Scalar resonances in the final state interactions of the decays $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$

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We investigate the scalar resonances in the processes $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$ based on the chiral unitary approach for the final state interaction. We start from singly Cabibbo-suppressed production diagrams which provide a primary quark pair to hadronize two pseudoscalar mesons in the D^0 decays. The resonances $f_0(500)$, $f_0(980)$, and $a_0(980)$ are dynamically produced from the final state interactions of the meson pairs. In our results, the experimental data for the $\pi^0\eta$ invariant mass spectrum of the $D^0 \to \pi^0\eta\eta$ decay can be described well. We also make the predictions for the $\pi^0\pi^0$ invariant mass spectrum of the $D^0 \to \pi^0\pi^0\pi^0$ decay, where the $f_0(980)$ can be found, and for the $\pi^0\pi^0$, $\pi^0\eta$ invariant mass spectra of the $D^0 \to \pi^0\pi^0\eta$ decay, where the states $f_0(500)$, $f_0(980)$, and $a_0(980)$ appear. Furthermore, the branching ratios of each decay channel are predicted. We expect more accurate measurements of these decays to better understand the nature of the states $f_0(500)$, $f_0(980)$, and $a_0(980)$.

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I. INTRODUCTION

The three-body charmed meson decays have become important sources for investigating the nature of low-lying scalar resonances. Due to the existence of three mesons in the final states, a large number of scalar resonances are produced in these processes. Recently, many experiments have reported the three-meson decay channels of D mesons. The LHCb Collaboration performed a Dalitz plot analysis of the doubly Cabibbo-suppressed decay $D^+ \to K^-K^+K^+$ for the first time in Ref. [1], where the structures of the $\phi(1020)$, $f_0(980)$, $f_0(1370)$, and $a_0(980)$ states in this decay process were studied. The BESIII Collaboration observed the singly Cabibbo-suppressed decay $D^+ \to \eta\eta\pi^+$ and accurately measured the branching fractions of $D^+ \to \eta\pi^+\pi^0$ and $D^0 \to \eta\pi^+\pi^-$ in Ref. [2], which offered an opportunity to investigate the decays $D \to \rho\eta$, $a_0(980)\pi$, $a_0(980)\eta$. In Ref. [3], the decay of $D^0 \to K^-\pi^+\eta$ was investigated via Dalitz plot analysis by the Belle Collaboration, where the contributions from the states $\bar{K}^*(892)^0$, $a_0(980)^+$, $a_2(1320)^+$, etc., were found, and the ratios of branching fractions in different decay channels were measured. In Ref. [4], the BESIII Collaboration had reported the amplitude analysis results and the most precise branching fraction measurement of $D_s^+ \to K^+K^-\pi^+$, which were consistent with those obtained in previous experiments [5,6]. Moreover, one can find plenty of the processes of D meson decay into three pseudoscalar mesons in the Particle Data Group (PDG) [7]. In the PDG, one can find that most of the final states of D meson three-body decays contain charged mesons, but there are a few processes that have three neutral particles in the final states. For the unique D decay with three neutral pseudoscalar mesons in the final states, the first search for the decay $D^0 \to \pi^0\pi^0\pi^0$ was done in 2006 by the CLEO Collaboration in Ref. [8], where the single tag method was used to obtain a branching fraction upper limit of 3.5×10^{-4} at the 90% confidence level. The decays of $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta, \eta\eta\eta$ were investigated by the BESIII Collaboration in Ref. [9]. The corresponding branching fractions were measured to

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be $\mathcal{B}(D^0 \to \pi^0\pi^0\pi^0) = (2.0 \pm 0.4 \pm 0.3) \times 10^{-4}$, $\mathcal{B}(D^0 \to \pi^0\pi^0\eta) = (3.8 \pm 1.1 \pm 0.7) \times 10^{-4}$, and $\mathcal{B}(D^0 \to \pi^0\eta\eta) = (7.3 \pm 1.6 \pm 1.5) \times 10^{-4}$, respectively. The $D^0 \to \eta\eta\eta$ signal was not observed and the upper limit on its decay branching fraction was $\mathcal{B}(D^0 \to \eta\eta\eta) < 1.3 \times 10^{-4}$ at the 90% confidence level. The $a_0(980)$ state was found in the $\pi^0\eta$ invariant mass distribution of the $D^0 \to \pi^0\eta\eta$ decay. Recently, the CP violation had been looked for by the Belle Collaboration [10] in the processes $D^0 \to \pi^+\pi^-\eta$, $D^0 \to K^+K^-\eta$, and $D^0 \to \phi\eta$, where the corresponding branching fractions were measured. These experimental measurements provided an opportunity for studying the contribution of scalar resonances.

It is challenging to study the three-body decays of D mesons theoretically. As done in Ref. [11] for the scalar form factors of the $D_{(s)} \to f_0(980)$ transition using a covariant quark model, Ref. [12] studied the properties of the $f_0(980)$ resonance in the $D_{(s)} \to f_0(980)\pi/K$ decays and made predictions for the $B_{(s)} \to f_0(980)\pi/K$ decays. In Ref. [13], the CP asymmetries in the three-body decays $D^0 \to K^+K^-\pi^0$, $D^0 \to \pi^+\pi^-\pi^0$, $D^+ \to K^+K_S\pi^0$, and $D_s^+ \to K^0\pi^+\pi^0$ were analyzed through intermediate vector resonances within the framework of the topological amplitude approach for tree amplitudes and the QCD factorization approach for penguin amplitudes. The three-body decay processes $D^0 \to \pi^0\pi^0\pi^0$ and $D^0 \to \pi^0\pi^0\eta$, and the other decay modes, were researched in Ref. [14], where the decay width differences between the two physical eigenstates of the $D^0 - \bar{D}^0$ system were studied. With the analysis of the first measurement of D^0 and D^+ semileptonic decays $D^0 \to a_0(980)^- e^+\nu_e$, $D^+ \to a_0(980)^0 e^+\nu_e$ by the BESIII Collaboration [15], Ref. [16] found no constituent two-quark component in the $a_0(980)$ wave function, where its four-quark production in the semileptonic decay $D \to \eta\pi e^+\nu_e$ was investigated in a further work of [17]. Performing the analysis of semileptonic decays $D \to \pi^+\pi^- e^+\nu_e$, $D_s \to \pi^+\pi^- e^+\nu_e$ from the BESIII and CLEO data [18,19], Ref. [20] supported the interpretation of four-quark nature for the resonances $f_0(500)$ and $f_0(980)$, which were also discussed recently in detail in Ref. [21] for their four-quark nature based on the BESIII data of the decay $J/\psi \to \gamma\pi^0\pi^0$ [22]. Moreover, more discussions on the four-quark nature of the states $f_0(500)$, $f_0(980)$, and $a_0(980)$ can be referred to Refs. [23–27]. Taking into account the final state interactions with the chiral unitary approach (ChUA) [28–33], the decays $D^0 \to \bar{K}^0\pi^+\pi^-$ and $D^0 \to \bar{K}^0\pi^0\eta$ were investigated in Ref. [34], where the contributions of the low-lying scalar resonances $f_0(500)$, $f_0(980)$ in $\pi^+\pi^-$ components and $a_0(980)$ in $\pi^0\eta$ were reproduced and a ratio obtained with their contributions to the branching fractions was in good agreement with the experiment measurement. Reference [35] studied the $f_0(980)$ production in the decays $D_s^+ \to \pi^+\pi^+\pi^-$ and $D_s^+ \to \pi^+K^+K^-$, where the $f_0(980)$ signal in both the $\pi^+\pi^-$ and K^+K^- distributions was found. In Ref. [36], the decay process $D_s^+ \to \pi^+\pi^0\eta$ was studied, and it was found that the $a_0(980)$ resonance could be produced via W-internal emission, but no need to invoke the W-annihilation process, which solved the puzzle of the abnormally large decay rate observed for this decay mode. A continuation of similar work in this direction is done in Refs. [37,38]. More theoretical studies on three-body decays of D mesons with the ChUA can be seen in Refs. [39–43].

In the present work, we study the final state interactions of the singly Cabibbo-suppressed decays of $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$ with the ChUA, where we first get the potential kernel for the hadron-hadron interaction from chiral Lagrangians [44], then solve the Bethe-Salpeter equation in coupled channels. Furthermore, one can calculate the branching fractions of decay channels and make predictions for the invariant mass spectra. Within the ChUA, the scalar resonances are dynamically generated from the hadron-hadron interaction and qualified as molecular states. In our cases, we will consider the contributions of the states $f_0(500)$, $f_0(980)$, and $a_0(980)$ in the two-body final state interactions, and look for their signals in the subsystems $\pi^0\pi^0$ and $\pi^0\eta$. Furthermore, the experimental results indicated that the decays of $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$ were dominated by the W-internal emission and the W-exchange mechanism [9]. According to the analysis in Refs. [36,45,46], the order of weak decay strengths based on topological classification is followed as W-external emission, W-internal emission, W-exchange, W-annihilation, horizontal W-loop and vertical W-loop. Since the W-external emission has no contribution to these decays, we will only consider the contribution of the W-internal emission and omit the W-exchange mechanism.

This paper is organized as follows. In Sec. II, we will introduce the formalism of final state interactions under the ChUA. Then, we will show the results of the $\pi^0\pi^0$ and $\pi^0\eta$ invariant mass distributions and the ratios of branching fractions for the decays $[D^0 \to f_0(980)\pi^0, f_0(980) \to \pi^0\pi^0]$, $[D^0 \to a_0(980)\eta, a_0(980) \to \pi^0\eta]$, $[D^0 \to a_0(980)\pi^0, a_0(980) \to \pi^0\eta]$, $[D^0 \to f_0(500)\eta, f_0(500) \to \pi^0\pi^0]$, and $[D^0 \to f_0(980)\eta, f_0(980) \to \pi^0\pi^0]$ in Sec. III. The conclusion is made in Sec. IV.

II. FORMALISM

The weak decays $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$ can be described by the Feynman diagrams at quark level by means of the W-internal emission mechanism as shown in Fig. 1. We consider all the cases in which the final states contain a π^0 or η meson, and distinguish these three decay processes for analyzing the amplitudes below. First, let us look at Fig. 1(a), the c quark in the D^0 meson produces a d quark and a W^+ boson, while the $\bar{u}$ quark remains a spectator, then the W^+ boson goes to u and $\bar{d}$ quarks. The final $d\bar{d}$ pair quarks can form a π^0 or η meson and the $u\bar{u}$ quarks hadronize by adding an extra $q\bar{q}(\bar{u}u + \bar{d}d + \bar{s}s)$ with the quark pairs created from the vacuum as depicted in Fig. 1(a). This hadronization process can be expressed as

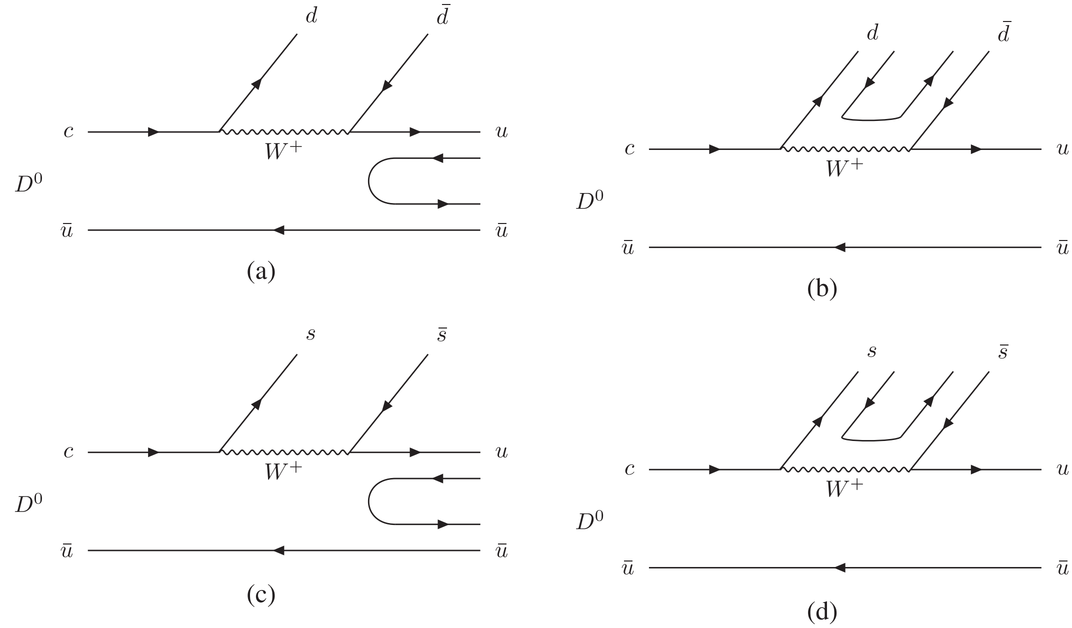

FIG. 1. Dominant diagrams for the D^0 decays with W-internal emission. (a) The creations of $d\bar{d}$ and $u\bar{u}$ quarks, then $u\bar{u}$ pair hadronizes into a final pseudoscalar meson pair. (b) Similar to (a), but $d\bar{d}$ pair hadronizes into a final pseudoscalar meson pair. (c) The creations of $s\bar{s}$ and $u\bar{u}$ quarks, then $u\bar{u}$ pair hadronizes into a final pseudoscalar meson pair. (d) Analogous to (c), but $s\bar{s}$ pair hadronizes into a final pseudoscalar meson pair.

$$H^{(a)} = V_P V_{cd} V_{ud}\left(\left(d\bar{d} \to \frac{-1}{\sqrt{2}}\pi^0\right)[u\bar{u} \to u\bar{u}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)] + \left(d\bar{d} \to \frac{1}{\sqrt{6}}\eta\right)[u\bar{u} \to u\bar{u}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)]\right). \quad (1)$$

Contrarily, the $u\bar{u}$ pair quarks also can merge into a π^0 or η meson and the $d\bar{d}$ quarks hadronize, which is shown in Fig. 1(b). The hadronization process is formulated as

$$H^{(b)} = V_P V_{cd} V_{ud}\left(\left(u\bar{u} \to \frac{1}{\sqrt{2}}\pi^0\right)[d\bar{d} \to d\bar{d}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)] + \left(u\bar{u} \to \frac{1}{\sqrt{6}}\eta\right)[d\bar{d} \to d\bar{d}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)]\right). \quad (2)$$

In Fig. 1(c), the $\bar{u}$ quark also remains a spectator, and the c quark decays into an s quark and a W^+ boson. Then the W^+ boson generates to a u quark and an $\bar{s}$ quark and the $s\bar{s}$ quark pair can merge into an η meson. It should be noted that unlike other diagrams, the $s\bar{s}$ quarks cannot form π^0 in this case. The $u\bar{u}$ quark pair hadronizes with the quark pairs produced from the vacuum $q\bar{q}(\bar{u}u + \bar{d}d + \bar{s}s)$, written as

$$H^{(c)} = V_P V_{cs} V_{us}\left(s\bar{s} \to \frac{-2}{\sqrt{6}}\eta\right)[u\bar{u} \to u\bar{u}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)]. \quad (3)$$

Similar to the process of Fig. 1(b), in Fig. 1(d), the $u\bar{u}$ quark pair merges into a π^0 or η meson and the $s\bar{s}$ quarks hadronize, and we get

$$H^{(d)} = V_P V_{cs} V_{us}\left(\left(u\bar{u} \to \frac{1}{\sqrt{2}}\pi^0\right)[s\bar{s} \to s\bar{s}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)] + \left(u\bar{u} \to \frac{1}{\sqrt{6}}\eta\right)[s\bar{s} \to s\bar{s}\cdot(\bar{u}u + \bar{d}d + \bar{s}s)]\right). \quad (4)$$

In Eqs. (1)–(4), V_P contains all the dynamical factors, which is common to all reactions because of the similar production dynamics, and is called the production vertex. We take this production vertex factor as a constant in the calculation as done in Refs. [47,48]. The $V_{q_1q_2}$ is the element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix from the transition q_1 to q_2 quark. Note that, according to the QCD factorization [49,50] and perturbative QCD [51,52], the dynamics of the production vertexes for the weak decay processes are complicated and in principle energy dependent. Since we only concern the flavor structure of the weak decay processes, one can approximately treat these vertexes as an energy independent constant [53], as we take a universal V_P for the similar production dynamics of these decay processes in Fig. 1. Thus, the differences of these decay processes are specified by the different CKM matrix elements $V_{q_1q_2}$, as shown in Eqs. (1)–(4). Moreover, the factors $1/\sqrt{2}$, $-1/\sqrt{2}$ of π^0 and $1/\sqrt{6}$, $-2/\sqrt{6}$ of η in Eqs. (1)–(4) are due to the prefactor of the flavor component of the π^0 and η, which are taken from

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}|(u\bar{u} - d\bar{d})\rangle, \qquad |\eta\rangle = \frac{1}{\sqrt{6}}|(u\bar{u} + d\bar{d} - 2s\bar{s})\rangle. \tag{5}$$

Then we define the matrix M for the $q\bar{q}$ elements,

$$M = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}, \tag{6}$$

so we can easily get the following formulae for the hadronization processes:

$$u\bar{u} \cdot (\bar{u}u + \bar{d}d + \bar{s}s) = (M \cdot M)_{11}, \tag{7}$$

$$d\bar{d} \cdot (\bar{u}u + \bar{d}d + \bar{s}s) = (M \cdot M)_{22}, \tag{8}$$

$$s\bar{s} \cdot (\bar{u}u + \bar{d}d + \bar{s}s) = (M \cdot M)_{33}. \tag{9}$$

In terms of the pseudoscalar meson fields, the SU(3) matrix M is given by

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{10}$$

where we take $\eta \equiv \eta_8$, and the singlet of SU(3) components η_1 is removed since it does not lead to any interaction in chiral perturbation theory [54]. The hadronization process in quark level in Eqs. (7)–(9) can be accomplished to the hadron level in terms of two pseudoscalar mesons, given by

$$(M \cdot M)_{11} = (\Phi \cdot \Phi)_{11} = \pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{\sqrt{3}}\pi^0\eta + K^+K^- + \frac{1}{6}\eta\eta, \tag{11}$$

$$(M \cdot M)_{22} = (\Phi \cdot \Phi)_{22} = \pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 - \frac{1}{\sqrt{3}}\pi^0\eta + K^0\bar{K}^0 + \frac{1}{6}\eta\eta, \tag{12}$$

$$(M \cdot M)_{33} = (\Phi \cdot \Phi)_{33} = K^+K^- + K^0\bar{K}^0 + \frac{2}{3}\eta\eta. \tag{13}$$

Then, after the hadronization, we get the final states with π^0 or η as follows:

$$H^{(a)} = V_P V_{cd} V_{ud}\left(\left(\frac{-1}{\sqrt{2}}\pi^0\right)\left(\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{\sqrt{3}}\pi^0\eta + K^+K^- + \frac{1}{6}\eta\eta\right) + \frac{1}{\sqrt{6}}\eta\left(\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{\sqrt{3}}\pi^0\eta + K^+K^- + \frac{1}{6}\eta\eta\right)\right), \tag{14}$$

$$H^{(b)} = V_P V_{cd} V_{ud}\left(\frac{1}{\sqrt{2}}\pi^0\left(\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 - \frac{1}{\sqrt{3}}\pi^0\eta + K^0\bar{K}^0 + \frac{1}{6}\eta\eta\right) + \frac{1}{\sqrt{6}}\eta\left(\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 - \frac{1}{\sqrt{3}}\pi^0\eta + K^0\bar{K}^0 + \frac{1}{6}\eta\eta\right)\right), \tag{15}$$

$$H^{(c)} = V_P V_{cs} V_{us}\left(\frac{-2}{\sqrt{6}}\eta\right)\left(\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{\sqrt{3}}\pi^0\eta + K^+K^- + \frac{1}{6}\eta\eta\right), \tag{16}$$

$$H^{(d)} = V_P V_{cs} V_{us} \left(\frac{1}{\sqrt{2}} \pi^0 \left(K^+ K^- + K^0 \bar{K}^0 + \frac{2}{3} \eta\eta \right) + \frac{1}{\sqrt{6}} \eta \left(K^+ K^- + K^0 \bar{K}^0 + \frac{2}{3} \eta\eta \right) \right). \tag{17}$$

Note that the elements of the CKM matrix are $V_{cd} = -V_{us}$, $V_{ud} = V_{cs}$ [7], leading to $V_{cd}V_{ud} = -V_{us}V_{cs}$. Thus, we get the total contributions for Figs. 1(a)–1(d),

$$H = H^{(a)} + H^{(b)} + H^{(c)} + H^{(d)} = C\left(-\sqrt{2}\pi^0 K^+ K^- + \frac{4}{\sqrt{6}} \pi^+ \pi^- \eta + \frac{2}{\sqrt{6}} \eta K^+ K^- \right), \tag{18}$$

where C is a global factor, which absorbs the production vertex V_P and the elements of the CKM matrix $V_{cd}V_{ud}$ or $V_{cs}V_{us}$, and also includes the normalization factor used to match the events of the experimental data. Note that there are no final states $\pi^0\pi^0\pi^0$, $\pi^0\pi^0\eta$, or $\pi^0\eta\eta$ directly produced that we want in the D^0 decay, since these final states are cancelled by each other in the summation of Eqs. (14)–(17). However, upon the rescattering of the terms in Eq. (18), we can get them via the final state interactions, as depicted in Fig. 2. Then we get the amplitudes for the $D^0 \to \pi^0\pi^0\pi^0$ decay,

$$t_{D^0 \to \pi^0\pi^0\pi^0} = -\sqrt{2} C G_{K^+K^-}(s_{\pi^0\pi^0}) T_{K^+K^- \to \pi^0\pi^0}(s_{\pi^0\pi^0}), \tag{19}$$

for the $D^0 \to \pi^0\pi^0\eta$ decay,

$$t_{D^0 \to \pi^0\pi^0\eta} = C\left[\frac{4}{\sqrt{6}} G_{\pi^+\pi^-}(s_{\pi^0\pi^0}) T_{\pi^+\pi^- \to \pi^0\pi^0}(s_{\pi^0\pi^0}) + \frac{2}{\sqrt{6}} G_{K^+K^-}(s_{\pi^0\pi^0}) T_{K^+K^- \to \pi^0\pi^0}(s_{\pi^0\pi^0}) - \sqrt{2} G_{K^+K^-}(s_{\pi^0\eta}) T_{K^+K^- \to \pi^0\eta}(s_{\pi^0\eta}) \right], \tag{20}$$

and the one for the $D^0 \to \pi^0\eta\eta$ decay,

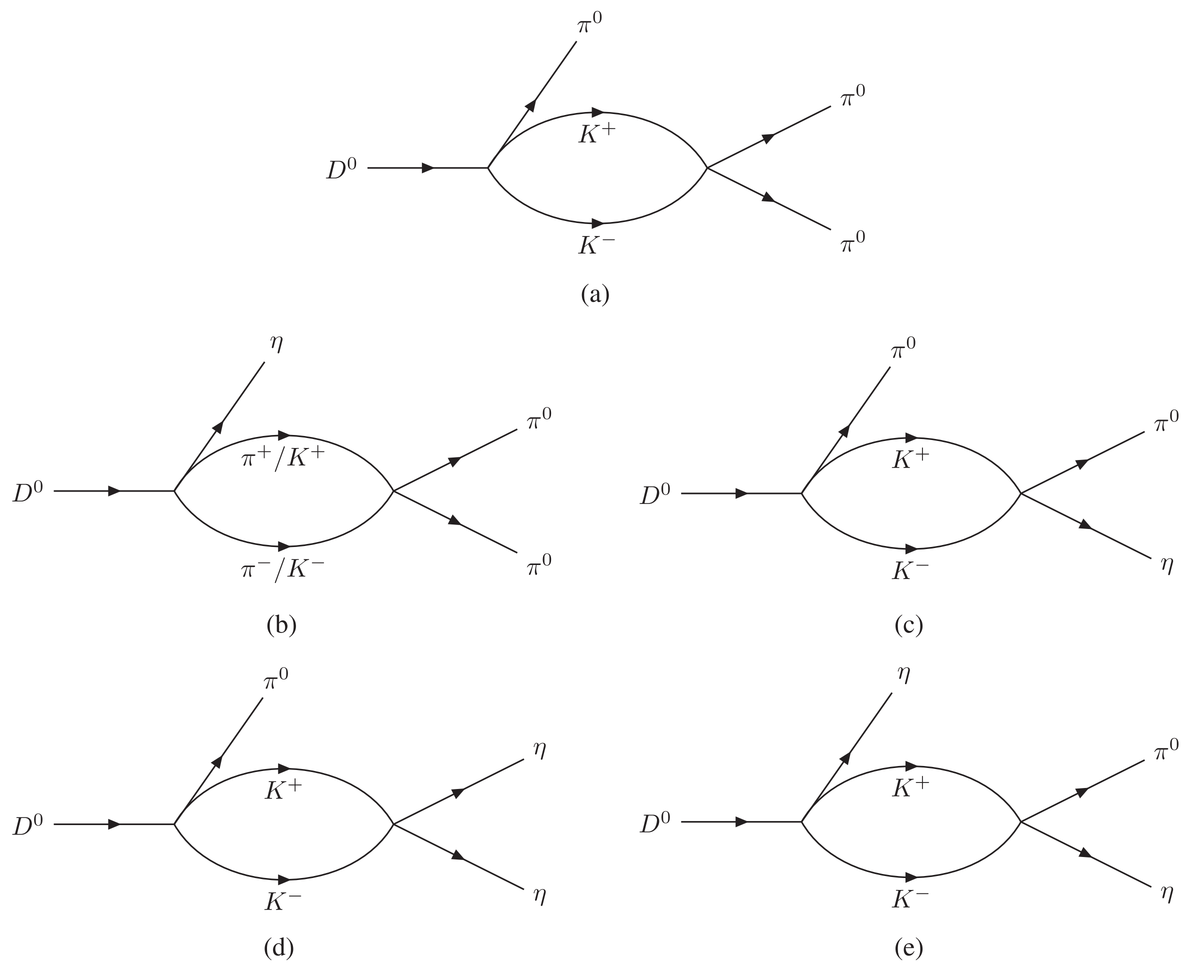

FIG. 2. Diagrammatic representation for the final state interactions of meson pairs. (a) For the $D^0 \to \pi^0\pi^0\pi^0$ decay. (b) For the $D^0 \to \pi^0\pi^0\eta$ decay. (c) For the $D^0 \to \pi^0\pi^0\eta$ decay. (d) For the $D^0 \to \pi^0\eta\eta$ decay. (e) For the $D^0 \to \pi^0\eta\eta$ decay.

$$t_{D^0\to\pi^0\eta\eta} = C\left[-\sqrt{2}G_{K^+K^-}(s_{\eta\eta})T_{K^+K^-\to\eta\eta}(s_{\eta\eta}) + \frac{2}{\sqrt{6}}G_{K^+K^-}(s_{\pi^0\eta})T_{K^+K^-\to\pi^0\eta}(s_{\pi^0\eta})\right]. \quad (21)$$

For isospin $I = 0$, we consider five coupled channels, $\pi^+\pi^-$ (1), $\pi^0\pi^0$ (2), K^+K^- (3), $K^0\bar{K}^0$ (4), and $\eta\eta$ (5). For isospin $I = 1$, we consider the contribution of three coupled channels, K^+K^- (1), $K^0\bar{K}^0$ (2), and $\pi^0\eta$ (3). Thus, the $\pi^+\pi^-$, $\pi^0\pi^0$, and $\eta\eta$ channels only contribute to $I = 0$ [41], and the $\pi^0\eta$ channel only contributes to $I = 1$. The K^+K^- and $K^0\bar{K}^0$ channels contribute to both isospins, $I = 0$ and $I = 1$, taking into account the isospin decomposition of the $K\bar{K}$ channel,

$$|K^+K^-\rangle = -\frac{1}{\sqrt{2}}|K\bar{K}\rangle_{I=1,I_3=0} - \frac{1}{\sqrt{2}}|K\bar{K}\rangle_{I=0,I_3=0}, \quad (22)$$

$$|K^0\bar{K}^0\rangle = \frac{1}{\sqrt{2}}|K\bar{K}\rangle_{I=1,I_3=0} - \frac{1}{\sqrt{2}}|K\bar{K}\rangle_{I=0,I_3=0}, \quad (23)$$

where we have used the convention that $|K^+\rangle = -|1/2, 1/2\rangle$ for the isospin basis [28]. Thus, the final state interaction amplitudes in Eqs. (19)–(21) are given by

$$T_{K^+K^-\to\pi^0\pi^0} = \frac{1}{2}(T_{K^0\bar{K}^0\to\pi^0\pi^0} + T_{K^+K^-\to\pi^0\pi^0}), \quad (24)$$

$$T_{K^+K^-\to\eta\eta} = \frac{1}{2}(T_{K^0\bar{K}^0\to\eta\eta} + T_{K^+K^-\to\eta\eta}), \quad (25)$$

$$T_{K^+K^-\to\pi^0\eta} = \frac{1}{2}(T_{K^+K^-\to\pi^0\eta} - T_{K^0\bar{K}^0\to\pi^0\eta}). \quad (26)$$

In addition, the $G_{PP'}$ in Eqs. (19)–(21) is the loop function of two-meson (PP') propagators, which is given by

$$G_{PP'}(s) = i\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m_1^2 + i\varepsilon} \times \frac{1}{(q_1 + q_2 - q)^2 - m_2^2 + i\varepsilon}, \quad (27)$$

where q_1 and q_2 are the four-momenta of the two initial particles, respectively, m_1 and m_2 are the masses of the two intermediate particles (PP'), and $s = (q_1 + q_2)^2$. The integral of this equation is logarithmically divergent, and we take the formula of the dimensional regularization method to solve this singular integral [55–58],

$$\begin{aligned}G_{PP'}(s) = \frac{1}{16\pi^2}\Big\{&a_{PP'}(\mu) + \ln\frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + s}{2s}\ln\frac{m_2^2}{m_1^2} \\ &+ \frac{q_{cm}(s)}{\sqrt{s}}[\ln(s - (m_2^2 - m_1^2) + 2q_{cm}(s)\sqrt{s}) \\ &+ \ln(s + (m_2^2 - m_1^2) + 2q_{cm}(s)\sqrt{s}) \\ &- \ln(-s - (m_2^2 - m_1^2) + 2q_{cm}(s)\sqrt{s}) \\ &- \ln(-s + (m_2^2 - m_1^2) + 2q_{cm}(s)\sqrt{s})]\Big\}, \end{aligned} \quad (28)$$

where μ is the regularization scale, $a_{PP'}(\mu)$ the subtraction constant, and we take $\mu = 0.6$ GeV from Ref. [37]. As discussed in Ref. [37], following Eq. (17) of Ref. [32], one has

$$a_{PP'}(\mu) = -2\log\left(1 + \sqrt{1 + \frac{m_1^2}{\mu^2}}\right) + \cdots, \quad (29)$$

where m_1 is the mass of a larger-mass meson in the coupled channels.[1] Then we get the values of the subtraction constants $a_{\pi^+\pi^-} = -1.41$, $a_{\pi^0\pi^0} = -1.41$, $a_{K^+K^-} = -1.66$, $a_{K^0\bar{K}^0} = -1.66$, $a_{\eta\eta} = -1.71$, and $a_{\pi^0\eta} = -1.71$. Besides, $q_{cm}(s)$ is the three-momentum of the particle in the center-of-mass frame, given by

$$q_{cm}(s) = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}, \quad (30)$$

with the usual Källén triangle function $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$.

Besides, T_{ij} is an element of the scattering amplitude matrices for the transitions of channel $i \to j$ in the ChUA evaluated by the coupled channel Bethe-Salpeter equation

$$T = [1 - VG]^{-1}V, \quad (31)$$

where the matrix V is constructed by the scattering potentials of each coupled channel and obtained from the lowest order chiral Lagrangians. For the $I = 0$ sector, it is a 5×5 symmetric matrix, of which the elements are given by [54],

[1]Note that in Ref. [32] m_1 is the mass of the baryon in meson-baryon coupled channels.

$$V_{11} = -\frac{1}{2f^2}s, \quad V_{12} = -\frac{1}{\sqrt{2}f^2}(s - m_\pi^2), \quad V_{13} = -\frac{1}{4f^2}s,$$
$$V_{14} = -\frac{1}{4f^2}s, \quad V_{15} = -\frac{1}{3\sqrt{2}f^2}m_\pi^2, \quad V_{22} = -\frac{1}{2f^2}m_\pi^2,$$
$$V_{23} = -\frac{1}{4\sqrt{2}f^2}s, \quad V_{24} = -\frac{1}{4\sqrt{2}f^2}s, \quad V_{25} = -\frac{1}{6f^2}m_\pi^2,$$
$$V_{33} = -\frac{1}{2f^2}s, \quad V_{34} = -\frac{1}{4f^2}s,$$
$$V_{35} = -\frac{1}{12\sqrt{2}f^2}(9s - 6m_\eta^2 - 2m_\pi^2), \quad V_{44} = -\frac{1}{2f^2}s,$$
$$V_{45} = -\frac{1}{12\sqrt{2}f^2}(9s - 6m_\eta^2 - 2m_\pi^2),$$
$$V_{55} = -\frac{1}{18f^2}(16m_K^2 - 7m_\pi^2), \tag{32}$$

and the one for the $I = 1$ sector is a 3×3 symmetric matrix [34], having

$$V_{11} = -\frac{1}{2f^2}s, \quad V_{12} = -\frac{1}{4f^2}s,$$
$$V_{13} = -\frac{\sqrt{3}}{12f^2}\left(3s - \frac{8}{3}m_K^2 - \frac{1}{3}m_\pi^2 - m_\eta^2\right), \quad V_{22} = -\frac{1}{2f^2}s,$$
$$V_{23} = \frac{\sqrt{3}}{12f^2}\left(3s - \frac{8}{3}m_K^2 - \frac{1}{3}m_\pi^2 - m_\eta^2\right), \quad V_{33} = -\frac{1}{3f^2}m_\pi^2, \tag{33}$$

where f is the pion decay constant, and we take $f = 0.093$ GeV [28].

Finally, the formula of double differential width for a three-body decay process is given by [7,41]

$$\frac{d^2\Gamma}{d\sqrt{s_{12}}d\sqrt{s_{23}}} = \frac{1}{(2\pi)^3}\frac{\sqrt{s_{12}}\sqrt{s_{23}}}{8m_{D^0}^3}\frac{1}{N}|t_{D^0\to P_1P_2P_3}|^2, \tag{34}$$

where N is the number of identical particles in the final states [41], like $N = 3$ for the decay $D^0 \to \pi^0\pi^0\pi^0$. For the decays $D^0 \to \pi^0\pi^0\eta$ and $D^0 \to \pi^0\eta\eta$, there are two cases, $N = 2$ for the final states of $\pi^0\eta$ components in these two decays, $N = 1$ for the final states of $\pi^0\pi^0$ components in the $D^0 \to \pi^0\pi^0\eta$ decay and $\eta\eta$ components in the $D^0 \to \pi^0\eta\eta$ decay. Also, $t_{D^0\to P_1P_2P_3} \equiv t_{D^0\to P_1P_2P_3}(s_{12}, s_{13}, s_{23})$ with P_i $(i = 1, 2, 3)$ representing the pseudoscalar mesons, which depends on the invariant masses of two components s_{12}, s_{13}, s_{23}, where the indices 1 to 3 denote the final meson state accordingly and only two of these variables are independent, because of

$$s_{12} + s_{23} + s_{13} = m_{D^0}^2 + m_1^2 + m_2^2 + m_3^2. \tag{35}$$

Then, the $d\Gamma/d\sqrt{s_{12}}$ and $d\Gamma/d\sqrt{s_{23}}$ can be obtained by integrating Eq. (34) over each of the invariant mass variables. Furthermore, one can obtain $d\Gamma/d\sqrt{s_{13}}$ through Eq. (35). Note that, for these final states $\pi^0\pi^0\pi^0$, $\pi^0\pi^0\eta$, and $\pi^0\eta\eta$, there is no vector meson in the p-wave contributed, and thus, we do not need to consider the primary vector meson produced in the weak decay processes in the present work. In principle, one can have more complicated decay chains, such as $D^0 \to \rho^-(\to \pi^0\pi^-)\pi^+ \to \pi^0\pi^-\pi^+ \to \pi^0\pi^0\pi^0$ with the final state interactions at the end, which will be more suppressed and can be ignored. Furthermore, since the subsystems $\pi\pi$ and $\pi\eta$ with the strangeness $S = 0$ in the final states $\pi^0\pi^0\pi^0$, $\pi^0\pi^0\eta$, and $\pi^0\eta\eta$ can not couple to the $S = 1$ channels $K\pi$ and $K\eta$, there is no contribution from the $K_0^*(700)$ resonance (called κ) in these final state interactions, which can be dynamically generated in the coupled channel interactions of the channels $K\pi$ and $K\eta$. Our cases are different from the case of the $D^0 \to K^-\pi^+\eta$ decay as discussed in Ref. [40], where the $K_0^*(700)$ resonance contributed to the final state interactions.

III. RESULTS

In our model, we have only one parameter C for the global normalization in Eqs. (19)–(21). In the introduction, we have mentioned that the BESIII Collaboration had reported the decay of $D^0 \to \pi^0\eta\eta$, where the $\pi^0\eta$ invariant mass spectrum was given in Ref. [9]. We first fit the invariant mass spectrum to determine the value of parameter C. It is worth emphasizing that the parameter C only determines the overall strength, but does not affect the trend of the curve. Our result is shown in Fig. 3, which clearly shows that the fitting result is in a good agreement with the

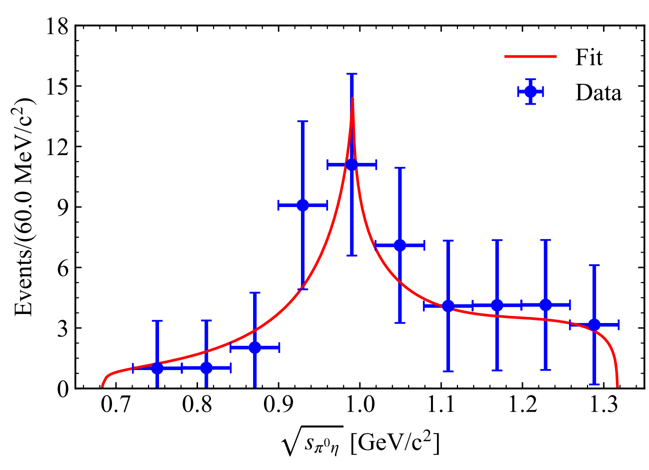

FIG. 3. The $\pi^0\eta$ invariant mass distribution of the $D^0 \to \pi^0\eta\eta$ decay. The parameter $C = 918.33$ is obtained with the reduced chi-square $\chi^2/\mathrm{dof.} = 1.82/(10 - 1) = 0.20$. Data is taken from [9].

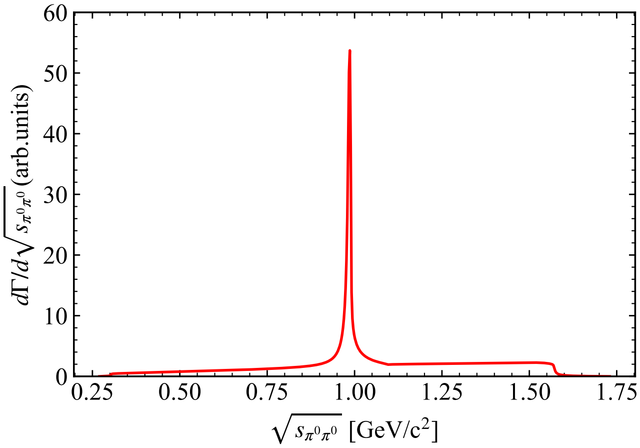

FIG. 4. The $\pi^0\pi^0$ invariant mass distribution of the $D^0 \to \pi^0\pi^0\pi^0$ decay.

experimental data. We do not add the resonant state $a_0(980)$ to the theoretical formula by hand. The structure of the peak near the threshold of $K\bar{K}$ in Fig. 3 is dynamically generated with the ChUA, where the $a_0(980)$ state can be well reproduced. It should be noted that the result shows the typical cusp effect for the $a_0(980)$, which is consistent with many calculations [36,40,59–61]. In high precision experimental measurements, this kind of cusp structure appears quite evidently for the $a_0(980)$ resonance [62–65]. A similar situation is also visible in recent lattice QCD simulations [36]. The experimental data of the $D^0 \to \pi^0\eta\eta$ decay from BESIII Collaboration did not show this characteristic clearly, because the sampling intervals and errors are very large. It is expected that future experiments can give more accurate measurements.

We show the $\pi^0\pi^0$ invariant mass distribution of the decay $D^0 \to \pi^0\pi^0\pi^0$ in Fig. 4, where the peak of $f_0(980)$ rises near the $K\bar{K}$ threshold with no signal of $f_0(500)$ in the invariant mass spectrum. From the results in the ChUA [28,61], we know that the resonance $f_0(980)$ is the bound state of $K\bar{K}$ components, and the $f_0(500)$ state is mainly contributed by the $\pi\pi$ channel. The $\pi^0\pi^0$ invariant mass spectrum of the decay $D^0 \to \pi^0\pi^0\pi^0$ is contributed by the amplitude of $K^+K^- \to \pi^0\pi^0$ as shown in Eq. (19). Thus, the absence of the $f_0(500)$ in this result is not surprising, and indicates the different nature of these two states.

In Fig. 5, we show the $\pi^0\pi^0$ invariant mass distribution in the $D^0 \to \pi^0\pi^0\eta$ decay in subfigure (a), and the one of $\pi^0\eta$ in subfigure (b). One can see the $a_0(980)$ signal in the $\pi^0\eta$ invariant mass spectrum, and the $f_0(980)$ and $f_0(500)$ signal in the one with $\pi^0\pi^0$ components. Note that both the $\pi^0\pi^0$ and $\pi^0\eta$ components have significant contributions in the final state interactions of the $D^0 \to \pi^0\pi^0\eta$ decay. This is different from the case of the $D^0 \to \pi^0\pi^0\pi^0$ decay, where there is only $\pi^0\pi^0$ invariant mass distribution, and the case of the $D^0 \to \pi^0\eta\eta$ decay, where the interaction of $\pi^0\eta$ components is little affected by the ones of $\eta\eta$. The Dalitz plot for the $D^0 \to \pi^0\pi^0\eta$ decay is shown in Fig. 6, where the red solid line (vertical one) is the position of the $a_0(980)$ state, the blue solid line (upper horizontal one) is the position of the $f_0(980)$ resonance, and the green solid line (lower horizontal one) is the $f_0(500)$ state, where the PDG values for the masses of each particle [7] are used in the plot. The $a_0(980)$ state contributes in the region of $0.15 < s_{\pi^0\pi^0} < 1.7\ \mathrm{GeV}^2/\mathrm{c}^4$ of the $\pi^0\pi^0$ invariant mass distribution, the $f_0(980)$ state contributes in the region of $0.5 < s_{\pi^0\eta} < 2.4\ \mathrm{GeV}^2/\mathrm{c}^4$, and the $f_0(500)$ resonance contributes in the region of $0.6 < s_{\pi^0\eta} < 3.0\ \mathrm{GeV}^2/\mathrm{c}^4$ of the $\pi^0\eta$ invariant mass distribution. Then we analyze the contributions of the $\pi^0\pi^0$ and $\pi^0\eta$ components to the $\pi^0\pi^0$ and $\pi^0\eta$ invariant mass spectra, which are shown in Fig. 7. The broad peak of the $f_0(500)$ resonance is obvious in Fig. 7(a), which should be a contribution from the transition $\pi^+\pi^- \to \pi^0\pi^0$ in Eq. (20), and the small peak near the $K\bar{K}$ threshold is the $f_0(980)$ state, which should be a contribution from the transition $K^+K^- \to \pi^0\pi^0$ in Eq. (20). In Fig. 7(b), there is no obvious peak structure in the $\pi^0\pi^0$

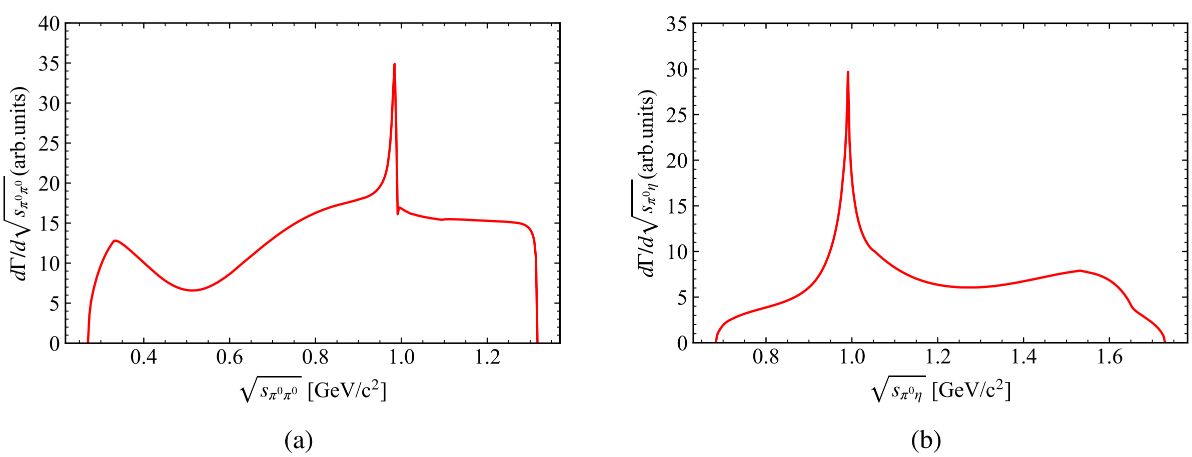

FIG. 5. $\pi^0\pi^0$ (a) and $\pi^0\eta$ (b) invariant mass distributions of the $D^0 \to \pi^0\pi^0\eta$ decay.

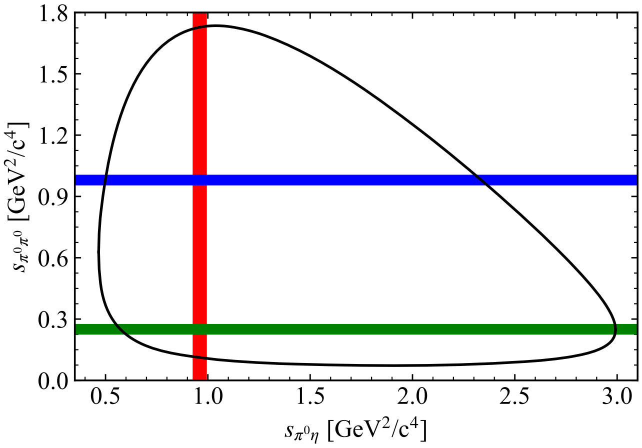

FIG. 6. Dalitz plot of the $D^0 \to \pi^0\pi^0\eta$ decay.

invariant mass spectrum from the contribution of $\pi^0\eta$ parts. Thus, we can confirm that in Fig. 5(a), the low energy region is the $f_0(500)$ resonance, and the peak near the $K\bar{K}$ threshold is the $f_0(980)$ state, which is enhanced by the interference effect with the contribution of the $\pi^0\eta$ components. Then, from the results shown in Figs. 7(c) and 7(d), we know the peak structure near the $K\bar{K}$ threshold in Fig. 5(b) is the $a_0(980)$ resonance with no $f_0(980)$ contribution.

We then make some predictions for the ratios of branching fractions in different decay processes. In our theoretical model, the parameter of the production vertex V_P in Eqs. (14)–(17) is unknown (see more discussion in Ref. [48]). Thus we calculate the ratios of branching fractions for different decay channels as follows, where the unknown production vertex V_P can be cancelled. Therefore, the results for these ratios are independent with parameter V_P and more reliable. By integrating the invariant mass variables in the decays $D^0 \to \pi^0\pi^0\pi^0$ and $D^0 \to \pi^0\eta\eta$ over the invariant mass distributions, we find

$$\frac{\mathcal{B}[D^0 \to f_0(980)\pi^0, f_0(980) \to \pi^0\pi^0]}{\mathcal{B}[D^0 \to a_0(980)\eta, a_0(980) \to \pi^0\eta]} = 1.01^{+0.10}_{-0.10}, \tag{36}$$

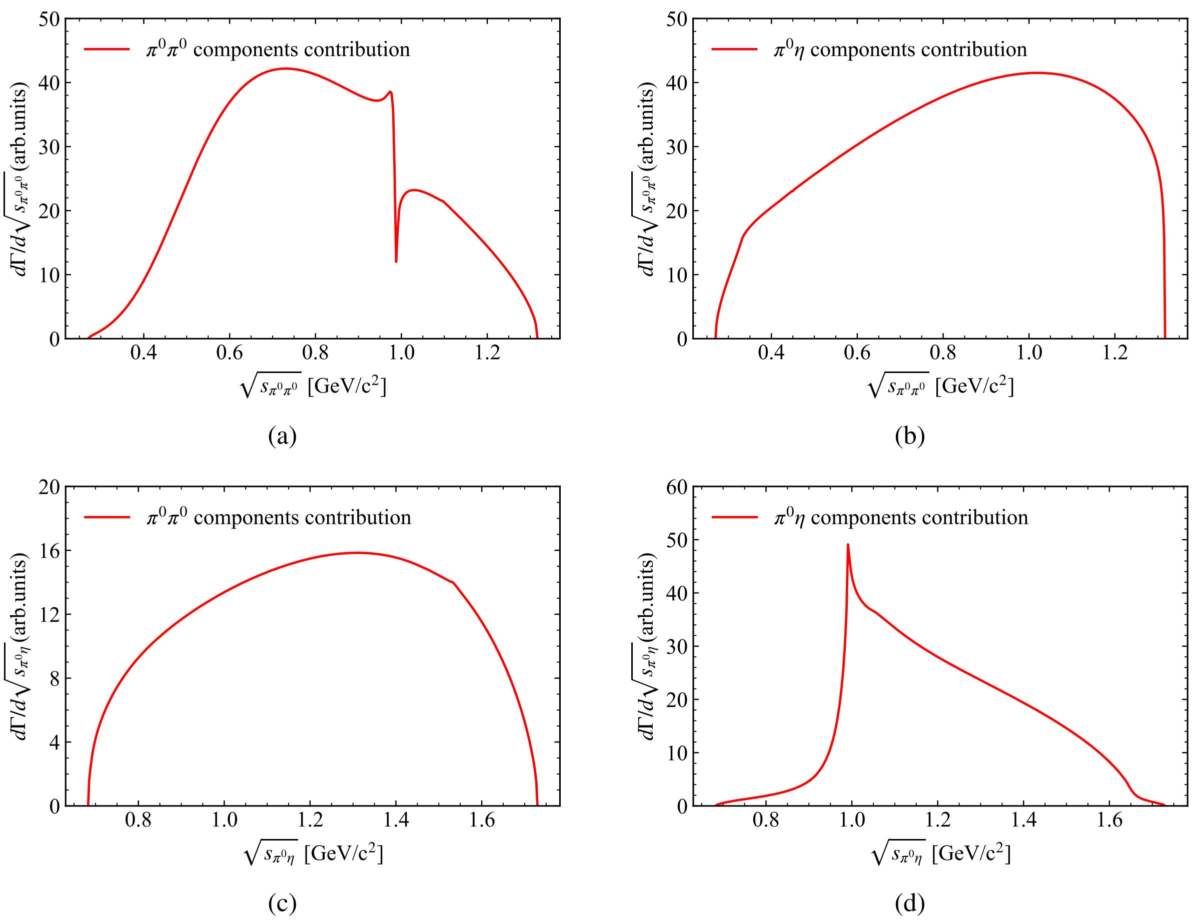

FIG. 7. Different components contributed to the $\pi^0\pi^0$ and $\pi^0\eta$ invariant mass distributions of the $D^0 \to \pi^0\pi^0\eta$ decay. (a) $\pi^0\pi^0$ components contributed to the $\pi^0\pi^0$ invariant mass distributions. (b) $\pi^0\eta$ components contributed to the $\pi^0\pi^0$ invariant mass distributions. (c) $\pi^0\pi^0$ components contributed to the $\pi^0\eta$ invariant mass distributions. (d) $\pi^0\eta$ components contributed to the $\pi^0\eta$ invariant mass distributions.

where the integral limits are $[2m_{\pi^0}, 1.2]$ GeV and $[m_{\pi^0} + m_\eta, 1.2]$ GeV for $D^0 \to f_0(980)\pi^0$ and $D^0 \to a_0(980)\eta$, respectively, with the uncertainties from the integrated upper limit being 1.2 ± 0.05 GeV. Analogously, we get

$$\frac{\mathcal{B}[D^0 \to a_0(980)\pi^0, a_0(980) \to \pi^0\eta]}{\mathcal{B}[D^0 \to a_0(980)\eta, a_0(980) \to \pi^0\eta]} = 1.87^{+0.22}_{-0.23}, \quad (37)$$

$$\frac{\mathcal{B}[D^0 \to f_0(500)\eta, f_0(500) \to \pi^0\pi^0]}{\mathcal{B}[D^0 \to a_0(980)\eta, a_0(980) \to \pi^0\eta]} = 3.50^{+0.54}_{-0.53}, \quad (38)$$

$$\frac{\mathcal{B}[D^0 \to f_0(980)\eta, f_0(980) \to \pi^0\pi^0]}{\mathcal{B}[D^0 \to a_0(980)\eta, a_0(980) \to \pi^0\eta]} = 2.55^{+0.48}_{-0.50}, \quad (39)$$

where the integral limits and uncertainties for the $D^0 \to a_0(980)\pi^0$ decay are the same as $D^0 \to a_0(980)\eta$. For the decays $D^0 \to f_0(500)\eta$ and $D^0 \to f_0(980)\eta$, the integral limits are $[2m_{\pi^0}, 0.9]$ GeV and [0.9, 1.2] GeV, respectively, where the uncertainties are obtained from the integrated limit of 0.9 ± 0.05 GeV, as done in Ref [48]. As one can see from the results in Eqs. (36)–(39), the ratios of the branching fractions of these decay channels are at the same order of magnitude, of which the different values are from about 1 to 3.5. The branching fractions of the decays $D^0 \to \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$, measured by BESIII Collaboration, are also at the same order of magnitude [9], $\mathcal{B}(D^0 \to \pi^0\pi^0\pi^0) = (2.0 \pm 0.4 \pm 0.3) \times 10^{-4}$, $\mathcal{B}(D^0 \to \pi^0\pi^0\eta) = (3.8 \pm 1.1 \pm 0.7) \times 10^{-4}$, and $\mathcal{B}(D^0 \to \pi^0\eta\eta) = (7.3 \pm 1.6 \pm 1.5) \times 10^{-4}$, which implies that the scalar resonances $f_0(500)$, $f_0(980)$, or $a_0(980)$ are dominant in these D^0 meson decay processes. We hope that their contributions can be measured in the future experiments.

IV. CONCLUSIONS

In the present work, we make a theoretical study of the singly Cabibbo-suppressed processes $D^0 \to \pi^0\pi^0\pi^0$, $\pi^0\pi^0\eta, \pi^0\eta\eta$ by taking into account the final state interactions. We have presented the $\pi^0\pi^0$ and $\pi^0\eta$ invariant mass distributions of these decay processes, where the scalar resonances $f_0(500)$, $f_0(980)$, and $a_0(980)$ are dynamically generated in the S-wave interactions with the ChUA. The results are in a good agreement with the experimental data with only one parameter. Indeed, the final states $\pi^0\pi^0\pi^0$, $\pi^0\pi^0\eta$, and $\pi^0\eta\eta$ are not possibly produced at the tree level [see Eqs. (19)–(21)], and all the contributions come from the rescattering of the two-body final state interactions. For the $D^0 \to \pi^0\pi^0\pi^0$ decay, the dominant contributions come from the $I = 0$ resonance $f_0(980)$. For the $D^0 \to \pi^0\pi^0\eta$ decay, the contributions come from the $I = 0$ states $f_0(500)$ and $f_0(980)$ in the $\pi^0\pi^0$ components, and the $I = 1$ state $a_0(980)$ in the ones of $\pi^0\eta$. For the $D^0 \to \pi^0\eta\eta$ decay, the dominant contributions are contributed by the $I = 1$ resonance $a_0(980)$. With the analysis of the invariant mass spectra and the corresponding amplitudes, we find that the main components of $f_0(500)$ are the $\pi\pi$ and the dominant components of $f_0(980)$ are the $K\bar{K}$, which are consistent with the analysis of Ref. [61]. These results indicate that these resonances are dynamically generated from the final state interactions of the pseudoscalar meson pairs, which show the molecular nature of these resonances. Moreover, we also calculate the ratios of the corresponding branching fractions. Finally, we hope that our predicted $\pi^0\pi^0$ invariant mass distribution for the decay of $D^0 \to \pi^0\pi^0\pi^0$, and $\pi^0\pi^0$, $\pi^0\eta$ invariant mass distributions for the decay of $D^0 \to \pi^0\pi^0\eta$, can be measured by future experiments.

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