Primordial perturbations and inflation in a holography inspired Gauss-Bonnet cosmology

Nicolas R. Bertini[1,*] Neven Bilić[1,2,†] and Davi C. Rodrigues[1,‡]

[1]*Núcleo de Astrofísica e Cosmologia & Departamento de Física, Universidade Federal do Espírito Santo.*
Av. Fernando Ferrari 514, 29075-910 Vitória, ES, Brazil
[2]*Division of Theoretical Physics, Rudjer Bošković Institute, 10002 Zagreb, Croatia*

We consider an action for gravity that, in addition to the Einstein-Hilbert term, contains a function of the Ricci scalar and the Gauss-Bonnet invariant. The specific form of the function considered is motivated by holographic cosmology. At background level the field equations imply modified Friedmann equations of the same form as those in the holographic cosmology. We calculate the cosmological perturbations and derive the corresponding power spectra assuming a general k-inflation. We find that the resulting power spectra differ substantially from those obtained in both holographic and standard cosmology. The estimated spectral index and tensor-to-scalar ratio are confronted with the Planck results.

I. INTRODUCTION

A modified Gauss-Bonnet (MGB) gravity [1–5] is a class of modified gravity models in which the gravitational action is a general function of two variables: the Ricci scalar R and the Gauss-Bonnet invariant

$$\mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}. \tag{1}$$

The functional dependence on R and $\mathcal{G}$ can be further constrained by physical requirements. In a cosmological context it is natural to require that the second Friedmann equation is linear in $\dot{H}$. Then, in addition to the Einstein-Hilbert term the gravitational Lagrangian can contain a function of R and $\mathcal{G}$ of the form $f(J)$ depending only on one invariant [6]

$$J = \frac{1}{\sqrt{12}}\left(-R + \sqrt{R^2 - 6\mathcal{G}}\right)^{1/2} \tag{2}$$

normalized so that $J^2 = H^2$ for a spatially flat cosmology at background level. The Friedmann equations in this case become very simple: the left-hand side of the first Friedmann equation, in addition to the usual H^2 term, contains a general function of H.

In this paper we study in particular a MGB gravity with $f(J) \propto J^4$. In the following we will refer to this theory as the MGB model. In this model one obtains the cosmology equivalent to that on the holographic braneworld [7–10] at

*nicolas.bertini@cosmo-ufes.org
†bilic@irb.hr
‡davi.rodrigues@cosmo-ufes.org

background level. This equivalence between the two models poses a natural question if the equivalence goes beyond background cosmology. In particular, it would be of considerable interest to check if the two models produce similar spectra of the primordial cosmological perturbations.

At this point it is worth mentioning a few related works in the context of inflation. In Ref. [11] inflationary models were studied with arbitrary functions of H added to the left-hand side of the first Friedmann equation. Chackraborty *et al.* [12] have studied inflation in a model with the Gauss-Bonnet term coupled to a scalar field. Basilacos *et al.* [13] have shown that a modification of the Friedmann equation with a quartic term $\propto H^4$ is obtained in a string theory inspired model with a Kalb-Ramond term in the Lagrangian. In spite of some similarities, these examples are not equivalent to the model considered here.

Inflation in the modified gravity models in which the gravitational Lagrangian is a function $F(R, \mathcal{G})$ have been studied in Refs. [14,15]. De Laurentis *et al.* [14] have studied inflation in a kind of Starobinski extended model of the type $F(R, \mathcal{G}) = -R + aR^2 + b\mathcal{G}$ where a and b are constants. Odintsov *et al.* [15] have studied inflation in models of the type $F(R, \mathcal{G}) = -R + a\mathcal{G}^\alpha$ where $\alpha \neq 1$. In both Refs. [14,15] the action does not contain matter fields and inflation is driven solely by the geometry. In our approach, in contrast, inflation is driven by a scalar field coupled to the modified gravity with Lagrangian of the type $F(R, \mathcal{G}) = -R + (\ell^2/2)J^4$ where the invariant J is given by (2) and ℓ is a constant of dimension of length.

In a recent paper [10] we presented the calculations of the cosmological perturbations for a k-essence field theory in the holographic braneworld in the context of inflation. We demonstrated that the perturbations produce the power

023509-1

spectra as in the standard k-inflation in general relativity. Here we calculate the perturbations in the MGB model with background equations identical to the holographic cosmology and find a substantial departure from the general-relativistic results.

As a side issue, it is important to address the ghost instability problem in MGB models which is somewhat controversial. It was argued that the modified gravity models in which the gravitational action is a general function $F(R, \mathcal{G})$ are ghost free [16,17]. However, in a recent paper [18] it was demonstrated that, with a few exceptions, there is an instability in the scalar sector of $F(R, \mathcal{G})$ models. We present a brief review of these issues in Appendix D, where we also point out why the analysis of Ref. [18] does not apply to the model considered here.

The remainder of the paper is organized as follows. In Sec. II we introduce the MGB model and using the scalar-tensor representation formalism we derive the background field equations from which we derive the corresponding Friedmann equations. In Sec. III we derive the spectra of the cosmological perturbations for the MGB model with k-essence. In Sec. IV we calculate the power spectra and spectral indices. Concluding remarks are given in Sec. V. In Appendix A we justify some approximations made in Sec. III. In Appendices B and C we present details of the calculations of the scalar and tensor perturbations, respectively. Appendix D is devoted to ghost issues in general $F(R, \mathcal{G})$ theories and to the particular case considered here.

II. FIELD EQUATIONS IN THE MGB MODEL

A. The action

Consider the MGB action of the form

$$S_{\text{MGB}} = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_{\text{N}}} (-R + f(J)) + \mathcal{L} \right], \quad (3)$$

where G_{N} is the Newtonian constant and f is a smooth function of the invariant J defined by (2). We will assume that the value of G_{N} is provided by the measurements in the Solar System since the modifications of gravity should be relevant only for short distances. Matter is represented by a Lagrangian $\mathcal{L}$ as a general function of the scalar field θ and kinetic term

$$X \equiv g^{\mu\nu} \theta_{,\mu} \theta_{,\nu}. \quad (4)$$

This type of scalar field theories, dubbed k-essence [19,20], is very general and includes the canonical scalar field theory as a particular case. A k-essence is dynamically equivalent to a generally nonisentropic and nonbarotropic potential fluid flow, whereas a purely kinetic k-essence is equivalent to a barotropic potential flow [21,22].

For a general Friedmann-Lemaître-Robertson-Walker (FLRW) metric with line element

$$ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu$$
$$= dt^2 - a^2(t) \left[d\chi^2 + \frac{\sin^2(\sqrt{\kappa}\chi)}{\kappa} (d\vartheta^2 + \sin^2\vartheta d\varphi^2) \right], \quad (5)$$

one finds

$$J^2 = H^2 + \frac{\kappa}{a^2}, \quad (6)$$

where $\kappa = 1, -1$, or 0 for closed, open hyperbolic, or open flat space, respectively. Then applying the Euler-Lagrange formalism (for some efficient methods see Refs. [6,23]), we find a modified first Friedmann equation in the form

$$J^2 + \frac{1}{6} \left(f(J) - H \frac{\partial f}{\partial H} \right) = \frac{8\pi G_{\text{N}}}{3} \rho. \quad (7)$$

Hence, the left-hand side is a function of a and H only and the second Friedmann equation will be linear in $\dot{H}$. The above equation extends the $\kappa = 0$ result of Ref. [6] to arbitrary κ values (see also Ref. [23]). In the next section, the Friedmann equations are derived directly from the field equations with a specific function $f(J)$.

An interesting particular case is obtained for

$$f(J) = \frac{1}{2} \ell^2 J^4, \quad (8)$$

where ℓ is a coupling constant of dimension of length. In this case the Friedman equation (7) takes the form obtained in the spatially flat holographic cosmology [7,8,10] if we identify the constant ℓ with the AdS$_5$ curvature radius. In the following, we will study the action (3) with (8), i.e.,

$$S = \frac{1}{16\pi G_{\text{N}}} \int d^4x \sqrt{-g} F(R, \mathcal{G}) + \int d^4x \sqrt{-g} \mathcal{L}, \quad (9)$$

where

$$F(R, \mathcal{G}) = -R + \frac{\ell^2}{288} \left(-R + \sqrt{R^2 - 6\mathcal{G}} \right)^2. \quad (10)$$

B. Scalar-tensor representation

It is well known that $f(R)$ gravity can be described by a dual scalar-tensor action with a single scalar (for a review see, e.g., [24]). More general extended gravity theories, as is the action (9), may need additional scalars [25,26]. Here we follow the approach of [26] to find the dual action. For a general gravitational action with an arbitrary dependence on some metric invariants X^a (e.g., R and $\mathcal{G}$)

$$S_1 = \int d^4x \sqrt{-g} F(X_a), \qquad (11)$$

we can form a dual action by making use of a Legendre transformation

$$S_2 = \int d^4x \sqrt{-g}(\psi_a X^a - V(\psi^a)), \qquad (12)$$

where ψ_a are scalar fields that satisfy the following relations:

$$\psi_a = \frac{\partial F}{\partial X^a}; \qquad X^a = \frac{\partial V}{\partial \psi_a}. \qquad (13)$$

For the action (9) with (10) the dual version is

$$S_D = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}(\psi_1 R + \psi_2 \mathcal{G} - V(\psi_1, \psi_2))$$
$$+ \int d^4x \sqrt{-g} \mathcal{L}, \qquad (14)$$

where

$$\psi_1 = \frac{\partial F(R, \mathcal{G})}{\partial R} = -1 - \frac{\ell^2(-R + \sqrt{R^2 - 6\mathcal{G}})^2}{144\sqrt{R^2 - 6\mathcal{G}}}, \qquad (15)$$

$$\psi_2 = \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}} = \frac{\ell^2}{48}\left(-1 + \frac{R}{\sqrt{R^2 - 6\mathcal{G}}}\right). \qquad (16)$$

Then, the second set of equations in (13) reads

$$R = \frac{\partial V}{\partial \psi_1}, \qquad \mathcal{G} = \frac{\partial V}{\partial \psi_2}, \qquad (17)$$

and by integrating these we obtain

$$V(\psi_1, \psi_2) = -\frac{3(\psi_1 + 1)^2(\psi_2 + \ell^2/48)}{2\psi_2^2}. \qquad (18)$$

The variation of S_D with respect to the metric leads to modified Einstein's equations

$$-\psi_1 G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}V(\psi_1, \psi_2) - g_{\mu\nu}\Box\psi_1 + \nabla_\mu\nabla_\nu\psi_1$$
$$+ 4(\Box\psi_2)G_{\mu\nu} + 2(\nabla_\mu\nabla_\nu\psi_2)R - 4(\nabla_\rho\nabla_\mu\psi_2)R^\rho_\nu$$
$$- 4(\nabla_\rho\nabla_\nu\psi_2)R^\rho_\mu + 4g_{\mu\nu}(\nabla_\rho\nabla_\sigma\psi_2)R^{\sigma\rho}$$
$$- 4(\nabla^\rho\nabla^\sigma\psi_2)R_{\mu\rho\nu\sigma} = 8\pi G_N T_{\mu\nu}, \qquad (19)$$

where the energy-momentum tensor $T_{\mu\nu}$ is associated with the matter Lagrangian $\mathcal{L}$. Note that the variation of $\mathcal{G}$ in the above expression is in agreement with Ref. [27]. One can

easily check that the variation with respect to ψ_1 and ψ_2 yields a pair of equations equivalent to (15) and (16).

C. Background equations

Now we specify the background metric to the FLRW form (5) and we assume

$$T^\mu_\nu = \text{diag}(\rho, -p, -p, -p). \qquad (20)$$

Then, using the modified Einstein equations (19) we obtain the following modified Friedmann equations:

$$3\left(H^2 + \frac{\kappa}{a^2}\right) - \frac{\ell^2}{4}\left(H^2 + \frac{\kappa}{a^2}\right)\left(3H^2 - \frac{\kappa}{a^2}\right) = 8\pi G_N \rho, \qquad (21)$$

$$\dot{H}\left[1 - \frac{\ell^2}{6}\left(\frac{\kappa}{a^2} + 3H^2\right)\right] - \frac{\kappa}{a^2}\left[1 - \frac{\ell^2}{6}\left(H^2 - \frac{\kappa}{a^2}\right)\right]$$
$$= -4\pi G_N(p + \rho). \qquad (22)$$

Of course, Eq. (21) agrees with Eq. (7) for $f(J)$ given by (8). It is easy to show that Eqs. (21) and (22) imply

$$\dot{\rho} + 3H(p + \rho) = 0, \qquad (23)$$

which also follows from energy-momentum conservation

$$T^{\mu\nu}_{;\nu} = 0. \qquad (24)$$

In the following we adopt the usual assumption that the early Universe is spatially flat. Then, Eqs. (21) and (22) with $\kappa = 0$ reduce to

$$H^2 - \frac{\ell^2}{4}H^4 = \frac{8\pi G_N}{3}\rho, \qquad (25)$$

$$\dot{H}(1 - \ell^2 H^2/2) = -4\pi G_N(p + \rho), \qquad (26)$$

precisely as in the spatially flat holographic cosmology [7,8,10].

The pressure p and energy density ρ are derived from $\mathcal{L}$ using the usual prescription

$$p = \mathcal{L}, \qquad \rho = 2X\mathcal{L}_{,X} - \mathcal{L}, \qquad (27)$$

where the kinetic term X is defined in (4) and the subscript $_{,X}$ denotes a partial derivative with respect to X. The energy-momentum tensor is then given by

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}, \qquad (28)$$

where

$$u_\mu = \frac{\theta_{,\mu}}{\sqrt{X}}. \tag{29}$$

The MGB cosmology has interesting properties. Solving the first Friedmann equation (25) as a quadratic equation for H^2 we find

$$H^2 = \frac{2}{\ell}\left(1 \pm \sqrt{1 - \frac{8\pi G_N \ell^2}{3}\rho}\right). \tag{30}$$

Now, by demanding that Eq. (30) reduces to the standard Friedmann equation in the low density limit, i.e., in the limit when $G_N \ell^2 \rho \ll 1$, we are led to keep only the $(-)$ sign solution in (30) and discard the $(+)$ sign solution as unphysical. Then, it follows that the physical range of the Hubble expansion rate is between zero and the maximal value $H_{max} = \sqrt{2}/\ell$ corresponding to the maximal energy density $\rho_{max} = 3/(8\pi G_N \ell^2)$ [7,11]. Assuming no violation of the weak energy condition $p + \rho \geq 0$, the expansion rate will, according to (26), be a monotonously decreasing function of time.

Our ambition here is by no means an attempt to explain the very beginning of the Universe. Nevertheless, it is worth noting that if the evolution starts from $t = 0$ with an initial $H_i \leq H_{max}$ the initial energy density and cosmological expansion scale will be both finite. Hence, as already noted by Gao [6], in the modified cosmology described by the Friedmann equations (25) and (26), the big bang singularity is avoided.

The expansion of the early Universe is conveniently described using the so-called slow-roll parameters. We use the following recursive definition of the slow-roll parameters [28,29]:

$$\varepsilon_{i+1} = \frac{\dot{\varepsilon}_i}{H\varepsilon_i}, \tag{31}$$

starting with

$$\varepsilon_1 = -\frac{\dot{H}}{H^2}. \tag{32}$$

The beginning of inflation is characterized by the slow-roll regime with slow-roll parameters satisfying $\varepsilon_i \ll 1$.

D. Speed of sound

The adiabatic speed of sound is given by

$$c_s^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_\theta = \frac{p_{,X}}{\rho_{,X}} = \frac{p_{,X}}{p_{,X} + 2Xp_{,XX}} = \frac{p + \rho}{2X\rho_{,X}}. \tag{33}$$

In the slow-roll regime, the sound speed deviates slightly from unity and may be expressed in terms of the slow-roll parameters ε_i. First, by making use of the definition (32)

and modified Friedman equations (25), (26) with (27), we can express the variable X in the slow-roll regime as

$$X = -\frac{2p(2 - h^2)}{3p_{,X}(4 - h^2)}\varepsilon_1 + \mathcal{O}(\varepsilon_i^2), \tag{34}$$

where we have abbreviated

$$h \equiv \ell H. \tag{35}$$

Then from (33) we find

$$c_s^2 = 1 + \frac{4(2 - h^2)}{3(4 - h^2)}\frac{p\, p_{,XX}}{p_{,X}^2}\varepsilon_1 + \mathcal{O}(\varepsilon_i^2). \tag{36}$$

For example, in the tachyon model with Lagrangian $\mathcal{L} = -V\sqrt{1 - X}$ one finds [9]

$$c_s^2 = 1 - \frac{4(2 - h^2)}{3(4 - h^2)}\varepsilon_1 + \mathcal{O}(\varepsilon_i^2). \tag{37}$$

III. PERTURBATIONS IN MGB GRAVITY

Here we derive the spectra of the cosmological perturbations for the MGB cosmology with matter represented by a general k-essence. We shall closely follow Garriga and Mukhanov [30] and adjust their formalism to account for the modification of the Einstein equations.

A. Scalar perturbations

Assuming a spatially flat background with line element (5) with $\kappa = 0$, we introduce the perturbed line element in the Newtonian gauge

$$ds^2 = (1 + 2\Psi)dt^2 - (1 - 2\Phi)a^2(t)(dr^2 + r^2 d\Omega^2). \tag{38}$$

Inserting the above metric components in the field equations (19) we obtain a set of equations for Φ and Ψ derived in Appendix B. The relevant equations are (B21), (B22), and the off-diagonal part of (B23). Owing to $\delta T^i_j = \delta^i_j \delta p$ the off-diagonal part of Eq. (B23) in momentum space can be written as

$$\left[1 - \frac{h^2}{3}\left(\frac{H^2}{2\dot{H}} - \frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H\dot{H}} - \frac{\ddot{H}^2}{\dot{H}^3} + \frac{\dddot{H}}{2\dot{H}^2}\right)\right]\Phi$$
$$- \left[1 - \frac{h^2}{6}\left(\frac{H\ddot{H}}{\dot{H}^2} - \frac{H^2}{\dot{H}} - 2\right)\right]\Psi + \frac{h^2}{9H^2}\frac{k^2}{a^2}\Phi + \frac{h^2}{18\dot{H}}\frac{k^2}{a^2}\Psi$$
$$- \frac{h^2}{3}\left(\frac{H}{\dot{H}} - \frac{1}{H}\right)\dot{\Phi} - \frac{h^2 H}{6\dot{H}}\dot{\Psi} = 0, \tag{39}$$

where $h = \ell H$. Hence, the slip parameter defined in momentum space as

$$\eta \equiv \frac{\Phi}{\Psi} \tag{40}$$

is in general a function of k and t and can be calculated numerically for a specific inflation model. However, making use of the slow-roll parameters will prove helpful to develop a model independent estimate on η at horizon crossing (i.e., $k = aH$). First, with the help of (31) and (32), Eq. (39) with $k = aH$ becomes

$$\left(1 + \frac{3 + \varepsilon_1(2 - 3\varepsilon_2) - 3\varepsilon_2^2 + 3\varepsilon_2\varepsilon_3}{18\varepsilon_1}h^2\right)\Phi$$
$$- \left(1 - \frac{2 - 3\varepsilon_2}{18\varepsilon_1}h^2\right)\Psi + \frac{1 + \varepsilon_1}{3\varepsilon_1 H}h^2\dot{\Phi}$$
$$+ \frac{1}{6\varepsilon_1 H}h^2\dot{\Psi} = 0. \tag{41}$$

In the slow-roll regime, it is reasonable to assume

$$\dot{f} \simeq \mathcal{O}(\varepsilon_1)Hf, \tag{42}$$

where f stands for an arbitrary smooth and slow varying function of time. For example, $\dot{H} = -\varepsilon_1 H^2$, $\dot{\varepsilon}_1 = \varepsilon_2 H\varepsilon_1$, etc. In view of the above relation, we introduce arbitrary parameters ε_Φ and ε_Ψ of order $\mathcal{O}(\varepsilon_1)$ and write

$$\dot{\Phi} = \varepsilon_\Phi H\Phi, \qquad \dot{\Psi} = \varepsilon_\Psi H\Psi. \tag{43}$$

Then we find

$$\eta = \frac{18\varepsilon_1 - (2 - 3\varepsilon_2 + 3\varepsilon_\Psi)h^2}{18\varepsilon_1 - [(-2 + 3\varepsilon_2 - 6\varepsilon_\Phi)\varepsilon_1 + 3(-1 + \varepsilon_2^2 - \varepsilon_2\varepsilon_3 - 2\varepsilon_\Phi)]h^2}. \tag{44}$$

The above expression is exact on the proviso that Φ and Ψ satisfy (43). Note that $\eta \to 1$ as $h \to 0$, as expected. Now we make an approximation by taking all epsilons to be nearly equal, i.e., $\varepsilon_1 \simeq \varepsilon_2 \simeq \varepsilon_3 \simeq \varepsilon_\Phi \simeq \varepsilon_\Psi \simeq \varepsilon$. Then we obtain

$$\eta \simeq \frac{18\varepsilon - 2h^2}{18\varepsilon + (3 + 8\varepsilon + 3\varepsilon^2)h^2}. \tag{45}$$

The above function $\eta = \eta(\varepsilon, h)$ has a single minimum and a single maximum which yields the lower and upper bounds on η as

$$-\frac{2}{3} < \eta < 1. \tag{46}$$

The maximum is found for $h \to 0$, while the minimum for $\varepsilon \to 0$. If we require $\eta > 0$, Eq. (45) implies

$$\varepsilon > \frac{h^2}{9}. \tag{47}$$

For sufficiently small ε we have

$$\eta \simeq -\frac{2}{3} + \frac{2}{9}(45 + 8h^2)\frac{\varepsilon}{h^2} + O(\varepsilon^2). \tag{48}$$

In the intermediate slow-roll regime ($\varepsilon_i \sim h^2$) one can calculate η numerically for a specific model of k-essence. However, it is possible to obtain a rough model independent estimate in the intermediate slow-roll regime assuming as above $\varepsilon_1 \simeq \varepsilon_2 \simeq \varepsilon_3 \simeq \varepsilon_\Phi \simeq \varepsilon_\Psi \simeq \varepsilon$. Let ε_h denote the value of ε close to h^2. Then

$$\eta|_{\varepsilon \sim h^2} \simeq \frac{16}{21 + 8\varepsilon_h + 3\varepsilon_h^2}, \tag{49}$$

and hence, we have $\eta|_{\varepsilon \sim h^2} > 0$. Moreover, assuming that inflation ends when $\varepsilon \sim 1$, then $\eta|_{\varepsilon \sim h^2} > 1/2$.

The above estimates rely on the assumption that all of the epsilons are nearly equal. Nonetheless, it provides a simple and illustrative analytical description. In the following we will consider yet another approximation: we will adopt the simplification that during inflation η can be taken to be a constant between 0 and 1.

B. Scalar power spectrum

Using the definition (40) we can express the remaining perturbation equations in terms of Φ and η. The perturbations of the stress tensor components δT_ν^μ are induced by the perturbations of the scalar field $\theta(t, x) = \theta(t) + \delta\theta(t, x)$ and the perturbation of the metric. Using the energy conservation (23) and the definition (4) of X one finds

$$\delta T_0^0 = \frac{p + \rho}{c_s^2}\left[\left(\frac{\delta\theta}{\dot{\theta}}\right)^{\cdot} - \Psi\right] - 3H(p + \rho)\frac{\delta\theta}{\dot{\theta}}, \tag{50}$$

$$\delta T_i^0 = (p + \rho)\left(\frac{\delta\theta}{\dot{\theta}}\right)_{,i}, \tag{51}$$

where the adiabatic sound speed c_s is defined by (33). Using (51), Eq. (B22) becomes

$$(2 - h^2)(\dot{\Phi} + H\Psi) - \frac{h^2}{9}\left(\frac{H}{\dot{H}} + \frac{\ddot{H}}{\dot{H}^2} - \frac{4}{H}\right)\frac{\nabla^2\Phi}{a^2}$$
$$+ \frac{h^2}{9\dot{H}}\frac{\nabla^2(\dot{\Phi} + H\Psi)}{a^2} = 8\pi G_N(p + \rho)\frac{\delta\theta}{\dot{\theta}}. \tag{52}$$

Multiplying this by $3H$ and adding to (B21) with (50) we obtain

$$2\left(1-\frac{h^2}{6}\right)\frac{\nabla^2\Phi}{a^2} - \frac{Hh^2}{3\dot{H}}\frac{\nabla^2(\dot{\Phi}+H\Psi)}{a^2}$$
$$-\frac{h^2}{3}\left(\frac{H}{\dot{H}}+\frac{\ddot{H}}{\dot{H}^2}-\frac{4}{H}\right)\frac{\nabla^2\Phi}{a^2} - \frac{h^2}{9\dot{H}}\frac{k^2\nabla^2\Phi}{a^4}$$
$$= 8\pi G_N \frac{p+\rho}{c_s^2}\left[\left(\frac{\delta\theta}{\dot{\theta}}\right)^{\cdot} - \Psi\right]. \tag{53}$$

Here, in the last term on the left-hand side we have made a replacement $\nabla^2 \to -k^2$. Next, employing the slow-roll condition (42) we neglect $\dot{\Phi}$ in the second term, use the horizon crossing relation $k = aH$, replace Ψ by Φ/η and approximate η by a constant, as discussed at the end of Sec. III A. Then, by making use of the definitions (31) and (32), from (53) we obtain

$$\gamma\frac{\nabla^2\Phi}{a^2} = 4\pi G_N \frac{p+\rho}{c_s^2}\left[\left(\frac{\delta\theta}{\dot{\theta}}\right)^{\cdot} - \frac{\Phi}{\eta}\right], \tag{54}$$

where

$$\gamma = 1 + \frac{h^2}{6}\left(1+\frac{\epsilon_2}{\epsilon_1}\right) + \left(\frac{2}{9}+\frac{1}{6\eta}\right)\frac{h^2}{\epsilon_1}. \tag{55}$$

This is our first basic equation. The second equation is obtained from (B22) in which we replace $\nabla^2 \to -a^2H^2$. Then we find

$$(\alpha\dot{\Phi} + \beta H\Phi)_{,i} = 4\pi G_N \delta T_i^0, \tag{56}$$

where

$$\alpha = 1 - \frac{h^2}{6} + \frac{h^2}{18}\frac{1}{\epsilon_1}, \tag{57}$$

$$\beta = \frac{1}{\eta} - h^2\left(\frac{1}{9}+\frac{1}{2\eta}\right) - \frac{h^2}{18}\left(1-\frac{1}{\eta}\right)\frac{1}{\epsilon_1} - \frac{h^2}{18}\frac{\epsilon_2}{\epsilon_1}. \tag{58}$$

Next, by noting that $\beta/\alpha = \mathcal{O}(1)$ and employing the slow-roll condition (42) we can neglect the first term in brackets on the left-hand side of (56). Furthermore we use

$$(a\Phi)^{\cdot} = aH\Phi + a\dot{\Phi} \simeq aH\Phi \tag{59}$$

and finally obtain

$$\beta(a\Phi)^{\cdot} \simeq 4\pi G_N a(p+\rho)\frac{\delta\theta}{\dot{\theta}}. \tag{60}$$

Now, we can proceed in a way similar to Ref. [30] (for more details see also [10] and the Appendix of [9]). Introducing

$$\tilde{c}^2 = \frac{\gamma}{\beta}c_s^2, \tag{61}$$

Eqs. (54) and (60) can be put in the form

$$\left(\frac{\delta\theta}{\dot{\theta}}\right)^{\cdot} = \frac{\Phi}{\eta} + \frac{\tilde{c}^2}{4\pi G_N a^2(p+\rho)}\beta\nabla^2\Phi, \tag{62}$$

$$(a\Phi)^{\cdot} = 4\pi G_N a(p+\rho)\frac{1}{\beta}\frac{\delta\theta}{\dot{\theta}}. \tag{63}$$

As shown in Appendix A, we can neglect the first term on the right-hand side of Eq. (62). With this, we find two equations

$$a(H\xi)^{\cdot} = z^2\tilde{c}^2\chi, \tag{64}$$

$$a\dot{\chi} = z^{-2}H\nabla^2\xi, \tag{65}$$

where

$$\xi = \frac{a\Phi}{4\pi G_N H}, \qquad \chi = \frac{\delta\theta}{\dot{\theta}} \tag{66}$$

and

$$z = \frac{a}{\tilde{c}}\sqrt{\frac{p+\rho}{\beta}} = \frac{aH}{c_s}\sqrt{\frac{(1-h^2/2)\epsilon_1}{4\pi G_N\gamma}}. \tag{67}$$

In conformal time $\tau = \int dt/a$ Eqs. (64) and (65) yield a second order differential equation

$$v'' - \tilde{c}^2\nabla^2 v - \frac{z''}{z}v = 0, \tag{68}$$

where

$$v = z\chi. \tag{69}$$

The function v is related to the gauge invariant quantity

$$\zeta = \Phi + H\frac{\delta\theta}{\dot{\theta}}, \tag{70}$$

introduced in Ref. [30]. Indeed, using (66), (69), and (70) we have

$$v = z\left(\frac{\zeta}{H} - \frac{4\pi G_N}{a}\xi\right) \simeq \frac{z\zeta}{H}, \tag{71}$$

where we have neglected the second term in brackets being of higher order in ϵ_i as shown in Appendix A. The quantity ζ measures the spatial curvature of comoving (or constant-θ) hypersurfaces.

As usual, Eq. (68) is solved in momentum space where it reads

$$v_k'' + \left(\tilde{c}^2 k^2 - \frac{z''}{z} \right) v_k = 0. \tag{72}$$

The quantity z''/z can be easily calculated up to the second order in ε_i. However, as we have systematically neglected the terms of order $\mathcal{O}(\varepsilon_i^2)$ it is consistent to keep only the dominant contribution to z''/z. In the slow-roll regime one can use the relation [31]

$$\tau = -\frac{1 + \varepsilon_1}{aH} + \mathcal{O}(\varepsilon_1^2), \tag{73}$$

which follows from the definition (32) expressed in terms of the conformal time. Keeping the terms up to the first order one finds

$$\frac{z''}{z} = \frac{1}{\tau^2} (\nu^2 - 1/4) + \mathcal{O}(\varepsilon_i^2), \tag{74}$$

where

$$\nu^2 = \frac{9}{4} + 3 \left(1 + \frac{h^2}{2 - h^2} \right) \varepsilon_1 + 3\varepsilon_2. \tag{75}$$

We look for a solution to (72) which satisfies the positive frequency asymptotic limit

$$\lim_{\tau \to -\infty} v_k = \frac{e^{-i\tilde{c}k\tau}}{\sqrt{2\tilde{c}k}}. \tag{76}$$

Then the properly normalized solution to (68) which up to a phase agrees with (76) is

$$v_k = \frac{\sqrt{\pi}}{2} (-\tau)^{1/2} H_\nu^{(1)}(-\tilde{c}k\tau), \tag{77}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind of rank ν. In the limit of the de Sitter background all ε_i vanish so $\nu = 3/2$ in which case the solution to (72) is given by

$$v_k = \frac{e^{-i\tilde{c}k\tau}}{\sqrt{2\tilde{c}k}} \left(1 - \frac{i}{\tilde{c}k\tau} \right). \tag{78}$$

Applying the standard canonical quantization [32] the field v_k is promoted to an operator and the power spectrum of the field $\zeta_k = v_k/z$ is obtained from the two-point correlation function

$$\langle \hat{\zeta}_k \hat{\zeta}_{k'} \rangle = \langle \hat{v}_k \hat{v}_{k'} \rangle / z^2 = (2\pi)^3 \delta(\boldsymbol{k} + \boldsymbol{k}') |\zeta_k|^2. \tag{79}$$

The dimensionless spectral density

$$\mathcal{P}_S(k) = \frac{k^3}{2\pi^2} |\zeta_k|^2 = \frac{k^3 H^2}{2\pi^2 z^2} |v_k|^2, \tag{80}$$

with z given by (67), characterizes the primordial scalar fluctuations. Next, we evaluate the scalar spectral density at the horizon crossing, i.e., for a wave number satisfying $k = aH$. Following Refs. [29,33] we make use of the expansion of the Hankel function in the limit $\tilde{c}k\tau \to 0$

$$H_\nu^{(1)}(-\tilde{c}k\tau) \simeq -\frac{i}{\pi} \Gamma(\nu) \left(\frac{-\tilde{c}k\tau}{2} \right)^{-\nu}, \tag{81}$$

where the conformal time $\tau < 0$ and k is the comoving wave number. Using this we find at the lowest order in ε_1 and ε_2

$$\mathcal{P}_S \simeq \frac{G_N H^2}{\pi c_s \varepsilon_1} \frac{(\beta/\gamma)^{3/2} \gamma}{1 - h^2/2} \left[1 - \left(2 + 2K - \ln\frac{\beta}{\gamma} \right) \varepsilon_1 \right. \\ \left. - \left(2K - \ln\frac{\beta}{\gamma} \right) \left(\varepsilon_2 + \frac{h^2}{2 - h^2} \varepsilon_1 \right) \right], \tag{82}$$

where $K = \gamma_E - 2 + \ln 2 \simeq -0.730$ and γ_E is the Euler constant.

It is worth comparing this expression with the standard k-inflation result [30]

$$\mathcal{P}_S \simeq \frac{G_N H^2}{\pi c_s \varepsilon_1} [1 - 2(1 + K)\varepsilon_1 - K\varepsilon_2]. \tag{83}$$

In the regime where $h^2 \ll \varepsilon_i, \beta \to 1$, and $\gamma \to 1$, we recover the standard result apart from a difference by a factor of 2 in the ε_2 correction in square brackets. The reason for this discrepancy is due to the linear dependence of z on ε_1 as opposed to $\sqrt{\varepsilon_1}$ dependence in the standard case. Although the field equations of MGB gravity become identical to the field equations of general relativity (GR) in the limit $h \to 0$, MGB gravity is appreciably different from GR for small ε_1 (when ρ is expected to be large). Hence, GR need not have been recovered if one first expands in ε_1 and then takes the limit $h \to 0$, as in the mentioned regime $h^2 \ll \varepsilon_i$.

In the ultra-slow-roll regime where $\varepsilon_i \ll h^2$ we find a substantial enhancement with respect to the standard result by the factor $\gamma \simeq (4\eta + 3)h^2/(18\eta\varepsilon_1)$.

C. Tensor perturbations

The tensor perturbations are related to the production of gravitational waves during inflation. The metric perturbations are defined as

$$ds^2 = dt^2 - a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \tag{84}$$

where h_{ij} is traceless and transverse. Inserting the metric components in the field equations (19) yields an equation for the perturbation h_{ij} which we derive in Appendix C. Assuming as usual no contribution from matter we write the equation for h_{ij}, Eq. (C21), in the form

$$A\ddot{h}_{ij} + B3H\dot{h}_{ij} + DH^2 h_{ij} - C\frac{\nabla^2 h_{ij}}{a^2} = 0, \tag{85}$$

where A, B, C, and D are functions of H and its derivatives which we can be expressed in terms of ε_i. Using (31) and (32) we find

$$A = 1 + \frac{h^2}{6\varepsilon_1}(1 + \varepsilon_2), \qquad (86)$$

$$B = 1 + \frac{h^2}{6\varepsilon_1}\left(1 - \frac{2}{3}\varepsilon_1 + \frac{2}{3}\varepsilon_2 + \frac{8}{3}\varepsilon_1^2 - \varepsilon_2^2 - \frac{2}{3}\varepsilon_1\varepsilon_2 + \frac{1}{3}\varepsilon_2\varepsilon_3\right), \qquad (87)$$

$$C = 1 + \frac{h^2}{6\varepsilon_1}(1 - \varepsilon_2^2 - \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3), \qquad (88)$$

$$D = \frac{h^2}{3\varepsilon_1}\varepsilon_2. \qquad (89)$$

We now proceed as in Sec. III B and divide (85) by A

$$\ddot{h}_{ij} + \frac{B}{A}3H\dot{h}_{ij} + \frac{D}{A}H^2 h_{ij} - \frac{C}{A}\frac{\nabla^2 h_{ij}}{a^2} = 0. \qquad (90)$$

At the beginning and at the end of inflation the coefficient D/A tends to zero and B/A and C/A both tend to unity. At quadratic order in ε_i we find

$$\frac{B}{A} = 1 - \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 + \left(\frac{8}{3} + \frac{4(h^2 - 9)}{h^4}\right)\varepsilon_1^2$$
$$- \frac{2}{3}\varepsilon_2^2 + \frac{2}{h^2}\varepsilon_1\varepsilon_2 + \frac{1}{3}\varepsilon_2\varepsilon_3, \qquad (91)$$

$$\frac{C}{A} = 1 - \varepsilon_2 - \frac{36}{h^4}\varepsilon_1^2 + \left(-1 + \frac{6}{h^2}\right)\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3, \qquad (92)$$

$$\frac{D}{A} = 2\varepsilon_2 - 2\varepsilon_2^2 - \frac{12}{h^2}\varepsilon_1\varepsilon_2. \qquad (93)$$

Now we proceed by solving Eq. (90) in the usual way. Keeping the linear order in ε_i, Eq. (90) becomes

$$\ddot{h}_{ij} + H(3 - 2\varepsilon_1 - \varepsilon_2)\dot{h}_{ij} + 2H^2\varepsilon_2 h_{ij} - (1 - \varepsilon_2)\frac{\nabla^2 h_{ij}}{a^2} = 0. \qquad (94)$$

To solve this one uses the standard Fourier decomposition in conformal time τ

$$h_{ij}(\tau, \mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3 k e^{i\mathbf{k}\mathbf{x}}\sum_s h_k^s(\tau)e_{ij}^s(k), \qquad (95)$$

where the polarization tensor e_{ij}^s satisfies $k^i e_{ij}^s = 0$, and $e_{ij}^s e_{ij}^{s'} = 2\delta_{ss'}$ with comoving wave number k and two polarizations $s = +, \times$. The amplitude $h_k^s(\tau)$ then satisfies

$$h_k'' + 2aHh_k' - (2\varepsilon_1 + \varepsilon_2)aHh_k' + 2a^2 H^2 \varepsilon_2 h_k$$
$$+ (1 - \varepsilon_2)k^2 h_k = 0, \qquad (96)$$

where we have suppressed the dependence on s for simplicity bearing in mind that we have to sum over two polarizations in the final expression. Note that the third term may be neglected as it is suppressed by a factor $\mathcal{O}(\varepsilon_i^2)$ with respect to the last term. This may be seen by estimating the ratio $aHh_k'/(k^2 h_k)$. Employing the trick (42) we estimate $\dot{h}_k \simeq \mathcal{O}(\varepsilon_1)Hh_k$ and find

$$\frac{aHh_k'}{k^2 h_k} \simeq \frac{a^2 H^2}{k^2}\mathcal{O}(\varepsilon_1) \simeq \mathcal{O}(\varepsilon_1). \qquad (97)$$

In the second equality we have used the value

$$k \simeq aH \qquad (98)$$

near the horizon crossing. Thus, neglecting the suppressed term and introducing a canonically normalized amplitude

$$v_k = \frac{a}{16\pi G_N}h_k \qquad (99)$$

we obtain the equation

$$v_k'' + \left(k^2 - \frac{a''}{a} + a^2 H^2 \varepsilon_2\right)v_k = 0. \qquad (100)$$

This equation is of the same form as (72) with $c_s = 1$ and z''/z replaced by $a''/a - a^2 H^2 \varepsilon_2$. As before, using the relations (73) and (98) we find a properly normalized solution

$$v_k = \frac{\sqrt{\pi}}{2}(-\tau)^{1/2}H_\nu^{(1)}(-k\tau), \qquad (101)$$

with

$$\nu^2 = 9/4 + 3\varepsilon_1 - \varepsilon_2. \qquad (102)$$

The spectral density of the primordial tensor fluctuations is then given by

$$\mathcal{P}_T(k) = \frac{k^3}{\pi^2}|h_k|^2 = \frac{k^3}{\pi^2}\left|\frac{16\pi G_N}{a}v_k\right|^2, \qquad (103)$$

with v_k given by (101). Then, at the horizon crossing, using the approximation (81) we find

$$\mathcal{P}_T \simeq \frac{16GH^2}{\pi}\left[1 - 2(1 + K)\varepsilon_1 + \frac{2}{3}K\varepsilon_2\right]. \qquad (104)$$

IV. SCALAR SPECTRAL INDEX AND TENSOR-TO-SCALAR RATIO

The scalar spectral index n_S and tensor to scalar ratio r are given by

$$n_S - 1 = \frac{d \ln \mathcal{P}_S}{d \ln k} \simeq \frac{1}{H(1 - \varepsilon_1)} \frac{d \ln \mathcal{P}_S}{dt}, \qquad (105)$$

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_S}, \qquad (106)$$

where $\mathcal{P}_S$ and $\mathcal{P}_T$ are evaluated at the horizon crossing. The second equality in (105) is obtained with the help of (73) and the horizon crossing relation $k = aH$.

To be consistent with our approximation, in the calculation of n_S and r we keep only the lowest order corrections to the leading term. From (82) and (104) we find at linear order

$$n_S = 1 - 2\left(1 + \frac{2}{2 - h^2}\right)\varepsilon_1 - 2\varepsilon_2 + \mathcal{O}(\varepsilon_i^2) \qquad (107)$$

and

$$r = \frac{144(2 - h^2)}{h^2} \frac{\eta(4\eta + 3)^{1/2}}{(1 - \eta)^{3/2}} \varepsilon_1^2 [1 + a\varepsilon_1 + b\varepsilon_2 + \mathcal{O}(\varepsilon_i^2)], \qquad (108)$$

where

$$a = \frac{2h^2}{2 - h^2} K + \frac{2}{2 - h^2} \ln \frac{4\eta + 3}{1 - \eta} - \frac{9(\eta^2 + 11\eta + 9)}{h^2(4\eta + 3)(1 - \eta)} + \frac{3(7\eta^2 + 43\eta + 27)}{2(4\eta + 3)(1 - \eta)} + \frac{2(2 - h^2)}{3(4 - h^2)} \frac{pp_{,XX}}{p_{,X}^2}, \qquad (109)$$

$$b = \frac{8}{3} K + \ln \frac{4\eta + 3}{1 - \eta} + \frac{3\eta(3\eta + 4)}{2(4\eta + 3)(1 - \eta)}. \qquad (110)$$

For comparison, it is worth quoting the results we have obtained in holographic cosmology (HC) [10]

$$r|_{HC} = 16\varepsilon_1 \left[1 + K\varepsilon_2 + \frac{2(2 - h^2)}{3(4 - h^2)} \frac{pp_{,XX}}{p_{,X}^2} \varepsilon_1\right] \qquad (111)$$

and

$$n_S|_{HC} = 1 - 2\varepsilon_1 - \varepsilon_2 - \left(2 + \frac{8h^2}{3(4 - h^2)^2} \frac{pp_{,XX}}{p_{,X}^2}\right)\varepsilon_1^2$$
$$- \left(3 + 2K + \frac{2(2 - h^2)}{3(4 - h^2)} \frac{pp_{,XX}}{p_{,X}^2}\right)\varepsilon_1\varepsilon_2$$
$$- K\varepsilon_2\varepsilon_3, \qquad (112)$$

which, in the limit $h^2 \to 0$, coincide with the results obtained in the standard k-inflation [33] in general relativistic cosmology. Clearly, there is a significant deviation from the standard cosmology: the leading term in (108) is suppressed by a factor

$$\frac{9(2 - h^2)}{h^2} \frac{\eta(4\eta + 3)^{1/2}}{(1 - \eta)^{3/2}} \varepsilon_1$$

compared with (111) and the first order corrections to n_S in (107) are enhanced roughly by a factor of 2 compared with (112). It is interesting to note that a similar suppression of the leading term in r is obtained in a recent model [34] based on $f(R)$ gravity.

Our analysis and the results obtained so far have been basically model independent. To make a comparison with observation, e.g., to plot r versus n_S, we need to specify a model. A model which can be easily treated is the tachyon condensate [35] which has been extensively studied in the context of inflation [29,36–45]. Tachyon models are of particular interest as in these models inflation is driven by the tachyon field originating in M or string theory. The basics of the tachyon condensation are contained in an effective field theory [35] with Lagrangian of the Dirac-Born-Infeld (DBI) form

$$\mathcal{L}_{DBI} = -\ell^{-4} V(\theta/\ell) \sqrt{1 - g^{\mu\nu} \theta_{,\mu} \theta_{,\nu}}. \qquad (113)$$

The dimensionless potential V is a positive function of θ with a unique local maximum at $\theta = 0$ and a global minimum at $\theta = \infty$ at which V vanishes. A simple potential which satisfies the above requirements is the exponential potential

$$V = e^{-\omega\theta}, \qquad (114)$$

where ω is a parameter of dimension of mass. This potential has been studied in Ref. [9] in the context of holographic braneworld inflation.

For the tachyon model with exponential potential we have

$$\frac{pp_{,XX}}{p_{,X}^2} = -1 \qquad (115)$$

and

$$\varepsilon_2 \simeq 2\varepsilon_1 \left(1 - \frac{2h^2}{(2 - h^2)(4 - h^2)}\right), \qquad (116)$$

where $h \equiv \ell H$ is a monotonously decreasing function of time according to the second Friedmann equation (26). Owing to (30), the initial value of h at $t = 0$ must satisfy the restriction

$$h_{\mathrm{i}} \leq \sqrt{2}. \tag{117}$$

The choice of ω and h_{i} affects the e-fold number defined as

$$N = \int_0^{t_0} H dt, \tag{118}$$

where t_0 is the duration of the slow-roll regime fixed by the requirement $\varepsilon_1(t_0) = 1$. Hence, N is an implicit function of ω and h_{i} only. Our basic results (107) and (108) also depend on the slip parameter η which, according to (39), is generally a function of k and t. Our estimate at the horizon crossing allows us to assume that η, being a smooth function of t, is roughly constant during the slow-roll regime. For simplicity, in the following, we will treat η as a free parameter with values in the interval $0 < \eta < 1$.

The numerical calculations proceed as follows. For chosen ω, h_i, and η we evolve our background with t ($0 < t < t_0$) to get $h(t)$ and $\varepsilon_1(t)$ and produce $h = h(\varepsilon_1)$ as a parametric function. For each η and initial h_i the value of t_0 is fixed by $\varepsilon_1(t_0) = 1$. This gives N as a function of ω for each fixed η and h_i. This function can be numerically inverted to obtain $\omega = \omega(N)$. In this way, for fixed h_i and η we can produce a set of curves $r = r(n_{\mathrm{S}})$ each labeled by a value of N. Similarly, for fixed N and h_i we can produce another set of curves $r = r(n_{\mathrm{S}})$ each labeled by a value of η.

In Fig. 1 the theoretical plot for fixed initial value $h_{\mathrm{i}} = 1$ and $\eta = 0.5$ is superimposed on the observational constraints taken from the Planck Collaboration 2018 [46]. The parameter ω is allowed to vary so that the e-fold number N varies in the range $50 \leq N \leq 90$. The central point where the lines $N = 50$ and $N = 90$ cross corresponds to $n_{\mathrm{S}} = 0.965$ and $r = 0.0205$, in excellent agreement with observations. Similarly, the theoretical plot for varying
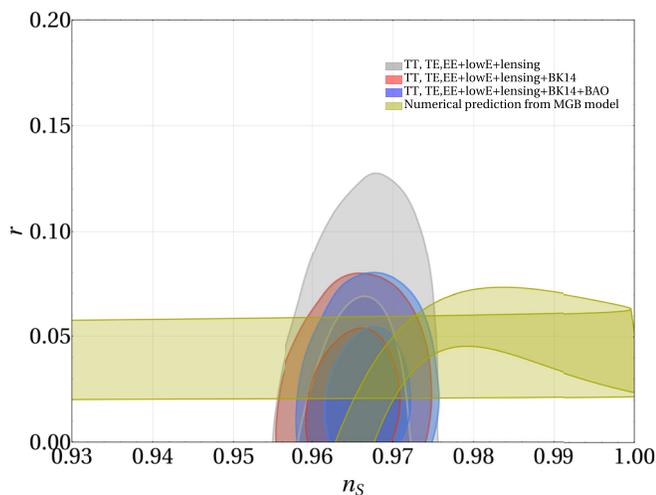
FIG. 1. r versus n_{s} diagram with observational constraints from Ref. [46]. The yellow shaded strips represent theoretical predictions for fixed $h_{\mathrm{i}} = 1$ and $\eta = 0.5$ and N ranging from 50 (top boundary of the strips) to 90 (bottom boundary of the strips).
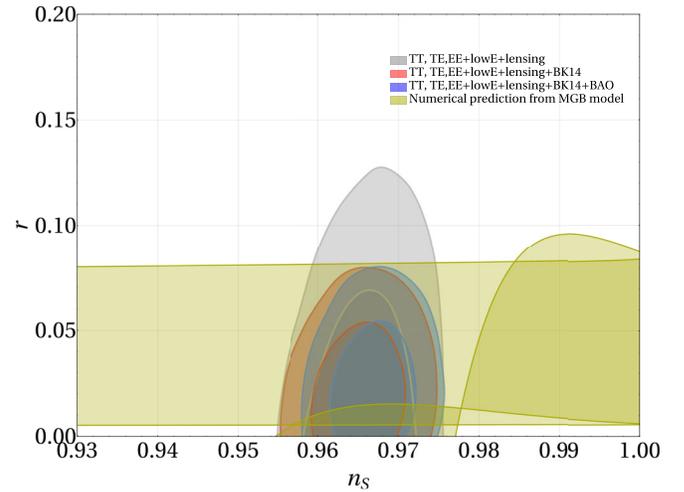
FIG. 2. r versus n_{s} diagram with observational constraints from Ref. [46]. The yellow shaded strips represent theoretical predictions for fixed $h_{\mathrm{i}} = 1$ and $N = 75$ and η ranging from 0.2 (bottom boundary of the strips) to 0.7 (top boundary of the strips).

slip parameter η in the range $0.2 \leq \eta \leq 0.7$ is presented in Fig. 2. In this plot, the e-fold number $N = 75$ and initial $h_{\mathrm{i}} = 1$ are kept fixed. Both figures demonstrate that there exist a reasonable set of parameters for which the theoretical prediction is in good agreement with observations.

V. SUMMARY AND CONCLUSIONS

We have studied the early Universe cosmology by means of a modified gravity model in which the gravity action consists of a function of the Ricci scalar and the Gauss-Bonnet invariant in addition to the Einstein-Hilbert term. The field equations are obtained by making use of the scalar-tensor representation of the action. We have specified the functional form of the action so that the modified Friedmann equations have the same form as those obtained in the holographic cosmology scenario. Furthermore, we have developed the formalism for calculating cosmological perturbations for scalar and tensor modes with matter represented by a general k-essence field theory. Using this, we have derived the scalar and tensor power spectra and calculated the scalar-to-tensor ratio r and spectral index n_{S}.

To confront our model with observations we have calculated n_{S} and r for a particular tachyon type k-essence with exponential potential. Our numerical results (see Fig. 1) show that the predictions of the MGB model are consistent with the Planck observational constraints. In this comparison we have fixed the initial expansion rate to $h_{\mathrm{i}} = 1$ and the slip parameter to $\eta = 0.5$, and the only remaining free parameter, the parameter ω in the potential, has been allowed to vary in such a way that the e-fold number N varies in the physically acceptable range $50 \leq N \leq 90$.

One of our aims has been to compare the MGB model with the holographic cosmology. We have recently studied inflation within the latter model [10], where a modest

departure from standard GR with k-essence has been found: for an inflationary scenario with k-essence, the differences in the power spectrum are only found at the second order in the slow-roll parameters (due to a change in the speed of sound). The expression (108) shows that the tensor-to-scalar ratio r departs from zero only at second order in the slow parameters in contrast to the holographic cosmology or the standard GR cosmology with k-essence where the departure from zero is at first order.

APPENDIX A: JUSTIFICATION FOR THE APPROXIMATIONS MADE

Here we justify the approximations of neglecting the term Φ/η in Eq. (62) and the term $4\pi G_N \xi/a$ in Eq. (71). First, we estimate the magnitude of the second term on the right-hand side of (62) in momentum space. Using the second Friedmann equation, the definition of ε_1, and the approximate value in the ultra-slow-roll regime $\gamma \simeq (4\eta + 3)h^2/(18\eta\varepsilon_1)$ we find

$$\frac{c_s^2 \gamma k^2 \Phi}{4\pi G_N a^2 (p + \rho)} = \frac{(4\eta + 3)h^2 c_s^2 k^2 \Phi}{18\eta a^2 H^2 \varepsilon_1^2 (1 - h^2/2)}. \quad (A1)$$

To make an order of magnitude estimate we can use the value $k \simeq aH/c_s$ near the acoustic horizon crossing. With this we find

$$\frac{c_s^2 \gamma k^2 \Phi}{4\pi G_N a^2 (p + \rho)} \simeq \frac{(4\eta + 3)h^2 \Phi}{9\eta(2 - h^2)\varepsilon_1^2} \gg \Phi, \quad (A2)$$

which justifies the approximation of neglecting the term Φ/η in Eq. (62).

Next, we estimate the order of magnitude of $4\pi G_N \xi/a$ in comparison with $\chi \simeq \zeta/H$ in Eq. (71). Applying (42) to Eq. (64) we find

$$\mathcal{O}(\varepsilon_1)H^2\xi \simeq \frac{1}{a}z^2\tilde{c}^2\chi = \frac{aH^2 9\eta(2 - h^2)\varepsilon_1^2}{4\pi G_N(1 - \eta)h^2}\chi \quad (A3)$$

and hence

$$\frac{4\pi G_N}{a}\xi \simeq \frac{9\eta(2 - h^2)}{(1 - \eta)h^2}\frac{\varepsilon_1^2}{\mathcal{O}(\varepsilon_1)}\chi \ll \chi. \quad (A4)$$

This justifies the approximation made to obtain the second equality in Eq. (71).

APPENDIX B: SCALAR PERTURBATIONS

In this Appendix we explicitly derive the equations for linear order scalar mode perturbations in the Newtonian gauge for the MGB model (9), (10). We apply the usual metric formalism and assume that Christoffel symbols are related to the metric through the Levi-Civita connection. Using the perturbed components of Christoffel symbols in the expressions for Ricci and Riemann tensor one can obtain all the perturbed quantities of Eq. (19). For the sake of completeness, in Sec. B 1 we provide the expressions for all these geometric quantities together with the perturbed Ricci scalar and the Gauss-Bonnet invariant. In Sec. B 2 we derive the expressions for the auxiliary fields ψ_1 and ψ_2, and the final equations for the scalar perturbations.

1. Perturbed geometric quantities

For the line element (38), the metric components are

$$g_{00} = 1 + 2\Psi, \quad (B1)$$

$$g_{ij} = -a^2(t)(1 - 2\Phi)\delta_{ij}. \quad (B2)$$

Plugging the above metric components into the Cristoffel symbols we find

$$\Gamma^0_{00} = \dot{\Psi}, \quad \Gamma^i_{00} = \frac{\delta^{ij}}{a^2}\partial_j\Psi, \quad \Gamma^0_{ij} = a^2[H - 2H(\Phi + \Psi) - \dot{\Phi}]\delta_{ij},$$

$$\Gamma^i_{0j} = (H - \dot{\Phi})\delta^i_j, \quad \Gamma^i_{jk} = \delta^i_k\partial_k\Psi + \delta^i_j\partial_k\Psi - \delta^{il}\delta_{jk}\partial_l\Psi. \quad (B3)$$

The components of the Riemann and Ricci tensor are

$$R_{0i0j} = a^2(H^2 + \dot{H})\delta_{ij} - \partial_i\partial_j\Psi$$
$$- a^2\delta_{ij}[2(H^2 + \dot{H})\Phi + H(2\dot{\Phi} + \dot{\Psi}) + \ddot{\Phi}], \quad (B4)$$

$$R_{ijkl} = -a^4 H^2(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj})$$
$$+ a^2(\delta_{li}\delta^n_k\delta^m_j - \delta_{jl}\delta^n_k\delta^m_i - \delta_{ki}\delta^m_l\delta^n_j + \delta_{kj}\delta^m_l\delta^n_i)\partial_m\partial_n\Phi$$
$$+ (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})[2a^4 H^2(\Psi + 2\Phi) + 2a^4 H\dot{\Psi}], \quad (B5)$$

$$R_{00} = -3(H^2 + \dot{H}) + \frac{1}{a^2}\nabla^2\Psi + 3H(2\dot{\Phi} + \dot{\Psi}) + 3\ddot{\Phi}, \quad (B6)$$

$$R_{0i} = 2\partial_i(\dot{\Phi} + H\Psi), \quad (B7)$$

$$R_{ij} = a^2(3H^2 + \dot{H})\delta_{ij} + \partial_i\partial_j(\Phi - \Psi) + \delta_{ij}\nabla^2\Phi$$
$$- a^2\delta_{ij}[2(3H^2 + \dot{H})(\Phi + \Psi) + H(6\dot{\Phi} + \dot{\Psi}) + H\ddot{\Phi}]. \quad (B8)$$

With the help of the above quantities the perturbed Ricci scalar and Gauss-Bonnet invariant can be calculated yielding

$$R = -6(2H^2 + \dot{H}) + \frac{2}{a^2}\nabla^2\Psi - \frac{4}{a^2}\nabla^2\Phi + 6[2(2H^2 + \dot{H})\Psi + H(4\dot{\Phi} + \dot{\Psi}) + \ddot{\Phi}] \tag{B9}$$

and

$$\mathcal{G} = 24H^2(H^2 + \dot{H}) + \frac{16}{a^2}(H^2 + \dot{H})\nabla^2\Phi - \frac{8}{a^2}H^2\nabla^2\Psi - 96H^2(H^2 + \dot{H})\Psi - 48H(2H^2 + \dot{H})\dot{\Phi}$$
$$- 24H^3\dot{\Psi} - 24H^2\ddot{\Phi}. \tag{B10}$$

2. Perturbed field equations

Scalar modes induce fluctuations in all quantities in (19). Inserting the scalar perturbations of the metric in the Newtonian gauge and the field perturbations $\psi_a \to \psi_a(t) + \delta\psi_a$, the components of the linear part of Eq. (19) become

$$\frac{1}{a^2}[\nabla^2\delta\psi_1 - 4H^2\nabla^2\delta\psi_2 - 2(\psi_1 - 4H\dot{\psi}_2)\nabla^2\Phi] + 3(H^2 + \dot{H})(\delta\psi_1 - 4H^2\delta\psi_2)$$
$$+ 6H(\dot{\psi}_1 + H(\psi_1 - 8H\dot{\psi}_2))\Psi - 3H\delta\dot{\psi}_1 + 12H^3\delta\dot{\psi}_2 + 3(\dot{\psi}_1 + 2H(\psi_1 - 6H\dot{\psi}_2))\dot{\Phi} = 8\pi G\delta T^0_0, \tag{B11}$$

$$\partial_i[-2(\psi_1 - 4H\dot{\psi}_2)\dot{\Phi} + (12H^2\dot{\psi}_2 - 2H\psi_1 - \dot{\psi}_1)\Psi - H\delta\psi_1 + 4H^3\delta\psi_2 + \delta\dot{\psi}_1 - 4H^2\delta\dot{\psi}_2] = 8\pi G\delta T^0_i, \tag{B12}$$

$$\frac{1}{a^2}[\nabla^2\delta\psi_1 - 4(H^2 + \dot{H})\nabla^2\delta\psi_2 - (\psi_1 - 4\ddot{\psi}_2)\nabla^2\Phi + (\psi_1 - 4H\dot{\psi}_2)\nabla^2\Psi]\delta^i_j$$
$$+ [(3H^2 + \dot{H})\delta\psi_1 - 12H^2(H^2 + \dot{H})\delta\psi_2 + 2(2\psi_1\dot{H} - 16H^2\dot{\psi}_2 + 2H(\dot{\psi}_1 - \dot{H}\dot{\psi}_2) + \ddot{\psi}_2 + H^2(3\psi_1 - 8\ddot{\psi}_2))\Psi$$
$$- 2H\delta\dot{\psi}_1 + 8H(H^2 + \dot{H})\delta\dot{\psi}_2 + 2(\dot{\psi}_1 - 12H^2\dot{\psi}_2 - 4\dot{H}\dot{\psi}_2 + H(3\psi_1 - 4\ddot{\psi}_2))\dot{\Phi}$$
$$+ (\dot{\psi}_1 + 2H(\psi_1 - 6H\dot{\psi}_2))\dot{\Psi} + 2(\psi_1 - 4H\dot{\psi}_2)\ddot{\phi} - \delta\ddot{\psi}_1 - 4H^2\delta\ddot{\psi}_2]\delta^i_j$$
$$- \frac{1}{a^2}\partial^i\partial_j\delta\psi_1 + \frac{4}{a^2}(H^2 + \dot{H})\partial^i\partial_j\delta\psi_2 + \frac{1}{a^2}(\psi_1 - 4\ddot{\psi}_2)\partial^i\partial_j\Phi - \frac{1}{a^2}(\psi_1 - 4H\dot{\psi}_2)\partial^i\partial_j\Psi = 8\pi G\delta T^i_j. \tag{B13}$$

The perturbed auxiliary fields are functions of the invariants X^a and hence

$$\psi_b(X^a + \delta X^a) \approx \psi_b(X^a) + \frac{\partial\psi_b}{\partial X^c}\delta X^c. \tag{B14}$$

Then, from (13) it follows

$$\delta\psi_b = \frac{\partial^2 F}{\partial X^c \partial X^b}\delta X^c. \tag{B15}$$

Using (10) together with the spatially flat background metric the linear parts of the auxiliary fields become

$$\delta\psi_1 = \frac{\ell^2 H^4(2H^2 + 3\dot{H})}{36\dot{H}^2}\delta R + \frac{\ell^2 H^2(H^2 + \dot{H})}{72\dot{H}^2}\delta\mathcal{G}, \tag{B16}$$

$$\delta\psi_2 = \frac{\ell^2 H^2(H^2 + \dot{H})}{72\dot{H}^2}\delta R + \frac{\ell^2(2H^2 + \dot{H})}{144\dot{H}^3}\delta\mathcal{G}, \tag{B17}$$

where we have used the background expressions

$$\psi_1 = -1 + \frac{\ell^2 H^4}{6\dot{H}}, \qquad \psi_2 = \frac{\ell^2 H^2}{24\dot{H}}. \tag{B18}$$

Then, in terms of the Ψ and Φ fields we find

$$\psi_1(t) + \delta\psi_1 = -1 + \frac{\ell^2 H^4}{6\dot{H}} + \left(\frac{H^4}{\dot{H}^2} + \frac{2H^2}{\dot{H}}\right)\frac{\ell^2}{9a^2}\nabla^2\Phi + \frac{\ell^2 H^4}{18\dot{H}^2 a^2}\nabla^2\Psi + \frac{\ell^2 H^4}{6\dot{H}^2}(\ddot{\Phi} + H\dot{\Psi}) - \frac{\ell^2 H^3}{3\dot{H}}(2\dot{\Phi} + H\Psi),$$ (B19)

$$\psi_2(t) + \delta\psi_2 = \frac{\ell^2 H^2}{24\dot{H}} + \left(\frac{H^2}{\dot{H}^2} + \frac{1}{\dot{H}}\right)\frac{\ell^2}{36a^2}\nabla^2\Phi + \frac{\ell^2 H^2}{72\dot{H}a^2}\nabla^2\Psi + \frac{\ell^2 H^3}{24\dot{H}^2}\dot{\Psi} - \frac{\ell^2 H}{12\dot{H}}\dot{\Phi} + \frac{\ell^2 H^2}{24\dot{H}^2}\ddot{\Phi}.$$ (B20)

and with that the components of Eq. (19) are

$$-6H\left(1 - \frac{h^2}{2}\right)(H\Psi + \dot{\Phi}) + 2\left(1 - \frac{h^2}{6}\right)\frac{\nabla^2\Phi}{a^2} - \frac{2Hh^2}{3\dot{H}}\frac{\nabla^2(\dot{\Phi} + H\Psi)}{a^2} + \frac{h^2}{9\dot{H}}\frac{\nabla^2\nabla^2\Phi}{a^4} = 8\pi G_N \delta T^0_0,$$ (B21)

$$\partial_i\left[2\left(1 - \frac{h^2}{2}\right)(\dot{\Phi} + H\Psi) - \frac{h^2}{9}\left(\frac{H}{\dot{H}} + \frac{\ddot{H}}{\dot{H}^2} - \frac{4}{H}\right)\frac{\nabla^2\Phi}{a^2} + \frac{h^2}{9\dot{H}}\frac{\nabla^2(\dot{\Phi} + H\Psi)}{a^2}\right] = 8\pi G_N \delta T^0_i,$$ (B22)

$$\left\{-2\left[3H^2 + 2\dot{H} - \frac{h^2}{2}(3H^2 + 4\dot{H})\right]\Psi - 2H\left(1 - \frac{h^2}{2}\right)\dot{\Psi} - 6H\left[1 - \frac{h^2}{6}\left(\frac{2\dot{H}}{H^2} + 3\right)\right]\dot{\Phi} - 2\left(1 - \frac{h^2}{2}\right)\ddot{\Phi}\right.$$
$$+ \left[1 - \frac{h^2}{9}\left(\frac{H^2}{2\dot{H}} + \frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H\dot{H}} - \frac{\ddot{H}^2}{\dot{H}^3} + \frac{\dddot{H}}{2\dot{H}^2} + 1\right)\right]\frac{\nabla^2\Phi}{a^2} - \left[1 - \frac{h^2}{9}\left(\frac{5H\ddot{H}}{2\dot{H}^2} - \frac{5H^2}{2\dot{H}} - 6\right)\right]\frac{\nabla^2\Psi}{a^2}$$
$$\left. - \frac{h^2}{9H^2}\frac{\nabla^2\nabla^2\Phi}{a^4} - \frac{h^2}{18\dot{H}}\frac{\nabla^2\nabla^2\Psi}{a^4} + \frac{h^2}{9}\left(\frac{\ddot{H}}{\dot{H}^2} - \frac{3H}{\dot{H}} - \frac{3}{H}\right)\frac{\nabla^2\dot{\Phi}}{a^2} - \frac{5h^2 H}{18\dot{H}}\frac{\nabla^2\dot{\Psi}}{a^2} - \frac{5h^2}{18\dot{H}}\frac{\nabla^2\ddot{\Phi}}{a^2}\right\}\delta^i_j$$
$$- \left[1 - \frac{h^2}{3}\left(\frac{H^2}{2\dot{H}} - \frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H\dot{H}} - \frac{\ddot{H}^2}{\dot{H}^3} + \frac{\dddot{H}}{2\dot{H}^2}\right)\right]\frac{\partial^i\partial_j\Phi}{a^2} + \left[1 - \frac{h^2}{6}\left(\frac{H\ddot{H}}{\dot{H}^2} - \frac{H^2}{\dot{H}} - 2\right)\right]\frac{\partial^i\partial_j\Psi}{a^2}$$
$$+ \frac{h^2}{9H^2}\frac{\partial^i\partial_j\nabla^2\Phi}{a^4} + \frac{h^2}{18\dot{H}}\frac{\partial^i\partial_j\nabla^2\Psi}{a^4} + \frac{h^2}{3}\left(\frac{H}{\dot{H}} - \frac{1}{H}\right)\frac{\partial^i\partial_j\dot{\Phi}}{a^2} + \frac{h^2 H}{6\dot{H}}\frac{\partial^i\partial_j\dot{\Psi}}{a^2} = 8\pi G_N \delta T^i_j.$$ (B23)

In the Fourier space Eqs. (B21) and (B22) can be written respectively in the form

$$\left(-6 + 3h^2 + \frac{2h^2 k^2}{3a^2\dot{H}}\right)H(\dot{\Phi} + H\Psi) + \left(-2 + \frac{h^2}{3} + \frac{h^2 k^2}{9a^2\dot{H}}\right)\frac{k^2}{a^2}\Phi = 8\pi G_N \delta T^0_0,$$ (B24)

$$\left[\left(2 - h^2 - \frac{h^2 k^2}{9a^2\dot{H}}\right)(\dot{\Phi} + H\Psi) + \frac{h^2}{9}\left(\frac{H}{\dot{H}} + \frac{\ddot{H}}{\dot{H}^2} - \frac{4}{H}\right)\frac{k^2}{a^2}\right](ik_i) = 8\pi G_N \delta T^0_i.$$ (B25)

APPENDIX C: TENSOR PERTURBATIONS

This Appendix deals with tensor mode perturbations and has a structure similar to the one in Appendix B. For tensor modes, the perturbed line element is given by

$$ds^2 = dt^2 - a^2(\delta_{ij} + h_{ij})dx^i dx^j, \qquad (C1)$$

where h_{ij} is a traceless and transverse tensor. In the next section we will provide the expressions for the geometric quantities, and in Sec. C 2 we derive the perturbed field equations.

1. Perturbed geometric quantities

The elements of the metric tensor and its inverse can be straightforwardly obtained from (C1). Plugging them into the metric connection one gets the following expressions for the non-null components of Christoffel symbols:

$$\Gamma^0_{ij} = a^2 H\delta_{ij} + a^2 Hh_{ij} + \frac{a^2}{2}\dot{h}_{ij}, \qquad (C2)$$

$$\Gamma^i_{0j} = H\delta^i_j + \frac{\delta^{ik}}{2}\dot{h}_{kj}, \qquad (C3)$$

$$\Gamma^i_{jk} = \frac{\delta^{in}}{2}(\partial_j h_{kn} + \partial_k h_{nj} - \partial_n h_{jk}). \qquad (C4)$$

The linear parts of the covariant, contravariant, and mixed components of interest of the Ricci tensor are

$$\delta R_{ij} = a^2(3H^2 + \dot{H})h_{ij} + \frac{3}{2}a^2 H\dot{h}_{ij} + \frac{a^2}{2}\ddot{h}_{ij} - \frac{1}{2}\nabla^2 h_{ij}, \qquad (C5)$$

$$\delta R^{ij} = \frac{\delta^{ik}\delta^{jl}}{a^2}\left(-(3H^2+\dot{H})h_{kl} + \frac{3}{2}H\dot{h}_{kl} + \frac{1}{2}\ddot{h}_{kl} - \frac{1}{2a^2}\nabla^2 h_{kl}\right), \tag{C6}$$

$$\delta R^i_j = \frac{\delta^{ik}}{2}\left(\frac{1}{a^2}\nabla^2 h_{kj} - 3H\dot{h}_{kj} - \ddot{h}_{kj}\right), \tag{C7}$$

and the linear parts of the Riemann tensor are

$$\delta R_{0i0j} = a^2\left((H^2+\dot{H})h_{ij} + H\dot{h}_{ij} + \frac{1}{2}\ddot{h}_{ij}\right), \tag{C8}$$

$$\delta R^i_{0j0} = -\delta^{ik}\left(H\dot{h}_{kj} + \frac{1}{2}\ddot{h}_{kj}\right), \tag{C9}$$

$$\delta R^i_{jkl} = \frac{\delta^{in}}{2}(\partial_k\partial_j h_{ln} - \partial_k\partial_n h_{jl} - \partial_l\partial_j h_{kn} + \partial_l\partial_n h_{jk}) + \frac{\delta^{in}\dot{h}_{nk}}{2}a^2 H\delta_{lj} + H\delta^i_k\left(a^2 H h_{lj} + a^2\frac{\dot{h}_{lj}}{2}\right)$$
$$- \frac{\delta^{in}\dot{h}_{nl}}{2}a^2 H\delta_{kj} - H\delta^i_l\left(a^2 H h_{kj} + a^2\frac{\dot{h}_{kj}}{2}\right), \tag{C10}$$

$$\delta R_{mjkl} = -a^4 H(h_{mk}\delta_{lj} - h_{ml}\delta_{kj}) - a^2\delta_{im}\delta R^i_{jkl}, \tag{C11}$$

where we have used the background expressions for the geometric quantities as in Sec. B 1.

2. Perturbed field equations

From the equations derived in Sec. C 1 one can easily conclude that tensor modes do not induce fluctuations in the Ricci scalar, Gauss-Bonnet invariant, and $\Box\psi_b$. Therefore, the space-space component of (19) are

$$-\psi_1\delta R^i_j + \delta(g^{i\sigma}\nabla_\sigma\nabla_j\psi_1) + 4(\Box\psi_2)\delta R^i_j + 2\delta(g^{i\sigma}\nabla_\sigma\nabla_j\psi_2)R - 4\delta(g^{i\sigma}\nabla_\rho\nabla_\sigma\psi_2)R^\rho_j$$
$$- 4(g^{i\sigma}\nabla_\rho\nabla_\sigma\psi_2)\delta R^\rho_j - 4\delta(\nabla_\rho\nabla_j\psi_2)R^{\rho i} - 4(\nabla_\rho\nabla_j\psi_2)\delta R^{\rho i} - 4\delta(\nabla^\rho\nabla^\sigma\psi_2)R^i_{\rho j\sigma}$$
$$+ 4\delta^i_j[\delta(\nabla_\rho\nabla_\sigma\psi_2)R^{\rho\sigma} + (\nabla_\rho\nabla_\sigma\psi_2)\delta R^{\rho\sigma}] - 4(\nabla^\rho\nabla^\sigma\psi_2)\delta R^i_{\rho j\sigma} = 8\pi G\delta T^i_j. \tag{C12}$$

For tensor modes $\delta(\nabla_\alpha\nabla_\beta\psi_b) = -\delta\Gamma^0_{\alpha\beta}\dot{\psi}_b$. Using this, the terms in the above equation can be expressed as

$$\delta(g^{i\sigma}\nabla_\sigma\nabla_j\psi_1) = \frac{\dot{\psi}_1}{2}\delta^{ik}\dot{h}_{kj}, \tag{C13}$$

$$\delta(g^{i\sigma}\nabla_\rho\nabla_\sigma\psi_2)R^\rho_j = -\frac{\dot{\psi}_2}{2}(3H^2+\dot{H})\delta^{ik}\dot{h}_{kj}, \tag{C14}$$

$$(g^{i\sigma}\nabla_\rho\nabla_\sigma\psi_2)\delta R^\rho_j = H\dot{\psi}_2\delta R^i_j, \tag{C15}$$

$$\delta(\nabla_\rho\nabla_j\psi_2)R^{\rho i} = -\dot{\psi}_2(3H^2+\dot{H})\delta^{ik}\left(Hh_{kj} + \frac{1}{2}\dot{h}_{kj}\right), \tag{C16}$$

$$(\nabla_\rho\nabla_j\psi_2)\delta R^{\rho i} = -a^2 H\dot{\psi}_2\delta_{kj}\delta R^{ki}, \tag{C17}$$

$$\delta(\nabla^\rho\nabla^\sigma\psi_2)R^i_{\rho j\sigma} = -H^2\dot{\psi}_2\delta^{ik}\left(Hh_{kj} - \frac{\dot{h}_{kj}}{2}\right), \tag{C18}$$

$$(\nabla^\rho\nabla^\sigma\psi_2)\delta R^i_{\rho j\sigma} = \frac{H\dot{\psi}_2}{2a^2}\delta^{ik}\nabla^2 h_{kj} - H\left(\ddot{\psi}_2 + \frac{H\dot{\psi}_2}{2}\right)\delta^{ik}\dot{h}_{kj} + \dot{\psi}_2 H^3\delta^{ik}h_{kj} - \frac{\ddot{\psi}_2}{2}\delta^{ik}\ddot{h}_{kj}. \tag{C19}$$

Then, using these expressions in Eq. (C12) we obtain

$$\left[\frac{1}{2}(\psi_1 - 4H\dot{\psi}_2)\ddot{h}_{kj} + \frac{1}{2}(3H\psi_1 + \dot{\psi}_1 - 4H\ddot{\psi}_2 - 4(3H^2 + \dot{H})\dot{\psi}_2)\dot{h}_{kj} - 8H^3\dot{\psi}_2 h_{kj} - \frac{1}{2}(\psi_1 - 4\ddot{\psi}_2)\frac{\nabla^2 h_{kj}}{a^2}\right]\delta^{ik} = 8\pi G\delta T^i_j. \quad \text{(C20)}$$

By substituting the background expressions (B18) for the auxiliary fields we obtain

$$\left[-\frac{1}{2}\left(1 - \frac{h^2}{6}\left(\frac{H^2}{\dot{H}} + \frac{H\ddot{H}}{\dot{H}^2} - 2\right)\right)\ddot{h}_{kj} - \frac{3}{2}H\left(1 - \frac{h^2}{6}\left(\frac{H^2}{\dot{H}} - \frac{4\dot{H}}{3H^2} + \frac{2H\ddot{H}}{3\dot{H}^2} + \frac{\ddot{H}}{H\dot{H}} - \frac{2\ddot{H}^2}{3\dot{H}^3} + \frac{\dddot{H}}{3\dot{H}^2} - \frac{2}{3}\right)\right)\dot{h}_{kj}\right.$$

$$\left. - H^2\frac{h^2}{3}\left(2 - \frac{H\ddot{H}}{\dot{H}^2}\right)h_{kj} + \frac{1}{2}\left(1 - \frac{h^2}{6}\left(\frac{H^2}{\dot{H}} - \frac{2\dot{H}}{H^2} + \frac{2\ddot{H}}{H\dot{H}} - \frac{2\ddot{H}^2}{\dot{H}^3} + \frac{\dddot{H}}{\dot{H}^2}\right)\right)\frac{\nabla^2 h_{kj}}{a^2}\right]\delta^{ik} = 8\pi G\delta T^i_j. \quad \text{(C21)}$$

In the limit $h^2 \equiv \ell^2 H^2 \to 0$ this equation takes the usual general relativity form.

APPENDIX D: GHOST INSTABILITIES IN $F(R,\mathcal{G})$ THEORY

A general action

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}F(R,P,Q)$$
$$+ \int d^4x\sqrt{-g}\mathcal{L}_{\text{matt}}, \quad \text{(D1)}$$

where

$$P = R^{\mu\nu}R_{\mu\nu}, \qquad Q = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \quad \text{(D2)}$$

can be expanded around a background up to second order in the fluctuations. It has been shown [47,48] that such an expansion will be identical to that obtained from

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left(-2\Lambda + aR + \frac{b}{2}R^2 - \frac{c}{6}C^2\right)$$
$$+ \int d^4x\sqrt{-g}\mathcal{L}_{\text{matt}}, \quad \text{(D3)}$$

where

$$C^2 \equiv C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = Q - 2P + \frac{1}{3}R^2 \quad \text{(D4)}$$

is the Weyl tensor squared and the coefficients Λ, a, b, and c depend on the background.

The gravity theory with the four-derivative terms R^2 and C^2 was extensively studied [16,17,47–51] and it was found that there appear new degrees of freedom in addition to the spin-2 massless graviton: a massive spin-0 field [with mass $m_0^2 \propto 1/b$ corresponding to the R^2 term and a massive spin-2 field (with mass $m_2^2 \propto 1/c$) corresponding to the Weyl squared term]. Moreover, the massive spin-2 field is known

to have a wrong sign of the kinetic term and thus has negative energy: a ghost field.

Consider first the Minkowski background. If we linearize gravity using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the traceless part of the metric perturbation $\bar{h}_{\mu\nu}$ in momentum space will satisfy

$$\left(k^2 - \frac{k^4}{m_2^2}\right)\bar{h}_{\mu\nu} = 0, \quad \text{(D5)}$$

where

$$m_2^2 = -\frac{F_R}{F_P + 4F_Q}, \quad \text{(D6)}$$

with $F_R = \partial F/\partial R$ etc., and it is understood that these derivatives are evaluated on the background. Note that the propagator for $\bar{h}_{\mu\nu}$ can be written as [51]

$$G(k) \propto \frac{1}{k^2 - k^4/m_2^2} = \frac{1}{k^2} - \frac{1}{k^2 - m_2^2}. \quad \text{(D7)}$$

The first term on the right-hand side corresponds to the massless graviton whereas the second term corresponds to the massive spin-2 field with mass m_2. The second term has the opposite sign, which indicates the presence of a ghost. Hence, ghost terms may occur for a general $F(R,P,Q)$ theory and is parametrized by the k^4 mode. However, for a theory of the form $F = F(R, 4P - Q)$ we have $F_P + 4F_Q = 0$ and Eq. (D5) simplifies to $k^2\bar{h}_{\mu\nu} = 0$. Clearly in this case the mentioned spin-2 ghost is absent.

Next, assume that the background is a constant curvature maximally symmetric spacetime. In this case we have $R_{\mu\nu\rho\sigma} = (R/12)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ so $P = R^2/4$ and $Q = R^2/6$. The propagator of the graviton has the same structure as in (D7) with the ghost mass $m_2^2 \propto -(F_P + 4F_Q)^{-1}$, similar to (D6). As before, the presence of the Weyl term implies the presence of a ghost

field. Again, for the theory where $F = F(R, 4P - Q)$ we have $c = 0$, and the C^2 term in (D3) is absent. We are left with an effective R^2 theory in which there are no ghosts. This applies also to a general $F(R, \mathcal{G})$ theory including our model

$$F(R, \mathcal{G}) = -R + f(J), \tag{D8}$$

where J is defined in Eq. (2).

There is still some ambiguity concerning a possible presence of instabilities in the scalar sector. De Felice and Suyama [18] (hereafter DFS) argue that an instability can arise in vacuum for the scalar modes of the cosmological perturbations if the background is not de Sitter. The cause of instability is a term proportional to k^4, which, apart from a few exceptions (examples are provided in their Table I), appears in their master equation (see below). In contrast, Navarro and Van Acoleyen [17] find no such k^4 instability in the scalar sector for a general $F(R, 4P - Q)$. They derive a propagator for the scalar field but their derivation in an Appendix is restricted to de Sitter space although they argue that in a generic FLRW background the results will be qualitatively similar. This casts a certain doubt on the validity of their result.

The analysis of DFS is based on the following equation:

$$\frac{1}{a^3 Q} \partial_t(a^3 Q \dot{\hat{\Phi}}) + B_1 \frac{k^2}{a^2} \hat{\Phi} + B_2 \frac{k^4}{a^4} \hat{\Phi} = 0, \tag{D9}$$

dubbed "master equation" (Eq. (59) in [18]). The quantities Q, B_1, and B_2 are time dependent background functions and $\hat{\Phi}$ is a gauge invariant scalar perturbation in the spatial slices of constant time. The quantity $\hat{\Phi}$ is one of the three gauge invariant perturbations introduced in Ref. [18] to describe general scalar perturbations for generic $F(R, \mathcal{G})$ theories. In our notation

$$\hat{\Phi} = \Phi + \frac{H(\delta\psi_1 - 4H^2\delta\psi_2)}{\dot{\psi}_1 - 4H^2\dot{\psi}_2}, \tag{D10}$$

where ψ_1 and ψ_2 are auxiliary fields defined in Sec. II B. By manipulating Eqs. (B11)–(B13) it is possible to eliminate two other scalars in favor of $\hat{\Phi}$, thus yielding the master equations (D9). If a solution to (D9) exhibits an instability (e.g., exponential growth of $\hat{\Phi}$), this implies that the scalar perturbations of the $F(R, \mathcal{G})$ model are plagued by an instability.

The problematic term in Eq. (D9) is the one that contains the k^4 term. For models where this term vanishes, i.e., when $B_2 = 0$, we have a standard wave equation. For a general $F(R, \mathcal{G})$ model, $B_2 = 0$ if the background is de Sitter.

Besides, coefficient B_2 is identically zero for some particular cases of $F(R, \mathcal{G})$, e.g., for $F = f(R) + \text{const.}\mathcal{G}$ or $F = f(R + \text{const.}\mathcal{G})$ [18]. Note that the models of inflation considered in Refs. [14,15] belong to the class of models in which B_2 does not vanish identically.

For short wavelengths, Eq. (D9) is solved using the WKB approximation and the properties of the solution are discussed in detail [18]. In a nutshell, if B_2 is negative $\hat{\Phi}$ grows exponentially with time, implying that the perturbations in a FLRW universe is unstable on small scales. These instabilities grow to the point where linear perturbations are not valid. If B_2 is positive, the perturbations propagate with group velocity

$$v_g(k) \approx 2\sqrt{B_2}\frac{k}{a}. \tag{D11}$$

The group velocity exceeds the speed of light for modes above a critical $k_{\text{cr}} \approx a/(2\sqrt{B_2})$ so the short wavelength propagation modes will be superluminal for $k > k_{\text{cr}}$. However, a superluminal behavior of the cosmological perturbations is not necessarily unphysical [52]. Note that a superluminal behavior is absent in a theory in which the critical k_{cr} exceeds the cutoff of the theory Λ_{cutoff}.

Finally, we investigate the coefficients B_1 and B_2 for models of the type (D8). We have computed these factors for a general $F(R, \mathcal{G})$ model in Newtonian gauge and our results agree with those obtained in Ref. [18] where the precise expressions can be found. The point essential for our analysis is that both B_1 and B_2 can be expressed as fractions, the denominators of which contain a factor

$$\begin{aligned} D = {} & 2\dot{H}(2H^2\dot{\psi}_1 + \dot{H}\dot{\psi}_1 - 48H\dot{H}^3\psi_{1,\mathcal{G}} - 8H^4\dot{\psi}_2) \\ & + H\ddot{H}(\dot{\psi}_1 - 4H^2\dot{\psi}_2). \end{aligned} \tag{D12}$$

It may be explicitly verified that this factor vanishes identically for models of the type (D8). To see this, consider the background quantities

$$\psi_1 = \frac{\partial F}{\partial R} = -1 + \frac{\partial f}{\partial J}\frac{\partial J}{\partial R}, \qquad \psi_2 = \frac{\partial F}{\partial \mathcal{G}} = \frac{\partial f}{\partial J}\frac{\partial J}{\partial \mathcal{G}}. \tag{D13}$$

Inserting the time derivatives of ψ_1 and ψ_2 and the derivative of ψ_1 with respect to $\mathcal{G}$ into (D12) one finds that the factor D is identically zero. This implies that the coefficients B_1 and B_2 are ill defined and the master equation (D9) cannot serve as a check for ghost instabilities in a model of the type (D8). In other words, the DFS analysis is inconclusive with regard to the presence of instabilities in models of the type (D8).

[1] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **73,** 084007 (2006).

[2] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rev. D **99,** 044050 (2019).

[3] S. Nojiri, S. D. Odintsov, V. K. Oikonomou, N. Chatzarakis, and T. Paul, Eur. Phys. J. C **79,** 565 (2019).

[4] S. Nojiri, S. Odintsov, and V. Oikonomou, Phys. Rev. D **99,** 044050 (2019).

[5] E. Elizalde, S. Odintsov, V. Oikonomou, and T. Paul, Nucl. Phys. **B954,** 114984 (2020).

[6] C. Gao, Phys. Rev. D **86,** 103512 (2012).

[7] N. Bilić, Phys. Rev. D **93,** 066010 (2016).

[8] P. S. Apostolopoulos, G. Siopsis, and N. Tetradis, Phys. Rev. Lett. **102,** 151301 (2009).

[9] N. Bilic, D. D. Dimitrijevic, G. S. Djordjevic, M. Milosevic, and M. Stojanovic, J. Cosmol. Astropart. Phys. 08 (2019) 034.

[10] N. R. Bertini, N. Bilic, and D. C. Rodrigues, Phys. Rev. D **102,** 123505 (2020).

[11] S. del Campo, J. Cosmol. Astropart. Phys. 12 (2012) 005.

[12] S. Chakraborty, T. Paul, and S. SenGupta, Phys. Rev. D **98,** 083539 (2018).

[13] S. Basilakos, N. E. Mavromatos, and J. Solà Peracaula, Phys. Lett. B **803,** 135342 (2020).

[14] M. De Laurentis, M. Paolella, and S. Capozziello, Phys. Rev. D **91,** 083531 (2015).

[15] S. D. Odintsov, V. K. Oikonomou, and S. Banerjee, Nucl. Phys. **B938,** 935 (2019).

[16] D. Comelli, Phys. Rev. D **72,** 064018 (2005).

[17] I. Navarro and K. Van Acoleyen, J. Cosmol. Astropart. Phys. 03 (2006) 008.

[18] A. De Felice and T. Suyama, J. Cosmol. Astropart. Phys. 06 (2009) 034.

[19] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, Phys. Lett. B **458,** 209 (1999).

[20] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, Phys. Rev. D **63,** 103510 (2001).

[21] F. Arroja and M. Sasaki, Phys. Rev. D **81,** 107301 (2010).

[22] O. F. Piattella, J. C. Fabris, and N. Bilić, Classical Quant. Grav. **31,** 055006 (2014).

[23] A. Casalino, L. Sebastiani, L. Vanzo, and S. Zerbini, Phys. Dark Universe **29,** 100594 (2020).

[24] V. Faraoni and S. Capozziello, *Beyond Einstein Gravity*, Fundamental Theories of Physics, Vol. 170 (Springer, Dordrecht, 2010).

[25] D. Wands, Classical Quant. Grav. **11,** 269 (1994).

[26] D. C. Rodrigues, F. de O.Salles, I. L. Shapiro, and A. A. Starobinsky, Phys. Rev. D **83,** 084028 (2011).

[27] S. Nojiri, S. D. Odintsov, and M. Sasaki, Phys. Rev. D **71** (2005), 123509.

[28] D. J. Schwarz, C. A. Terrero-Escalante, and A. A. Garcia, Phys. Lett. B **517,** 243 (2001).

[29] D. A. Steer and F. Vernizzi, Phys. Rev. D **70,** 043527 (2004).

[30] J. Garriga and V. F. Mukhanov, Phys. Lett. B **458,** 219 (1999).

[31] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys. **69,** 373 (1997).

[32] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215,** 203 (1992).

[33] J. c. Hwang and H. Noh, Phys. Rev. D **66,** 084009 (2002).

[34] S. D. Odintsov and V. K. Oikonomou, Phys. Lett. B **807,** 135576 (2020).

[35] A. Sen, J. High Energy Phys. 10 (1999) 008; Mod. Phys. Lett. A **17,** 1797 (2002).

[36] M. Fairbairn and M. H. G. Tytgat, Phys. Lett. B **546,** 1 (2002); A. Feinstein, Phys. Rev. D **66,** 063511 (2002).

[37] A. V. Frolov, L. Kofman, and A. A. Starobinsky, Phys. Lett. B **545,** 8 (2002).

[38] G. Shiu and I. Wasserman, Phys. Lett. B **541,** 6 (2002).

[39] M. Sami, P. Chingangbam, and T. Qureshi, Phys. Rev. D **66,** 043530 (2002).

[40] G. Shiu, S. H. H. Tye, and I. Wasserman, Phys. Rev. D **67,** 083517 (2003); P. Chingangbam, S. Panda, and A. Deshamukhya, J. High Energy Phys. 02 (2005) 052; S. del Campo, R. Herrera, and A. Toloza, Phys. Rev. D **79,** 083507 (2009); S. Li and A. R. Liddle, J. Cosmol. Astropart. Phys. 03 (2014) 044.

[41] L. Kofman and A. D. Linde, J. High Energy Phys. 07 (2002) 004.

[42] J. M. Cline, H. Firouzjahi, and P. Martineau, J. High Energy Phys. 11 (2002) 041.

[43] F. Salamate, I. Khay, A. Safsafi, H. Chakir, and M. Bennai, Moscow Univ. Phys. Bull. **73,** 405 (2018).

[44] N. Barbosa-Cendejas, R. Cartas-Fuentevilla, A. Herrera-Aguilar, R. R. Mora-Luna, and R. da Rocha, J. Cosmol. Astropart. Phys. 01 (2018) 005.

[45] D. M. Dantas, R. da Rocha, and C. A. S. Almeida, Phys. Lett. B **782,** 149 (2018).

[46] Y. Akrami *et al.* (Planck Collaboration), Astron. Astrophys. **641,** A10 (2020).

[47] A. Hindawi, B. A. Ovrut, and D. Waldram, Phys. Rev. D **53,** 5597 (1996).

[48] T. Chiba, J. Cosmol. Astropart. Phys. 03 (2005) 008.

[49] K. S. Stelle, Gen. Relativ. Gravit. **9,** 353 (1978).

[50] I. de Martino, M. De Laurentis, and S. Capozziello, Phys. Rev. D **102,** 063508 (2020).

[51] C. Bogdanos, S. Capozziello, M. De Laurentis, and S. Nesseris, Astropart. Phys. **34,** 236 (2010).

[52] E. Babichev, V. Mukhanov, and A. Vikman, J. High Energy Phys. 02 (2008) 101.