# Unitarily inequivalent quantum cosmological bouncing models

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By quantizing the background as well as the perturbations in a simple one fluid model, we show that there exists an ambiguity in the choice of relevant variables (assuming factor ordering to have been taken care of), potentially leading to incompatible observational physical predictions. There are, in such models, two fundamental ambiguities, a well-known one (factor ordering) which can be removed by an actual choice, and a new one, which depends on the choice of variables themselves. In a classical inflationary background, the exact same canonical transformations lead to unique predictions, so the ambiguity we put forward demands a background with a sufficiently strong departure from classical evolution. The latter condition happens to be satisfied in bouncing scenarios, which may thus be having predictability issues. Inflationary models could evade such a problem because of the monotonic behavior of their scale factor; they do, however, initiate from a singular state which bouncing scenarios aim at solving.

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## I. INTRODUCTION

Cosmological perturbations are usually studied on a classical background in the framework of inflation [1]. Most models of bouncing alternatives are either based on a classical background [2,3] or it is assumed that the semiclassical approximation ensures similar behavior for the perturbations. The purpose of this paper is to show that there might be some important caveat that should be taken into account as an unsolved ambiguity, not to be mistaken with that due to operator ordering (also present but fixed independently), can emerge in a quantum bouncing scenario. It is worth mentioning that already in classical backgrounds, the notion of the initial vacuum state depends on the choice of perturbation variables for quantization as noted e.g., in [4]. Herein, we show that once the background is quantized, the physical ambiguity gets much stronger and concerns the dynamics of mode functions as well. A similar point was considered in Refs. [5,6] for an inflationary background, leading to a vanishingly small effect.

Before going on, we would like to introduce some terminology. We define as "classical" a set of $c$–numbers-valued variables following the underlying dynamical equations derived from the Euler-Lagrange variations of the action. In the case of general relativity (GR), this means that a classical solution not only permits to actually reconstruct a full 4-dimensional spacetime, but one that satisfies Einstein equations.

We shall call "semiclassical" a set of $c$–numbers-valued variables (expectation values of quantum operators in a given state for instance) not following the classical underlying equations of the relevant theory. We thus emphasize on the classical properties such trajectories have and merely consider functions of these variables as those that would be obtained if they were actually classical; quantum uncertainties are here assumed to be negligible. For GR, this means a regular 4-dimensional spacetime, seen as a classical object, but now solving quantum corrected equations of motion instead of Einstein equations. In the framework of cosmological perturbation theory, plugging such a solution into the perturbation action does not lead to any ambiguity as one then merely quantizes the perturbation modes, and the (classical or quantum [7]) canonical transformation on those does not require the background equations of motion to hold but only demand that the trajectory be well defined functions of time.

Finally, finding a trajectory as proposed above might lead to new, purely quantum, effects, when the uncertainties cannot be neglected. To avoid any confusion with the semiclassical case as defined above, we coined the neologism "semiquantum" in what follows for such trajectories, to emphasize their quantum nature. It is those we want to discuss in what follow, and we argue that a fundamental ambiguity prevents a straightforward use of such semiquantum trajectories. In practice, once a

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trajectory approximation is proposed, one needs to check explicitly if it corresponds to a semiclassical or a semiquantum one before using it in the appropriate way.

To illustrate our point, we examine a simple model based on canonical quantization of GR in which the matter content is represented by a perfect fluid with constant equation of state $w \in [0, 1[$. This model is of course classically singular, all trajectories being either expanding from a singularity (vanishing scale factor) or contracting toward one. We first recall the classical model in its Hamiltonian formulation both for the background universe, before moving to a quantum approach aiming at resolving the classical singularity: instead of using the Wheeler-De Witt equation to estimate the probabilities to connect contracting and expanding branches, we propose a semiquantum approximation leading to regular, bouncing behavior. With these, one can then proceed to evaluating the behavior of perturbations.

The paper is organized as follows. Section II considers the background. The classical model consists of general relativity sourced by a simple constant equation-of-state fluid, for which we define the Hamiltonian dynamics and make explicit the singular solutions. A general quantization scheme is then discussed in which the factor ordering ambiguity is fully taken care of by inserting appropriate unknown constants in the symmetrized operator version of the classical $c$–numbers. This section ends with the semiquantum approximation leading to effective bouncing (regular) trajectories.

In Sec. III, we move to the usual treatment of classical perturbations over the classical background, which we classify according to the times relevant for their description, namely the fluid or conformal times, the relation between these descriptions being based on a simple canonical transformation. We move on to quantizing these perturbations in Sec. IV by assuming a very general quantization procedure allowing to account for self-adjointness issues on the half line; the perturbations are then treated in the usual (canonical) way. The ensuing quantum dynamics is examined in Sec. V where the semiquantum approximation for the background is found to yield two different, and incompatible, equations of motion for the perturbation modes, leading to a potential ambiguity in the observational predictions. Our conclusions are followed by an Appendix showing an explicit example of quantization based on coherent states with definite fiducial vectors.

## II. BACKGROUND EVOLUTION

In this section we provide the definition of the classical model at the background level. The physical phase space for the model is introduced together with the physical Hamiltonian that generates its dynamics with respect to an internal clock. The background solution to the classical dynamics is briefly discussed.

### A. Classical dynamics

We assume the universe to be spatially compact, $\mathcal{M} \simeq \mathbb{R} \times \mathbb{T}^3$, with coordinate volume we note $\mathcal{V}_0$ below. Its evolution is supposed to be driven by a perfect fluid that satisfies a barotropic equation of state $p = w\rho$, with $-\frac{1}{3} < w < 1$. The fully canonical formalism for the perturbed Friedmann universe that can be easily adapted to the present case can be found in [8]: we start from the Einstein-Hilbert-Schutz action [9,10]

$$\mathcal{S}_{\rm EHS} = \underbrace{\frac{1}{16\pi G_{\rm N}} \int \mathrm{d}^4 x \sqrt{-g} R}_{\mathcal{S}_{\rm EH}} + \underbrace{\int \mathrm{d}^4 x \sqrt{-g} P(w, \phi)}_{\mathcal{S}_{\rm s}}, \quad (1)$$

where $P = w\rho$ is the pressure of the cosmic fluid while $\phi$ defines its flow. The action $\mathcal{S}_{\rm EHS}$ is first expanded to second order around the flat Friedmann universe. Next the Hamiltonian description is obtained in which the truly physical degrees of freedom are identified and the remaining ones removed.

#### 1. Hamiltonian evolution

Let us consider the usual Einstein-Hilbert action $\mathcal{S}_{\rm EH}$ at zeroth order, omitting the integrated term

$$\frac{1}{16\pi G_{\rm N}} \int \mathrm{d}^4 x \sqrt{-g} R = -\frac{1}{2\kappa} \int \mathrm{d}\tau N a^3 \underbrace{\int \sqrt{\gamma} \mathrm{d}^3 x}_{\mathcal{V}_0} \underbrace{\frac{6\dot{a}^2}{a^2 N^2}}_{R}, \quad (2)$$

in which we used the background isotropic and homogeneous flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -N^2(\tau)\mathrm{d}\tau^2 + a^2(\tau)\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j, \quad (3)$$

a dot meaning a derivative with respect to the coordinate time $\tau$, later to be identified with the fluid clock variable. Written as $\mathcal{S}_{\rm EH} = \int L^{(0)}(a, \dot{a})\mathrm{d}\tau$, with Lagrangian $L^{(0)} = 3\mathcal{V}_0 a\dot{a}^2/(N\kappa)$, this yields the canonically conjugate momentum $p_a = \partial L^{(0)}/\partial\dot{a} = 6\mathcal{V}_0 a\dot{a}/(\kappa N)$, and the gravitational Hamiltonian at zeroth order $H_{\rm G}^{(0)}$ reads

$$H_{\rm G}^{(0)} = -\frac{\kappa N}{12\mathcal{V}_0 a} p_a^2, \quad (4)$$

which can also be expressed in terms of the canonical variables,

$$q = \frac{4\sqrt{6}}{3(1-w)\sqrt{1+w}} a^{\frac{3}{2}(1-w)} \equiv \gamma a^{\frac{3}{2}(1-w)}, \quad (5)$$

thereby defining the constant $\gamma$, and

$$p = \frac{\sqrt{6(1+w)}}{2\kappa_0} a^{\frac{3}{2}(1+w)} H, \tag{6}$$

where $H = \dot{a}/(Na)$ is the Hubble rate and $\kappa_0 = \kappa/\mathcal{V}_0$. The Hamiltonian $H_{\rm G}^{(0)}$ reads

$$H_{\rm G}^{(0)} = -\frac{2\kappa_0 N}{(1+w)a^{3w}} p^2 = -2\kappa_0 p^2, \tag{7}$$

where in the last equality, we made the choice of the lapse $N = (1+w)a^{3w}$. It can be shown that for this particular choice of the lapse the matter Hamiltonian $H_{\rm M}^{(0)}$ obtained from the Schutz action $\mathcal{S}_{\rm S}$ equals the cosmic fluid conjugate momentum (see, e.g., [11] for details). Therefore, the total Hamiltonian generates a uniform motion in the cosmic fluid variable. It is a standard procedure at this point to promote the cosmic fluid variable to the role of internal clock while removing it and its conjugate momentum from the phase space. The physical Hamiltonian that generates the dynamics of the background geometry with respect to the fluid variable is thus simply $H_{\rm G}^{(0)}$. However, we find it convenient to inverse the direction of time with respect to the fluid variable in order to have the positive physical Hamiltonian,

$$H^{(0)} = -H_{\rm G}^{(0)} = 2\kappa_0 p^2. \tag{8}$$

We shall denote the internal clock by $\tau$ and assume it coincides with the FLRW time set in (3) [1]. It can be shown that the Hamiltonian $H^{(0)} = (1+w)E_{\rm f}|_{a=1}$ equals $(1+w)$ times the energy of the fluid contained in the universe when $a = 1$ (we choose a dimensionless scale factor, so that the canonical variable $q$ is also dimensionless).

### 2. Singular solutions

The background Hamilton equations stemming from Eq. (8) read

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = 4\kappa_0 p \quad \text{and} \quad \frac{\mathrm{d}p}{\mathrm{d}\tau} = 0, \tag{9}$$

and are easily solved by

$$q(\tau) = \sqrt{8\kappa_0 H^{(0)}}(\tau - \tau_{\rm s}) \quad \text{and} \quad p(\tau) = \sqrt{\frac{H^{(0)}}{2\kappa_0}}, \tag{10}$$

where $H^{(0)}$ is the value of the zeroth-order Hamiltonian, a constant by virtue of its definition (8) and the equation of motion (9). The phase space trajectories that either terminate at or emerge from the singularity at time $\tau_{\rm s}$ are straight lines in phase space $\{q, p\}$ with constant $p$ [12], shown as straight lines in Fig. 1 below. Note that in order to assign the correct trajectory to the background universe, one needs to know the value of the energy of the fluid in the whole universe when $a = 1$. This value can be determined only when one knows the size of the universe, size which can be fixed by demanding that the volume of the observable patch be a given ratio (less than unity) of the size of the full universe.

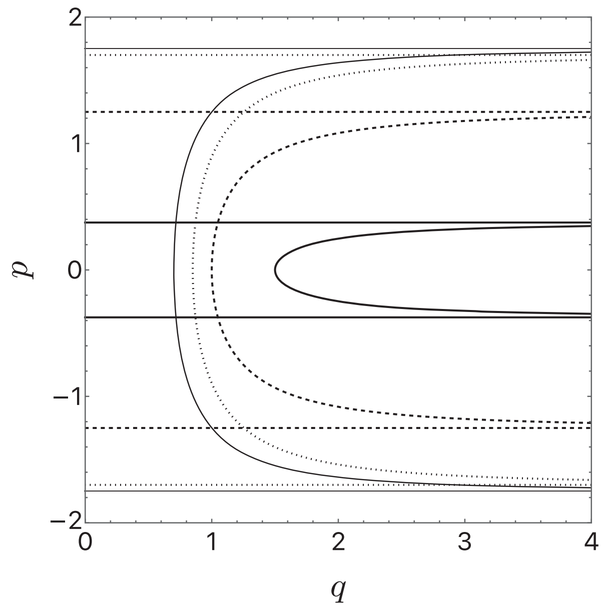


FIG. 1. Background phase space evolutions: the straight lines represent Eqs. (10), either going to or emerging from a singularity ($q \to 0$), while the curves are the solutions (25) leading to the same asymptotic classical lines. The semiquantum solution are seen to consist of a bounce smoothly joining expanding ($p > 0$) and contracting ($p < 0$) classical universes.

Given our choice for the lapse function $N = (1+w)a^{3w}$, the conformal time $\eta$, defined by $N\mathrm{d}\tau = a\mathrm{d}\eta$ is found to be given by

$$\mathrm{d}\eta = Z^2 \mathrm{d}\tau = (1+w)\left(\frac{q}{\gamma}\right)^{\frac{2(3w-1)}{3(1-w)}} \mathrm{d}\tau, \tag{11}$$

where use has been made of Eq. (5) and we have defined the function

$$Z(\tau) \equiv \sqrt{1+w}\left(\frac{q}{\gamma}\right)^{\frac{3w-1}{3(1-w)}}. \tag{12}$$

Anticipating the quantum solution (25), we write the classical solution as $q = q_{\rm B}\omega\tau$ (setting the singularity time to $\tau_{\rm s} \to 0$), and therefore $p = q_{\rm B}\omega/(4\kappa_0)$. Equation (11) with this solution permits to integrate explicitly for the conformal time $\eta$, also assuming $\eta \to 0$ for $\tau \to 0$. One then finds the "classical" conformal time to read

$$\eta = \frac{1+w}{r_1 + r_2}\left(\frac{q_{\rm B}\omega}{\gamma}\right)^{2r_1} \tau^{r_1 + r_2}, \tag{13}$$

which is straightforwardly inverted to yield $\tau(\eta)$, and finally

$$q(\eta) = q_{\rm B}\omega\left[\frac{r_1+r_2}{1+w}\left(\frac{q_{\rm B}\omega}{\gamma}\right)^{-2r_1}\eta\right]^{1/(r_1+r_2)} \propto \eta^{\frac{3(1-w)}{1+3w}}, \quad (14)$$

where we have set

$$r_1 = \frac{3w-1}{3(1-w)} \quad \text{and} \quad r_2 = r_1 + 1 = \frac{2}{3(1-w)}. \quad (15)$$

### B. A quantum background

The phase space for the cosmological background is the half-plane rather than the full plane and hence the usual canonical quantization rules seem to be inadequate. There exist many quantization methods (see, e.g., [13] for a comprehensive review) some of which one could find more suitable in the present context. In order to account for this issue, we introduce a family of quantum models, all of which in correspondence with the underlying classical model. They are given by a set of free parameters that can be computed, for instance, in the framework of the so-called affine quantization (see [14] for details) that has been proposed for a consistent quantum gravity program [15,16]; we briefly recap what is relevant for our purposes of this method in Appendix. This approach enables us to free ourselves from a particular quantization of the background geometry, to take care of issues such as ordering ambiguity in a straightforward way and, finally, to emphasize the universal character of the ambiguity that we discuss in the next sections.

Given the existing ambiguities due to factor ordering when going from classical to quantum, we propose the following set of operators to replace the Hamiltonian (8):

$$H^{(0)} \mapsto \hat{H}^{(0)} = 2\kappa_0(\hat{P}^2 + \hbar^2\mathfrak{c}_0\hat{Q}^{-2}), \quad (16)$$

where $\mathfrak{c}_0 \geq 0$ is a free parameter. The value $\mathfrak{c}_0 = 0$ corresponds to the "canonical quantization", whereas the values $\mathfrak{c}_0 > 0$ can be justified in various ways, for instance by using the affine group as the symmetry of quantization [11,17]. In the latter case, the repulsive potential $\propto \hat{Q}^{-2}$, of quantum geometric origin, naturally prevents the universe from reaching the singular point $q = 0$ by reversing its motion from contraction to expansion. If $\mathfrak{c}_0 \geq \frac{3}{4}$, then $\hat{H}^{(0)}$ is essentially self-adjoint and no boundary condition needs be imposed at $\hat{Q} = 0$ to ensure a unique and unitary dynamics (see, e.g., Ref. [18] and references therein). The only way to determine the right value of the parameter $\mathfrak{c}_0$ is to compare the predictions of the model with the actual observations of the Universe.

We will need quantum operators to replace other zeroth-order quantities appearing in the Hamiltonians relevant for describing perturbations (28) and (34) below. We propose the following replacements

$$q^\alpha \mapsto \mathfrak{l}(\alpha)\hat{Q}^\alpha, \quad (17a)$$

$$q^\alpha p^2 \mapsto \mathfrak{a}(\alpha)\hat{Q}^\alpha\hat{P}^2 + i\hbar\mathfrak{b}(\alpha)\hat{Q}^{\alpha-1}\hat{P} + \hbar^2\mathfrak{c}(\alpha)\hat{Q}^{\alpha-2}, \quad (17b)$$

where $\hat{Q}$ and $\hat{P}$ are the "position" and "momentum" operators on the half-line, satisfying the usual commutation relation $[\hat{Q},\hat{P}] = i\hbar$, and therefore $[\hat{Q}^\alpha,\hat{P}] = i\hbar\alpha\hat{Q}^{\alpha-1}$, so that $\mathfrak{b}(\alpha) = -\alpha\mathfrak{a}(\alpha)$ in order to ensure that the second-line operator (17b) is symmetric, i.e., so that it reads

$$q^\alpha p^2 \mapsto \mathfrak{a}(\alpha)\hat{P}\hat{Q}^\alpha\hat{P} + \hbar^2\mathfrak{c}(\alpha)\hat{Q}^{\alpha-2}; \quad (18)$$

the power-depending numbers $\mathfrak{l}(\alpha)$, $\mathfrak{a}(\alpha)$ and $\mathfrak{c}(\alpha)$ are assumed positive and dimensionless.

It should be emphasized at this stage that the usual factor ordering ambiguity is fully taken care of in this framework by merely providing actual numbers for the gothic-style parameters appearing in Eqs. (16) to (18). Assuming knowledge of these (e.g., by comparison with some relevant experimental result), one expects the ensuing predictions to be unambiguous from the point of view of factor ordering; whatever remaining ambiguity, as the one detailed below, cannot follow from it.

### C. Phase space semiquantum approximation

We now introduce a semiquantum approximation (as suggested in the introduction) to the quantum dynamics of the background geometry. It should be noted that any ambiguous effect such as the one we obtain here at a semiquantum level may only be enhanced if a fully quantum description of the background were to be used. We carefully construct the semiquantum description with the use of coherent states.

It is very useful to have at disposal background solutions $|\psi_{\rm B}(\tau)\rangle$ corresponding to various energies and with various spreads in $\hat{Q}$ and $\hat{P}$. One can find a wide class of solutions by approximating the Hilbert space with a family of coherent states built from a single wave function, the so-called fiducial vector; this construction is presented in Appendix. For the present purpose, suffice it to note that any fixed family of coherent states is given by state vectors $(q,p) \mapsto |q,p\rangle$ in one-to-one correspondence with the phase space. In practice, from a fiducial state $|\tilde{\xi}\rangle$, for which $\langle\tilde{\xi}|\hat{Q}|\tilde{\xi}\rangle = 1$ (recall $q$, and therefore $\hat{Q}$, is dimensionless) and $\langle\tilde{\xi}|\hat{P}|\tilde{\xi}\rangle = 0$, one builds the coherent state through [16]

$$|q(\tau),p(\tau)\rangle = \mathrm{e}^{ip(\tau)\hat{Q}/\hbar}\mathrm{e}^{-i\ln q(\tau)\hat{D}/\hbar}|\tilde{\xi}\rangle, \quad (19)$$

where $\hat{D} = \frac{1}{2}(\hat{Q}\hat{P} + \hat{P}\hat{Q})$ is the dilation operator. The expectation values of $\hat{Q}$ and $\hat{P}$ in $|q(\tau),p(\tau)\rangle$ are respectively $q(\tau)$ and $p(\tau)$.

The dynamics confined to the vectors $|q(\tau),p(\tau)\rangle$ can be deduced from the quantum action

$$\mathcal{S}_{\rm B} = \int \langle q(\tau), p(\tau)| \left( i\hbar \frac{\partial}{\partial \tau} - \hat{H}^{(0)} \right) |q(\tau), p(\tau)\rangle \mathrm{d}\tau, \tag{20}$$

which, upon using the properties of the state (19), can be transformed into

$$\mathcal{S}_{\rm B} = \int \{\dot{q}(\tau)p(\tau) - H_{\rm sem}[q(\tau), p(\tau)]\} \,\mathrm{d}\tau, \tag{21}$$

with the semiquantum Hamiltonian given by

$$H_{\rm sem} = \langle q, p|\hat{H}^{(0)}|q, p\rangle, \tag{22}$$

from which one derives the ordinary Hamilton equations

$$\dot{q} = \frac{\partial H_{\rm sem}}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H_{\rm sem}}{\partial q}. \tag{23}$$

Given the quantum Hamiltonian (16), we find that the semiquantum background Hamiltonian reads, by virtue of our ordering choice (18) (with $\alpha = 0$)

$$H_{\rm sem} = 2\kappa_0 \left( p^2 + \frac{\hbar^2 \mathfrak{K}}{q^2} \right), \tag{24}$$

where the constant $\mathfrak{K}$ is positive ($\mathfrak{K} > 0$), irrespective of whether $\mathfrak{c}_0 = 0$ or $\mathfrak{c}_0 > 0$. The actual value of $\mathfrak{K}$ depends on the choice of family of coherent states, as illustrated in Appendix. We find the solution to (23) to read

$$q = q_{\rm B}\sqrt{1 + (\omega\tau)^2}, \tag{25a}$$

$$p = \frac{q_{\rm B}\omega^2}{4\kappa_0} \frac{\tau}{\sqrt{1 + (\omega\tau)^2}}, \tag{25b}$$

where $q_{\rm B}^2 = 2\kappa_0\hbar^2\mathfrak{K}/H_{\rm sem}$ and $\omega = 2H_{\rm sem}/(\hbar\sqrt{\mathfrak{K}})$. We display in Fig. 1 a few trajectories in the phase space illustrating these solutions and comparing them with their classical counterparts (10).

With this semiquantum solution, one can also integrate (11) to yield the conformal time $\eta$, as a function of $\tau$

$$\eta = (1 + w)\tau \left( \frac{q_{\rm B}}{\gamma} \right)^{2r_1} \mathcal{F}\left[ \frac{1}{2}, -r_1; \frac{3}{2}; -(\omega\tau)^2 \right], \tag{26}$$

where $\mathcal{F}(a, b; c; z)$ is the hypergeometric function (see Sec. 15 of Ref. [19]). As expected, one recovers the classical power law (13) in the large-time classical limit $\tau \gg \omega^{-1}$, up to a constant depending on the equation of state $w$ and vanishing for $w = \frac{1}{3}$. Figure 2 shows the classical and semiquantum relationships $\eta(\tau)$.

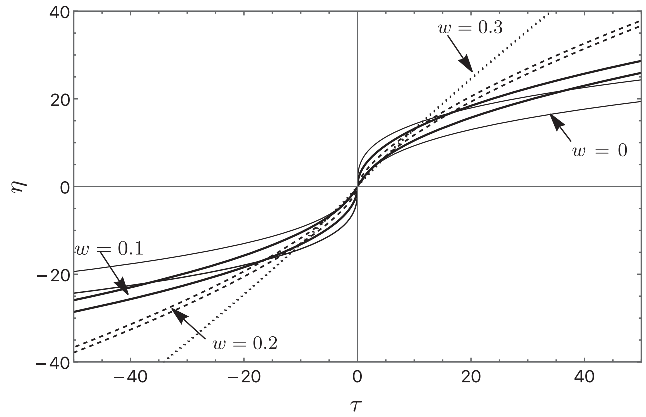


FIG. 2. Conformal time $\eta$ as a function of $\tau$, for the classical (13) and semiquantum (26) solutions for $w = 0$ (thin line), $w = 0.1$ (thick), $w = 0.2$ (dashed) and $w = 0.3$ (dotted). The quantum conformal time tends to the classical one up to a constant factor, which vanishes for $= \frac{1}{3}$.

## III. CLASSICAL PERTURBATIONS

Having clarified the status of the background evolution and found a way to regularize it through quantization, we now move on to identifying the truly physical degrees of freedom also at linear order. We restrict attention in this section to classical perturbations over the classical background, following the terminology introduced in the introduction.

### A. Fluid perturbations

We write the full classical Hamiltonian $H_{\rm full}$ as

$$H_{\rm full} = H^{(0)} - \sum_{\boldsymbol{k}} H_{\boldsymbol{k}}^{(2)}, \tag{27}$$

where the second-order Hamiltonian $H_{\boldsymbol{k}}^{(2)}$, depending only on the discrete (recall the Universe considered is compact) wave vector $\boldsymbol{k}$, reads

$$H_{\boldsymbol{k}}^{(2)} = \frac{1}{2}|\pi_{\phi,\boldsymbol{k}}|^2 + \frac{1}{2}w(1 + w)^2 \left( \frac{q}{\gamma} \right)^{4r_1} k^2|\phi_{\boldsymbol{k}}|^2, \tag{28}$$

with $\gamma$ defined in (5) and $r_1$ in (15) above. The Fourier component $\phi_{\boldsymbol{k}}$ of the perturbation field is a combination of the fluid perturbation[1] $\delta\phi_{\boldsymbol{k}}$ and the intrinsic curvature perturbation $\delta R_{\boldsymbol{k}}$, namely [8]

$$\phi_{\boldsymbol{k}} = \frac{p^{\frac{1-w}{1+w}}}{\sqrt{2w(1 + w)\kappa_0}} \delta\phi_{\boldsymbol{k}} + \sqrt{\frac{3}{w\kappa_0}} \frac{a^{-\frac{3w-7}{2}}}{4k^2} \delta R_{\boldsymbol{k}}, \tag{29}$$

[1]The background fluid time $\tau$ is actually a combination of the fluid variable and its momentum, $(1 + w)\tau = \phi|p_\phi|^{-1/w}$. For more details, see e.g., [8].

with $k \equiv |\boldsymbol{k}|$ the amplitude of the wave vector; note that since the FLRW background (3) is isotropic, it is expected, as usual, that the initial conditions, and therefore the solutions of the perturbation evolution equation should depend only on the amplitude $k$ and not on its direction $\boldsymbol{k}/k$. Given our conventions, the physical dimensions are $[\phi_{\boldsymbol{k}}] = \sqrt{M}L$ and $[\pi_{\phi,\boldsymbol{k}}] = \sqrt{M}$. The Poisson bracket reads $\{\phi_{\boldsymbol{k}_1}, \pi_{\phi,-\boldsymbol{k}_2}\} = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$. The equation of motion expressed in the conformal time $\eta$ defined by Eq. (11), is found to read

$$\phi_{\boldsymbol{k}}'' + \left(\frac{q}{\gamma}\right)^{4r_1} w(1+w)^2 k^2 \phi_{\boldsymbol{k}} = 0. \tag{30}$$

It shows that for radiation, i.e., for $w = \frac{1}{3}$, which implies $r_1 = 0$, the dynamics of $\phi_{\boldsymbol{k}}$ becomes decoupled from the dynamical background.

There exist infinitely many parametrizations of the reduced phase space of perturbations and the pair $(\phi, \pi_\phi)$ can be seen as merely one example. As the relevant time for that description is $\tau$, which stems from the fluid, in terms of which the kinetic term in Eq. (27) is canonical, we shall call it the fluid-parametrization.

### B. Conformal perturbations

Another example of canonical fields is provided by the pair $(v, \pi_v)$, that is commonly used for solving the dynamics of scalar perturbations. It is defined by the canonical transformation

$$v_{\boldsymbol{k}} = Z\phi_{\boldsymbol{k}}, \tag{31a}$$

$$\pi_{v,\boldsymbol{k}} = Z^{-1}\pi_{\phi,\boldsymbol{k}} + \frac{\dot{Z}}{Z^2}\phi_{\boldsymbol{k}}, \tag{31b}$$

where the function $Z$ is defined in (12) above.

It can be noted that in the comoving gauge, one has $\delta\phi_{\boldsymbol{k}} = 0$, and thus $v_{\boldsymbol{k}} = -\sqrt{\frac{3(1+w)}{16w\kappa_0}}a\Psi_{\boldsymbol{k}}$, where $\Psi_{\boldsymbol{k}} = -a^2\delta R_{\boldsymbol{k}}/k^2$ is the comoving curvature.

We easily obtain the second-order Hamiltonian $H^{(2)}_{\boldsymbol{k}}$ in terms of $(v, \pi_v)$, namely

$$H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}Z^2\{|\pi_{v,\boldsymbol{k}}|^2 + [wk^2 - \mathcal{V}_{\rm cl}(\tau)]|v_{\boldsymbol{k}}|^2\}, \tag{32}$$

with the potential $\mathcal{V}_{\rm cl}$ defined through

$$\mathcal{V}_{\rm cl} = \frac{1}{Z^4}\left[\frac{\ddot{Z}}{Z} - 2\left(\frac{\dot{Z}}{Z}\right)^2\right] \tag{33}$$

which can be written explicitly in terms of the background canonical variables $q$ and $p$ as

$$H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}(1+w)\left(\frac{q}{\gamma}\right)^{2r_1}\left\{|\pi_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \frac{8}{9q^2}\left(\frac{q}{\gamma}\right)^{-4r_1}\frac{(2\kappa_0)^2(1-3w)p^2}{(1+w)^2(1-w)^2}\right]|v_{\boldsymbol{k}}|^2\right\}, \tag{34}$$

where we used the background equations of motion by assuming $p \to$ const.

The coefficient in front of the Hamiltonian (34) can be removed by switching to the internal conformal time $\eta$ (11) [12], in terms of which the potential (33) takes the simpler and usual form $\mathcal{V}_{\rm cl} = Z''/Z$, where a prime means a derivative with respect to the conformal time $\eta$ ($Z' \equiv \mathrm{d}Z/\mathrm{d}\eta$). The second-order Hamiltonian $Z^{-2}H^{(2)}_{\boldsymbol{k}}$ is then found to generate

$$v_{\boldsymbol{k}}'' + \left[wk^2 - \frac{8}{9q^2Z^4}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2}p^2\right]v_{\boldsymbol{k}} = 0, \tag{35}$$

which can be written in the usual Mukhanov-Sasaki form

$$v_{\boldsymbol{k}}'' + [wk^2 - \mathcal{V}_{\rm cl}(\eta)] = v_{\boldsymbol{k}}'' + \left(wk^2 - \frac{z''}{z}\right)v_{\boldsymbol{k}} = 0, \tag{36}$$

thereby identifying the classical potential

$$\mathcal{V}_{\rm cl}(\eta) = \frac{8}{9q^2Z^4}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2}p^2 = \frac{z''}{z}, \tag{37}$$

where the last equality is obtained by applying the classical equations of motion Eq. (9) below and we have used the generic function $z$, as there are in fact two different and equivalent choices that can be made, namely $z_1 = q^{r_1}$ and $z_2 = q^{r_2}$, namely

$$\mathcal{V}_{\rm cl} = \frac{(q^{r_1})''}{q^{r_1}} = \frac{(q^{r_2})''}{q^{r_2}} = \frac{2(1-3w)}{(1+3w)^2\eta^2}, \tag{38}$$

as usual for a background dominated by a perfect fluid with constant equation of state. These two power laws stem from the fact that although what enters into (32) is $Z''/Z$, with $Z \propto z_1$, one can then just as well choose the second solution of $z''/z = Z''/Z$, namely $z_2 \propto Z\int \mathrm{d}\eta/Z^2 = Z\int \mathrm{d}\tau = Z\tau$ which, taking the background solution $q \propto \tau$ [see Eq. (10)] yields $z_2 \propto Zq = z_1 q = q^{r_1+1}$, as indeed one has $r_2 = r_1 + 1$.

The internal conformal time provides a convenient form of the equation of motion for perturbations. We shall, however, quantize the dynamics of both the background and the perturbations reduced with respect to a unique internal time, the internal fluid time. The term $z''/z$ is usually referred to as the potential for the perturbations, as Eq. (36) is mathematically identical to a time-independent Schrödinger equation in such a potential [20]. As

$$\mathcal{V}_{\rm cl} = \frac{z''}{z} = \frac{1-3w}{2}\mathcal{H}^2 \tag{39}$$

has the clear physical meaning of the conformal Hubble rate $\mathcal{H}$ squared ($w < \frac{1}{3}$), the conformal Hubble rate determines the coordinate scale at which the amplification of perturbations starts to take place.

We shall call the set of variables $(v_{\boldsymbol{k}}, \pi_{v,\boldsymbol{k}})$ the conformal parametrization, as it involves naturally the conformal time. It differs from the fluid parametrization (28) by the nontrivial coefficient standing in front of the entire expression as well as the frequency that now depends on both background variables, $q$ and $p$.

### C. Solutions for perturbations

The two parametrizations described above, $(\phi, \pi_\phi)$ and $(v, \pi_v)$, being related by a canonical transformation, are physically equivalent and therefore it is sufficient to consider just one of them, e.g., the conformal one, in order to determine the dynamics of perturbations. It is also true at the quantum level [7] provided the background evolution is described by a classical or semiclassical trajectory.

Using the definition (11) to derive the power-law behavior of $q(\eta)$ in (10), the potential $z''/z$ in Eq. (36) is found to yield the specific form (38) (independently of the choice $z = q^{r_1}$ or $z = q^{r_2}$), so that the classical evolution of perturbation modes is

$$\frac{\mathrm{d}^2 v_{\boldsymbol{k}}}{\mathrm{d}\eta^2} + \left[wk^2 - \frac{2(1-3w)}{(1+3w)^2\eta^2}\right]v_{\boldsymbol{k}} = 0. \tag{40}$$

Clearly, the potential $\mathcal{V}_{\rm cl} \propto \eta^{-2}$ is singular at the singularity $\eta \to 0$. The solution can be expressed in terms of Hankel functions, namely

$$v_{\boldsymbol{k}}(\eta) = \sqrt{\eta}[c_1(\boldsymbol{k})H_\nu^{(1)}(\sqrt{w}k\eta) + c_2(\boldsymbol{k})H_\nu^{(2)}(\sqrt{w}k\eta)], \tag{41}$$

where $\nu = \frac{3(1-w)}{2(3w+1)}$ and $c_1(\boldsymbol{k})$, $c_2(\boldsymbol{k})$ are constants depending on the comoving wave vector $\boldsymbol{k}$ through the initial conditions; for isotropic initial conditions as those used for quantum vacuum fluctuation, they can depend only on the amplitude $k$ and not on the direction $\boldsymbol{k}/k$. The solution is finite but discontinuous at $\eta = 0$. Therefore, the comoving curvature $\Psi_{\boldsymbol{k}} \propto v_k/a$ in general blows up at $\eta = 0$ where the scale factor reaches the singularity $a \to 0$; see Ref. [21] for a full treatment of the relevant cases.

## IV. QUANTUM PERTURBATIONS

In the present section, we quantize the Hamiltonian (27) in the two classically equivalent parametrizations we introduced above. Next we apply some approximations in order to integrate the dynamics. We find that the two classically equivalent parametrizations lead to two unitarily inequivalent quantum theories. This dependence on parametrization is a natural consequence of the nonlinearity of the theory of gravity. The subsequent ambiguity is not a consequence of ordering of operators associated with non commutative sets at the quantum level, but rather the result of a choice of such sets. In fact, it follows directly from the fact that the quantum-regularized trajectories are semiquantum and not semiclassical.

Recall that Dirac's "Poisson bracket → commutator" quantization rule [22] works only for simplest observables. It is well-known that there exists no quantization of any given classical system that is an isomorphism between the Poisson and commutator algebras (there is actually one known exception that nevertheless is irrelevant in the present context, see [23] for an exhaustive review). As a result, a quantized observable is in general unitarily inequivalent if its quantization is made with a different choice of basic observables. Note that this "obstruction" is absent when quantum perturbations evolve linearly in classical or semiclassical backgrounds as in the latter case all the possible sets of basic variables are related by linear transformations that enjoy unique unitary representations consistent with Dirac's rule.

### A. Fluid parametrization

The canonical perturbation variables of the fluid parametrization satisfy the reality condition $\phi_{\boldsymbol{k}}^* = \phi_{-\boldsymbol{k}}$ and $\pi_{\phi,\boldsymbol{k}}^* = \pi_{\phi,-\boldsymbol{k}}$ and it is possible to promote their real and imaginary parts to canonical operators in $L^2[\mathbb{R}^2, \frac{i}{2}\mathrm{d}\phi_{\boldsymbol{k}}\mathrm{d}\phi_{\boldsymbol{k}}^*]$ for each direction $\boldsymbol{k}$. It is, however, more convenient to work with the Fock representation,

$$\phi_{\boldsymbol{k}} \mapsto \hat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}[a_{\boldsymbol{k}}\phi_k^*(\tau) + a_{-\boldsymbol{k}}^\dagger \phi_k(\tau)], \tag{42}$$

where the time-dependent mode functions $\phi_k(\eta)$ are assumed to be isotropic and $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ are fixed annihilation and creation operators that satisfy $[a_{\boldsymbol{k}_1}, a_{\boldsymbol{k}_2}^\dagger] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ (recall the compactness of space implies discrete eigenvectors $\boldsymbol{k}$). As shown later, it follows that the mode functions must satisfy a suitable normalization condition. Note that the whole evolution of the operators $\hat{\phi}_{\boldsymbol{k}}$ and $\hat{\pi}_{\phi,\boldsymbol{k}}$ in the Heisenberg picture is encoded into the mode functions.

Combining the background quantization with the quantization of perturbations, using the definition (15) of the classical power laws, yields the quantized Hamiltonian (28) in the fluid parametrization (henceforth dubbed F-parametrization)

$$\hat{H}^{(2)}_{\boldsymbol{k}} = \frac{1}{2}|\hat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\mathrm{Q}}}{2} w(1+w)^2 \left(\frac{\hat{Q}}{\gamma}\right)^{4r_1} k^2 |\hat{\phi}_{\boldsymbol{k}}|^2, \quad (43)$$

where $\mathfrak{L}_{\mathrm{Q}} = \mathfrak{l}(4r_1)$ [see Eq. (17a)] is a free parameter of the quantization.

### B. Conformal parametrization

We repeat the same quantization for the conformal parametrization (C-parametrization in what follows),

$$v_{\boldsymbol{k}} \mapsto \hat{v}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}[a_{\boldsymbol{k}}\bar{v}_k(\tau) + a^{\dagger}_{-\boldsymbol{k}} v_k(\tau)], \quad (44)$$

and obtain the quantum Hamiltonian derived from (34) as

$$\hat{H}^{(2)}_{\boldsymbol{k}} = \frac{1}{2}(1+w)\left(\frac{\hat{Q}}{\gamma}\right)^{2r_1} \mathfrak{M}_{\mathrm{Q}} H^{(2)}_{\boldsymbol{k},\mathrm{eff}}, \quad (45)$$

with

$$\hat{H}^{(2)}_{\boldsymbol{k},\mathrm{eff}} = |\hat{\pi}_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \frac{8\mathfrak{M}_{\mathrm{Q}}^{-1}}{9\hat{Q}^2}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2(1+w)^2}\left(\frac{\hat{Q}}{\gamma}\right)^{-4r_1}(\mathfrak{N}_{\mathrm{Q}}\hat{P}^2 + i\hbar\mathfrak{R}_{\mathrm{Q}}\hat{Q}^{-1}\hat{P} + \hbar^2\mathfrak{T}_{\mathrm{Q}}\hat{Q}^{-2})\right]|\hat{v}_{\boldsymbol{k}}|^2, \quad (46)$$

where $\mathfrak{M}_{\mathrm{Q}} = \mathfrak{l}(2r_1)$, $\mathfrak{N}_{\mathrm{Q}} = \mathfrak{a}(-2r_2)$, $\mathfrak{R}_{\mathrm{Q}} = \mathfrak{b}(-2r_2) = 2r_2\mathfrak{N}_{\mathrm{Q}}$ and $\mathfrak{T}_{\mathrm{Q}} = \mathfrak{c}(-2r_2)$ are free parameters in the quantization map [see Eqs. (17)]. Note that there are more free parameters and hence more quantization ambiguities in the C-parametrization.

## V. SEMIQUANTUM DYNAMICS

A general approach to solving the dynamics of quantum perturbations in quantum spacetime was recently given in [24]. In what follows, we assume the full state vector to be a product of background and perturbation states, i.e.,

$$|\psi(\tau)\rangle = |\psi_{\mathrm{B}}(\tau)\rangle \cdot |\psi_{\mathrm{P}}(\tau)\rangle. \quad (47)$$

The canonical formalism for cosmological perturbations has been developed under the assumption that the perturbations do not backreact on the background spacetime. Therefore, the dynamics of $|\psi_{\mathrm{B}}(\tau)\rangle$ should be determined independently of the state $|\psi_{\mathrm{P}}(\tau)\rangle$.

Given that the dynamics of the background state is fixed by $|\psi_{\mathrm{B}}\rangle \to |q(\tau), p(\tau)\rangle$, the dynamics of the perturbation state $|\psi_{\mathrm{P}}(\tau)\rangle$ can be deduced from the quantum action at second order $\mathcal{S}^{(0)+(2)} = \mathcal{S}_{\mathrm{B}} + \mathcal{S}_{\mathrm{P}}$

$$\mathcal{S}^{(0)+(2)} = \int \langle\psi(\tau)|\left(i\hbar\frac{\partial}{\partial\tau} - \hat{H}^{(0)} + \sum_{\boldsymbol{k}}\hat{H}^{(2)}_{\boldsymbol{k}}\right)|\psi(\tau)\rangle \mathrm{d}\tau, \quad (48)$$

with the state vector given by (47). Extracting the zeroth order action $\mathcal{S}_{\mathrm{B}}$, one finds

$$\mathcal{S}_{\mathrm{P}} = \int \langle\psi_{\mathrm{P}}|\left(i\hbar\frac{\partial}{\partial\tau} + \sum_{\boldsymbol{k}}\hat{H}^{(2)}_{\boldsymbol{k}}\right)|\psi_{\mathrm{P}}\rangle \mathrm{d}\tau, \quad (49)$$

and setting $|\psi_{\mathrm{P}}\rangle = \prod_{\boldsymbol{k}}|\psi_{\boldsymbol{k}}\rangle$ with $\langle\psi_{\boldsymbol{k}_1}|\psi_{\boldsymbol{k}_2}\rangle = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$, one gets the associated Schrödinger equation for each Fourier mode $|\psi_{\boldsymbol{k}}\rangle$ (up to an irrelevant phase factor), namely

$$i\hbar\frac{\partial}{\partial\tau}|\psi_{\boldsymbol{k}}\rangle = \tilde{H}_{\boldsymbol{k}}|\psi_{\boldsymbol{k}}\rangle, \quad (50)$$

where the operator $\tilde{H}_{\boldsymbol{k}} \equiv -\langle q,p|\hat{H}^{(2)}_{\boldsymbol{k}}|q,p\rangle$ is obtained from either (43) or (45) depending on the choice of parametrization. We discuss those in turn below.

### A. Fluid modes

In the F-parametrization case, the second-order Hamiltonian generating the dynamics of perturbations reads

$$\langle q,p|\hat{H}^{(2)}_{\boldsymbol{k}}|q,p\rangle = \frac{1}{2}|\hat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\mathrm{S}}}{2}w(1+w)^2\left(\frac{q}{\gamma}\right)^{4r_1}k^2|\hat{\phi}_{\boldsymbol{k}}|^2, \quad (51)$$

where the value of $\mathfrak{L}_{\mathrm{S}}$ depends on the value of $\mathfrak{L}_{\mathrm{Q}}$ and the family of coherent states used to approximate the background dynamics.

The Heisenberg equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\phi}_{\boldsymbol{k}} = -\hat{\pi}_{\phi,\boldsymbol{k}}, \quad (52a)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\pi}_{\phi,\boldsymbol{k}} = \mathfrak{L}_{\mathrm{S}} w(1+w)^2\left(\frac{q}{\gamma}\right)^{4r_1}k^2\hat{\phi}_{\boldsymbol{k}}, \quad (52b)$$

and it follows from (52a) that

$$\hat{\pi}_{\phi,\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}[a_{\boldsymbol{k}}\dot{\phi}^*_k(\tau) + a^{\dagger}_{-\boldsymbol{k}}\dot{\phi}_k(\tau)],$$

and hence the canonical commutation rule, namely $[\hat{\phi}_{-\boldsymbol{k}}, \hat{\pi}_{\phi,\boldsymbol{k}}] = i\hbar$, implies the normalization condition on the mode functions $\dot{\phi}_k\phi^*_k - \phi_k\dot{\phi}^*_k = 2i$. By combining the above equations, we may obtain the second-order dynamical equation for $\hat{\phi}_{\boldsymbol{k}}$, which must also be obeyed by the mode function $\phi_k$. We switch to the internal conformal clock given by Eq. (11) and rescale the mode functions, $v_k^{\rm F} = Z\phi_k$, where $v_k^{\rm F}$ is the Mukhanov-Sasaki variable. The superscript "F" indicates that its dynamics is generated by the fluid Hamiltonian. More specifically, we find that the dynamics of $v_k^{\rm F}$ generated by the Hamiltonian (51) reads

$$\frac{{\rm d}^2 v_k^{\rm F}}{{\rm d}\eta^2} + [k_{\rm F}^2 - \mathcal{V}_{\rm F}(\eta)]v_k^{\rm F} = 0, \tag{53}$$

with the effective wave number $k_{\rm F} \equiv \sqrt{\mathfrak{L}_{\rm S} w}k$, and the fluid potential given by

$$\mathcal{V}_{\rm F} = \frac{8}{9q^2Z^4}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2}\left[p^2 - \frac{3(1-w)\mathfrak{K}}{2q^2}\right]. \tag{54}$$

Note that for large $q$, i.e., away from the bounce, the quantum correction becomes negligible so that the semiquantum potential (53) approaches the classical one (34). Indeed, using $\dot{Z}/Z = r_1\dot{q}/q$ and $q' = \dot{q}/Z^2$, one finds

$$\frac{Z''}{Z} = \frac{r_1}{Z^4}\left[\frac{\ddot{q}}{q} - (1+r_1)\left(\frac{\dot{q}}{q}\right)^2\right],$$

and replacing the function $q(\tau)$ by the solution (25) for the background semiquantum trajectory, it is straightforward to check that, for all times, the potential $\mathcal{V}_{\rm F}$ can be given the familiar form $\mathcal{V}_{\rm F} = Z''/Z = (q^{r_1})''/q^{r_1}$. Since the semiquantum trajectory (25) is asymptotic to the classical one (10) for $\omega\tau \to \infty$, i.e., for $\eta \to \infty$, the fluid potential satisfies

$$\lim_{\eta\to\infty}\mathcal{V}_{\rm F}(\eta) = \mathcal{V}_{\rm cl}(\eta),$$

where $\mathcal{V}_{\rm cl}$ is given by (38); it is illustrated in Fig. 3.

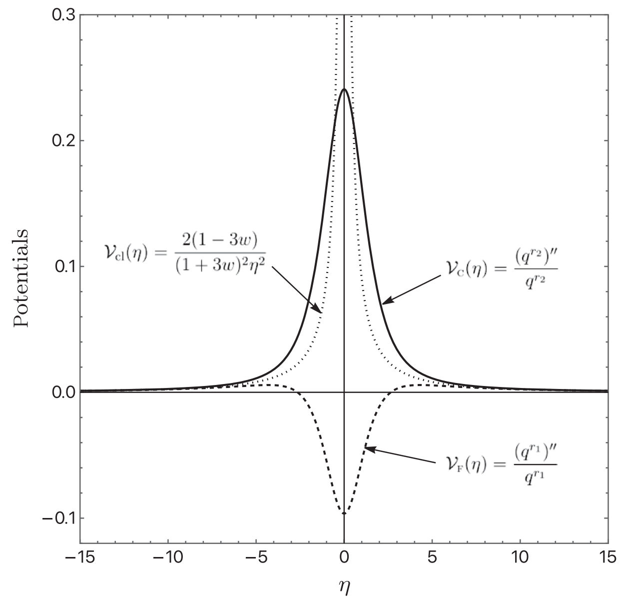


FIG. 3. The gravitational potentials $V_{\rm C}$ (full line), from (57), and $V_{\rm F}$ (dashed line), from (53), as functions of the conformal time $\eta$; the parameter values are chosen as $q_{\rm B} \to 1$, $\omega \to 1$, $\kappa_0 \to 1$ and $w = 0.2$ for the purpose of illustration. These potentials are deduced from the quantum fluid (51) and conformal (55) Hamiltonians and the classical Hamiltonian. They all asymptotically decay as $\eta^{-2}$ far from the bounce where they are well-approximated by their classical counterpart given by $V_{\rm cl} = \frac{2(1-3w)}{(1+3w)^2\eta^2}$ (dotted line) [cf. Eq. (40)].

## B. Conformal modes

The same procedure applied to the conformal parametrization yields

$$\langle q,p|\hat{H}_{\boldsymbol{k}}^{(2)}|q,p\rangle = \frac{1}{2}Z^2\mathfrak{M}_{\rm S}(|\hat{\pi}_{v,k}|^2 + \Omega_v^2|\hat{v}_k|^2), \tag{55}$$

with

$$\Omega_v^2 = wk^2 - \frac{8\mathfrak{M}_{\rm S}^{-1}}{9q^2Z^4}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2}\left(\mathfrak{N}_{\rm S}p^2 + \frac{\hbar^2\mathfrak{T}_{\rm S}}{q^2}\right), \tag{56}$$

where $\mathfrak{M}_{\rm S}$, $\mathfrak{N}_{\rm S}$, $\mathfrak{T}_{\rm S}$ depend on the family of coherent states used to approximate the background dynamics and on the values of $\mathfrak{M}_{\rm Q}$, $\mathfrak{N}_{\rm Q}$, $\mathfrak{R}_{\rm Q}$ and $\mathfrak{T}_{\rm Q}$ and $\mathfrak{N}_{\rm Q}$, respectively. For the following discussion, one should bear in mind that all the quantities $\mathfrak{K}_{\rm S}$, $\mathfrak{L}_{\rm S}$, $\mathfrak{M}_{\rm S}$, $\mathfrak{N}_{\rm S}$ and $\mathfrak{T}_{\rm S}$ are positive definite. The canonical commutation rule implies the normalization condition on the mode functions

$$\dot{v}_k v_k^* - v_k\dot{v}_k^* = 2i(1+w)\left(\frac{q}{\gamma}\right)^{-2r_1}\mathfrak{M}_{\rm S} = 2iZ^2\mathfrak{M}_{\rm S}.$$

After switching to the internal conformal clock, the normalization condition reads $v_k'v_k^* - v_kv_k^{*\prime} = 2i\mathfrak{M}_{\rm S}$ and the Hamiltonian (55) is found to generate the following dynamics of the mode function $v_k^{\rm C}$ (the subscript "C" now indicating that its dynamics is generated by the conformal Hamiltonian)

$$\frac{{\rm d}^2 v_k^{\rm C}}{{\rm d}\eta^2} + [\mathfrak{M}_{\rm S}^2 wk^2 - \mathfrak{M}_{\rm S}\mathfrak{N}_{\rm S}\mathcal{V}_{\rm C}(\eta)]v_k^{\rm C} = 0, \tag{57}$$

where the potential, shown in Fig. 4 for different numerical values of the relevant parameter, reads

$$\mathcal{V}_{\rm C} = \frac{8}{9q^2Z^4}\frac{(2\kappa_0)^2(1-3w)}{(1-w)^2}\left(p^2 + \frac{\hbar^2\mathfrak{T}_{\rm S}/\mathfrak{N}_{\rm S}}{q^2}\right), \tag{58}$$

whose limit for large $q$ yields back the classical case (37). The usual Mukhanov-Sasaki equation is recovered from (57) provided one defines a rescaled conformal time $\varsigma$ through $\varsigma = \sqrt{\mathfrak{M}_{\rm S}\mathfrak{N}_{\rm S}}\eta$, leading to

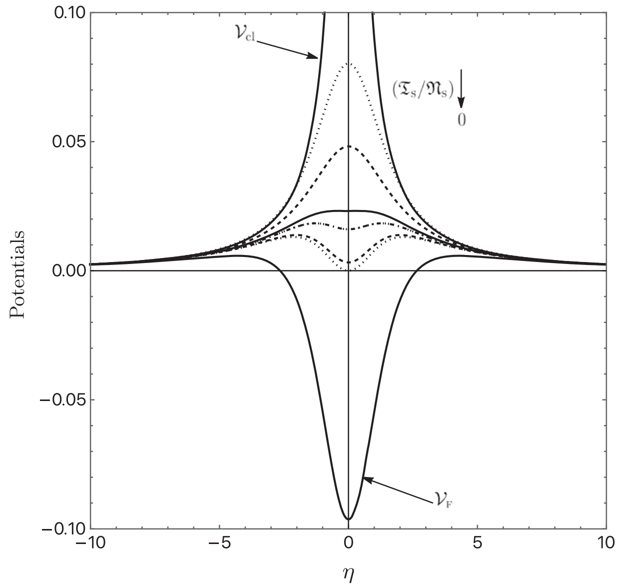


FIG. 4. Shape of the conformal potential $\mathcal{V}_{\rm C}$ (58) for $w = 0.2$ and various values of $\mathfrak{T}_{\rm S}/\mathfrak{N}_{\rm S}$, decreasing to 0 according to the arrow, compared with the classical $\mathcal{V}_{\rm cl}$ (38) and fluid $\mathcal{V}_{\rm F}$ (54) potentials. The special value (we assume $\mathfrak{K} \to 1$) $\mathfrak{T}_{\rm S}/\mathfrak{N}_{\rm S} = 3(1-w)/(1-3w) \simeq 0.375$ (not shown) corresponds to that in Fig. 3 for which $\mathcal{V}_{\rm C} = (q^{r_2})''/q^{r_2}$. Shown are $\mathfrak{T}_{\rm S}/\mathfrak{N}_{\rm S} = 0.125$ (dotted line), 0.075 (dashed), 0.036 (full), 0.025 (dot-dashed), 0.005 (dashed) and 0 (dotted). The full line represents a critical point above which the potential has only one maximum. For $\mathfrak{T}_{\rm S}/\mathfrak{N}_{\rm S} = 0$, the potential is minimum at the bounce where it vanishes.

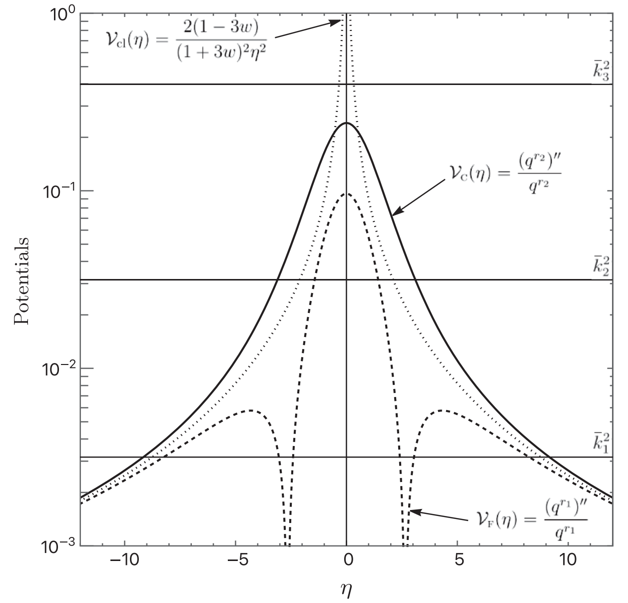


FIG. 5. Same as Fig. 3 in logarithmic scale for the potentials, with different wave numbers ($\bar{k}$ standing for either $k_{\rm F}$ or $k_{\rm C}$ depending on the case at hand), illustrating the various possible predictions. For $\bar{k} \sim \bar{k}_3$, the quantum potentials is not felt by the perturbations, and only the classical potential induce a nontrivial spectrum. In the region of wavelengths around $\bar{k} \sim \bar{k}_2$, the perturbations enter the potentials at different points, but the characteristic behavior is more or less comparable; one would expect in this regime to have different amplitudes and even perhaps power indices, but an overall similar shape. For $\bar{k} \sim \bar{k}_1$ on the other hand, the number of entries and exits of the perturbation in and out of the potentials $V_{\rm F}$ and $V_{\rm C}$ being different, predictions between the two models could radically differ, e.g., with superimposed oscillations changing the shape of the primordial power spectrum.

$$\frac{\mathrm{d}^2 v_k^{\rm C}}{\mathrm{d}\varsigma^2} + [k_{\rm C}^2 - \mathcal{V}_{\rm C}(\varsigma)] v_k^{\rm C} = 0, \tag{59}$$

as expected; in Eq. (59), the effective wave number is $k_{\rm C} \equiv k\sqrt{w\mathfrak{M}_{\rm S}/\mathfrak{N}_{\rm S}}$.

We have seen above that $\mathcal{V}_{\rm F} = (q^{r_1})''/q^{r_1}$. Let us see under what conditions the potential $\mathcal{V}_{\rm C}$ can also be put in a similar form $X''/X = (q^r)''/q^r$ for a given function $X(\eta) = q^r$ with a power $r$ to be determined. Straightforward calculation yields

$$\frac{X''}{X} = \frac{r}{Z^4}\left[\frac{\ddot{q}}{q} + (r - 2r_1 - 1)\left(\frac{\dot{q}}{q}\right)^2\right] = \frac{4(2\kappa_0)^2}{Z^4 q^2} r(r - 2r_1 - 1)\left[p^2 + \frac{\mathfrak{K}}{(r - 2r_1 - 1)q^2}\right],$$

where in the second equality we have made use of the semiquantum solution (25). In order to recover the classical limit (35), the power $r$ should satisfy $r(r - 2r_1 + 1) = \frac{2}{9}(1-3w)/(1-w)^2$, whose two roots happen to coincide with $r_1$ and $r_2$. Setting $r = r_1$ yields (54), with a negative coefficient in the $q^{-2}$ term (we assume $0 < w < 1$), as could have been anticipated. The second root $r = r_2$ yields instead a positive coefficient in the $q^{-2}$ term, and reproduces (58) if we demand that $w < \frac{1}{3}$ and

$$\frac{\mathfrak{T}_{\rm S}}{\mathfrak{N}_{\rm S}} = \frac{3\mathfrak{K}(1-w)}{1-3w} \Rightarrow \mathcal{V}_{\rm C} \to \frac{(q^{r_2})''}{q^{r_2}}. \tag{60}$$

Both potentials are shown in Figs. 3 and 5.

It is clear from (54) and (58) that the two equivalent parametrizations of the classical model induce two inequivalent quantum theories, as is clear from Figs. 3 and 5 showing a comparison of the respective gravitational potentials. The difference is perhaps even clearer when the gravitational potentials are given in the familiar form based in the configuration space and the semiclassical variable $q$ is raised to two distinct powers, i.e., $r_1 = \frac{3w-1}{3(1-w)}$ and $r_2 = \frac{2}{3(1-w)}$. In some sense these two parametrizations are exhaustive in regard to the quantization ambiguity as these are the only powers possible for theories that satisfy the classical limit, as follows from our discussion below (36).

The source of the ambiguity is the nonlinearity of the theory of gravity. Since the quantization concerns both the linear perturbations and the background variables, the transformation of the perturbation variables (31) is nonlinear, contrary to the situation of Ref. [7], and therefore, it leads to unitarily inequivalent theories.

In our framework, the non-equivalence is responsible for the discrepancy between the two semiquantum F-potential (54) and C-potential (58). The formula (33) that is used to derive the F-potential depends on the expectation value of $Z$ that is a function of $\hat{Q}$ only. On the other hand, the C-potential comes from the expectation value of a compound observable, involving both $\hat{Q}$ and $\hat{P}$, and given in (46). These two potentials cannot coincide because the classical relations between basic and compound observables do not apply to the expectation values of the respective operators due to the quantum uncertainty.

## VI. CONCLUSIONS

We have suggested a finite cosmological model in which quantum gravitational effects play a leading role, resolving the classically expected singularity to a bouncing scenario. Our model consists in adding to general relativity a perfect fluid with constant equation of state $w$. Classically, the FLRW solution initiates out of or contracts to a singularity at which the scale factor $a$ vanishes. The perturbations around such a background also tend to diverge at the singularity.

Upon quantizing the background, factor ordering ambiguities permit to add to the zeroth order Hamiltonian a repulsive potential term, whose strength is thus undetermined. Choosing the canonical ordering removes it altogether. The fact that the trajectories are nonsingular results from our definition of these trajectories as expectation values. For coherent states, that leads to Eq. (25). The ordering ambiguity also translates into the fact that the coefficients appearing in this equation, i.e., the minimum scale factor $q_{\mathrm{B}}$ and its acceleration $\omega$ at the bounce, are free parameters which cannot be calculated from first principles. In that sense, the ordering ambiguity is always present in our model and, at the perturbation level, is conveniently encoded in the free parameters $\mathfrak{L}_{\mathrm{Q}}$, $\mathfrak{M}_{\mathrm{Q}}$, $\mathfrak{N}_{\mathrm{Q}}$, $\mathfrak{R}_{\mathrm{Q}}$ and $\mathfrak{T}_{\mathrm{Q}}$. Assuming a coherent state to describe the evolution in terms of an actual spacetime, i.e., a trajectory $a(\tau)$ for the scale factor, one can then calculate a phase space trajectory which, thanks to the quantum effective potential, smoothly connects the contracting and expanding solutions, avoiding the singularity in the process.

Most model-building approaches would then identify these bouncing trajectories as semiclassical, and would then go on to quantize the perturbations on top. By doing so, one would then be allowed whatever canonical transformation on the perturbation variables, leading to classically and quantum mechanically undistinguishable theories.[2] Here however, we take seriously the quantum nature of the background time development and show that the classically harmless canonical transformations become unitarily inequivalent theories with potentially different physical predictions: the bouncing trajectories are semiquantum and not semiclassical.

Summarizing, we found that upon quantizing the background to regularize the classical singularity, one finds two qualitatively different perturbation theories. It is important to note that had the background dynamics been given by a classical or semiclassical trajectory, singular or nonsingular, the relation between the two quantum perturbation theories would be unitary as the change of perturbation variables would be given by a linear (time-dependent) canonical transformation. However, the introduction of a background wave function and the subsequent replacement of the background variables with the respective expectation values is not equivalent to the background following an actual trajectory. One should not be misled by the existence of semiquantum trajectories in Fig. 1, representing expectation values of $q = \langle \hat{Q} \rangle$ and $p = \langle \hat{P} \rangle$ only; they cannot be assumed to provide a semiclassical dynamics, and therefore cannot be used to determine the other expectation values that are involved in the transformation (31) between the two sets of perturbation variables.

In other words, in this instance, the notion of a classical or even semiclassical spacetime in which quantum perturbations evolve needs be replaced by a more general "quantum spacetime." Somehow, the perturbative expansion breaks down and the transformation between the sets of variables should be generalized to account for the uncertainties of the background in order to permit unambiguous predictions.

## ACKNOWLEDGMENTS

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## APPENDIX: AFFINE COHERENT STATES AND AFFINE QUANTIZATION

In what follows we discuss the affine coherent states and how they can be used as a means to implement affine quantization as well as to provide useful trajectories [11,25–27].

[2] The calculations we showed concern the scalar part of the perturbation, but is not restricted to it, the tensor component being also presumably affected by a similar ambiguity.

### 1. Coherent states and quantization

The background phase space $(q, p)$ is the half-plane that is not invariant under the usual group of $q$–and $p$–translations. For this reason the application of "canonical quantization" based on the unitary and irreducible representation of the group of translations, the Weyl-Heisenberg group, is problematic. It is however possible to consider a more general quantization that is based on any minimal group of canonical transformations that enjoys a nontrivial unitary representation, the so-called covariant integral quantization. In the case of the half-plane the natural choice is the affine group of a real line, $(q, p) \in \mathbb{R}^+ \times \mathbb{R}$,

$$(q', p')\circ(q, p) = \left(q'q, \frac{p}{q'} + p'\right). \tag{A1}$$

Its unitary, irreducible and square-integrable representation in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+, \mathrm{d}x)$ reads

$$\langle x|U(q, p)|\psi\rangle = \langle x|q, p\rangle = \frac{\mathrm{e}^{ipx/\hbar}}{\sqrt{q}}\psi\left(\frac{x}{q}\right), \tag{A2}$$

where $\psi(x) = \langle x|\psi\rangle \in \mathcal{H}$.

Let us consider a particular example of the covariant integral quantization that is based on coherent states. In the present case, they are the affine coherent states,

$$\mathbb{R}^+ \times \mathbb{R} \ni (q, p) \mapsto |q, p\rangle \coloneqq U(q, p)|\xi\rangle \in \mathcal{H}, \tag{A3}$$

where $|\xi\rangle$ is the so-called fiducial vector, a fixed normalized vector in Hilbert space such that $\mathcal{N} = \rho(0) < \infty$, with

$$\rho(\alpha) = \int \frac{|\xi(x)|^2}{x^{\alpha+1}}\mathrm{d}x,$$

and the operator $U(q, p)$ is given by Eq. (19). The resolution of unity is

$$\int \frac{\mathrm{d}q\mathrm{d}p}{2\pi\hbar\mathcal{N}}|q, p\rangle\langle q, p| = \mathbb{1}, \tag{A4}$$

as can be verified in a straightforward manner using Eq. (A2) and applying the above operator on two arbitrary states $\langle\phi_1|$ and $|\phi_2\rangle$:

$$\int \frac{\mathrm{d}q\mathrm{d}p}{2\pi\hbar\mathcal{N}}\langle\phi_1|q, p\rangle\langle q, p|\phi_2\rangle = \int \mathrm{d}x\phi_1^*(x)\phi_2(x) = \langle\phi_1|\phi_2\rangle,$$

using the usual closure relation

$$\int \mathrm{d}x|x\rangle\langle x| = \mathbb{1}$$

and the property

$$\delta(x - y) = \int \frac{\mathrm{d}p}{2\pi\hbar}\mathrm{e}^{ip(x-y)/\hbar}$$

for the Dirac distribution.

The affine coherent state quantization is obtained by substituting functions of $q$ and $p$ by

$$f(q, p) \mapsto A_f \coloneqq \int_{\mathbb{R}^+\times\mathbb{R}} \frac{\mathrm{d}q\mathrm{d}p}{2\pi\hbar\mathcal{N}}|q, p\rangle f(q, p)\langle q, p|, \tag{A5}$$

with $\mathcal{N}$ the normalization constant. Let us also introduce

$$\sigma(\alpha) = \int \left|\frac{\mathrm{d}\xi(x)}{\mathrm{d}x}\right|^2 \frac{\mathrm{d}x}{x^{\alpha+1}}, \tag{A6}$$

which is the same as $\rho$ with the function $\xi(x)$ replaced by its derivative $\xi'(x)$.

One may easily find the affine coherent state quantization (A5) of the following observables through (see, e.g., Appendixes of [27] or [28] for explicit computations)

$$A_1 = \mathbb{1}, \tag{A7a}$$

$$A_{q^\alpha} = \mathfrak{a}(\alpha)\hat{Q}^\alpha, \tag{A7b}$$

$$A_p = \hat{P}, \tag{A7c}$$

$$A_{q^\alpha p^2} = \mathfrak{a}(\alpha)\hat{Q}^\alpha\hat{P}^2 - i\alpha\hbar\mathfrak{a}(\alpha)\hat{Q}^{\alpha-1}\hat{P} + \mathfrak{c}(\alpha)\hbar^2\hat{Q}^{\alpha-2}, \tag{A7d}$$

where $\hat{Q}$ and $\hat{P}$ are the "position" and "momentum" operators on the half-line: Eqs. (A7b) and (A7c) are to be understood as $\langle x|A_{q^\alpha}|\phi\rangle = \mathfrak{a}(\alpha)x^\alpha\phi(x)$ and $\langle x|A_p|\phi\rangle = -i\hbar\mathrm{d}\phi/\mathrm{d}x$, where $\phi(x) \coloneqq \langle x|\phi\rangle$.

The parameters

$$\mathfrak{a}(\alpha) = \frac{\rho(\alpha)}{\rho(0)}$$

and

$$\mathfrak{c}(\alpha) = \frac{1}{2}\alpha(1-\alpha)\mathfrak{a}(\alpha) + \frac{\sigma(\alpha-2)}{\rho(0)}$$

are calculable for any real fiducial vector $\xi(x)$, which should be chosen such that $\mathfrak{a}(1) = 1$, i.e., $\rho(1) = \rho(0)$ in order to ensure that $A_q = \hat{Q}$ so that Eqs. (A7b) and (A7c) implement the required usual commutation relation $[A_q, A_p] = [\hat{Q}, \hat{P}] = i\hbar$.

From the above, it follows that the application of the affine quantization (A5) to the background Hamiltonian (8) yields

$$H^{(0)} \mapsto \hat{H}^{(0)} = 2\kappa(\hat{P}^2 + \hbar^2\mathfrak{c}_0\hat{Q}^{-2}), \tag{A8}$$

with $\mathfrak{c}_0 = \mathfrak{c}(0) = \sigma(-2)/\rho(0)$.

Furthermore, using again (15), one may easily calculate the various constants appearing in the perturbation Hamiltonians, namely

$$\mathfrak{L}_{\mathrm{Q}} = \frac{\rho(4r_1)}{\rho(0)} \tag{A9}$$

for (43), as well as

$$\begin{aligned} \mathfrak{M}_{\mathrm{Q}} &= \frac{\rho(2r_1)}{\rho(0)}, \\ \mathfrak{N}_{\mathrm{Q}} &= \frac{\rho(-2r_2)}{\rho(0)}, \\ \mathfrak{R}_{\mathrm{Q}} &= 2\hbar r_2 \mathfrak{N}_{\mathrm{Q}}, \end{aligned} \tag{A10}$$

and

$$\mathfrak{T}_{\mathrm{Q}} = -r_2(1+2r_2)\mathfrak{N}_{\mathrm{Q}} + \frac{\sigma(-2r_2-2)}{\rho(0)}, \tag{A11}$$

which appear in (45). Obviously, these parameters are to a large extent free as the affine quantization depends on the fiducial vector $|\xi\rangle$. One might think about the coherent state quantization based on the fiducial vector as a convenient method for parametrizing natural ordering ambiguities.

### 2. Coherent state expectation values

The most important application of the affine coherent states in the present work is to derive a useful trajectory description. As discussed around Eq. (19), one needs to ensure the so-called physical centering condition $\langle\hat{Q}\rangle = 1$, where the expectation value is taken in the fiducial state. This condition may not be satisfied by the state $|\xi\rangle$, already normalized to enforce the canonical commutation relation, and so we introduce a new real fiducial vector $|\tilde{\xi}\rangle$ and the associated moments $\tilde{\rho}(\alpha) = \int_{\mathbb{R}^+} \frac{\mathrm{d}x}{x^{\alpha+1}} |\tilde{\xi}|^2$ and $\tilde{\sigma}(\alpha) = \int_{\mathbb{R}^+} \frac{\mathrm{d}x}{x^{\alpha+1}} |\tilde{\xi}'|^2$. We find

$$\langle q,p|\hat{Q}^\alpha \hat{P}^2|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha p^2 + i\alpha\tilde{\rho}(-\alpha)q^{\alpha-1}p + \left[\tilde{\sigma}(-\alpha-1) + \frac{\alpha(1-\alpha)}{2}\tilde{\rho}(-\alpha+1)\right]q^{\alpha-2}, \tag{A12a}$$

$$\langle q,p|\hat{Q}^\alpha \hat{P}|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha p + i\frac{\alpha}{2}\tilde{\rho}(-\alpha)q^{\alpha-1}, \tag{A12b}$$

$$\langle q,p|\hat{Q}^\alpha|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha. \tag{A12c}$$

Note that the special case $\alpha = 0$ in (A12c) yields the normalization $\langle q,p|q,p\rangle = \tilde{\rho}(-1) = \langle\xi|\xi\rangle = 1$.

For the quantum Hamiltonian (A8), we introduce the following semiquantum Hamiltonian

$$H_{\mathrm{sem}} \coloneqq \langle q,p|\hat{H}^{(0)}|q,p\rangle = 2\kappa_0\left(p^2 + \frac{\hbar^2\mathfrak{K}}{q^2}\right), \tag{A13}$$

where the new constant $\mathfrak{K}$ is given by $\mathfrak{K} = \mathfrak{c}_0\tilde{\rho}(1) + \tilde{\sigma}(-2)$. As for perturbations, it is straightforward to determine the constant in (51), namely

$$\mathfrak{L}_{\mathrm{S}} = \mathfrak{L}_{\mathrm{Q}}\tilde{\rho}(-4r_1-1), \tag{A14}$$

whereas one gets

$$\begin{aligned} \mathfrak{M}_{\mathrm{S}} &= \mathfrak{M}_{\mathrm{Q}}\tilde{\rho}(-2r_1-1), \\ \mathfrak{N}_{\mathrm{S}} &= \mathfrak{N}_{\mathrm{Q}}\tilde{\rho}(2r_2-1), \end{aligned} \tag{A15}$$

and

$$\mathfrak{T}_{\mathrm{S}} = \mathfrak{N}_{\mathrm{Q}}\tilde{\sigma}(2r_2-1) + \mathfrak{T}_{\mathrm{Q}}\tilde{\rho}(2r_2+1) \tag{A16}$$

for those appearing in (55).

### 3. Fiducial vectors

For the sake of concreteness in the present discussion, let us consider some examples of fiducial vectors and the specific values of $\rho(\alpha)$, $\sigma(\alpha)$, $\tilde{\rho}(\alpha)$ and $\tilde{\sigma}(\alpha)$ that they produce. We use two distinct families of fiducial vectors, namely one for quantization and another one for the semi-quantum approximation. This is due to the fact that they satisfy special and distinct conditions. Namely, the fiducial vectors for quantization are such as to preserve the canonical commutation rule (on the half-line), whereas the fiducial vectors for semiquantum approximations are such as to yield the expectation values for the momentum and position operators in any coherent state, aligned with the phase space point to which a given coherent state is assigned.

We consider the following family of fiducial vectors for quantization

$$\xi_\nu(x) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp\left[-\frac{\nu}{2}\left(\ln x - \frac{3}{4\nu}\right)^2\right], \tag{A17}$$

where $\nu > 0$ is assumed, and for which we obtain the corresponding coefficients

$$\begin{aligned} \rho_\nu(\alpha) &= \exp\left[\frac{(\alpha-2)(\alpha+1)}{4\nu}\right], \\ \sigma_\nu(\alpha) &= \left[\frac{\nu}{2} + \left(\frac{\alpha+2}{2}\right)^2\right] \exp\left[\frac{\alpha(\alpha+3)}{4\nu}\right], \end{aligned} \tag{A18}$$

which are positive definite. As expected, one verifies that $\rho_\nu(1) = \rho_\nu(0) = \mathrm{e}^{-1/(2\nu)}$, as needed to ensure the correct commutation relation between the position variable and its associated canonical momentum. We also note that $\langle\xi|\hat{Q}|\xi\rangle = \rho_\nu(-2) = \mathrm{e}^{3/(2\nu)} \neq 1$, so the physical centering condition is not fulfilled by this fiducial state.

As for the semiquantum description, we consider the following family of fiducial vectors

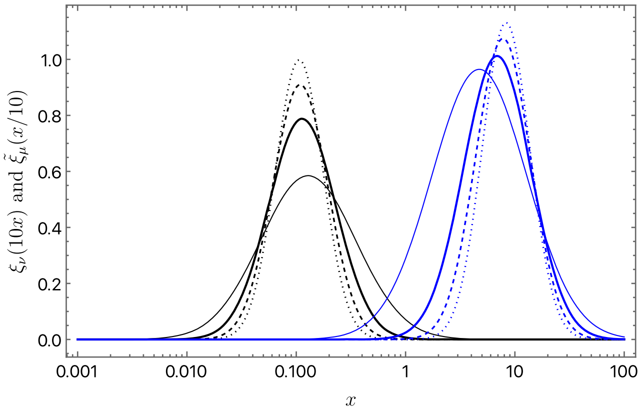


FIG. 6. Fiducial functions $\xi_\nu(10x)$ and $\tilde{\xi}_\mu(x/10)$ (blue), for $\nu$, $\mu = 1$ (thin line), 2 (full), 3 (dashed) and 4 (dotted). For better readability of the figure, the functions have been shifted so that $\xi_\nu$ appears centered around 0.1 and $\tilde{\xi}_\mu$ around 10. As functions of $x$, they should all be centered around $x = 1$.

$$\tilde{\xi}_\mu(x) = \left(\frac{\mu}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp\left[-\frac{\mu}{2}\left(\ln x + \frac{1}{4\mu}\right)^2\right], \quad \text{(A19)}$$

where now $\mu > 0$ is assumed. In this case, we obtain

$$\tilde{\rho}_\mu(\alpha) = \exp\left[\frac{(\alpha+1)(\alpha+2)}{4\mu}\right],$$
$$\tilde{\sigma}_\mu(\alpha) = \left[\frac{\mu}{2} + \left(\frac{\alpha+2}{2}\right)^2\right] \exp\left[\frac{(\alpha+3)(\alpha+4)}{4\mu}\right]. \quad \text{(A20)}$$

These are also positive definite as expected. It is now clear that $\tilde{\rho}_\mu(-2) = 1$, as expected for this description to satisfy the centering condition, but that now $\tilde{\rho}(1) = \mathrm{e}^{3/(2\mu)} \neq \mathrm{e}^{1/(2\mu)} = \tilde{\rho}(0)$ so that these fiducial vectors cannot be used for quantization. Some example functions $\xi_\nu$ and $\tilde{\xi}_\mu$ are displayed in Fig. 6.

The above relations permit to actually calculate the various coefficients appearing in the previous sections. First, one finds that $\mathfrak{c}_0 = \nu/2$, so that it suffices to demand $\nu \geq \frac{3}{2}$ to ensure self-adjointedness of the Hamiltonian (16). As for its semiclassical counterpart (24), one finds

$$\mathfrak{K} = \left(\frac{\nu}{2} + \frac{2\mu+1}{4}\right) \exp\left(\frac{3}{2\mu}\right),$$

whose minimum value $\mathfrak{K}_{\min}$ is reached for $\nu = 0$ and $\mu_{\min} = (3+\sqrt{21})/4 \approx 1.89$, at which point one has $\mathfrak{K}_{\min} \approx 2.64$.

Moving to the quantum corrections to the evolution of perturbations, we find

$$\frac{\mathfrak{T}_\mathrm{S}}{\mathfrak{N}_\mathrm{S}} = \left(\frac{1}{4} + \frac{\mu+\nu}{2}\right) \exp\left[\frac{17-9w}{6\mu(1-w)}\right],$$

so that the conformal potential can be cast into the usual $z''/z$ form if the equation

$$\left(\mu+\nu+\frac{1}{2}\right) \exp\left[\frac{17-9w}{6\mu(1-w)}\right] = \frac{3(1-w)}{1-3w}\left[\nu + \left(\mu+\frac{1}{2}\right)\exp\left(\frac{3}{2\mu}\right)\right]$$

has non trivial solutions for $\mu$, $\nu > 0$. This is solved for $\nu$ as a function of $\mu$ and $w$ through

$$\nu(w,\mu) = \frac{\exp\left(\frac{3}{2\mu}\right) - \frac{1-3w}{3(1-w)}\exp\left[\frac{17-9w}{6\mu(1-w)}\right]}{\frac{1-3w}{3(1-w)}\exp\left[\frac{17-9w}{6\mu(1-w)}\right] - 1}\left(\mu+\frac{1}{2}\right). \quad \text{(A21)}$$

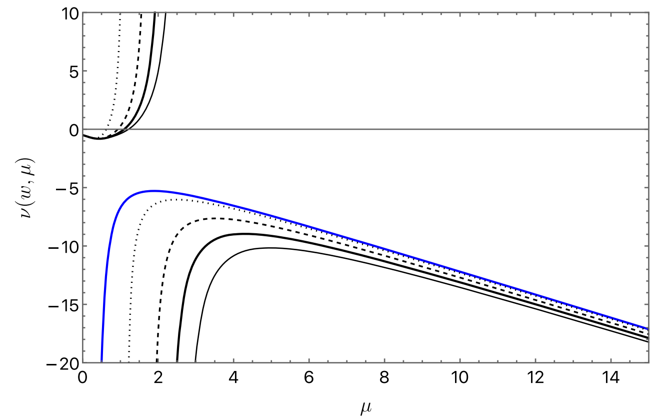


FIG. 7. Condition (A21) on $\nu(w,\mu)$ ensuring the potential $\mathcal{V}_\mathrm{C}$ in (59) takes the form $(q^{r_2})''/(q^{r_2})$; shown are $\nu(w,\mu)$ for $w = 0$ (thin line), $w = 0.1$ (full line), $w = 0.2$ (dashed line), $w = 0.3$ (dotted line) and $w = \frac{1}{3}$ (full blue line). As both $\mu$ and $\nu$ are positive definite, it is clear that for a given value of $w$, there is only a very limited range of $\mu$ satisfying the condition. For $w = \frac{1}{3}$, the positive branch disappears and there is no such solution.

Figure 7 illustrates the behavior of (A21) for various values of $w$. For the conformal radiation case $w = \frac{1}{3}$, Eq. (A21) may only be satisfied for $\nu < 0$, in contradiction to the assumption. As expected from the form (58) of the potential $\mathcal{V}_\mathrm{C}$, the limit $w = \frac{1}{3}$ yields an identically vanishing potential, and (60) is undefined unless $\mathfrak{K}$ vanishes, which does not happen with the basis used.

[1] P. Peter and J.-P. Uzan, *Primordial Cosmology*, Oxford Graduate Texts (Oxford University Press, Oxford, UK, 2013).

[2] D. Battefeld and P. Peter, A critical review of classical bouncing cosmologies, Phys. Rep. **571**, 1 (2015).

[3] R. Brandenberger and P. Peter, Bouncing cosmologies: Progress and problems, Found. Phys. **47**, 797 (2017).

[4] J. Grain and V. Vennin, Canonical transformations and squeezing formalism in cosmology, J. Cosmol. Astropart. Phys. 02 (2020) 022.

[5] C. Kiefer and M. Kramer, Quantum Gravitational Contributions to the Cosmic Microwave Background Anisotropy Spectrum, Phys. Rev. Lett. **108**, 021301 (2012).

[6] L. Chataignier and M. Krämer, Unitarity of quantum-gravitational corrections to primordial fluctuations in the Born-Oppenheimer approach, Phys. Rev. D **103**, 066005 (2021).

[7] J. Martin, *Inflationary Perturbations: The Cosmological Schwinger Effect* (Springer, Berlin, Heidelberg, 2008). 10.1007/978-3-540-74353-8_6

[8] P. Małkiewicz, Hamiltonian formalism and gauge-fixing conditions for cosmological perturbation theory, Classical Quantum Gravity **36**, 215003 (2019).

[9] B. F. Schutz, Perfect fluids in general relativity: Velocity potentials and a variational principle, Phys. Rev. D **2**, 2762 (1970).

[10] B. F. Schutz, Hamiltonian theory of a relativistic perfect fluid, Phys. Rev. D **4**, 3559 (1971).

[11] H. Bergeron, A. Dapor, J. P. Gazeau, and P. Małkiewicz, Smooth big bounce from affine quantization, Phys. Rev. D **89**, 083522 (2014).

[12] P. Małkiewicz, What is dynamics in quantum gravity?, Classical Quantum Gravity **34**, 205001 (2017).

[13] S. T. Ali and M. Englis, Quantization methods: A guide for physicists and analysts, Rev. Math. Phys. **17**, 391 (2005).

[14] J. P. Gazeau and R. Murenzi, Covariant affine integral quantization(s), J. Math. Phys. (N.Y.) **57**, 052102 (2016).

[15] J. R. Klauder, The Affine quantum gravity program, Classical Quantum Gravity **19**, 817 (2002).

[16] J. R. Klauder, *Enhanced Quantization: Particles, Fields and Gravity* (World Scientific, Hackensack, 2015).

[17] P. Małkiewicz, P. Peter, and S. D. P. Vitenti, Quantum empty Bianchi I spacetime with internal time, Phys. Rev. D **101**, 046012 (2020).

[18] A. Vilenkin, Quantum cosmology and the initial state of the Universe, Phys. Rev. D **37**, 888 (1988).

[19] F. Olver, D. Lozier, R. Boisvert, and C. Clark, *The NIST Handbook of Mathematical Functions* (Cambridge University Press, New York, NY, 2010).

[20] J. Martin and P. Peter, On the "Causality Argument" in Bouncing Cosmologies, Phys. Rev. Lett. **92**, 061301 (2004).

[21] P. Peter and N. Pinto-Neto, Has the Universe always expanded?, Phys. Rev. D **65**, 023513 (2001).

[22] P. A. M. Dirac, The fundamental equations of quantum mechanics, Proc. R. Soc. A **109**, 642 (1925).

[23] M. J. Gotay, Obstruction results in quantization theory, J. Nonlinear Sci. **6**, 469 (1996).

[24] P. Małkiewicz and A. Miroszewski, Dynamics of primordial fields in quantum cosmological spacetimes, Phys. Rev. D **103**, 083529 (2021).

[25] R. Fresneda and J. P. Gazeau, Integral quantizations with POVM and some applications, J. Phys. Conf. Ser. **597**, 012037 (2015).

[26] J.-P. Gazeau, *Coherent States in Quantum Physics* (Wiley-VCH, Berlin, 2009).

[27] H. Bergeron, J. P. Gazeau, and P. Małkiewicz, Primordial gravitational waves in a quantum model of big bounce, J. Cosmol. Astropart. Phys. 05 (2018) 057.

[28] C. R. Almeida, H. Bergeron, J. P. Gazeau, and A. C. Scardua, Three examples of quantum dynamics on the half-line with smooth bouncing, Ann. Phys. (Amsterdam) **392**, 206 (2018).