the horizon of an arbitrary metric of the form (3.1). Following Paper I, the stress tensor localized on the null horizon surface is [41]

$${}^{(\Sigma)}T^a_b e^{\alpha+\nu} = S^a_b \delta(r - R_H) \quad (7.1)$$

for $a, b = t, \theta, \phi$, where ${}^{(\Sigma)}T^a_b$ is a well-defined Dirac δ -distribution with respect to the volume integration measure $\sqrt{-g} dt dr d\theta d\phi$, analogously to (2.10).

Under transformations of the hypersurface coordinates (t, θ, ϕ) , $e^{\beta+\psi} S^a_b$ transforms as a tensor density, while under transformations involving the surface coordinates (θ, ϕ) only, S^A_B is a tensor, S^t_i is a scalar, and S^t_B, S^A_t are vectors, for $A, B = \theta, \phi$. The null-surface stress tensor has nonzero components [41]

$$8\pi S^t_t = -\omega_H[\mathcal{J}] \quad (7.2a)$$

$$8\pi S^t_\phi = [\mathcal{J}] \quad (7.2b)$$

$$8\pi S^\phi_t = -\omega_H[\kappa] - \omega_H^2[\mathcal{J}] \quad (7.2c)$$

$$8\pi S^\phi_\phi = [\kappa] + \omega_H[\mathcal{J}] \quad (7.2d)$$

$$8\pi S^\theta_\theta = [\kappa] \quad (7.2e)$$

where the surface gravity κ is to second order in the rotation

$$\kappa = \frac{1}{2} e^{-\alpha-\nu} \frac{\partial}{\partial r} e^{2\nu} = \frac{1}{2j} \frac{dh}{dr} \left\{ 1 + h_0 + h_2 P_2 - \left(\frac{m_0 + m_2 P_2}{r - 2m} \right) \right\} + \frac{h}{j} (h'_0 + h'_2 P_2) \quad (7.3)$$

and \mathcal{J} , which is first order in the rotation, is

$$\mathcal{J} = -\frac{1}{2} e^{2\psi} e^{-\alpha-\nu} \frac{\partial \omega}{\partial r} = -\frac{j}{2} r^2 \sin^2 \theta \frac{d\omega}{dr}. \quad (7.4)$$

The square brackets in (7.2) denote the discontinuities in these quantities at the horizon, i.e., for a function $F(r, \theta)$,

$$[F] = F_+ - F_- \quad (7.5)$$

with

$$F_\pm \equiv \lim_{r \rightarrow R_H^\pm} F(r, \theta). \quad (7.6)$$

With the interior and exterior perturbations given by the results of the previous two sections, namely the finiteness conditions $B_E = B_I = 0$, and the conditions (4.11), (5.7), (6.7), (6.8) on the horizon, the surface gravity as the horizon boundary at $r = R_H = R_S$ is approached from the interior and exterior is, to second order in the rotation

$$\kappa_- = \lim_{r \rightarrow R_H^-} \kappa(r) = -\frac{1}{j R_H} \left(1 + C - \frac{3\delta M_I}{R_H} + \frac{3j^2 W_2^2}{2R_H^4} \right) \quad (7.7a)$$

$$\kappa_+ = \lim_{r \rightarrow R_H^+} \kappa(r) = \frac{1}{2R_H} \left(1 - \frac{6J^2}{R_H^4} \right). \quad (7.7b)$$

The discontinuity of the surface gravity at the horizon is therefore

$$[\kappa] = \kappa_+ - \kappa_- = \left(\frac{1}{2} + \frac{1+C}{j} \right) \frac{1}{R_H} - \frac{3J^2}{R_H^5} - \frac{3\delta M_I}{j R_H^2} + \frac{3j W_2^2}{2R_H^5} \quad (7.8)$$

which for a gravastar ($j = 2$) becomes

$$[\kappa] = \frac{1}{R_H} + \frac{C}{2R_H} - \frac{3\delta M_I}{2R_H^2} - \frac{3J^2 - 3W_2^2}{R_H^5}. \quad (7.9)$$

Similarly \mathcal{J} approached from the interior and exterior is

$$\mathcal{J}_- = \lim_{r \rightarrow R_H^-} \mathcal{J}(r) = \frac{3j W_2}{2R_H^2} \sin^2 \theta \quad (7.10a)$$

$$\mathcal{J}_+ = \lim_{r \rightarrow R_H^+} \mathcal{J}(r) = \frac{3J}{R_H^2} \sin^2 \theta \quad (7.10b)$$

so that its discontinuity at the horizon is

$$[\mathcal{J}] = \mathcal{J}_+ - \mathcal{J}_- = \frac{3(2J - jW_2)}{2R_H^2} \sin^2\theta = \frac{3(J - W_2)}{R_H^2} \sin^2\theta \quad (7.11)$$

with the latter expression for a gravastar ($j = 2$).

Dropping the second $\mathcal{O}(J^3)$ term in (7.2c), Eqs. (7.2) then give for a gravastar,

$$8\pi\mathcal{S}^t_t = -\frac{6J(J - W_2)}{R_H^5} \sin^2\theta \quad (7.12a)$$

$$8\pi\mathcal{S}^t_\phi = \frac{3(J - W_2)}{R_H^2} \sin^2\theta \quad (7.12b)$$

$$8\pi\mathcal{S}^\phi_t = -\frac{2J}{R_H^4} \quad (7.12c)$$

$$8\pi\mathcal{S}^\phi_\phi = \frac{1}{R_H} + \frac{C}{2R_H} - \frac{3J^2}{R_H^5} - \frac{3\delta M_I}{2R_H^2} + \frac{3W_2^2}{R_H^5} + \frac{6J(J - W_2)}{R_H^5} \sin^2\theta \quad (7.12d)$$

$$8\pi\mathcal{S}^\theta_\theta = \frac{1}{R_H} + \frac{C}{2R_H} - \frac{3J^2}{R_H^5} - \frac{3\delta M_I}{2R_H^2} + \frac{3W_2^2}{R_H^5} \quad (7.12e)$$

in terms of the integration constants J , C , W_2 , δM_I , after using the conditions (4.11), (5.7), (6.7), (6.8) on the horizon at $r = R_H = R_S$. In the nonrotating limit, with J , C , W_2 , and δM_I set to zero, the surface stress-energy tensor of the nonrotating gravastar is recovered [11].

The surface stress components in (7.12) indicate that for general values of the constants W_2 , δM_I , and C , there are nonvanishing anisotropic surface stresses, an azimuthal current, and an energy density localized on the horizon surface. The \mathcal{S}^ϕ_ϕ and $\mathcal{S}^\theta_\theta$ terms are the azimuthal and latitudinal stresses, containing a zeroth order (nonrotating) term with second order corrections. Because the zeroth order term is associated with the spherically symmetric unperturbed system, it is the same in the θ and ϕ components, but the second order corrections may differ (equal only if $W_2 = J$). The azimuthal current terms \mathcal{S}^t_ϕ and \mathcal{S}^ϕ_t are first order in the rotation. \mathcal{S}^ϕ_t is nonzero for a rotating gravastar ($J \neq 0$), while \mathcal{S}^t_ϕ is nonzero in general and vanishing only if $W_2 = J$. The energy density term \mathcal{S}^t_t is second order in the rotation, and vanishes if $W_2 = J$.

VIII. GRAVASTAR MASS, ANGULAR MOMENTUM, AND MOMENT OF INERTIA

Section V of Paper I [41] gives the integrals for the Komar mass and angular momentum functions for an arbitrary metric of type (3.1). The Komar functions at

some radius are given by a surface integral at that radius, which may be decomposed by Stokes' theorem into the contribution from a surface of smaller radius and a volume integral of appropriate energy-momentum components.

A. Komar mass

The Komar mass function is [41]

$$M_K(r) = \frac{1}{4\pi} \int_{\partial V_+} (\kappa + \omega\mathcal{J}) dA \quad (8.1a)$$

$$= \int_V \sqrt{-g} (-T^t_t + T^r_r + T^\theta_\theta + T^\phi_\phi) dr d\theta d\phi + \frac{1}{4\pi} \int_{\partial V_-} (\kappa + \omega\mathcal{J}) dA \quad (8.1b)$$

where V is the three-volume at fixed t enclosed by two-surfaces ∂V_- and ∂V_+ at r_- and r_+ in coordinates (3.1).

In the exterior the Komar mass is easily found to be

$$M_K^{\text{ext}} = M + \delta M = \mathcal{M}, \quad r > R_H \quad (8.2)$$

which is the constant total mass of the exterior Kerr solution, as expected. Since the exterior is a vacuum solution with $T^\mu_\nu = 0$, there is no volume contribution in (8.1b) and no r dependence to the Komar mass function in the exterior. Since the exterior is Kerr, we may compare Eq. (8.2) to the Christodoulou formula for a Kerr BH [56]

$$M_K^{\text{ext}} = \sqrt{M_{irr}^2 + \frac{J^2}{4M_{irr}^2}} \approx M_{irr} + \frac{J^2}{8M_{irr}^3} \quad (8.3)$$

where M_{irr} is the irreducible mass and the final equality comes from expanding for small J at constant M_{irr} . With h_0 finite and δM given by (4.11), one can see from Eqs. (8.2) and (8.3) that the unperturbed mass of the nonrotating gravastar M equals the irreducible mass of the Kerr BH, as implied by the $R_H = R_S$ condition and the definition of the irreducible mass $4\pi R_H^2 = 16\pi M_{irr}^2$.

The two-surface integrals in (8.1) may be evaluated at fixed r with the induced area element

$$dA = e^{\beta+\psi} d\theta d\phi = r^2 \sin\theta [1 + 2k_2(r)P_2(\cos\theta)] d\theta d\phi \quad (8.4)$$

and the total area of the horizon is

$$\begin{aligned} A &= \int e^{\beta+\psi} d\theta d\phi \\ &= \int d\theta d\phi R_H^2 \sin\theta [1 + 2k_2(R_H)P_2(\cos\theta)] \\ &= 4\pi R_H^2 = 16\pi M_{irr}^2 = 16\pi M^2 \end{aligned} \quad (8.5)$$

since the quadrupole term does not contribute. The final equality of (8.5) makes use of the condition of finite h_0 and matching at $R_H = R_S$.

Substituting the perturbations h_0, m_0, h_2, m_2 for the interior solution (6.9), and performing the surface integral ∂V_+ at r , the quadrupole terms proportional to $P_2(\cos \theta)$ again do not contribute, and we obtain

$$\frac{1}{4\pi} \int_{\partial V_+} \kappa dA = -\frac{r^3}{jR_H^2} \left(1 + C + \frac{8\pi\mathcal{E}_0 R_H^2}{3} \right) - \frac{jW_2^2}{r^3} + \frac{\delta M_I}{j} \quad (8.6)$$

retaining terms up to second order in the rotation. Likewise for the $\omega\mathcal{J}$ term in (8.1a), we obtain

$$\frac{1}{4\pi} \int_{\partial V_+} \omega\mathcal{J} dA = jW_2 \left(W_1 + \frac{W_2}{r^3} \right) \quad (8.7)$$

correct to second order. Combining this with (8.6) we find the expression for the Komar mass in the interior,

$$M_K^{\text{int}}(r) = -\frac{r^3}{jR_H^2} \left(1 + C + \frac{8\pi\mathcal{E}_0 R_H^2}{3} \right) + \frac{j^2 W_1 W_2 + \delta M_I}{j}, \quad 0 < r < R_H. \quad (8.8)$$

The r dependent term in (8.8) can be identified as the volume contribution from (8.1b),

$$\begin{aligned} & \int_V \sqrt{-g} (-T^t_t + T^r_r + T^\theta_\theta + T^\phi_\phi) dr d\theta d\phi \\ &= -2(\rho + \mathcal{E}_0) \int_0^r dr \int d\theta d\phi \frac{r^2 \sin \theta}{j} \\ & \quad \times \left(1 + h_0 + h_2 P_2 + \frac{m_0 + m_2 P_2}{r(1 - H^2 r^2)} + 2k_2 P_2 \right) \\ &= -\frac{r^3}{jR_H^2} \left(1 + C + \frac{8\pi\mathcal{E}_0 R_H^2}{3} \right). \end{aligned} \quad (8.9)$$

Extending the latter integration over the whole interior volume $0 < r < R_H$ gives the total interior volume contribution to the Komar mass

$$M_K^{\text{volume}} = -\frac{R_H}{j} \left(1 + C - \frac{2\delta M_I}{R_H} + \frac{j^2 W_2^2}{2R_H^4} \right) \quad (8.10)$$

where $R_H = R_S$ and \mathcal{E}_0 from (5.7) has been used.

The r independent term in (8.8), i.e.,

$$M_K^{\text{origin}} = \frac{j^2 W_1 W_2 + \delta M_I}{j} \quad (8.11)$$

which is present at arbitrarily small r , can be ascribed to δ -function contribution to the volume integral in Eq. (8.1b) at the origin by

$$\begin{aligned} & \frac{1}{4\pi} \int d\theta d\phi \sqrt{-g} \square K^t_{(t)}|_{\text{origin}} \\ &= \int d\theta d\phi \sqrt{-g} (-T^t_t + T^r_r + T^\theta_\theta + T^\phi_\phi)|_{\text{origin}} \\ &= M_K^{\text{origin}} \delta(r) \end{aligned} \quad (8.12)$$

where $K^t_{(t)}$ is the t component of the Killing vector of time translation $\partial/\partial t$. Thus the general solution to the slow rotation perturbation equations allows for the constant contribution M_K^{origin} to the Komar mass in (8.11), much as integration of the mass function equation $m' = 4\pi r^2 \rho$ allows for an integration constant and mass at the origin. Requiring the absence of any such δ -function at the origin requires

$$\delta M_I = -j^2 W_1 W_2 \quad (8.13)$$

fixing one of the remaining three constants in the interior gravastar solution.

The contribution to the Komar mass of the arbitrarily thin surface layer stress energy localized on the horizon is

$$\begin{aligned} M_K^{\text{surface}} &= \int_0^\pi \int_0^{2\pi} \int_{R_H-\epsilon}^{R_H+\epsilon} d\theta d\phi dr \sqrt{-g} (-T^t_t + T^\theta_\theta + T^\phi_\phi) \\ &= 2\pi \int_0^\pi \int_{R_H-\epsilon}^{R_H+\epsilon} d\theta dr e^{\psi+\beta} \delta(r - R_H) (-S^t_t + S^\theta_\theta + S^\phi_\phi) \\ &= \frac{1}{4\pi} \int ([\kappa] + \omega_H [\mathcal{J}]) dA = M_K^{\text{ext}} - M_K^{\text{volume}} - M_K^{\text{origin}} \\ &= \left(\frac{1}{2} + \frac{1+C}{j} \right) R_H + \frac{J^2}{R_H^3} + \frac{jW_2^2}{2R_H^3} - \frac{3\delta M_I + j^2 W_1 W_2}{j} \end{aligned} \quad (8.14)$$

where $R_H = R_S$ and δM , \mathcal{E}_0 from (4.11) and (5.7) have been used.

The total mass in the exterior

$$M_K^{\text{ext}} = M_K^{\text{origin}} + M_K^{\text{volume}} + M_K^{\text{surface}} = \mathcal{M} \quad (8.15)$$

is the sum of the contributions from the interior volume, the possible δ -function at the origin, and the surface layer, independently of the conditions (8.13). If (8.13) is satisfied, the contribution from the origin is absent: $M_K^{\text{origin}} = 0$. Note also that changing the integration constant C changes the contributions to the interior volume and surface terms separately, but drops out of the sum (8.15).

B. Komar angular momentum

The Komar angular momentum is expressed similarly to the Komar mass as [41]

$$\begin{aligned} J_K(r) &= \frac{1}{8\pi} \int_{\partial V_+} \mathcal{J} dA \\ &= \int_V \sqrt{-g} T^t{}_\phi dr d\theta d\phi + \frac{1}{8\pi} \int_{\partial V_-} \mathcal{J} dA \end{aligned} \quad (8.16)$$

which shows that $\mathcal{J}/8\pi$ carries the interpretation of angular momentum per unit surface area. Using (7.4), the surface integral for the interior solution is

$$J_K^{\text{int}} = \frac{1}{8\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta r^2 \sin^3 \theta \left(\frac{3jW_2}{2r^2} \right) = \frac{jW_2}{2} \quad (8.17)$$

which is first order in the rotation. Since this is independent of r , and $T^t{}_\phi = 0$ in the interior, the volume integral term for angular momentum in (8.16) gives no contribution,

$$\begin{aligned} J_K^{\text{surface}} &= \int_0^\pi \int_0^{2\pi} \int_{R_H-\epsilon}^{R_H+\epsilon} d\theta d\phi dr \sqrt{-g} T^t{}_\phi = 2\pi \int_0^\pi \int_{R_H-\epsilon}^{R_H+\epsilon} d\theta dr e^{w+\beta} \delta(r - R_H) S^t{}_\phi \\ &= \frac{1}{8\pi} \int d\phi d\theta e^{w+\beta} [\mathcal{J}] = \frac{3(2J - jW_2)}{8R_H^2} \int_0^\pi \int_{R_H-\epsilon}^{R_H+\epsilon} d\theta dr r^2 \sin^3 \theta \delta(r - R_H) = J - \frac{jW_2}{2} \end{aligned} \quad (8.23)$$

to this order in rotation. This shows that the total Komar angular momentum of the gravastar is the sum of contributions from the surface and the origin,

$$J_K^{\text{ext}} = J_K^{\text{origin}} + J_K^{\text{surface}} = J. \quad (8.24)$$

If $W_2 = 0$, then $J_K^{\text{origin}} = 0$ and the entire angular momentum of the slowly rotating gravastar comes from the rotating surface layer localized on the horizon surface.

$$J_K^{\text{volume}} = 0 \quad (8.18)$$

but the constant contribution

$$J_K^{\text{origin}} = \frac{jW_2}{2} \quad (8.19)$$

can be ascribed to the volume term from Eq. (8.16) as a δ -function contribution at the origin,

$$\begin{aligned} -\frac{1}{4\pi} \int d\theta d\phi \sqrt{-g} \square K^t{}_{(\phi)}|_{\text{origin}} &= 2 \int d\theta d\phi \sqrt{-g} T^t{}_\phi|_{\text{origin}} \\ &= 2J_K^{\text{origin}} \delta(r) \end{aligned} \quad (8.20)$$

where $K^t{}_{(\phi)}$ is the t component of the Killing vector of azimuthal symmetry $\partial/\partial\phi$. Eliminating this δ -function contribution to the angular momentum requires $W_2 = 0$. Eliminating both the Komar angular momentum and Komar mass delta functions at the origin thus requires

$$\delta M_I = W_2 = 0 \quad (8.21)$$

so that the only remaining integration constant of the interior solution is C .

The Komar angular momentum function in the exterior is

$$J_K^{\text{ext}} = -\frac{1}{8} \int_0^\pi d\theta \sqrt{\frac{h}{f}} r^4 \sin^3 \theta \frac{d}{dr} \left(\frac{2J}{r^3} \right) = J \quad (8.22)$$

which is the expected constant total angular momentum of the Kerr exterior solution. Finally, we have the angular momentum contribution of the surface layer

C. Moment of inertia

The moment of inertia I of the rotating gravastar is

$$I = \frac{J}{\omega_H} = 4G^2 M^3 = MR_H^2 \quad (8.25)$$

which coincides with that of a Kerr BH. This might have been expected from the fact that the external geometry is identical to the Kerr BH geometry. With $W_2 = 0$, the interior de Sitter condensate carries no angular momentum from (8.18) and (8.19), and all the angular momentum is carried by the rotating horizon surface from (8.23). Unlike

a BH where there is no mass-energy at all at the horizon to rotate or to give rise to such a moment of inertia, the rotating gravastar has a well-defined stress energy (7.12) on its rotating surface where all the angular momentum resides, and it is this rotating mass-energy that gives rise to (8.25) by straightforward evaluation of the relevant surface integral (8.23).

IX. SUMMARY AND DISCUSSION

With the δ -functions contributions to the Komar mass and angular momentum at the origin set to zero ($M_K^{\text{origin}} = 0$ and $J_K^{\text{origin}} = 0$), and the condition that h_0 is finite at the gravastar surface, all interior integration constants for the slowly rotating gravastar are fixed by Eqs. (4.11), (5.7), (6.7), (6.8), (8.21), except for C . The interior slowly rotating gravastar solution in this case simplifies considerably, becoming

$$\omega = \frac{2J}{R_H^3} = \omega_H \quad (9.1a)$$

$$m_0 = 0 \quad (9.1b)$$

$$h_0 = C \quad (9.1c)$$

$$m_2 = -\frac{2J^2(r^2 - R_H^2)^2}{r^2 R_H^5} \quad (9.1d)$$

$$h_2 = \frac{2J^2}{rR_H} \left(\frac{1}{r^2} - \frac{1}{R_H^2} \right) \quad (9.1e)$$

$$k_2 = -\frac{2J^2}{rR_H} \left(\frac{1}{r^2} + \frac{1}{R_H^2} \right) \quad (9.1f)$$

with the exterior solution given by Eqs. (4.9). The surface stress tensor is obtained by setting $\delta M_I = W_2 = 0$ in Eqs. (7.12), i.e.,

$$8\pi S'_t = -\frac{6J^2}{R_H^5} \sin^2\theta \quad (9.2a)$$

$$8\pi S^t_\phi = \frac{3J}{R_H^2} \sin^2\theta \quad (9.2b)$$

$$8\pi S^\phi_t = -\frac{2J}{R_H^4} \quad (9.2c)$$

$$8\pi S^\phi_\phi = \frac{1}{R_H} + \frac{C}{2R_H} - \frac{3J^2}{R_H^5} + \frac{6J^2}{R_H^5} \sin^2\theta \quad (9.2d)$$

$$8\pi S^\theta_\theta = \frac{1}{R_H} + \frac{C}{2R_H} - \frac{3J^2}{R_H^5}. \quad (9.2e)$$

The constant C does not appear in the bulk ($r < R_H$) stress-energy tensor or curvature tensor of the interior solution. Referring to the metric line element (3.3), we observe that the constant $1 + C$ is a rescaling of the time coordinate of the interior, and thus appears to amount to a choice of coordinates. It is indeed similar to the constant $1 + C_E$ of (4.3b), which is a rescaling of the time coordinate of the exterior solution that is fixed by the condition that the asymptotically flat line element assumes the standard Minkowski form. It is also similar to the j factor, which rescales the interior g_{tt} component of the metric in the nonrotating gravastar. In that spherically symmetric case the value $j = 2$ is fixed by taking the horizon limit $R \rightarrow R_H$ of the Schwarzschild star [11]. Similarly, if the interior and exterior were continuously connected in an appropriate horizon limit, or if $f(r)$ were to be kept slightly different from zero near $r = R_H$ by a more detailed theory of the surface layer, we may expect that C would be fixed by the one condition on the Minkowski static time coordinate as $r \rightarrow \infty$.

As it is, in our present treatment because of the lightlike null surface intervening where $f(R_H) = 0$, the time coordinate of the interior solution becomes disconnected from the exterior, and C remains a free parameter, which appears in the interior bulk Komar mass (8.8), the surface stress tensor (7.12) and surface contribution (8.14) separately, but drops out in their sum \mathcal{M} for the total mass (8.15).

It is important to note that our results are contingent on the surface layer being located on the null horizon and infinitesimally thin, following the matching method presented in Paper I [41]. An infinitesimal surface which is located away from the null horizon leads to different results and has been studied by other authors, to which we may explicitly compare our results. Previous works were based on matching the interior Hartle-Thorne perturbed de Sitter to exterior geometries on a timelike [19–21] or spacelike [55] hypersurface $r = R = \text{const} \neq 2M$. These authors made a different choice of constants for both the exterior and interior perturbations. They use conventional time scaling in the interior $j = 1$ and do not consider \mathcal{E}_0 , but allow B_E, B_I to be nonzero, since they did not require the perturbations to be finite at the horizon. They set $A_I = W_2 = \delta M_I = 0$ in order to have finiteness of the perturbations at the origin $r = 0$. Comparing our solution in (5.2) with Eqs. (2.15)–(2.19) of Ref. [21], we see that the interior solution of Ref. [21] amounts to choosing $W_2 = \delta M_I = A_I = \mathcal{E}_0 = 0$, with arbitrary W_1, B_I, C in the present notation. With these choices, $k_2(r)$ diverges as $r_S \rightarrow 1/H$ unless it vanishes identically, i.e., also $B_I = 0$. The latter option would then set m_0, h_2, k_2, m_2 all to zero identically, and leave only $\omega = W_1, h_0 = C$ as constants in the interior. But this interior cannot be matched to the uniquely well behaved Kerr exterior on the horizon.

In contrast, finiteness and matching on the null horizon surface requires (5.5) and leaves a generally nontrivial

interior solution for the perturbations, which however may still diverge at the origin if either A_I or W_2 are non-vanishing. The junction conditions of matching on the horizon fix the relation (6.7), which precludes setting both A_I and W_2 to zero if $J \neq 0$.

The earlier authors [19–21,55] applied the Israel junction conditions, which do not apply in the limit $R \rightarrow 2M$, as they produce either a vanishing or a divergent result depending on whether the extrinsic curvature is computed with both lower contravariant indices, or with one of its indices raised, due to the fact that the induced metric on the null horizon is degenerate. The relation between the surface stress tensors of these previous approaches and the present one is given by (2.12) [41].

Another paper [57] uses a different perturbation method in Eddington-Finkelstein coordinates on rotating gravastars derived directly from the Schwarzschild star limit. This work finds regular perturbative solutions to second order, which fail only at third order. However, the analysis differs from ours in that the surface of the gravastar is allowed to shift away from being null in the rotating case, whereas we have performed the matching on a null surface in both the static and rotating cases.

Our analysis also differs from [40], where a sub-Buchdahl Schwarzschild star ($R < 9M/4$) was studied numerically in the Hartle formalism by matching the interior and exterior metrics continuously on a timelike hypersurface and approaching the horizon limit numerically. There it was suggested that the perturbative corrections were everywhere finite in the gravastar limit. However, below the Buchdahl bound the equation for frame dragging solved numerically, Eq. (50) of [40], does not follow from the change of variables given in the definitions in Eqs. (45)–(49) and the original frame-dragging equation in r coordinates, cf. Eq. (25). Hence the results in [40] would have to be recomputed before comparing them to our work, although the same result (8.25) for the moment of inertia was obtained.

A major result of our analysis is that although one may remove the δ -function contributions at the origin to both the Komar mass function M_K^{origin} and angular momentum J_K^{origin} by setting $\delta M_I = W_2 = 0$, the quadrupole perturbations m_2 , h_2 , and k_2 remain quadratically or cubically divergent at the origin for $J \neq 0$. The perturbations m_2 and k_2 in (6.9d) and (6.9f) become of order unity at

$$r_{\min} \sim \left(\frac{2J^2}{R_H} \right)^{\frac{1}{3}} \quad (9.3)$$

and hence the perturbative expansion breaks down for $r \lesssim r_{\min}$ (or possibly $r \lesssim \sqrt{|W_2|}$ if one allows for nonzero W_2). There are two possible reasons for this breakdown and corresponding possibilities for removing the divergences at $r = 0$.

The first possibility is that the perturbative expansion in slow rotation $J = aM$ is nonuniform in r , and invalid near the origin. If a diverging a^2/r^2 behavior is replaced by any smooth function $F(r/a)$ in a more complete solution, where F tends to unity for large values of its argument, but to zero rapidly enough as $r \rightarrow 0$, the divergence at the origin would be removed. For example

$$F\left(\frac{r}{a}\right) = \frac{a^2 r^{2n-2}}{(a^2 + r^2)^n} \quad (9.4)$$

for $n > 1$ is finite both for $a = 0$, and as $r \rightarrow 0$ for any $a > 0$, where it behaves like $(r/a)^{2n-2} \rightarrow 0$ near the origin. However, expanding this function to order a^2 yields a leading term, a^2/r^2 which diverges as $r \rightarrow 0$. Thus the diverging behavior at $r = 0$ in the slow rotation expansion may imply that the true solution for the interior metric of a rotating gravastar, though everywhere regular, does not admit a uniform regular expansion in the rotation parameter near the origin. We have not found an exact rotating metric in the literature with $p = -\rho$ equation of state and regular metric functions which expands into our solution in precisely this way, although there are solutions with some similar properties of $p = -\rho$ equation of state and diverging expansion, such as the modified Carter solution presented in [58].

Another method which has been used in the literature to generate possible interior solutions to the Kerr BH (see e.g., [59]) and manifests divergences on expansion is the Gürses-Gürsey generalization [60] of the Newman-Janis algorithm [61]. The Gürses-Gürsey rotating system has the line element

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2rm}{\Sigma} \right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ & + \sin^2\theta \left(\frac{2a^2 r m \sin^2\theta}{\Sigma} + a^2 + r^2 \right) d\phi^2 \\ & - \frac{4arms\sin^2\theta}{\Sigma} dt d\phi \end{aligned} \quad (9.5)$$

where $\Sigma(r, \theta) = r^2 + a^2 \cos^2\theta$, $\Delta(r) = r^2 + a^2 - 2rm(r)$, and $m(r)$ is the enclosed mass in the spherically symmetric system, which may be taken to be $H^2 r^3/2$ for the non-rotating de Sitter gravastar interior. Although this metric is completely regular at $r = 0$, expanding the $1/\Sigma$ and $1/\Delta$ functions to order a^2 will generate spurious divergences, similar to (9.4). However, this rotating geometry [62–65] is a solution of Einstein's equations with an energy-momentum tensor that does *not* satisfy the $p = -\rho$ dark energy equation of state [66].

This brings us to the second possibility for removing the divergence at the origin, which is that the preservation of the $p = -\rho$ equation of state may be too restrictive for an ultra-compact rotating gravastar. Our analysis is based on

the assumption, eminently reasonable for most fluids, that the local thermodynamic equilibrium equation of state would not be modified if the fluid sphere is set into slow rotation. However the dark energy equation of state, at the extreme limit of satisfying the weak energy condition $\rho + p \geq 0$, is not that of an ordinary fluid, and need not follow the expectation of local thermodynamic equilibrium under small perturbations familiar from classical fluids. If the equation of state is altered from dark energy form, as for instance in the Gürses-Gürsey system Eq. (9.5), then there may be angular momentum density distributed within the interior rather than purely on the surface as in our solution. Indeed if the interior of the gravastar is a gravitational condensate as originally proposed in [45,46], one might expect the development of vortex filaments first at the origin, and then a finite vortex density at higher values of cJ/GM^2 . Such vortices would alter the equation of state from its pure vacuum dark energy condensate form, altering the solution markedly, deep in the interior. One might speculate that even in this case the physical properties of the surface layer may not be radically changed at the horizon from that given in the present work, at least to second order in the rotation parameter cJ/GM^2 .

X. CONCLUSIONS

In this paper we have derived and solved analytically the Einstein equations for a slowly rotating gravastar by expanding the spherically symmetric gravastar solution of [11] to second order in the angular momentum, following the methods of Refs. [38,39,51], and assuming that the equation of state of the slowly rotating system is unchanged from that of the nonrotating solution.

In solving the perturbation equations for slow rotation, we applied the lightlike shell junction conditions of Paper I [41] to the concrete case of a rotating gravastar. Previous studies in the Hartle-Thorne formalism have focused on the case of matching on a timelike or spacelike boundary [19–21,55,67].

The surface stresses and Komar integrals work as expected, with the modifications appropriate for matching on a null hypersurface. The δ -function distribution stress tensor on the horizon surface (7.1)–(7.2) accounts for the difference between the exterior and interior Komar mass and angular momentum functions.

In the Komar description, the $p = -\rho$ equation of state cannot support any angular momentum density and the angular momentum must either be localized entirely on the horizon or in singular concentration at the origin. With the conditions (8.21) removing any δ -function contribution at the origin, the entire angular momentum J of the rotating solution is carried by the physical surface of the gravastar at the horizon.

Matching solutions on the horizon, we determine the interior metric and the surface stress tensor corrections of first and second order in the rotation. The exterior solution for the perturbations (4.9) yields exactly the same geometry

as that of a Kerr BH, expanded to second order in $a = J/M$, but expressed in Hartle-Thorne coordinates (3.3), which are related to more familiar Boyer-Lindquist coordinates by (4.12). Thus an important conclusion of this work is

- (1) Matching the interior of a slowly rotating gravastar to an asymptotically flat exterior vacuum solution of Einstein's equations at their horizons leads to an external geometry *identical* to that of a slowly rotating Kerr BH, notwithstanding a very different interior and the presence of nonzero energy density and stresses (9.2) localized on the physical surface, that takes the place of the Kerr BH horizon.

Since the exterior of the slowly rotating gravastar is precisely that of the Kerr BH to order a^2 , there is no possibility of detecting the difference to this order by any experiment or observation restricted to the large scale external geometry such as by accretion flows or light ring images. Like a classical Kerr BH, a slowly rotating gravastar with a null surface stress tensor also has “no hair” classically, up to possibly Planck scale corrections very near to the horizon surface. The fact that this surface remains at the horizon, and the moment of inertia (8.25) remains identical to that of a Kerr BH to this order also implies that a rotating gravastar defined this way will also have the same zero tidal deformability or Love number as a Kerr BH, at least to order a^2 .

If the δ -function surface stress-energies were to be replaced by a microscopically thin shell of radial extent Δr , several studies [68,69] indicate that the corrections to this null result may be of order $[\log(\Delta r/M)]^{-1}$, so that even a Δr of order the Planck length could potentially produce dimensionless Love numbers of a few tenths of a percent, which could lead to observable tidal effects that can distinguish ultra-compact objects from BHs in future gravitational wave detectors [21,68]. This question remains open for the universal null shell gravastars considered in this paper.

Assuming that the interior equation of state for a slowly rotating gravastar is unchanged from its nonrotating (3.5)–(3.7) form, the regular matching of the exterior Kerr and the interior spacetime at the horizon gives Eq. (6.7), which in our calculation implies a divergence of some perturbation functions at the origin. As explained in Sec. VI, since the horizons of the interior and exterior geometries are at singular points of the differential equation governing the quadrupolar perturbations, the four conditions:

- (i) Finiteness at the origin,
- (ii) Finiteness on the horizon,
- (iii) Finiteness at asymptotic infinity (i.e., asymptotic flatness), and

(iv) Matching (first junction condition) on the horizon, cannot be simultaneously satisfied if the equation of state in the interior is fixed to be $p = -\rho$. Thus a second important conclusion of this work is

- (2) There is no solution of the Hartle perturbation equations for a slowly rotating gravastar that is regular both at the horizon and the origin, and can be matched on the horizon to an asymptotically flat exterior, if the equation of state in the interior is fixed to be $p = -\rho$ of the nonrotating gravastar.

The divergences may indicate that the slow rotation expansion is nonuniform and breaks down as $r \rightarrow 0$, and/or that the assumption that the equation of state is unchanged from $p = -\rho$ is overly restrictive for ultracompact objects such as gravastars, and should be relaxed. If instead, the equation of state differs from the dark energy one anywhere in the interior, then the perturbation functions inside the gravastar will also be different and so their matching to the exterior functions at the null surface may become possible, while still remaining finite at the origin.

The corollary is that the slowly rotating gravastar model generated in this way by the Hartle-Thorne formalism [38,39] from the spherically symmetric case is necessarily incomplete, and not fully satisfactory. The present work represents an initial exploration of rotating gravastars with a physical surface localized on the BH horizon, whose results it is hoped will suggest approaches to a solution of the rotating gravastar interior, beyond the perturbative framework utilized in this paper. This requires a better understanding of the equation of state appropriate for rotating gravitational condensates.

No attempt has been made in this paper to present either a microscopic theory of the surface stresses, which requires the incorporation of quantum vacuum polarization effects such as those discussed in [9,10], nor a detailed model of the dynamical formation of rotating gravastars with $p = -\rho$ cores from positive pressure matter. See for example [70] for preliminary study of a model of gravastar formation. These important issues remain open to further investigation.

ACKNOWLEDGMENTS

The authors thank C. Posada for private communication confirming our understanding of the relation between his work and the present one, as discussed in Sec. IX. P. G. was partially supported by NSF Grant No. PHY-2014075, and is very grateful to Prof. Masahide Yamaguchi for his generous support under JSPS Grant-in-Aid for Scientific Research No. JP18K18764 at the Tokyo Institute of Technology. E. M. acknowledges support from LANL LDRD Grant No. 20200661ER.

APPENDIX A: EINSTEIN TENSOR AND EQS. TO SECOND ORDER IN THE ANGULAR MOMENTUM

With the form of the line element (3.3) the nonvanishing components of the Einstein tensor are:

$$G^t_r = -3H^2 + \left[-\frac{2m'_0}{r^2} + \frac{j^2 r^2}{6} \omega'^2 + \frac{j^2 r}{3} \omega(r\omega'' + 4\omega') \right] + 2P_2(\cos\theta) \left[hk'_2 + \frac{(rh' + 6h)}{2r} k'_2 - \frac{m'_2}{r^2} - \frac{2k_2}{r^2} - \frac{3m_2}{r^3 h} - \frac{j^2 r^2}{12} \omega'^2 - \frac{j^2 r}{6} \omega(r\omega'' + 4\omega') \right] \quad (\text{A1})$$

$$G^r_r = -3H^2 + \left[\frac{2h}{r} h'_0 - \frac{2(rh' + h)}{hr^3} m_0 + \frac{1}{6} j^2 r^2 \omega'^2 \right] + 2P_2(\cos\theta) \left[\frac{(rh' + 2h)}{2r} k'_2 + \frac{h}{r} h'_2 - \frac{3h_2}{r^2} - \frac{2k_2}{r^2} - \frac{(rh' + h)}{r^3 h} m_2 - \frac{j^2 r^2}{12} \omega'^2 \right] \quad (\text{A2})$$

$$\frac{G^r_\theta}{r^2 h} = G^\theta_r = \frac{3 \sin\theta \cos\theta}{r^2} \left[h'_2 + k'_2 + \frac{(rh' - 2h)}{2rh} h_2 - \frac{(rh' + 2h)}{2r^2 h^2} m_2 \right] \quad (\text{A3})$$

$$G^\theta_\theta = -3H^2 + \left[hh''_0 + \frac{(3rh' + 2h)}{2r} h'_0 - \frac{(rh' + 2h)}{2r^2 h} m'_0 - \frac{h_2}{r^2} - \frac{m_2}{r^3 h} + \left(\frac{r^2 h'^2}{h} - 2r^2 h'' - rh' + 2h \right) \frac{m_0}{2r^3 h} - \frac{j^2 r^2}{6} \omega'^2 \right] + P_2(\cos\theta) \left[h(h''_2 + k''_2) + \frac{(3rh' + 2h)}{2r} h'_2 + \frac{(rh' + 2h)}{r} k'_2 - \frac{(rh' + 2h)}{2r^2 h} m'_2 - \frac{2h_2}{r^2} + \left(\frac{r^2 h'^2}{h} - 2r^2 h'' - rh' + 2h - 4 \right) \frac{m_2}{2r^3 h} + \frac{j^2 r^2}{6} \omega'^2 \right] \quad (\text{A4})$$

$$\begin{aligned}
G^\phi_\phi = & -3H^2 + \left[hh''_0 + \frac{(3rh' + 2h)}{2r} h'_0 - \frac{(rh' + 2h)}{2r^2 h} m'_0 + \frac{h_2}{r^2} + \frac{m_2}{r^3 h} \right. \\
& + \left. \left(\frac{r^2 h^2}{h} - 2r^2 h'' - rh' + 2h \right) \frac{m_0}{2r^3 h} - \frac{j^2 r^2}{2} \omega'^2 - \frac{j^2 r}{3} \omega(r\omega'' + 4\omega') \right] \\
& + P_2(\cos \theta) \left[h(h''_2 + k''_2) + \frac{(3rh' + 2h)}{2r} h'_2 + \frac{(rh' + 2h)}{r} k'_2 - \frac{(rh' + 2h)}{2r^2 h} m'_2 - \frac{4h_2}{r^2} \right. \\
& + \left. \left(\frac{r^2 h^2}{h} - 2r^2 h'' - rh' + 2h - 8 \right) \frac{m_2}{2r^3 h} + \frac{j^2 r^2}{2} \omega'^2 + \frac{j^2 r}{3} \omega(r\omega'' + 4\omega') \right] \quad (\text{A5})
\end{aligned}$$

where the terms in the large square brackets are second order in the angular momentum J/M^2 , and

$$G^t_\phi = -\frac{j^2 r \sin^2 \theta}{2} (r\omega'' + 4\omega') \quad (\text{A6})$$

$$G^\phi_t = \frac{h}{2r} (r\omega'' + 4\omega') \quad (\text{A7})$$

are first order in this parameter and somewhat simpler. The difference

$$\begin{aligned}
G^\theta_\theta - G^\phi_\phi = & -\frac{2}{r^2} \left[h_2 + \frac{m_2}{rh} - \frac{j^2 r^4}{6} \omega'^2 \right. \\
& \left. - \frac{j^2 r^3}{6} \omega(r\omega'' + 4\omega') \right] (1 - P_2(\cos \theta)) \quad (\text{A8})
\end{aligned}$$

is particularly simple as well. In these expressions the prime $' \equiv d/dr$ and we have used the zeroth order non-rotating gravastar solution (2.2).

The Einstein equations $G^\mu_\nu = 8\pi T^\mu_\nu = -8\pi \mathcal{E} \delta^\mu_\nu$ for the stress energy tensor (3.5) are easily obtained from (A1)–(A8). Since $\nabla_\mu G^\mu_\nu = 0$, $\mathcal{E} = \rho + \mathcal{E}_0$ must be a constant. The terms in (A1) and (A2) with no $\cos \theta$ dependence give the monopole equations (3.8b) and (3.8c), the latter in the form

$$(r-2m) \frac{dh_0}{dr} = \frac{(1-2m')}{(r-2m)} m_0 - 4\pi r^2 \mathcal{E}_0 - \frac{j^2 r^4}{12} \left(\frac{d\omega}{dr} \right)^2 \quad (\text{A9})$$

after using $rh = r - 2m$ from (2.2), and with $m' \equiv dm/dr$. These equations may be compared to Eqs. (18) and (19) of Ref. [39], taking account of the constant values of j , $p = -\rho$, and \mathcal{E}_0 in the energy-momentum tensor (3.5), as well as $\varpi' = \omega'$. The vanishing of the difference $G^\theta_\theta - G^\phi_\phi$ in (A8), of the off-diagonal components G^θ_r in (A7), and of the $P_2(\cos \theta)$ terms of G^r_r in (A2) give

$$h_2 + \frac{m_2}{rh} - \frac{j^2 r^4}{6} \omega'^2 = 0 \quad (\text{A10a})$$

$$h'_2 + k'_2 + \frac{(rh' - 2h)}{2rh} h_2 - \frac{(rh' + 2h)}{2r^2 h^2} m_2 = 0 \quad (\text{A10b})$$

$$\begin{aligned}
& (rh' + 2h)k'_2 + 2hh'_2 - \frac{6h_2}{r} \\
& - \frac{4k_2}{r} - \frac{2(rh' + h)}{r^2 h} m_2 - \frac{1}{6} j^2 r^3 \omega'^2 = 0 \quad (\text{A10c})
\end{aligned}$$

since all of these must vanish by Einstein's equations for the stress tensor (3.5) with $\mathcal{E}_2 = 0$. Solving (A10a) for m_2 gives (3.8e), which when substituted into (A10b)–(A10c) allows these to be written

$$\begin{aligned}
& r(r-2m) \frac{d}{dr} (h_2 + k_2) + 2(m - rm') h_2 \\
& = \frac{1}{6} (r - m - rm') j^2 r^4 \omega'^2 \quad (\text{A11a})
\end{aligned}$$

$$\begin{aligned}
& r(r-2m) \frac{d}{dr} (h_2 + k_2) + r(m - rm') \frac{dk_2}{dr} \\
& - 2r(h_2 + k_2) - 2rm' h_2 = \frac{1}{12} (3 - 4m') j^2 r^5 \omega'^2 \quad (\text{A11b})
\end{aligned}$$

after rearranging. Introducing the notation $v_2 = h_2 + k_2$, eliminating k_2 in favor of $v_2 - h_2$, and using $(m - rm') = r(r - 2m) \nu'_0$, the two coupled first order Eqs. (A11) may also be expressed as Eqs. (3.8f) and (3.8g), which are directly comparable to Eqs. (25) and (26) in [39]. The remaining Einstein equations involving the second derivative terms in (A1)–(A5) are then satisfied automatically by the Bianchi identities.

APPENDIX B: WEYL SECTOR OF THE INTERIOR SOLUTION

The curvature tensors of our exterior solution are well known, since it is the Hartle-Thorne expansion of the Kerr metric. The curvature tensors of our interior solution are less well known, and we collect some information about them in this Appendix.

The Ricci sector of our interior solution is de Sitter like (albeit with a possible second order constant density shift \mathcal{E}_0), but the full Riemann tensor has differences of first and second order in the perturbations. Of the integration constants, the constants W_2 , A_I , δM_I , and B_I show up only in the Weyl sector, \mathcal{E}_0 shows up only in the Ricci

sector, and W_1 and C do not show up in either sector and hence do not contribute directly to the curvature of the interior solution.

A convenient method of examining the Weyl tensor is to use of the Petrov Q matrix [71,72]. We derive the Q matrix components for an arbitrary stationary axisymmetric metric

$$Q_{11} = \frac{6iW_2 \cos \theta}{r^4} + \frac{2(r^3 \delta M_I - W_2^2)}{r^6} - \frac{P_2(\cos \theta)}{r^6 R_H^2} [4A_I r R_H^2 (r^2 - 3R_H^2) + 2W_2^2 (15r^2 - 11R_H^2)] \quad (\text{B1a})$$

$$Q_{21} = \frac{3iW_2 \sin \theta \sqrt{R_H^2 - r^2}}{r^4 R_H} + \frac{3\sqrt{R_H^2 - r^2} \sin \theta \cos \theta}{r^6 R_H^3} [4A_I r R_H^4 - 5W_2^2 (r^2 - 2R_H^2)] \quad (\text{B1b})$$

$$Q_{22} = -\frac{3iW_2 \cos \theta}{r^4} + \frac{1}{r^6} \left[A_I r (R_H^2 - r^2) - r^3 \delta M_I + W_2^2 \left(6 - \frac{5r^2}{R_H^2} \right) \right] + \frac{P_2(\cos \theta)}{r^6 R_H^2} [4W_2^2 (5r^2 - 4R_H^2) + A_I r R_H^2 (3r^2 - 7R_H^2)] \quad (\text{B1c})$$

$$Q_{33} = -\frac{3iW_2 \cos \theta}{r^4} + \frac{1}{r^6} \left[A_I r (r^2 - R_H^2) - r^3 \delta M_I + \frac{5r^2 W_2^2}{R_H^2} - 4W_2^2 \right] + \frac{P_2(\cos \theta)}{r^6 R_H^2} [A_I r R_H^2 (r^2 - 5R_H^2) + 2W_2^2 (5r^2 - 3R_H^2)] \quad (\text{B1d})$$

To lowest order in the rotation, the Q matrix vanishes, in accordance with the fact that de Sitter space has a vanishing Weyl tensor. Note that W_2 causes a first order correction in the “magnetic” part of the Weyl tensor (imaginary part of the Q matrix). The correction to the “electric” part (real part of the Q matrix) is of second order. The Q matrix contains divergences at $r = 0$ for nonzero J , and it is impossible to choose W_2 and δM_I to cancel all diverging terms in the Weyl tensor at the origin. Since the Ricci tensor and scalar as determined by the stress-energy (3.5) is finite, the singular behavior of the Weyl tensor at the origin is the same as that of the corresponding components of the Riemann tensor. If one eliminates all the δ -function contributions at the origin to the Komar mass and angular momentum by setting $W_2 = \delta M_I = 0$ as in (8.13), and uses (6.7) for A_I , some of the divergences at $r = 0$ are removed and (B1) simplifies considerably to

$$Q_{11} = \frac{8J^2 P_2(\cos \theta) (r^2 - 3R_H^2)}{r^5 R_H^3} \quad (\text{B2a})$$

$$Q_{21} = -\frac{24J^2}{r^5 R_H^2} \sqrt{R_H^2 - r^2} \sin \theta \cos \theta \quad (\text{B2b})$$

$$Q_{22} = \frac{2J^2 (r^2 - R_H^2)}{r^5 R_H^3} + \frac{2J^2 P_2(\cos \theta) (7R_H^2 - 3r^2)}{r^5 R_H^3} \quad (\text{B2c})$$

of the form (3.1) in Appendix B of [41]. The full Q matrix components for our interior, and hence the Weyl tensor components, are functions of r , θ and of the integration constants W_2 , A_I , δM_I , and B_I . Setting $B_I = 0$ as in the finiteness condition (5.5), the nonzero independent components of the Q matrix are:

$$Q_{33} = \frac{2J^2 (R_H^2 - r^2)}{r^5 R_H^3} + \frac{2J^2 P_2(\cos \theta) (5R_H^2 - r^2)}{r^5 R_H^3}. \quad (\text{B2d})$$

APPENDIX C: NULL COORDINATES AND CONFORMAL DIAGRAM

In the spherically symmetric case, one can move to Eddington-Finkelstein coordinates v or u using

$$dv = dt + \frac{1}{\sqrt{fh}} dr, \quad \text{or} \quad du = dt - \frac{1}{\sqrt{fh}} dr, \quad (\text{C1})$$

respectively. Note that in Eddington-Finkelstein coordinates the metric

$$ds^2 = -f du^2 - 2\sqrt{\frac{f}{h}} du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (\text{C2})$$

has a discontinuity on the null $r = R_H$ hypersurface where $f(R_H) = 0$.

Because the interior is a de Sitter space and the exterior is Schwarzschild, we may construct a conformal diagram, which we sketch in Fig. 1. It is reminiscent of the conformal diagram for the extremal Reissner-Nordstrom solution, although now $r = 0$ is regular, but the null surface is associated with a Dirac δ -function energy-momentum distribution and hence is singular.

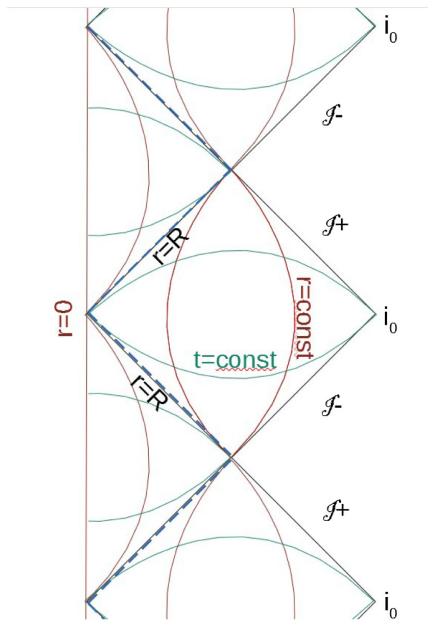


FIG. 1. Conformal diagram for the static gravastar. Schwarzschild exterior (“squares”) and de Sitter static patch interior (“triangles”) diagrams are joined at the $r = R_H$ null hypersurface and may be tiled in a structure similar to the extremal Reissner-Nordstrom black hole. Unlike the Reissner-Nordstrom spacetime, the static gravastar is regular at $r = 0$ but has a singularity associated with the surface at $r = R_H$ (shown in the diagram by a dashed blue line). Sketches of constant r hypersurfaces and of constant t hypersurfaces are shown.

- [1] R. Penrose, *Phys. Rev. Lett.* **14**, 57 (1965).
[2] S. W. Hawking and R. Penrose, *Proc. R. Soc. A* **314**, 529 (1970).
[3] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[4] S. W. Hawking, *Phys. Rev. D* **14**, 2460 (1976).
[5] G. 't Hooft, *Nucl. Phys. B, Proc. Suppl.* **43**, 1 (1995).
[6] W. G. Unruh and R. M. Wald, *Rep. Prog. Phys.* **80**, 092002 (2017).
[7] S. Mathur, *Fortschr. Phys.* **53**, 793 (2005).
[8] S. M. Christensen and S. A. Fulling, *Phys. Rev. D* **15**, 2088 (1977).
[9] E. Mottola and R. Vaulin, *Phys. Rev. D* **74**, 064004 (2006).
[10] E. Mottola, *Acta Phys. Pol. B* **41**, 2031 (2010).
[11] P. O. Mazur and E. Mottola, *Classical Quantum Gravity* **32**, 215024 (2015).
[12] G. Chapline, E. Hohlfeld, R. B. Laughlin, and D. I. Santiago, *Philos. Mag. B* **81**, 235 (2001).
[13] E. Poisson and W. Israel, *Classical Quantum Gravity* **5**, L201 (1988).
[14] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, *Phys. Lett. B* **216**, 272 (1989).
[15] I. Dymnikova, *Gen. Relativ. Gravit.* **24**, 235 (1992).
[16] I. Dymnikova, *Int. J. Mod. Phys. D* **12**, 1015 (2003).
[17] S. Ansoldi, in *Conference on Black Holes and Naked Singularities Milan, Italy, May 10-12, 2007* (2008), <https://arxiv.org/abs/0802.0330>.
[18] M. Visser and D. Wiltshire, *Classical Quantum Gravity* **21**, 1135 (2004).
[19] P. Pani, *Phys. Rev. D* **92**, 124030 (2015).
[20] N. Uchikata and S. Yoshida, *Classical Quantum Gravity* **33**, 025005 (2016).
[21] N. Uchikata, S. Yoshida, and P. Pani, *Phys. Rev. D* **94**, 064015 (2016).
[22] C. B. M. H. Chirenti and L. Rezzolla, *Classical Quantum Gravity* **24**, 4191 (2007).
[23] C. Chirenti and L. Rezzolla, *Phys. Rev. D* **94**, 084016 (2016).
[24] C. Cattoen, T. Faber, and M. Visser, *Classical Quantum Gravity* **22**, 4189 (2005).
[25] C. B. M. H. Chirenti and L. Rezzolla, *Phys. Rev. D* **78**, 084011 (2008).
[26] W. Israel, *Nuovo Cimento B* **44**, 1 (1966).
[27] C. Barrabès and W. Israel, *Phys. Rev. D* **43**, 1129 (1991).
[28] A. E. Broderick and R. Narayan, *Classical Quantum Gravity* **24**, 659 (2007).
[29] A. E. Broderick, A. Loeb, and R. Narayan, *Astrophys. J.* **701**, 1357 (2009).
[30] R. Carballo-Rubio, F. Di Filippo, S. Liberati, and M. Visser, *Phys. Rev. D* **98**, 124009 (2018).

- [31] B. P. Abbott *et al.* (LIGO and Virgo Scientific Collaborations), *Astrophys. J.* **896**, L44 (2020).
- [32] The EHT Collaboration, *Astrophys. J. Lett.* **875**, L1 (2019).
- [33] V. Cardoso, E. Franzin, and P. Pani, *Phys. Rev. Lett.* **116**, 171101 (2016).
- [34] V. Cardoso and P. Pani, *Living Rev. Relativity* **22**, 4 (2019).
- [35] V. Cardoso, S. Hopper, C. F. B. Macedo, C. Palenzuela, and P. Pani, *Phys. Rev. D* **94**, 084031 (2016).
- [36] J. Abedi, H. Dykaar, and N. Afshordi, *Phys. Rev. D* **96**, 082004 (2017).
- [37] A. B. Nielsen, C. D. Capano, O. Birnholtz, and J. Westerweck, *Phys. Rev. D* **99**, 104012 (2019).
- [38] J. B. Hartle and K. S. Thorne, *Astrophys. J.* **153**, 807 (1968).
- [39] S. Chandrasekhar and J. C. Miller, *Mon. Not. R. Astron. Soc.* **167**, 63 (1974).
- [40] C. Posada, *Mon. Not. R. Astron. Soc.* **468**, 2128 (2017).
- [41] P. Beltracchi, P. Gondolo, and E. Mottola, preceding paper, *Phys. Rev. D* **105**, 024001 (2022).
- [42] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
- [43] K. Schwarzschild, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.* **XVIII**, 424 (1916), <https://arxiv.org/abs/physics/9912033>.
- [44] J. L. Synge, *Relativity: The General Theory* (Interscience Publishers, New York, 1960).
- [45] P. O. Mazur and E. Mottola, [arXiv:gr-qc/0109035](https://arxiv.org/abs/gr-qc/0109035).
- [46] P. O. Mazur and E. Mottola, *Proc. Natl. Acad. Sci. U.S.A.* **101**, 9545 (2004).
- [47] H. A. Buchdahl, *Phys. Rev.* **116**, 1027 (1959).
- [48] T. Lewis and G. A. Schott, *Proc. R. Soc. A* **136**, 176 (1932).
- [49] A. Papapetrou, in *Annales de l'IHP Physique Théorique* (1966), Vol. 4, pp. 83–105.
- [50] J. B. Hartle and D. H. Sharp, *Astrophys. J.* **147**, 317 (1967).
- [51] J. B. Hartle, *Astrophys. J.* **150**, 1005 (1967).
- [52] J. M. Bardeen, *Astrophys. J.* **162**, 71 (1970).
- [53] S. Chandrasekhar and J. L. Friedman, *Astrophys. J.* **175**, 379 (1972).
- [54] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
- [55] N. Uchikata and S. Yoshida, *Phys. Rev. D* **90**, 064042 (2014).
- [56] D. Christodoulou, *Phys. Rev. Lett.* **25**, 1596 (1970).
- [57] M. Rutkowski, *Phys. Rev. D* **104**, 084041 (2021).
- [58] G. F. Chapline and P. Marecki, *Acta Phys. Pol. B* **45**, 1147 (2014).
- [59] A. Burinskii, E. Elizalde, S. R. Hildebrandt, and G. Magli, *Phys. Rev. D* **65**, 064039 (2002).
- [60] M. Gürses and F. Gürsey, *J. Math. Phys. (N.Y.)* **16**, 2385 (1975).
- [61] E. T. Newman and A. I. Janis, *J. Math. Phys. (N.Y.)* **6**, 915 (1965).
- [62] N. Ibohal, *Gen. Relativ. Gravit.* **37**, 19 (2005).
- [63] I. Dymnikova, *Phys. Lett. B* **639**, 368 (2006).
- [64] M. Azreg-Ainou, *Phys. Lett. B* **730**, 95 (2014).
- [65] E. J. Gonzalez de Urreta and M. Socolovsky, [arXiv:1504.01728](https://arxiv.org/abs/1504.01728).
- [66] P. Beltracchi and P. Gondolo, [arXiv:2104.02255](https://arxiv.org/abs/2104.02255) [Phys. Rev. D (to be published)].
- [67] B. Reina and R. Vera, *Classical Quantum Gravity* **32**, 155008 (2015).
- [68] V. Cardoso, E. Franzin, A. Maselli, P. Pani, and G. Raposo, *Phys. Rev. D* **95**, 084014 (2017).
- [69] C. Chirenti, C. Posada, and V. Guedes, *Classical Quantum Gravity* **37**, 195017 (2020).
- [70] P. Beltracchi and P. Gondolo, *Phys. Rev. D* **99**, 044037 (2019).
- [71] A. Z. Petrov, *Sci. Trans. Kazan State Univ.* **114**, 55 (1954), https://kpfu.ru/portal/docs/F1839109627/1954_114_8_petrov_5.pdf.
- [72] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2003).