Massive two-loop heavy particle diagrams

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We determine the master integrals for vertex and propagator diagrams that appear in effective field theories containing heavy fields. The integrals involve at least one heavy line, and the standard lines include an arbitrary mass scale. The evaluation is done analytically with modern techniques. We employ the methods of differential equations and dimensional recurrence relations to evaluate said integrals up to two-loop order.

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I. INTRODUCTION

Heavy particle effective theories (HPET) have a wide range of applicability, and its use cases have expanded dramatically in recent years. They are apparent when a field of arbitrary spin in a given theory is taken to have a large mass compared to other propagating massive degrees of freedom. HPETs were originally conceived in the context of quantum electrodynamics (QED) and quantum chromodynamics (QCD), such as in heavy quark effective theory (HQET), nonrelativistic (NR) QCD and QED and variations therein [1–3]. More recently, it has also been applied in the electroweak (EW) regime [4–10], as well as beyond the Standard Model (SM) such as in the context of heavy dark matter [11,12], Z' bosons [13,14] and black hole interactions [15–17].

When dealing with such theories beyond leading perturbative order, one is faced with loop diagrams containing eikonal lines. In this work, we determine these at two-loop order employing a set of modern techniques, in particular differential equations and dimensional recurrence relations, which have been successful in similar contexts [18,19]. We further include a nonzero mass-scale in the standard lines for theoretical models with massive propagating degrees of freedom. The mass scale bounds the infrared (IR) regime for the two- and three-point diagrams studied here. Even in theories with exclusively massless propagating degrees of freedom such as QED/QCD and gravity, the IR structure needs to be correctly understood [17,20]. The diagrams considered here are especially useful in the evaluation of form factors of a given model. The form factor is most well known for its uses in perturbative analyses of scattering processes occurring at the LHC and future colliders [21,22]. Form factors are of primary consideration instead of specific processes as they form the fundamental building blocks for a vast array of processes. For instance, they have been employed to study dijet, $\bar{t}t$, squark pair, and dark matter (DM) production in various models [8,11,23,24]. It is also the simplest amplitude that can be used to study the IR behavior of a theory of interest. For further reference in the context of the SM, the QCD form factors of quarks have been evaluated to three-loop order [20,25–28], and the EW corrections using both EFT and IR evolution equations are currently being studied to two-loop order [4,5,10,29–34].

On the other hand, there has also been significant progress in the realm of Feynman diagram evaluation. When previously, certain classes of multi-loop diagrams were intractable, they have now become determinable with the help of novel techniques. Most notably, diagrams with masses are now attainable with the differential equations method [35–39]. The basis of which is set upon differentiating the master integrals (MIs) of interest, forming a system of differential equations and reducing said system to so-called ε-form [40–42]. Given that such a reduction is achievable [42] and the obtained differential system is rational, the MIs are expressible in terms of multiple polylogarithms (MPLs) [43,44]. At present, we have also good progress in understanding equations not reducible to ε-form[45] and functions beyond multiple polylogarithms such as for example elliptical polylogarithms

(EPLs) [48–59], or entirely novel functions [59–67]. In our case, the diagrams we encounter are reducible to ε-form with rational kernels and thus, can be written in terms of MPLs. However, when we take external lines off-shell, the integrals are only reducible to $A + B\varepsilon$ form, as we will see. One must thus resort to EPLs to solve such MIs. This work provides results for the massive heavy-heavy, heavy-light and propagator diagrams at two-loop order. The results are explicitly given up to $\mathcal{O}(\varepsilon^2)$ working in $D = 4 - 2\varepsilon$ dimensions, which is the appropriate order for SM-like theories [6–8,10]. However, this order is arbitrary, as the results are simply attainable for any order in ε.

Outlining this paper, we begin in Sec. II by presenting our two-loop integral families and their associated differential equations. We then reduce each set of differential equations to either ε-form or $A + B\varepsilon$ form, illustrating the sequence of balance transformations required. In Sec. III we proceed to solve the differential equations and explicitly present results for each integral expanded up to an appropriate order in ε. In Appendix we further calculate the corresponding one-loop integrals for completeness.

II. INTEGRAL FAMILIES AND DIFFERENTIAL EQUATIONS

The master integrals for one loop HPET vertices and self-energy shown in Fig. 1 are easy to calculate. Nevertheless, for completeness, we present their calculation in Appendix. The massive HPET vertices and self-energy at two-loop level have MIs with topologies represented by Figs. 2 (a,b) and 3, respectively. We begin by considerinfg the prototype topologies for the heavy-heavy vertex in Figs. 2(a), the master integrals of which can be expressed in terms of a single integral family

$$J^{HH}_{\nu_1,\ldots,\nu_{10}} = \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \prod_{i=1}^{10} \frac{1}{(D_i + i0)^{\nu_i}}, \tag{1}$$

where

$$\begin{aligned} &D_1 = l_2 \cdot v_1, \quad D_2 = l_1 \cdot v_1, \quad D_3 = (l_1 - l_2)^2 - M^2, \\ &D_4 = l_2^2 - M^2, \qquad D_5 = l_1 \cdot v_2, \qquad D_6 = l_2 \cdot v_2, \\ &D_7 = (l_2 - l_1) \cdot v_2, \qquad D_8 = l_1^2 - M^2, \qquad D_9 = l_2^2, \\ &D_{10} = (l_1 - l_2)^2. \end{aligned} \tag{2}$$

Here, $v_{1,2}$ are the heavy particle velocities, and M is the mass of exchanged bosons. It is convenient to rescale integration momenta with respect to M and factor out the overall M dependence of the above integrals. So, in what follows, we will imply $M = 1$. After that, the integrals depend only on $w \equiv v_1 \cdot v_2$ scalar product.

Similarly, in the case of heavy-light vertex, the required prototype topologies in Figs. 2(b) can be assembled into the following single integral family

$$J^{HL}_{\nu_1,\ldots,\nu_{11}} = \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \prod_{i=1}^{11} \frac{1}{(D_i + i0)^{\nu_i}}, \tag{3}$$

where

$$\begin{aligned} &D_1 = (l_2 + p_2)^2 - m^2, \qquad D_2 = (l_1 + p_2)^2 - m^2, \\ &D_3 = l_1 \cdot v_1, \qquad D_4 = (l_1 - l_2) \cdot v_1, \qquad D_5 = l_2^2, \\ &D_6 = (l_1 - l_2)^2, \qquad D_7 = l_2 \cdot v_1, \qquad D_8 = l_1^2, \\ &D_9 = (l_2 + p_2)^2, \qquad D_{10} = l_2^2 - m^2, \\ &D_{11} = (l_1 - l_2)^2 - m^2. \end{aligned} \tag{4}$$

Here v_1 is the heavy field velocity, p_2 and m are the full theory field momentum and mass. Again, upon rescaling of integration momenta the overall dependence on m can be factored out and the rescaled integrals depend only on the scalar product $w \equiv v_1 \cdot p_2/m$ and we will again imply that in the above integral family definition $m = 1$.

Lastly, the self-energy diagrams which contribute to heavy field renormalization and residual mass term are examined [68]. The prototype topologies are shown in Fig. 3, and the corresponding single integral family is defined as

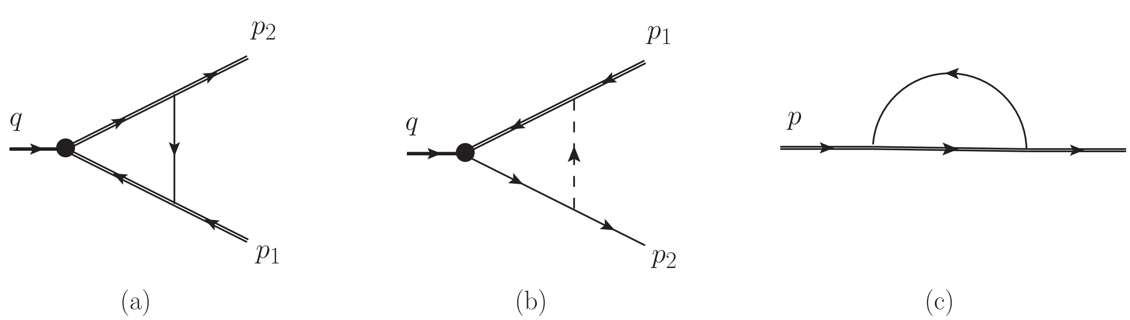

FIG. 1. Prototype topologies of one-loop vertex and self-energy diagrams: (a) is heavy-heavy vertex, (b) is heavy-light and (c) is self-energy. Solid lines represent massive particles, double lines represent heavy particles and dashed lines correspond to massless propagators. Arrows represent direction of momenta.

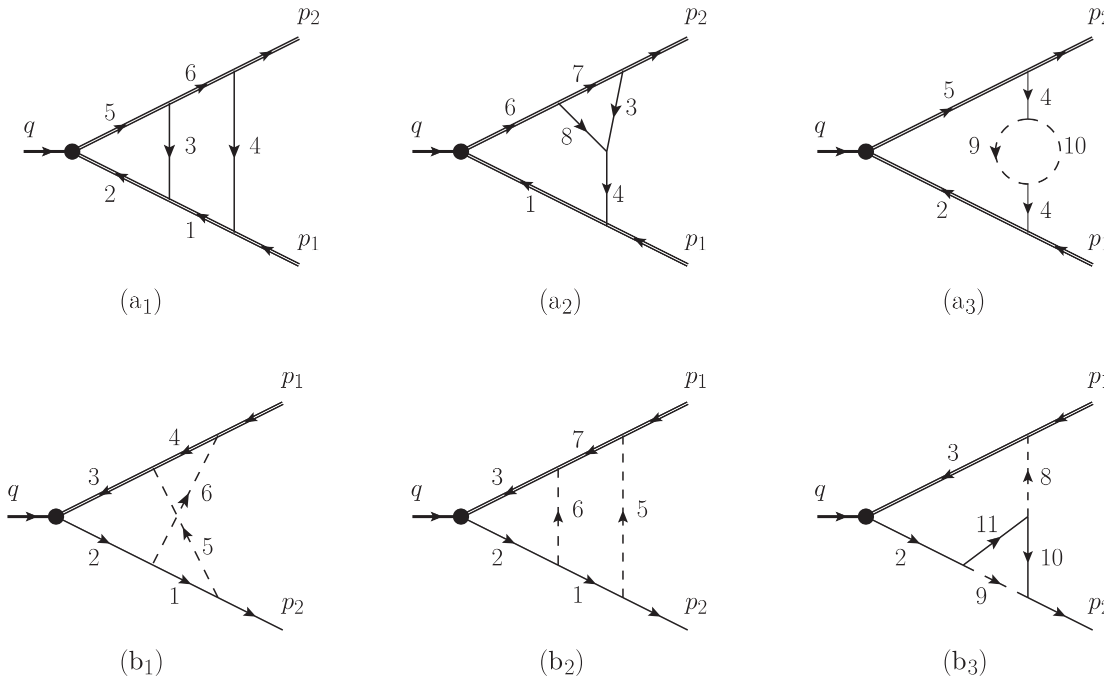

FIG. 2. Prototype topologies of two-loop vertex diagrams. Solid lines represent massive particles, double lines represent heavy particles, dashed lines correspond to massless propagators. Arrows represent direction of momenta. (a_i) and (b_i) correspond to heavy-heavy and heavy-light topologies. We also include the case of light self-energy insertions as is apparent in (a_3).

$$J^{SE}_{\nu_1,\ldots,\nu_8} = \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \prod_{i=1}^{8} \frac{1}{(D_i + i0)^{\nu_i}}, \tag{5}$$

where

$$D_1 = (p + l_1)\cdot v, \qquad D_2 = l_2 \cdot v, \qquad D_3 = l_1^2 - M^2,$$
$$D_4 = l_2^2 - M^2, \qquad D_5 = (l_1 - l_2)^2 - M^2,$$
$$D_6 = (p + l_1 + l_2)\cdot v, \qquad D_7 = l_2^2, \qquad D_8 = (l_1 - l_2)^2. \tag{6}$$

Here p and v are the heavy particle residual momentum and velocity and M is the mass of exchanged field. The overall M-dependence can be again factored out, so in what follows we will assume $M = 1$. The left integrals depend only on the scalar product $w \equiv v \cdot p/M$. Note, that the relation between the heavy field self-energy, $\Sigma(p)$, the bare field counterterm δZ_h and the residual heavy field mass, δm_h, are given by,

$$\delta Z_h = i\partial_{v\cdot p}\tilde{\Sigma}|_{v\cdot p=0} \tag{7}$$

$$\delta m_h = -i\tilde{\Sigma}|_{v\cdot p=0}. \tag{8}$$

To determine these quantities, one only requires the MIs on-shell at $v \cdot p = 0$, thus eliminating the momentum, p, from the propagators. The resulting MIs are, therefore, simple enough to evaluate with standard techniques. Maintaining $v \cdot p \neq 0$ is interesting in the case of off-shell studies. However, as we will see, in this case, we encounter integrals with elliptic structure.

With the use of IBP relations [69,70] all integrals in the described integral families can be reduced to the set of so called IBP master integrals. To evaluate the latter it is

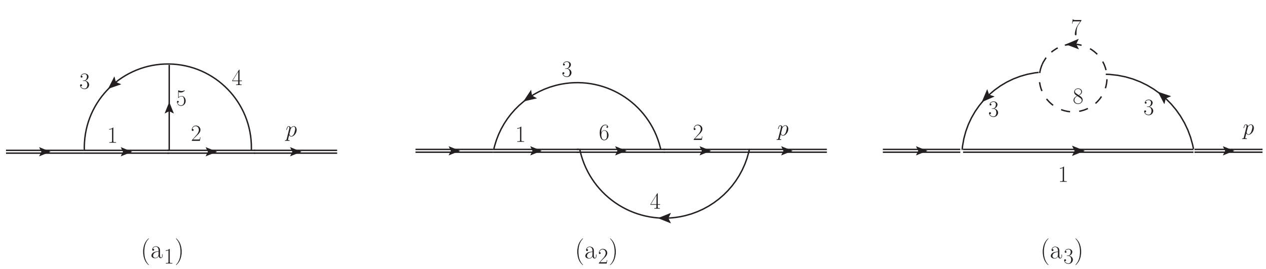

FIG. 3. Heavy field self-energy topologies. The MIs associated to other topologies are subsets of the MIs required for topologies illustrated.

convenient to use the method of differential equations [35–39]. As all master integrals we consider are dependent on a particular scalar product w, it is natural to consider differential equations for our master integrals with respect to w. To take the derivative with respect to an arbitrary scalar product $p \cdot q$, for two arbitrary vectors p and q, one can follow one of two equivalent ways,

$$\frac{\partial}{\partial(p\cdot q)} = \frac{(p\cdot q)p - p^2 q}{(p\cdot q)^2 - p^2 q^2}\cdot\frac{\partial}{\partial p} = \frac{(p\cdot q)q - q^2 p}{(p\cdot q)^2 - p^2 q^2}\cdot\frac{\partial}{\partial q} \tag{9}$$

In our study, we take derivatives with respect to the parameter w, defined in each integral family considered. Upon re-reducing the differentiated results with IBP identities, we obtain a linear combination of MIs, leading to a set of coupled differential equations. More precisely, the derivative of a given MI will inevitably lie in the same sector or sub-sector, meaning they contain the same set of non-zero ν_i, or a smaller set, compared to the original MI. Thus, one can combine all MIs and their derivatives into a linear system of differential equations. As we will see below, these systems can be further reduced either to ε or $A + B\varepsilon$ forms in the cases of vertex and self-energy integral families, respectively. From there, we solve each system iteratively at each order in a Laurent expansion about small ε.

A. Heavy-heavy vertex

In the heavy-heavy vertex case we have 26 master integrals shown in Figs. 4–6. The latter could be conveniently represented as a column vector $\boldsymbol{A}(w)$ with component $A_i(w)$ corresponding to $J^{HH}_{\nu_1,\ldots,\nu_{10}}(w)$ master #i. Upon differentiation of $\boldsymbol{A}(w)$ with respect to w and reduction by IBP identities, we have a differential system,

$$\partial_w \boldsymbol{A}(w) = \mathbb{M}(w,\varepsilon)\boldsymbol{A}(w). \tag{10}$$

with a 26×26 matrix, $\mathbb{M}(w,\varepsilon)$ which is neither Fuchsian nor in ε-form. Using balance transformations and the algorithm outlined in [41] this system can be however transformed into ε-form. First, we transform to ε-form diagonal blocks. The largest diagonal block (for masters #17, #19 and #20) is given by 3×3 matrix

$$M_3 = \begin{pmatrix} -\frac{2w^2-\varepsilon}{2(w-1)w(w+1)} & \frac{3}{4(w-1)w(w+1)} & \frac{(2w-1)(2w+1)}{4(w-1)w(w+1)\varepsilon} \\ \frac{\varepsilon^2}{(w-1)w(w+1)} & -\frac{2w^2-3\varepsilon}{2(w-1)w(w+1)} & -\frac{1}{2(w-1)w(w+1)} \\ \frac{2\varepsilon^2(2w^2\varepsilon-2w^2-2\varepsilon-1)}{(w-1)w(w+1)(2w-1)(2w+1)} & \frac{3\varepsilon(2w^2\varepsilon+2w^2-2\varepsilon-1)}{(w-1)w(w+1)(2w-1)(2w+1)} & -\frac{4w^4-4w^2\varepsilon-3w^2-2\varepsilon-1}{(w-1)w(w+1)(2w-1)(2w+1)} \end{pmatrix}. \tag{11}$$

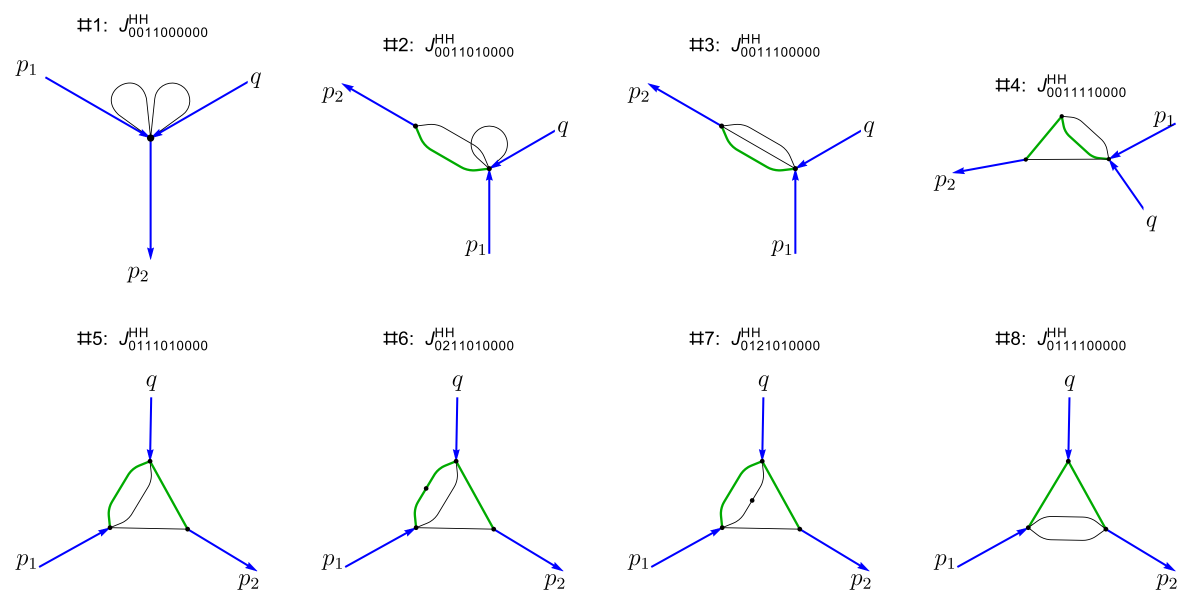

FIG. 4. Master integrals for heavy-heavy vertex (#: 1–8). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

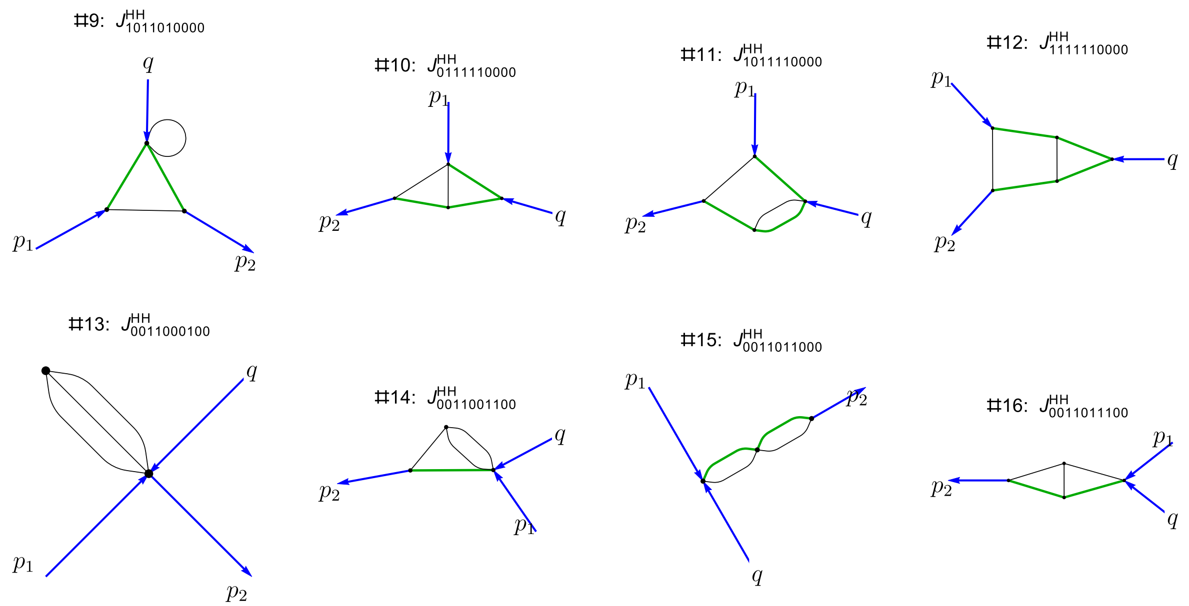

FIG. 5. Master integrals for heavy-heavy vertex (#: 9–16). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

Using `Libra` package [71] we can easily find the sequence of balance transformations to transform eigenvalues of M_3 matrix residues at all poles except at $w = \pm 1$ to $n\varepsilon$ form, where n is some integer. The required overall transformation matrix is given by

$$T_3 = \begin{pmatrix} \frac{1}{w+1} & 0 & 0 \\ 0 & \frac{1}{w-1} & 0 \\ \frac{2\varepsilon^2}{w+1} & -\frac{3\varepsilon}{(w-1)(4w^2-1)} & \frac{4w}{4w^2-1} \end{pmatrix}, \qquad (12)$$

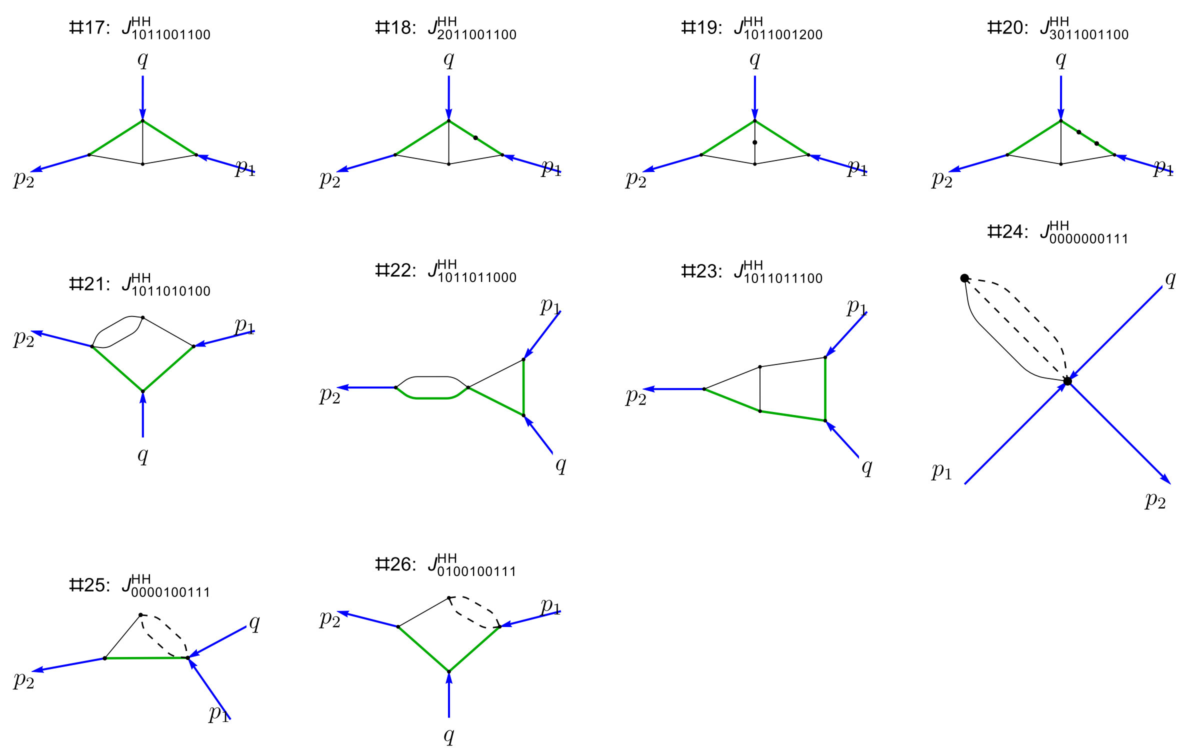

FIG. 6. Master integrals for heavy-heavy vertex (#: 17–26). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

The eigenvalues at $w = \pm 1$ have the form $\pm 1/2 + n\varepsilon$ and to normalize them further we require variable change. It is clear [72] that the appropriate variable employ is

$$w = \frac{1-\beta^2}{1+\beta^2}, \qquad \beta = \sqrt{\frac{1-w}{1+w}} \tag{13}$$

Now, we again use `Libra` to find the required sequence of balance transformations to normalize eigenvalues at $\beta = 0, \infty$ to $n\varepsilon$ form and find β-independent transformation to factor out the overall ε-dependence. The required transformation matrix was found to be

$$T_3' = \begin{pmatrix} \frac{1}{\beta} & 0 & 0 \\ 0 & \frac{\beta\varepsilon}{2} & 0 \\ 0 & 0 & \frac{\varepsilon^2}{4} \end{pmatrix}, \tag{14}$$

which together with previously found matrix T_3 reduces M_3 to ε-form,

$$S_3 = \varepsilon \begin{pmatrix} \frac{2w}{w^2-1} & 0 & \frac{\beta}{4(w-1)} \\ 0 & \frac{6w}{(w^2-1)(4w^2-1)} & \frac{\beta}{(w-1)(4w^2-1)} \\ \frac{16\beta}{w-1} & \frac{12\beta}{(w-1)(4w^2-1)} & -\frac{8w}{4w^2-1} \end{pmatrix}. \tag{15}$$

Next, we repeat these steps for other diagonal blocks. Finally, fuchsifying off-diagonal blocks and factoring out overall ε-dependence[73] the original system of differential equations is reduced to canonical or ε-form:

$$\partial_w \tilde{\boldsymbol{A}}(w) = \varepsilon \mathbb{S}(w) \tilde{\boldsymbol{A}}(w), \tag{16}$$

where $\boldsymbol{A} = \mathbb{T} \cdot \tilde{\boldsymbol{A}}$. In what follows we will refer to the vector of master integrals $\tilde{\boldsymbol{A}}$ as the vector of canonical master integrals. It turns out that in this case the only new variable appearing in the process of reduction is β and thus it is more convenient to consider differential system with respect to β

$$\partial_\beta \tilde{\boldsymbol{A}}(\beta) = \varepsilon \tilde{\mathbb{M}}(\beta) \tilde{\boldsymbol{A}}(\beta), \tag{17}$$

This way, as we will see in the next section, the corresponding solution of the differential system can be written in terms of MPLs. The corresponding expressions for the canonical $\tilde{\mathbb{M}}$ and transformation matrices $\mathbb{T}$ can be found in an arXiv ancillary file.

B. Heavy-light vertex

Here, we have 18 master integrals shown in Figs. 7 and 8. Writing the latter as a column vector $\boldsymbol{B}(w)$ the corresponding differential equations system can be written as

$$\partial_w \boldsymbol{B}(w) = \mathbb{M}(w, \varepsilon) \boldsymbol{B}(w). \tag{18}$$

with a 18×18 matrix, $\mathbb{M}(w, \varepsilon)$. The reduction to ε-form in the present case proceeds similar to the case of heavy-heavy vertex. That is, we first reduce to ε-form diagonal blocks. Then, we fuchsify off-diagonal blocks and finally factor out the overall ε-dependence. What is important is that in this case similar to the case of heavy-heavy vertex the only new variable introduced is β (13). For example, take 2×2 diagonal block (for masters #13 and #14)

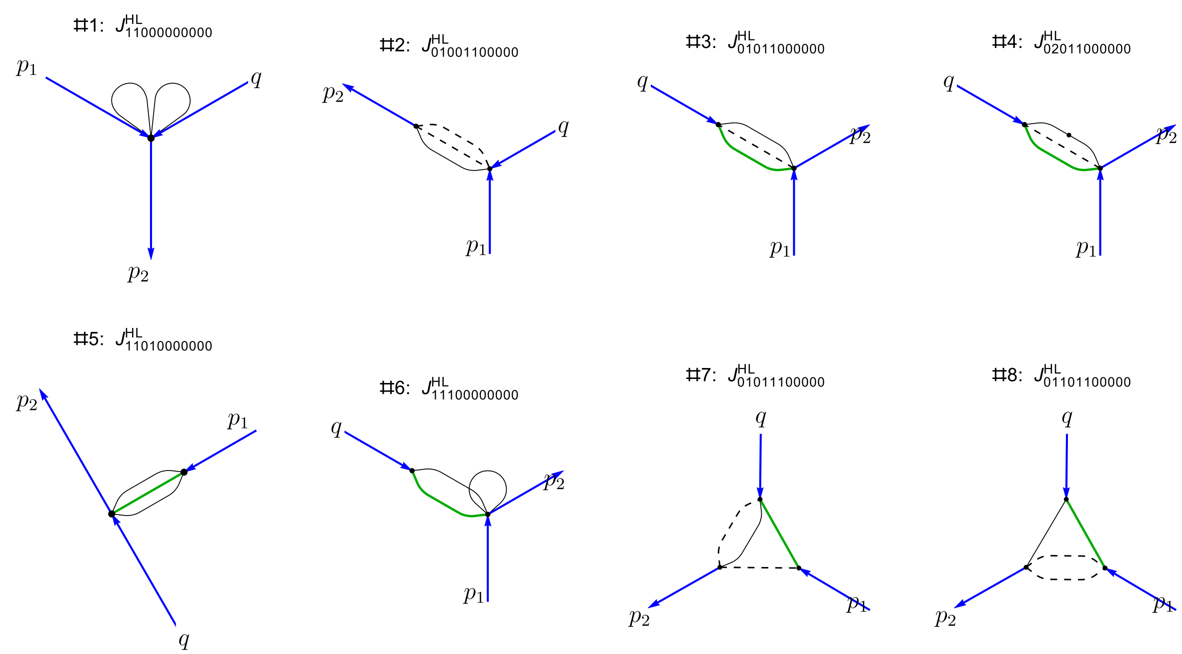

FIG. 7. Master integrals for heavy-light vertex (#: 1–8). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

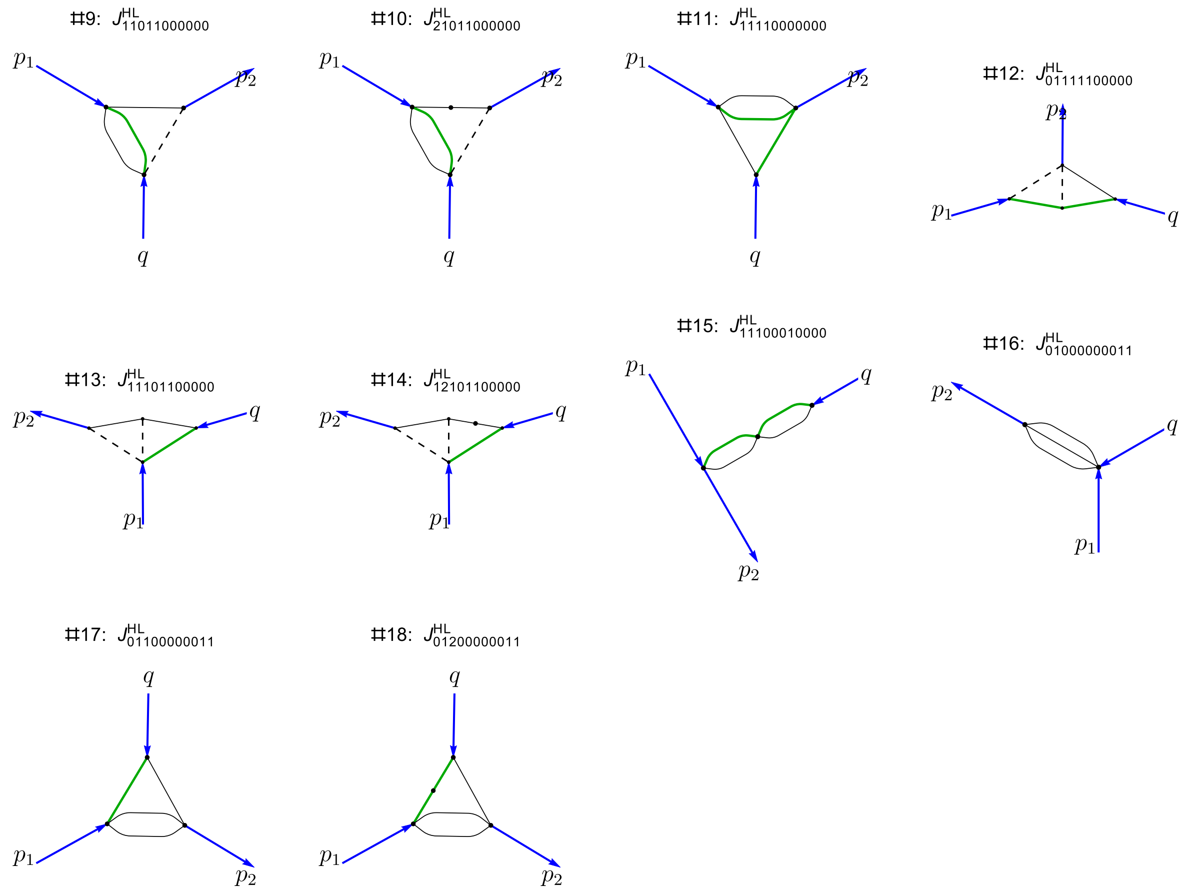

FIG. 8. Master integrals for heavy-light vertex (#: 9–18). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

$$M_2 = \begin{pmatrix} -\frac{2w^2\varepsilon+w^2-4\varepsilon}{(w-1)w(w+1)} & -\frac{4}{w} \\ -\frac{\varepsilon(4\varepsilon+1)}{(w-1)w(w+1)} & -\frac{w^2+4\varepsilon+1}{(w-1)w(w+1)} \end{pmatrix}. \tag{19}$$

Using balance transformations we again transform eigenvalues of M_2 matrix residues at all poles except at $w = \pm 1$ to $n\varepsilon$ form, where n is some integer. This can be done for example with transformation matrix

$$T_2 = \begin{pmatrix} \frac{1}{w+1} & 0 \\ -\frac{(w-2)\varepsilon}{2(w^2-1)} & \frac{w}{w^2-1} \end{pmatrix}. \tag{20}$$

To reduce eigenvalues at $w = \pm 1$ we require the same variable change to β-variable (13) as in the case of heavy-heavy vertex. Next, we again use `Libra` to find a sequence of balance transformations to normalize eigenvalues at $\beta = 0, \infty$ to $n\varepsilon$ form and find β-independent transformation to factor out the overall ε-dependence. The required transformation matrix was found to be

$$T_2' = \begin{pmatrix} \frac{1}{\beta} & 0 \\ -\frac{\varepsilon}{2\beta} & \frac{\varepsilon}{2} \end{pmatrix}, \tag{21}$$

which together with previously found matrix T_2 transforms M_2 to ε-form,

$$S_2 = \varepsilon \begin{pmatrix} \frac{2w}{w^2-1} & -\frac{2\beta}{w-1} \\ -\frac{4\beta}{w-1} & -\frac{4w}{w^2-1} \end{pmatrix}. \tag{22}$$

Altogether, in this case we can also write down the transformed differential system with respect to β in ε-form ($\boldsymbol{B} = \mathbb{T} \cdot \tilde{\boldsymbol{B}}$)

$$\partial_\beta \tilde{\boldsymbol{B}}(\beta) = \varepsilon \tilde{\mathbb{M}}(\beta) \tilde{\boldsymbol{B}}(\beta), \tag{23}$$

with the corresponding expressions for canonical $\tilde{\mathbb{M}}$ and transformation $\mathbb{T}$ matrices located in an arXiv ancillary file.

C. Heavy propagator

In the case of the heavy propagator, we have 12 master integrals as depicted in Fig. 9. The reduction of the corresponding differential system for a vector of master integrals $\boldsymbol{C}$ is achieved as in the cases of heavy-heavy and heavy-light vertices. In particular, we first reduce diagonal blocks where possible to ε-form and then fuchsify

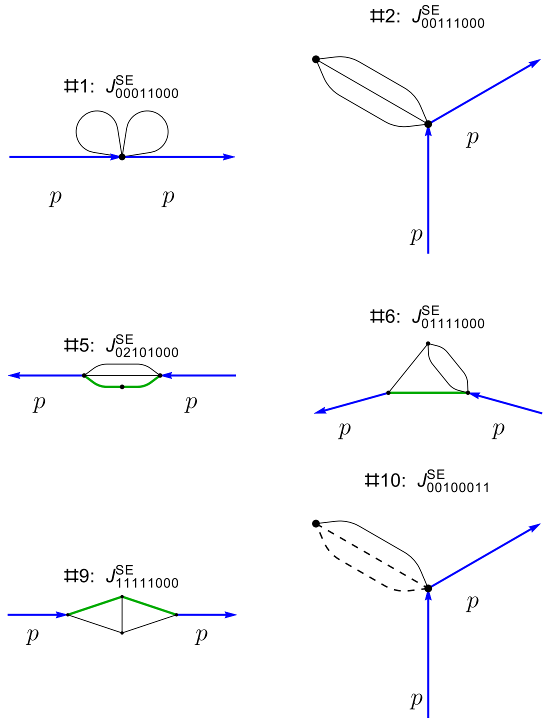

FIG. 9. Master integrals for heavy propagator (#: 1–12). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

off-diagonal blocks. Some of the diagonal blocks will also require a variable change to the β variable, as defined in (13). However, contrary to the vertex cases considered previously, we have elliptic subgraphs represented by elliptic sunsets (master integrals #4 and #5). This particular diagonal block can not be reduced to ε-form[74]. Still, one can reduce the latter to $A + B\varepsilon$ form. Also, instead of factoring out the overall ε-dependence in the final step, we find a ($\beta(w)$-independent) diagonal transformation matrix, which reduces the entire system to $A + B\varepsilon$ form. Altogether, in this case we write down the transformed differential system with respect to w in $A + B\varepsilon$ form ($\boldsymbol{C} = \mathbb{T} \cdot \tilde{\boldsymbol{C}}$):

$$\partial_w \tilde{\boldsymbol{C}}(w) = \tilde{\mathbb{M}}(w)\tilde{\boldsymbol{C}}(w), \tag{24}$$

with corresponding expressions for canonical $\tilde{\mathbb{M}}$ and transformation $\mathbb{T}$ matrices located in an arXiv ancillary file. Here, the matrix $\tilde{\mathbb{M}}$ does contain square roots in w induced by the variable β. As we already noted, for on-shell studies we need to know master integrals only for $w = 0$. Still, as will be shown in next section, we will also able to supply all the ingredients necessary to have Frobenius solution in terms of a generalized power series in w for $w \neq 0$.

III. SOLUTION OF DIFFERENTIAL EQUATIONS AND RESULTS

Given our reduced systems of differential equations to either ε or $A + B\varepsilon$ form in the previous section, we are now ready to solve them. In the case of heavy-heavy and heavy-light vertices, where we managed to reduce the corresponding differential systems to ε-form, the solution can be readily written in terms of $\mathcal{P}$-exponents as

$$\boldsymbol{J}(\beta) = \mathbb{T}(\beta)\tilde{\boldsymbol{J}}(\beta) = \mathbb{T}(\beta)\mathcal{P}\exp\left[\varepsilon\int_0^\beta \tilde{\mathbb{M}}(t)dt\right]\mathbb{L}\cdot\boldsymbol{c}, \tag{25}$$

where $\boldsymbol{J}$ is either vector of $\boldsymbol{A}$ or $\boldsymbol{B}$ masters, $\mathbb{T}$ is the transformation matrix to canonical basis and $\tilde{\mathbb{M}}$ is corresponding differential equations matrix in canonical basis. $\mathbb{L}$ is the adapter matrix converting the vector of constants $\boldsymbol{c}$ into boundary conditions for canonical master integrals. The expressions for said boundary conditions can be found in an arXiv ancillary file. The components of the vector of constants $\boldsymbol{c}$ have the form $c_i(\beta^j)$, where $c_i(\beta^j)$ is the coefficient of β^j in the β-expansion of the ith master. In the case of the heavy-heavy and heavy-light vertices, $\boldsymbol{c}$-vectors are given by

$$\begin{aligned}\boldsymbol{c}_{HH} = (&c_1(\beta^0), c_2(\beta^0), c_3(\beta^0), c_6(\beta^{-1+2\varepsilon}), c_5(\beta^0),\\ &c_5(\beta^{-1+2\varepsilon}), c_8(\beta^{-1}), c_9(\beta^{-1}), c_{10}(\beta^{-1}), c_{11}(\beta^{-1}),\\ &c_{12}(\beta^{-2}), c_{13}(\beta^0), c_{14}(\beta^0), c_{15}(\beta^0), c_{16}(\beta^0),\\ &c_{18}(\beta^{-1+2\varepsilon}), c_{17}(\beta^0), c_{17}(\beta^{-1+2\varepsilon}), c_{17}(\beta^{1+2\varepsilon}), c_{21}(\beta^{-1}),\\ &c_{22}(\beta^{-1}), c_{23}(\beta^{-1}), c_{24}(\beta^0), c_4(\beta^0), c_{25}(\beta^0), c_{26}(\beta^{-1}))^\top\end{aligned} \tag{26}$$

and

$$\boldsymbol{c}_{HL} = (c_1(\beta^0), c_2(\beta^0), c_3(\beta^0), c_3(\beta^{5-6\varepsilon}), c_5(\beta^0),\\ c_6(\beta^{1-2\varepsilon}), c_7(\beta^{-1+2\varepsilon}), c_8(\beta^{1-4\varepsilon}), c_9(\beta^{-1+2\varepsilon}),\\ c_{10}(\beta^{-4\varepsilon}), c_{11}(\beta^{2-4\varepsilon}), c_{12}(\beta^{-4\varepsilon}), c_{13}(\beta^{-4\varepsilon}), c_{13}(\beta^{-1+2\varepsilon}),\\ c_{15}(\beta^{2-4\varepsilon}), c_{16}(\beta^0), c_{17}(\beta^{-2\varepsilon}), c_{17}(\beta^{-1+2\varepsilon}))^\top. \tag{27}$$

It is easy to see, that all $c_i(\beta^j)$ constants other than $c_i(\beta^0)$ are zero either due to the regularity of master integrals at $\beta = 0$ or due to the absence of subgraphs in the large mass expansion (corresponding to the expansion at $\beta = 0$) of these vertices. The remaining constants will be calculated in the next subsection with the use of dimensional recursion relations.

In the heavy propagator case, due to the presence of elliptic subgraphs in its corresponding differential system, a $\mathcal{P}$-exponent solution is not attainable. However, we can still obtain a Frobenius solution in terms of a generalized power series in the w-variable. The solution can be written as

$$\boldsymbol{C}(w) = \mathbb{T}(w)\tilde{\boldsymbol{C}}(w) = \mathbb{T}(w)\mathcal{F}(w)\cdot\mathbb{L}\cdot\boldsymbol{c}, \tag{28}$$

where $\mathcal{F}(w)$ is the Frobenius solution for canonical master integrals. $\mathbb{L}$ is again the adapter matrix converting the vector of constants $\boldsymbol{c}$ into boundary conditions for canonical master integrals. the vector of constants $\boldsymbol{c}$ in this case is given by

$$\boldsymbol{c}_{SE} = (c_1(\beta^0), c_2(\beta^0), c_3(\beta^0), c_4(\beta^0), c_4(\beta^{2-2\varepsilon}),\\ c_6(\beta^0), c_7(\beta^0), c_8(\beta^0), c_9(\beta^0), c_{10}(\beta^0),\\ c_{11}(\beta^{3-4\varepsilon}), c_{12}(\beta^0))^\top. \tag{29}$$

Using Feynman parameters it is easy to see that $c_4(\beta^{2-2\varepsilon}) = 0$ and

$$c_{11}(\beta^{3-4\varepsilon}) = (-1)^{4\varepsilon}2^{4-4\varepsilon}\Gamma(1-\varepsilon)^2\Gamma(4\varepsilon-3). \tag{30}$$

Like the case of vertices, the remaining constants are more convenient to determine using dimensional recurrence relations. It is also possible to determine Frobenius solutions for canonical master integrals $\mathcal{F}(w)$, see for example [75–81]. Moreover, the `Libra` package [71] itself contains built-in tools for obtaining such a solution. The reader is advised to consult the accompanying arXiv ancillary notebook.

A. Boundary constants from dimensional recurrences

We need to calculate the remaining $c_i(\beta^0)$ constants in each case. First, we note that for $\beta = 0$ in the case of vertices, or $w = 0$ in the case of heavy propagators, the master integrals, as illustrated in Figs. 4–9, are no longer masters and additional reduction using partial fractioning and IBP identities is possible. In the case of the heavy-heavy vertex, additional boundary master integrals can be written in terms of a single integral family

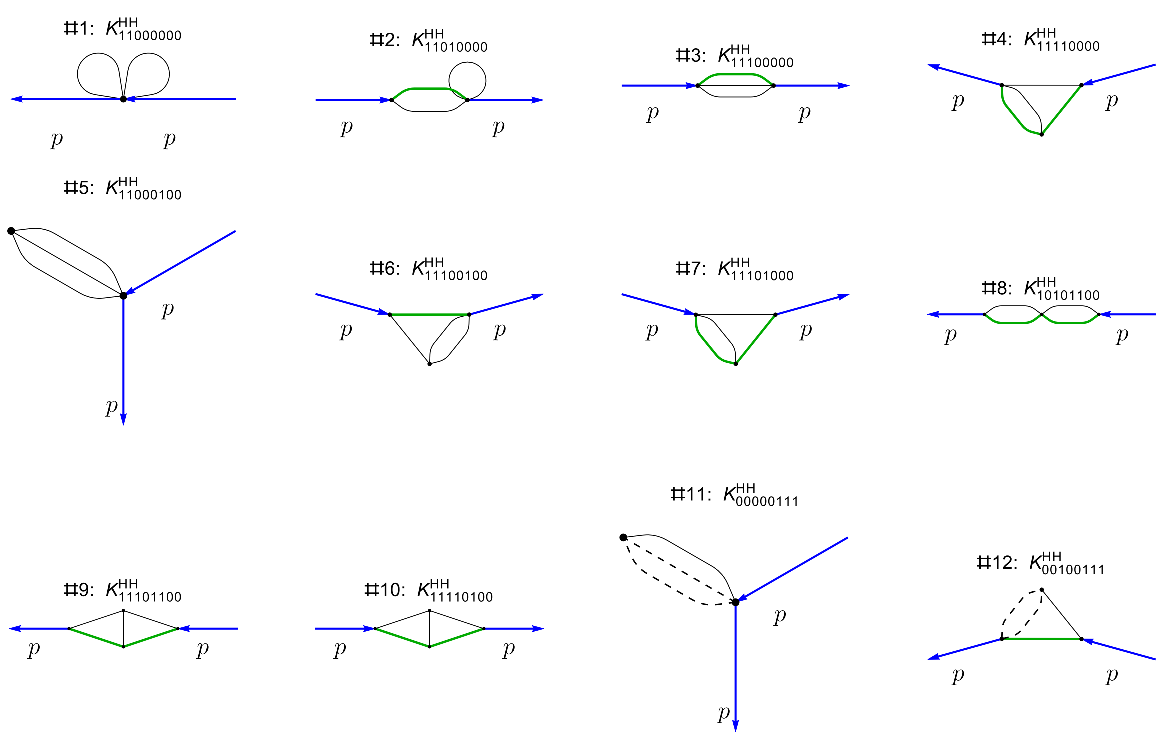

FIG. 10. Boundary master integrals for heavy-heavy vertex (#: 1–12). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

$$K^{HH}_{\nu_1,\ldots,\nu_8} = \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \prod_{i=1}^{8} \frac{1}{(D_i + i0)^{\nu_i}}, \quad (31)$$

where

$$D_1 = (l_1 - l_2)^2 - 1, \quad D_2 = l_2^2 - 1, \quad D_3 = l_1 \cdot v,$$
$$D_4 = l_2 \cdot v, \quad D_5 = (l_2 - l_1) \cdot v, \quad D_6 = l_1^2 - 1,$$
$$D_7 = l_2^2, \quad D_8 = (l_1 - l_2)^2 \quad (32)$$

and v is the heavy field velocity. Altogether in this case we have 12 master integrals shown in Fig. 10. Similarly, in the case of heavy-light vertex the single integral family is given by

$$K^{HL}_{\nu_1,\ldots,\nu_8} = \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \prod_{i=1}^{8} \frac{1}{(D_i + i0)^{\nu_i}}, \quad (33)$$

where

$$D_1 = (l_1 + v)^2 - 1, \quad D_2 = (l_1 - l_2) \cdot v, \quad D_3 = l_2^2,$$
$$D_4 = (l_1 - l_2)^2, \quad D_5 = (l_2 + v)^2 - 1, \quad D_6 = l_1^2,$$
$$D_7 = l_2^2 - 1, \quad D_8 = (l_1 - l_2)^2 - 1 \quad (34)$$

and the corresponding 5 boundary master integrals are shown in Fig. 11. Finally, for the heavy propagator case the single integral family takes the form

$$K^{SE}_{\nu_1,\ldots,\nu_7} = \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \prod_{i=1}^{7} \frac{1}{(D_i + i0)^{\nu_i}}, \quad (35)$$

where

$$D_1 = v \cdot l_1, D_2 = v \cdot l_2, \quad D_3 = l_1^2 - 1,$$
$$D_4 = l_2^2 - 1, \quad D_5 = (l_1 - l_2)^2 - 1, \quad D_6 = l_2^2,$$
$$D_7 = (l_1 - l_2)^2 \quad (36)$$

with 10 boundary master integrals shown in Fig. 12.

The simplest way to calculate the exact expressions in ε for these boundary master integrals is to use dimensional recurrence relations [82–85]. In all cases we present here, the recurrence relations are all first-order. Consider, for instance, the evaluation of the $K^{SE}_{0011100}(d)$ master integral. In this case, we have the following dimensional recurrence relation ($\nu = d/2$)

$$K^{SE}_{0011100}(\nu + 1) = \frac{3}{2\nu(2\nu - 1)} K^{SE}_{0011100}(\nu) + \frac{3}{2\nu(2\nu - 1)} K^{SE}_{0001100}(\nu), \quad (37)$$

where $K^{SE}_{0001100}(\nu)$ is the product of two one-loop tadpoles

$$K^{SE}_{0001100}(\nu) = \Gamma(1 - \nu)^2. \quad (38)$$

It is sufficient to consider the solution of this recurrence relation in a basic strip $\nu \in [1, 2)$. First, we determine the solution of the homogeneous difference equation,

$$J_h(\nu) = \frac{2 \cdot 3^{\nu-1} \pi \csc(2\pi\nu)}{\Gamma(2\nu - 1)}, \quad (39)$$

where an extra periodic factor $\csc(2\pi\nu)$ was chosen to be well-behaved at imaginary infinities and singularities close to those of the original master integral $K^{SE}_{0011100}(\nu)$ (see [84,85] for further details). The easiest way to see them is through numerical evaluation in a chosen strip with sector decomposition [86–92]. In particular, we used the function `SDAnalyse` from `Fiesta` [93]. Knowing the homogeneous solution, the particular solution can be found with substitution into original recurrence relation the ansatz

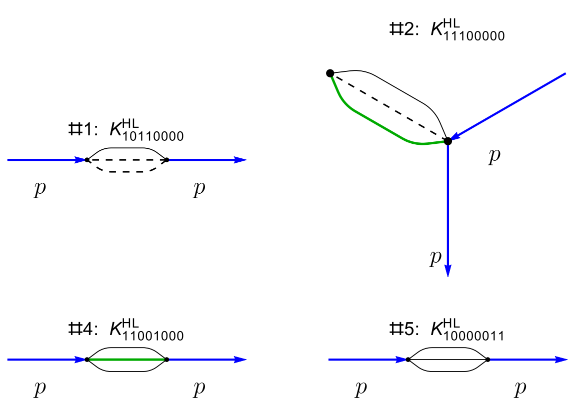

FIG. 11. Boundary master integrals for heavy-light vertex (#: 1–5). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

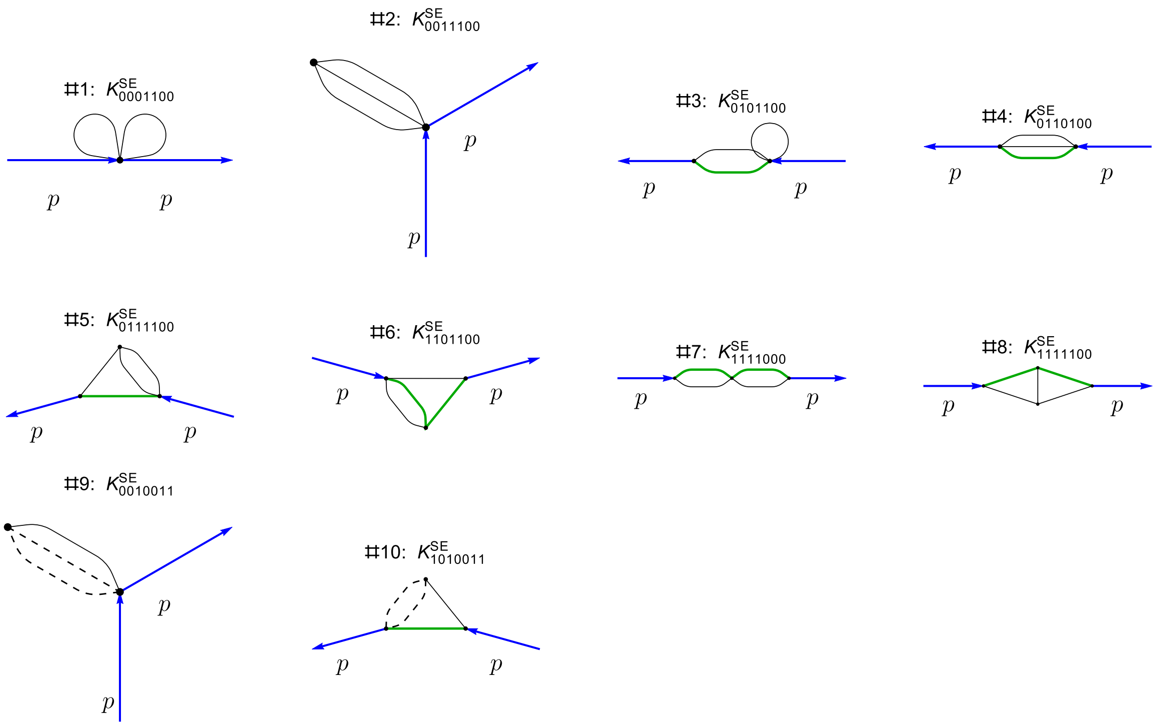

FIG. 12. Boundary master integrals for heavy propagator (#: 1–10). Green lines denote propagators for heavy particles, solid—propagators for massive particles and dashed—massless propagators.

$K^{SE}_{0011100}(\nu) = r(\nu)J_h(\nu)$ and obtaining the recurrence relation for $r(\nu)$

$$r(\nu+1) = r(\nu) + J_h^{-1}(\nu)\Gamma(1-\nu)^2. \tag{40}$$

This first order recurrence relation is easy to solve and we have

$$r(\nu) = -\sum_{i=0}^{\infty} \frac{3^{2+i-\nu}}{2} \frac{\Gamma(2+i-\nu)^2}{\Gamma(4+2i-2\nu)} \tag{41}$$

$$= -\frac{3^{2-\nu}\Gamma(2-\nu)^2}{2\Gamma(4-2\nu)}\, {}_2F_1\left(1, 2-\nu; \frac{5}{2}-\nu; \frac{3}{4}\right). \tag{42}$$

The general solution is then given by

$$K^{SE}_{0011100}(\nu) = J_h(\nu)r(\nu) + w(\nu)J_h(\nu), \tag{43}$$

where the periodic constant $w(\nu)$ is fixed from the leading term of the expansion at $\nu = 3/2$. Altogether, we have the exact solution,

$$K^{SE}_{0011100}(\nu) = \frac{4\cdot 3^{\nu-3/2}\pi^2 \csc(2\pi\nu)}{\Gamma(2\nu-1)} - \frac{3\pi\csc(2\pi\nu)\Gamma^2(2-\nu)}{\Gamma(4-2\nu)\Gamma(2\nu-1)} \times {}_2F_1\left(1, 2-\nu; \frac{5}{2}-\nu; \frac{3}{4}\right). \tag{44}$$

Employing the same technique, one can then calculate all other boundary master integrals. The corresponding results can be found in an arXiv ancillary file. Expanding the exact results at $d = 4-2\varepsilon$ and accounting for obvious $\exp(2\gamma_E\varepsilon)$ factors, we get.

1. Heavy-heavy:

$$a_1 = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \left(-\frac{2\zeta_3}{3} + 4 + \frac{\pi^2}{3}\right)\varepsilon + \left(-\frac{4\zeta_3}{3} + 5 + \frac{\pi^2}{2} + \frac{7\pi^4}{360}\right)\varepsilon^2 + \left(-2\zeta_3 - \frac{\pi^2\zeta_3}{9} - \frac{2\zeta_5}{5} + 6 + \frac{2\pi^2}{3} + \frac{7\pi^4}{180}\right)\varepsilon^3 + \left(-\frac{8\zeta_3}{3} - \frac{2\pi^2\zeta_3}{9} + \frac{2\zeta_3^2}{9} - \frac{4\zeta_5}{5} + 7 + \frac{5\pi^2}{6} + \frac{7\pi^4}{120} + \frac{31\pi^6}{15120}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \tag{45}$$

$$a_2 = -\frac{2\pi}{\varepsilon} + \pi(4l_2 - 6) + \pi\varepsilon\left(-14 - \frac{2\pi^2}{3} - 4l_2^2 + 12l_2\right) + \pi\varepsilon^2\left(\frac{16\zeta_3}{3} - 30 - 2\pi^2 + \frac{8l_2^3}{3} - 12l_2^2 + 28l_2 + \frac{4}{3}\pi^2 l_2\right) + \pi\varepsilon^3\left(16\zeta_3 - \frac{32}{3}\zeta_3 l_2 - 62 - \frac{14\pi^2}{3} - \frac{\pi^4}{5} - \frac{4l_2^4}{3} + 8l_2^3 - 28l_2^2 - \frac{4}{3}\pi^2 l_2^2 + 60l_2 + 4\pi^2 l_2\right) + \pi\varepsilon^4\left(\frac{112\zeta_3}{3} + \frac{16\pi^2\zeta_3}{9} + \frac{64\zeta_5}{5} + \frac{32}{3}\zeta_3 l_2^2 - 32\zeta_3 l_2 - 126 - 10\pi^2 - \frac{3\pi^4}{5} + \frac{8l_2^5}{15} - 4l_2^4 + \frac{56l_2^3}{3} + \frac{8}{9}\pi^2 l_2^3 - 60l_2^2 - 4\pi^2 l_2^2 + 124l_2 + \frac{28}{3}\pi^2 l_2 + \frac{2}{5}\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \tag{46}$$

$$a_3 = \frac{32\pi^2\varepsilon}{3} + \varepsilon^2\left(\frac{704\pi^2}{9} - \frac{256}{3}\pi^2 l_2\right) + \varepsilon^3\left(\frac{10880\pi^2}{27} + \frac{112\pi^4}{9} + \frac{1024}{3}\pi^2 l_2^2 - \frac{5632}{9}\pi^2 l_2\right) + \varepsilon^4\left(-\frac{1984\pi^2\zeta_3}{9} + \frac{147200\pi^2}{81} + \frac{2464\pi^4}{27} - \frac{8192}{9}\pi^2 l_2^3 + \frac{22528}{9}\pi^2 l_2^2 - \frac{87040}{27}\pi^2 l_2 - \frac{896}{9}\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \tag{47}$$

$$a_4 = 2\pi^2 + \varepsilon(8\pi^2 - 8\pi^2 l_2) + \varepsilon^2(24\pi^2 + \pi^4 + 16\pi^2 l_2^2 - 32\pi^2 l_2) + \varepsilon^3\left(-\frac{28\pi^2\zeta_3}{3} + 64\pi^2 + 4\pi^4 - \frac{64}{3}\pi^2 l_2^3 + 64\pi^2 l_2^2 - 96\pi^2 l_2 - 4\pi^4 l_2\right) + \varepsilon^4\left(-\frac{112\pi^2\zeta_3}{3} + \frac{112}{3}\pi^2\zeta_3 l_2 + 160\pi^2 + 12\pi^4 + \frac{5\pi^6}{12} + \frac{64}{3}\pi^2 l_2^4 - \frac{256}{3}\pi^2 l_2^3 + 192\pi^2 l_2^2 + 8\pi^4 l_2^2 - 256\pi^2 l_2 - 16\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \tag{48}$$

$$a_5 = \frac{3}{2\varepsilon^2} + \frac{9}{2\varepsilon} + \frac{1}{4}\left(8\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) + 42 + \pi^2\right) + \sum_{n=1}^{4} a_{1,n}\varepsilon^n + \mathcal{O}(\varepsilon^5), \tag{49}$$

$$a_6 = -\frac{2\pi}{\varepsilon} + \left(-8\pi - \frac{4\pi^2}{\sqrt{3}} + 4\pi l_2\right) + \sum_{n=1}^{4} a_{2,n}\varepsilon^n + \mathcal{O}(\varepsilon^5), \tag{50}$$

$$a_7 = 6\pi^2 + \varepsilon(24\pi^2 - 24\pi^2 l_2) + \varepsilon^2(72\pi^2 + 3\pi^4 + 48\pi^2 l_2^2 - 96\pi^2 l_2) + \varepsilon^3(-28\pi^2\zeta_3 + 192\pi^2 + 12\pi^4 - 64\pi^2 l_2^3 + 192\pi^2 l_2^2 - 288\pi^2 l_2 - 12\pi^4 l_2) + \varepsilon^4\left(-112\pi^2\zeta_3 + 112\pi^2\zeta_3 l_2 + 480\pi^2 + 36\pi^4 + \frac{5\pi^6}{4} + 64\pi^2 l_2^4 - 256\pi^2 l_2^3 + 576\pi^2 l_2^2 + 24\pi^4 l_2^2 - 768\pi^2 l_2 - 48\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \tag{51}$$

$$a_8 = 4\pi^2 + \varepsilon(16\pi^2 - 16\pi^2 l_2) + \varepsilon^2(48\pi^2 + 2\pi^4 + 32\pi^2 l_2^2 - 64\pi^2 l_2) + \varepsilon^3\left(-\frac{56\pi^2\zeta_3}{3} + 128\pi^2 + 8\pi^4 - \frac{128}{3}\pi^2 l_2^3 + 128\pi^2 l_2^2 - 192\pi^2 l_2 - 8\pi^4 l_2\right) + \varepsilon^4\left(-\frac{224\pi^2\zeta_3}{3} + \frac{224}{3}\pi^2\zeta_3 l_2 + 320\pi^2 + 24\pi^4 + \frac{5\pi^6}{6} + \frac{128}{3}\pi^2 l_2^4 - \frac{512}{3}\pi^2 l_2^3 + 384\pi^2 l_2^2 + 16\pi^4 l_2^2 - 512\pi^2 l_2 - 32\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \tag{52}$$

$$a_9 = -\frac{4\pi^2}{3\varepsilon} + \left(\frac{16}{3}\pi^2 l_3 - \frac{8\pi^2}{3}\right) + \sum_{n=1}^{4} a_{3,n}\varepsilon^n + \mathcal{O}(\varepsilon^5), \tag{53}$$

$$a_{10} = \frac{1}{2} a_9, \tag{54}$$

$$\begin{aligned} a_{11} =& \frac{1}{2\varepsilon^2} + \frac{3}{2\varepsilon} + \left(\frac{7}{2} + \frac{\pi^2}{4}\right) + \left(-\frac{4\zeta_3}{3} + \frac{15}{2} + \frac{3\pi^2}{4}\right)\varepsilon + \left(-4\zeta_3 + \frac{31}{2} + \frac{7\pi^2}{4} + \frac{7\pi^4}{80}\right)\varepsilon^2 \\ &+ \left(-\frac{28\zeta_3}{3} - \frac{2\pi^2\zeta_3}{3} - \frac{16\zeta_5}{5} + \frac{63}{2} + \frac{15\pi^2}{4} + \frac{21\pi^4}{80}\right)\varepsilon^3 + \left(-20\zeta_3 - 2\pi^2\zeta_3 + \frac{16\zeta_3^2}{9}\right. \\ &\left. - \frac{48\zeta_5}{5} + \frac{127}{2} + \frac{31\pi^2}{4} + \frac{49\pi^4}{80} + \frac{869\pi^6}{30240}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{55}$$

$$\begin{aligned} a_{12} =& -\frac{2\pi}{\varepsilon} + \pi(4l_2 - 8) + \pi\varepsilon\left(-24 - \frac{10\pi^2}{3} - 4l_2^2 + 16l_2\right) + \pi\varepsilon^2\left(\frac{28\zeta_3}{3} - 64 - \frac{40\pi^2}{3} + \frac{8l_2^3}{3}\right. \\ &\left. -16l_2^2 + 48l_2 + \frac{20}{3}\pi^2 l_2\right) + \pi\varepsilon^3\left(\frac{112\zeta_3}{3} - \frac{56}{3}\zeta_3 l_2 - 160 - 40\pi^2 - \frac{238\pi^4}{45} - \frac{4l_2^4}{3} + \frac{32l_2^3}{3}\right. \\ &\left. -48l_2^2 - \frac{20}{3}\pi^2 l_2^2 + 128l_2 + \frac{80}{3}\pi^2 l_2\right) + \pi\varepsilon^4\left(112\zeta_3 + \frac{140\pi^2\zeta_3}{9} + \frac{124\zeta_5}{5} + \frac{56}{3}\zeta_3 l_2^2\right. \\ & - \frac{224}{3}\zeta_3 l_2 - 384 - \frac{320\pi^2}{3} - \frac{952\pi^4}{45} + \frac{8l_2^5}{15} - \frac{16l_2^4}{3} + 32l_2^3 + \frac{40}{9}\pi^2 l_2^3 - 128l_2^2 \\ &\left. -\frac{80}{3}\pi^2 l_2^2 + 320l_2 + 80\pi^2 l_2 + \frac{476}{45}\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{56}$$

2. Heavy-light:

$$\begin{aligned} b_1 =& \frac{1}{2\varepsilon^2} + \frac{5}{4\varepsilon} + \left(\frac{11}{8} + \frac{5\pi^2}{12}\right) + \left(\frac{11\zeta_3}{3} - \frac{55}{16} + \frac{25\pi^2}{24}\right)\varepsilon + \left(\frac{55\zeta_3}{6} - \frac{949}{32} + \frac{55\pi^2}{48} + \frac{101\pi^4}{240}\right)\varepsilon^2 \\ &+ \left(\frac{121\zeta_3}{12} + \frac{55\pi^2\zeta_3}{18} + \frac{359\zeta_5}{5} - \frac{8575}{64} - \frac{275\pi^2}{96} + \frac{101\pi^4}{96}\right)\varepsilon^3 + \left(-\frac{605\zeta_3}{24} + \frac{275\pi^2\zeta_3}{36}\right. \\ &\left. + \frac{121\zeta_3^2}{9} + \frac{359\zeta_5}{2} - \frac{64189}{128} - \frac{4745\pi^2}{192} + \frac{1111\pi^4}{960} + \frac{3335\pi^6}{6048}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{57}$$

$$\begin{aligned} b_2 =& -\frac{1}{\varepsilon^2} - \frac{4}{3\varepsilon} + \frac{\pi^2}{2} + \frac{56}{9} + \left(-\frac{118\zeta_3}{3} + \frac{1520}{27} + \frac{2\pi^2}{3}\right)\varepsilon + \left(-\frac{472\zeta_3}{9} + \frac{24224}{81} - \frac{28\pi^2}{9}\right. \\ &\left. + \frac{299\pi^4}{120}\right)\varepsilon^2 + \left(\frac{6608\zeta_3}{27} + \frac{59\pi^2\zeta_3}{3} - \frac{6478\zeta_5}{5} + \frac{330176}{243} - \frac{760\pi^2}{27} + \frac{299\pi^4}{90}\right)\varepsilon^3 \\ &+ \left(\frac{179360\zeta_3}{81} + \frac{236\pi^2\zeta_3}{9} - \frac{6962\zeta_3^2}{9} - \frac{25912\zeta_5}{15} + \frac{4213376}{729} - \frac{12112\pi^2}{81} - \frac{2093\pi^4}{135}\right. \\ &\left. + \frac{91909\pi^6}{15120}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{58}$$

$$\begin{aligned} b_3 =& \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \left(-\frac{2\zeta_3}{3} + 4 + \frac{\pi^2}{3}\right)\varepsilon + \left(-\frac{4\zeta_3}{3} + 5 + \frac{\pi^2}{2} + \frac{7\pi^4}{360}\right)\varepsilon^2 + \left(-2\zeta_3\right. \\ &\left. -\frac{\pi^2\zeta_3}{9} - \frac{2\zeta_5}{5} + 6 + \frac{2\pi^2}{3} + \frac{7\pi^4}{180}\right)\varepsilon^3 + \left(-\frac{8\zeta_3}{3} - \frac{2\pi^2\zeta_3}{9} + \frac{2\zeta_3^2}{9} - \frac{4\zeta_5}{5} + 7 + \frac{5\pi^2}{6}\right. \\ &\left. + \frac{7\pi^4}{120} + \frac{31\pi^6}{15120}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{59}$$

$$b_4 = \frac{32\pi^2\varepsilon}{3} + \varepsilon^2\left(\frac{704\pi^2}{9} - \frac{256}{3}\pi^2 l_2\right) + \varepsilon^3\left(\frac{10880\pi^2}{27} + \frac{112\pi^4}{9} + \frac{1024}{3}\pi^2 l_2^2 - \frac{5632}{9}\pi^2 l_2\right)$$
$$+ \varepsilon^4\left(-\frac{1984\pi^2\zeta_3}{9} + \frac{147200\pi^2}{81} + \frac{2464\pi^4}{27} - \frac{8192}{9}\pi^2 l_2^3 + \frac{22528}{9}\pi^2 l_2^2 - \frac{87040}{27}\pi^2 l_2 - \frac{896}{9}\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5), \tag{60}$$

$$b_5 = \frac{3}{2\varepsilon^2} + \frac{17}{4\varepsilon} + \left(\frac{59}{8} + \frac{\pi^2}{4}\right) + \left(-\zeta_3 + \frac{65}{16} + \frac{49\pi^2}{24}\right)\varepsilon + \varepsilon^2\left(\frac{151\zeta_3}{6} - \frac{1117}{32} + \frac{475\pi^2}{48} + \frac{7\pi^4}{240}\right.$$
$$\left. - 8\pi^2 l_2\right) + \varepsilon^3\left(192\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{2125\zeta_3}{12} - \frac{\pi^2\zeta_3}{6} - \frac{3\zeta_5}{5} - \frac{13783}{64} + \frac{3745\pi^2}{96} - \frac{103\pi^4}{96}\right.$$
$$\left. + 8 l_2^4 + 16\pi^2 l_2^2 - 52\pi^2 l_2\right) + \varepsilon^4\left(1248\mathrm{Li}_4\left(\frac{1}{2}\right) + 1152\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{19255\zeta_3}{24} - \frac{361\pi^2\zeta_3}{36}\right.$$
$$+ \frac{\zeta_3^2}{3} - \frac{9317\zeta_5}{10} - \frac{114181}{128} + \frac{26563\pi^2}{192} - \frac{7073\pi^4}{960} + \frac{31\pi^6}{10080} - \frac{48 l_2^5}{5} + 52 l_2^4 - 32\pi^2 l_2^3$$
$$\left. + 104\pi^2 l_2^2 - 230\pi^2 l_2 + \frac{104}{15}\pi^4 l_2\right) + \mathcal{O}(\varepsilon^5). \tag{61}$$

3. Heavy propagator:

$$c_2 = a_5, \qquad c_5 = a_6, \qquad c_8 = \frac{1}{2}a_9 \tag{62}$$

$$c_1 = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \left(-\frac{2\zeta_3}{3} + 4 + \frac{\pi^2}{3}\right)\varepsilon + \left(-\frac{4\zeta_3}{3} + 5 + \frac{\pi^2}{2} + \frac{7\pi^4}{360}\right)\varepsilon^2$$
$$+ \left(-2\zeta_3 - \frac{\pi^2\zeta_3}{9} - \frac{2\zeta_5}{5} + 6 + \frac{2\pi^2}{3} + \frac{7\pi^4}{180}\right)\varepsilon^3 + \left(-\frac{8\zeta_3}{3} - \frac{2\pi^2\zeta_3}{9} + \frac{2\zeta_3^2}{9} - \frac{4\zeta_5}{5} + 7 + \frac{5\pi^2}{6} + \frac{7\pi^4}{120} + \frac{31\pi^6}{15120}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \tag{63}$$

$$c_3 = -\frac{2\pi}{\varepsilon} + \pi(4l_2 - 6) - \frac{2}{3}\varepsilon(\pi(21 + \pi^2 + 6l_2^2 - 18l_2)) + \frac{2}{3}\pi\varepsilon^2(8\zeta_3 - 45 - 3\pi^2 + 4l_2^3$$
$$- 18l_2^2 + 42l_2 + 2\pi^2 l_2) - \frac{1}{15}\varepsilon^3(\pi(-240\zeta_3 + 160\zeta_3 l_2 + 930 + 70\pi^2 + 3\pi^4 + 20l_2^4$$
$$- 120l_2^3 + 420l_2^2 + 20\pi^2 l_2^2 - 900l_2 - 60\pi^2 l_2)) + \frac{1}{45}\pi\varepsilon^4(1680\zeta_3 + 80\pi^2\zeta_3 + 576\zeta_5$$
$$+ 480\zeta_3 l_2^2 - 1440\zeta_3 l_2 - 5670 - 450\pi^2 - 27\pi^4 + 24l_2^5 - 180l_2^4 + 840l_2^3 + 40\pi^2 l_2^3$$
$$- 2700l_2^2 - 180\pi^2 l_2^2 + 5580l_2 + 420\pi^2 l_2 + 18\pi^4 l_2) + \mathcal{O}(\varepsilon^5), \tag{64}$$

$$c_4 = \frac{32\pi^2\varepsilon}{3} + \varepsilon^2\left(\frac{704\pi^2}{9} - \frac{256}{3}\pi^2 l_2\right) + \frac{16}{27}\varepsilon^3(680\pi^2 + 21\pi^4 + 576\pi^2 l_2^2 - 1056\pi^2 l_2)$$
$$- \frac{32}{81}\varepsilon^4(558\pi^2\zeta_3 - 4600\pi^2 - 231\pi^4 + 2304\pi^2 l_2^3 - 6336\pi^2 l_2^2 + 8160\pi^2 l_2$$
$$+ 252\pi^4 l_2) + \mathcal{O}(\varepsilon^5), \tag{65}$$

$$c_6 = 2\pi^2 + \varepsilon(8\pi^2 - 8\pi^2 l_2) + \varepsilon^2(24\pi^2 + \pi^4 + 16\pi^2 l_2^2 - 32\pi^2 l_2) + \varepsilon^3\left(-\frac{28\pi^2\zeta_3}{3} + 64\pi^2 + 4\pi^4\right.$$
$$\left. - \frac{64}{3}\pi^2 l_2^3 + 64\pi^2 l_2^2 - 96\pi^2 l_2 - 4\pi^4 l_2\right) + \frac{1}{12}\varepsilon^4(-448\pi^2\zeta_3 + 448\pi^2\zeta_3 l_2 + 1920\pi^2 + 144\pi^4$$
$$+ 5\pi^6 + 256\pi^2 l_2^4 - 1024\pi^2 l_2^3 + 2304\pi^2 l_2^2 + 96\pi^4 l_2^2 - 3072\pi^2 l_2 - 192\pi^4 l_2) + \mathcal{O}(\varepsilon^5), \tag{66}$$

$$c_7 = 4\pi^2 + \varepsilon(16\pi^2 - 16\pi^2 l_2) + \varepsilon^2(48\pi^2 + 2\pi^4 + 32\pi^2 l_2^2 - 64\pi^2 l_2) + \varepsilon^3\left(-\frac{56\pi^2\zeta_3}{3} + 128\pi^2 + 8\pi^4 - \frac{128}{3}\pi^2 l_2^3 + 128\pi^2 l_2^2 - 192\pi^2 l_2 - 8\pi^4 l_2\right) + \frac{1}{6}\varepsilon^4(-448\pi^2\zeta_3 + 448\pi^2\zeta_3 l_2 + 1920\pi^2 + 144\pi^4 + 5\pi^6 + 256\pi^2 l_2^4 - 1024\pi^2 l_2^3 + 2304\pi^2 l_2^2 + 96\pi^4 l_2^2 - 3072\pi^2 l_2 - 192\pi^4 l_2) + \mathcal{O}(\varepsilon^5), \tag{67}$$

$$c_9 = \frac{1}{2\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{1}{4}(14 + \pi^2) + \frac{1}{12}(-16\zeta_3 + 90 + 9\pi^2)\varepsilon + \left(-4\zeta_3 + \frac{31}{2} + \frac{7\pi^2}{4} + \frac{7\pi^4}{80}\right)\varepsilon^2 + \left(\pi^2\left(\frac{15}{4} - \frac{2\zeta_3}{3}\right) - \frac{28\zeta_3}{3} - \frac{16\zeta_5}{5} + \frac{63}{2} + \frac{21\pi^4}{80}\right)\varepsilon^3 + \left(\pi^2\left(\frac{31}{4} - 2\zeta_3\right) - 20\zeta_3 + \frac{16\zeta_3^2}{9} - \frac{48\zeta_5}{5} + \frac{127}{2} + \frac{49\pi^4}{80} + \frac{869\pi^6}{30240}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5), \tag{68}$$

$$c_{10} = -\frac{2\pi}{\varepsilon} + 4\pi(l_2 - 2) + \pi\varepsilon\left(-24 - \frac{10\pi^2}{3} - 4l_2^2 + 16l_2\right) + \frac{4}{3}\pi\varepsilon^2(7\zeta_3 - 48 - 10\pi^2 + 2l_2^3 - 12l_2^2 + 36l_2 + 5\pi^2 l_2) - \frac{2}{45}\varepsilon^3(\pi(-840\zeta_3 + 420\zeta_3 l_2 + 3600 + 900\pi^2 + 119\pi^4 + 30l_2^4 - 240l_2^3 + 1080l_2^2 + 150\pi^2 l_2^2 - 2880l_2 - 600\pi^2 l_2)) + \frac{4}{45}\pi\varepsilon^4(1260\zeta_3 + 175\pi^2\zeta_3 + 279\zeta_5 + 210\zeta_3 l_2^2 - 840\zeta_3 l_2 - 4320 - 1200\pi^2 - 238\pi^4 + 6l_2^5 - 60l_2^4 + 360l_2^3 + 50\pi^2 l_2^3 - 1440l_2^2 - 300\pi^2 l_2^2 + 3600l_2 + 900\pi^2 l_2 + 119\pi^4 l_2) + \mathcal{O}(\varepsilon^5). \tag{69}$$

where $\zeta_n = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function, $l_k = \ln k$. Here a_i constants correspond to #i boundary masters in Fig. 10, b_i constants to #i masters in Fig. 11 and c_i constants to #i masters in Fig. 12. The constants $a_{i,j}$ are given in an arXiv ancillary file. Note, that some of their analytical expressions contain derivatives of hypergeometric functions. The latter can be considered as new elliptic constants[94], which can be evaluated with very high precision using their triangle sum representation and SummerTime package [95], see ancillary arXiv file for details.

B. Results

The complete set of results for the canonical MIs up to $\mathcal{O}(\varepsilon^3)$ for the heavy-heavy vertex and $\mathcal{O}(\varepsilon^2)$ for the heavy-light vertex are given in an arXiv ancillary file. The same notebook also contains results for IBP MIs up to $\mathcal{O}(\varepsilon^2)$ in the case of heavy-heavy and heavy-light vertexes and up to $\mathcal{O}(\varepsilon^2, w)$ in the case of the heavy propagator. Moreover, we provide exact results for the boundary integrals and associated asymptotic coefficient vectors, $\boldsymbol{c}$, as well as the set of matrices, $\{\mathbb{M}, \mathbb{T}, \mathbb{L}\}$ for all cases. The interested reader can employ these to reproduce our results. Due to the size of the expressions for the canonical masters, we only present terms up to an appropriate order. Also, in the case of the heavy propagator, we only present results for IBP masters. We omit overall pre-factors of mass scale as these can be determined by inspection.

1. Heavy-heavy:

$$\tilde{A}_1 = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \frac{1}{3}(-2\zeta_3 + 12 + \pi^2)\varepsilon + \frac{1}{360}(-480\zeta_3 + 1800 + 180\pi^2 + 7\pi^4)\varepsilon^2 + \left(-\frac{1}{9}\pi^2(\zeta_3 - 6) - 2\zeta_3 - \frac{2\zeta_5}{5} + 6 + \frac{7\pi^4}{180}\right)\varepsilon^3 + \mathcal{O}(\varepsilon^4), \tag{70}$$

$$\tilde{A}_2 = \frac{35\pi}{3} + \frac{35}{18}\pi\varepsilon(44 - 12l_2) + \frac{35}{54}\pi\varepsilon^2(680 + 6\pi^2 + 36l_2^2 - 264l_2) - \frac{35}{81}\varepsilon^3(4\pi(18\zeta_3 - 1150 + 9l_2^3 - 99l_2^2 + 510l_2) + \pi^3(18l_2 - 66)) + \mathcal{O}(\varepsilon^4), \tag{71}$$

$$\tilde{A}_3 = \frac{32\pi^2\varepsilon}{3} + \frac{64}{9}\pi^2\varepsilon^2(11-12l_2) + \frac{16}{27}\varepsilon^3(21\pi^4+8\pi^2(85+72l_2^2-132l_2)) + \mathcal{O}(\varepsilon^4), \tag{72}$$

$$\begin{aligned}\tilde{A}_4 = &-\frac{2\pi}{\varepsilon} + 4\pi(l_2-2) - \frac{2}{3}\varepsilon(\pi(36+5\pi^2+6l_2^2-24l_2))) + \frac{4}{3}\pi\varepsilon^2(7\zeta_3-48+2l_2^3\\ &-12l_2^2+5\pi^2(l_2-2)+36l_2) - \frac{2}{45}\varepsilon^3(\pi(30(-28\zeta_3+2(7\zeta_3-48)l_2+120\\ &+l_2^4-8l_2^3+36l_2^2)+119\pi^4+150\pi^2(6+l_2^2-4l_2))) + \mathcal{O}(\varepsilon^4),\end{aligned} \tag{73}$$

$$\begin{aligned}\tilde{A}_5 = &-\frac{i(G(-i;\beta)-G(i;\beta))}{18\varepsilon} - \frac{1}{9}i(G(-i;\beta)-G(i;\beta)+G(0,-i;\beta)\\ &-G(0,i;\beta)-G(-i,-i;\beta)+G(-i,i;\beta)-G(i,-i;\beta)+G(i,i;\beta)) + \mathcal{O}(\varepsilon^1),\end{aligned} \tag{74}$$

$$\begin{aligned}\tilde{A}_6 = &-20i\varepsilon(\pi G(-i;\beta)-\pi G(i;\beta)) + \varepsilon^2(40\pi^2G(-1;\beta)-40\pi^2G(1;\beta)\\ &+80i\pi G(-1,-i;\beta)-80i\pi G(-1,i;\beta)-40i\pi G(0,-i;\beta)+40i\pi G(0,i;\beta)\\ &-40i\pi G(-i,-i;\beta)+40i\pi G(-i,i;\beta)-40i\pi G(i,-i;\beta)+40i\pi G(i,i;\beta)\\ &+80i\pi G(1,-i;\beta)-80i\pi G(1,i;\beta)+\frac{10}{3}i\pi(12l_2-44)G(-i;\beta)\\ &-\frac{10}{3}i\pi(12l_2-44)G(i;\beta)) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{75}$$

$$\begin{aligned}\tilde{A}_7 = &-\frac{1}{9\varepsilon^2} - \frac{2}{9\varepsilon} + \frac{1}{54}(-48G(-i,-i;\beta)+48G(-i,i;\beta)+48G(i,-i;\beta)\\ &-48G(i,i;\beta)-7\pi^2-18) + \mathcal{O}(\varepsilon^1),\end{aligned} \tag{76}$$

$$\tilde{A}_8 = \frac{i(G(-i;\beta)-G(i;\beta))}{3\varepsilon} + \frac{2}{3}i(G(-i;\beta)-G(i;\beta)) + \varepsilon\frac{1}{18}i(18+\pi^2)(G(-i;\beta)-G(i;\beta)) + \mathcal{O}(\varepsilon^2), \tag{77}$$

$$\tilde{A}_9 = \frac{i(G(-i;\beta)-G(i;\beta))}{\varepsilon} + 2i(G(-i;\beta)-G(i;\beta)) + \varepsilon\frac{1}{6}i(18+\pi^2)(G(-i;\beta)-G(i;\beta)) + \mathcal{O}(\varepsilon^2), \tag{78}$$

$$\begin{aligned}\tilde{A}_{10} = &\,\varepsilon\left(5i\pi G(-i;\beta)-5i\pi G(i;\beta)+\frac{5\pi}{2}\right) + \varepsilon^2\left(-10i\pi G(0,-i;\beta)+10i\pi G(0,i;\beta)\right.\\ &+10i\pi G(-i,-i;\beta)-10i\pi G(-i,i;\beta)+10i\pi G(i,-i;\beta)-10i\pi G(i,i;\beta)\\ &\left.-\frac{5}{6}i\pi(12l_2-44)G(-i;\beta)+\frac{5}{6}i\pi(12l_2-44)G(i;\beta)+\pi\left(\frac{95}{6}-5l_2\right)\right) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{79}$$

$$\begin{aligned}\tilde{A}_{11} = &\,\frac{35\pi\varepsilon}{6} + \frac{35}{18}\varepsilon^2(12i\pi G(0,-i;\beta)-12i\pi G(0,i;\beta)+25\pi-6\pi l_2)\\ &+\frac{35}{54}\varepsilon^3(-72\pi^2G(0,-1;\beta)+264i\pi G(0,-i;\beta)-264i\pi G(0,i;\beta)\\ &+72\pi^2G(0,1;\beta)-144i\pi G(0,-1,-i;\beta)+144i\pi G(0,-1,i;\beta)\\ &+72i\pi G(0,0,-i;\beta)-72i\pi G(0,0,i;\beta)+72i\pi G(0,-i,-i;\beta)\\ &-72i\pi G(0,-i,i;\beta)+72i\pi G(0,i,-i;\beta)-72i\pi G(0,i,i;\beta)\\ &-144i\pi G(0,1,-i;\beta)+144i\pi G(0,1,i;\beta)-72i\pi l_2G(0,-i;\beta)\\ &+72i\pi l_2G(0,i;\beta)+3\pi^3+415\pi+18\pi l_2^2-150\pi l_2) + \mathcal{O}(\varepsilon^4),\end{aligned} \tag{80}$$

$$\begin{aligned}\tilde{A}_{12} = &(G(-i,-i;\beta)-G(-i,i;\beta)-G(i,-i;\beta)+G(i,i;\beta))+2\varepsilon(G(-i,-i;\beta)\\ &-G(-i,i;\beta)-G(i,-i;\beta)+G(i,i;\beta)-G(-i,0,-i;\beta)+G(-i,0,i;\beta)\\ &+G(-i,-i,-i;\beta)-G(-i,-i,i;\beta)+G(-i,i,-i;\beta)-G(-i,i,i;\beta)\\ &+G(i,0,-i;\beta)-G(i,0,i;\beta)-G(i,-i,-i;\beta)+G(i,-i,i;\beta)\\ &-G(i,i,-i;\beta)+G(i,i,i;\beta))+\mathcal{O}(\varepsilon^2),\end{aligned} \tag{81}$$

$$\begin{aligned}\tilde{A}_{13} = &\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}+\frac{1}{12}\left(8\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\right)+18+\pi^2\right)+\frac{1}{12}\varepsilon\Bigg(4a_{1,1}\\ &-8\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\right)-\pi^2-66\Bigg)+\mathcal{O}(\varepsilon^2),\end{aligned} \tag{82}$$

$$\begin{aligned}\tilde{A}_{14} = &\frac{35\pi}{9}+\frac{10}{27}\pi\varepsilon(77+(7\sqrt{3}-12)\pi-21l_2)+\frac{5}{162}\varepsilon^2(-63a_{2,1}+8\pi^2(-195+35\sqrt{3}\\ &+144l_2)-56\pi(15l_2-58))+\mathcal{O}(\varepsilon^3),\end{aligned} \tag{83}$$

$$\begin{aligned}\tilde{A}_{15} = &\frac{4\pi^2}{9}-\frac{8}{9}\varepsilon(\pi^2(2l_2-1))+\frac{2}{9}\pi^2\varepsilon^2(6+\pi^2+16l_2^2-16l_2)-\frac{4}{27}\varepsilon^3(2\pi^2(7\zeta_3-6+16l_2^3\\ &-24l_2^2+18l_2)+\pi^4(6l_2-3))+\mathcal{O}(\varepsilon^4),\end{aligned} \tag{84}$$

$$\tilde{A}_{16} = -\frac{2\pi^2}{3\varepsilon}+\frac{4}{3}\pi^2(2l_3-1)+\frac{1}{2}\varepsilon a_{3,1}+\frac{1}{2}\varepsilon^2 a_{3,2}+\frac{1}{2}\varepsilon^3 a_{3,3}+\mathcal{O}(\varepsilon^4), \tag{85}$$

$$\begin{aligned}\tilde{A}_{17} = &-\frac{1}{6}i\varepsilon(\pi^2(G(-i;\beta)-G(i;\beta))+6(G(-i,-i,-i;\beta)-G(-i,-i,i;\beta)\\ &-G(-i,i,-i;\beta)+G(-i,i,i;\beta)-G(i,-i,-i;\beta)+G(i,-i,i;\beta)+G(i,i,-i;\beta)\\ &-G(i,i,i;\beta)))+\mathcal{O}(\varepsilon^2),\end{aligned} \tag{86}$$

$$\begin{aligned}\tilde{A}_{18} = &-\frac{35}{3}\pi\varepsilon^2(2\pi G(-\sqrt{3};\beta)-2\pi G(\sqrt{3};\beta)-3i(G(-i,-i;\beta)-G(-i,i;\beta)\\ &+G(i,-i;\beta)-G(i,i;\beta)-G(-\sqrt{3},-i;\beta)+G(-\sqrt{3},i;\beta)-G(\sqrt{3},-i;\beta)\\ &+G(\sqrt{3},i;\beta)))+\mathcal{O}(\varepsilon^3),\end{aligned} \tag{87}$$

$$\begin{aligned}\tilde{A}_{19} = &\frac{\varepsilon}{3\sqrt{3}}\Bigg(\pi^2\left(-G\left(-\frac{1}{\sqrt{3}};\beta\right)+G\left(\frac{1}{\sqrt{3}};\beta\right)+4G(-\sqrt{3};\beta)-4G(\sqrt{3};\beta)\right)\\ &+6\Bigg(-3G\left(-\frac{1}{\sqrt{3}},-i,-i;\beta\right)+3\left(G\left(-\frac{1}{\sqrt{3}},-i,i;\beta\right)+G\left(-\frac{1}{\sqrt{3}},i,-i;\beta\right)\right.\\ &-G\left(-\frac{1}{\sqrt{3}},i,i;\beta\right)+G\left(\frac{1}{\sqrt{3}},-i,-i;\beta\right)-G\left(\frac{1}{\sqrt{3}},-i,i;\beta\right)-G\left(\frac{1}{\sqrt{3}},i,-i;\beta\right)\\ &+G\left(\frac{1}{\sqrt{3}},i,i;\beta\right)+G(-\sqrt{3},-i,-i;\beta)-G(-\sqrt{3},-i,i;\beta)-G(-\sqrt{3},i,-i;\beta)\\ &+G(-\sqrt{3},i,i;\beta)-G(\sqrt{3},-i,-i;\beta)+G(\sqrt{3},-i,i;\beta)+G(\sqrt{3},i,-i;\beta)\\ &\left.-G(\sqrt{3},i,i;\beta)\right)+2i\Im\left(\mathrm{Li}_2\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\right)(G(-i;\beta)-G(i;\beta))\Bigg)\Bigg)+\mathcal{O}(\varepsilon^2),\end{aligned} \tag{88}$$

$$\tilde{A}_{20} = -6G(-i,-i;\beta)+6G(-i,i;\beta)+6G(i,-i;\beta)-6G(i,i;\beta)-\frac{17\pi^2}{12}+\mathcal{O}(\varepsilon), \tag{89}$$

$$\tilde{A}_{21} = \frac{i(G(-i;\beta) - G(i;\beta))}{6\varepsilon} + \frac{1}{3} i(G(-i;\beta) - G(i;\beta))$$
$$+ \frac{1}{36} i\varepsilon \left(24\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) + 18 + \pi^2 \right)(G(-i;\beta) - G(i;\beta))$$
$$- \frac{1}{36} i\varepsilon^2 (G(-i;\beta) - G(i;\beta))(-12a_{1,1} + 24\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) - 8\zeta_3$$
$$+ 7\pi^2 + 246) + \mathcal{O}(\varepsilon^3), \tag{90}$$

$$\tilde{A}_{22} = \frac{35}{3} i\varepsilon(\pi G(-i;\beta) - \pi G(i;\beta)) + \varepsilon^2 \left(\frac{35}{18} i\pi(12l_2 - 44)G(i;\beta) \right.$$
$$\left. - \frac{35}{18} i\pi(12l_2 - 44)G(-i;\beta) \right) + \varepsilon^3 \left(\frac{35}{54} i\pi(680 + 6\pi^2 + 36l_2^2 - 264l_2)G(-i;\beta) \right.$$
$$\left. - \frac{35}{54} i\pi(680 + 6\pi^2 + 36l_2^2 - 264l_2)G(i;\beta) \right) + \mathcal{O}(\varepsilon^4), \tag{91}$$

$$\tilde{A}_{23} = -\frac{35}{6}\pi\varepsilon^3 (2\pi G(0, -\sqrt{3};\beta) - 2\pi G(0, \sqrt{3};\beta) - 3i(G(0, -i, -i;\beta) - G(0, -i, i;\beta)$$
$$+ G(0, i, -i;\beta) - G(0, i, i;\beta) - G(0, -\sqrt{3}, -i;\beta) + G(0, -\sqrt{3}, i;\beta)$$
$$- G(0, \sqrt{3}, -i;\beta) + G(0, \sqrt{3}, i;\beta))) + \mathcal{O}(\varepsilon^4), \tag{92}$$

$$\tilde{A}_{24} = \frac{1}{2\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{1}{4}(14 + \pi^2) + \frac{1}{12}(-16\zeta_3 + 90 + 9\pi^2)\varepsilon + \frac{1}{80}(-320\zeta_3 + 1240 + 140\pi^2$$
$$+ 7\pi^4)\varepsilon^2 + \frac{1}{240}(-2240\zeta_3 - 160\pi^2\zeta_3 - 768\zeta_5 + 7560 + 900\pi^2 + 63\pi^4)\varepsilon^3 + \mathcal{O}(\varepsilon^4), \tag{93}$$

$$\tilde{A}_{25} = 2\pi^2 - 8\varepsilon(\pi^2(l_2 - 1)) + \varepsilon^2(\pi^4 + 8\pi^2(3 + 2l_2^2 - 4l_2)) - \frac{4}{3}\varepsilon^3(\pi^2(7\zeta_3 - 48 + 16l_2^3$$
$$- 48l_2^2 + 72l_2 + \pi^2(3l_2 - 3))) + \mathcal{O}(\varepsilon^4), \tag{94}$$

$$\tilde{A}_{26} = \frac{i(G(-i;\beta) - G(i;\beta))}{2\varepsilon} + \frac{3}{2} i(G(-i;\beta) - G(i;\beta)) + \varepsilon\frac{1}{4} i(14 + \pi^2)(G(-i;\beta)$$
$$- G(i;\beta)) + \varepsilon^2 \frac{1}{12} i(-16\zeta_3 + 90 + 9\pi^2)(G(-i;\beta) - G(i;\beta)) + \mathcal{O}(\varepsilon^3). \tag{95}$$

2. Heavy-light:

$$\tilde{B}_1 = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \frac{1}{3}(-2\zeta_3 + 12 + \pi^2)\varepsilon + \frac{1}{360}(-480\zeta_3 + 1800$$
$$+ 180\pi^2 + 7\pi^4)\varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{96}$$

$$\tilde{B}_2 = \frac{7}{60\varepsilon^2} + \frac{7}{30\varepsilon} + \left(\frac{7}{20} + \frac{7\pi^2}{72}\right) + \frac{7}{180}(22\zeta_3 + 12 + 5\pi^2)\varepsilon$$
$$+ \frac{7(1760\zeta_3 + 600 + 300\pi^2 + 101\pi^4)\varepsilon^2}{7200} + \mathcal{O}(\varepsilon^3), \tag{97}$$

$$\tilde{B}_3 = \frac{2}{105\varepsilon^2} + \frac{4}{105\varepsilon} + \frac{1}{105}(-16G(-i,-i;\beta) + 16G(-i,i;\beta) + 16G(i,-i;\beta) -16G(i,i;\beta) - \pi^2 + 6) - \frac{2}{315}\varepsilon(48G(-i,-i;\beta) - 48G(-i,i;\beta) -48G(i,-i;\beta) + 48G(i,i;\beta) - 144G(-i,0,-i;\beta) + 144G(-i,0,i;\beta) +144G(-i,-i,-i;\beta) - 144G(-i,-i,i;\beta) + 144G(-i,i,-i;\beta) -144G(-i,i,i;\beta) + 144G(i,0,-i;\beta) - 144G(i,0,i;\beta) - 144G(i,-i,-i;\beta) +144G(i,-i,i;\beta) - 144G(i,i,-i;\beta) + 144G(i,i,i;\beta) - 118\zeta_3 +3\pi^2 - 12) + \mathcal{O}(\varepsilon^2), \tag{98}$$

$$\tilde{B}_4 = -\frac{8i(G(-i;\beta) - G(i;\beta))}{105\varepsilon} - \frac{16}{105}i(G(-i;\beta) - G(i;\beta) - 3G(0,-i;\beta) + 3G(0,i;\beta) + 3G(-i,-i;\beta) - 3G(-i,i;\beta) + 3G(i,-i;\beta) - 3G(i,i;\beta)) + \mathcal{O}(\varepsilon^1), \tag{99}$$

$$\tilde{B}_5 = -\frac{64\pi^2}{105} + \frac{128}{105}\pi^2\varepsilon(4l_2 - 1) - \frac{32}{315}\varepsilon^2(\pi^2(18 + 7\pi^2 + 192l_2^2 - 96l_2)) + \mathcal{O}(\varepsilon^3), \tag{100}$$

$$\tilde{B}_6 = \frac{4i(G(-i;\beta) - G(i;\beta))}{3\varepsilon} + \frac{8}{3}i(G(-i;\beta) - G(i;\beta) - G(0,-i;\beta) + G(0,i;\beta) + G(-i,-i;\beta) - G(-i,i;\beta) + G(i,-i;\beta) - G(i,i;\beta)) + \mathcal{O}(\varepsilon^1), \tag{101}$$

$$\tilde{B}_7 = \frac{14}{45}i\varepsilon(\pi^2(G(-i;\beta) - G(i;\beta)) + 6(G(-i,-i,-i;\beta) - G(-i,-i,i;\beta) -G(-i,i,-i;\beta). + G(-i,i,i;\beta) - G(i,-i,-i;\beta) + G(i,-i,i;\beta) + G(i,i,-i;\beta) -G(i,i,i;\beta))) + \mathcal{O}(\varepsilon^2), \tag{102}$$

$$\tilde{B}_8 = -\frac{4i(G(-i;\beta) - G(i;\beta))}{15\varepsilon} - \frac{8}{15}i(G(-i;\beta) - G(i;\beta) - 2G(0,-i;\beta) + 2G(0,i;\beta) + 2G(-i,-i;\beta) - 2G(-i,i;\beta) + 2G(i,-i;\beta) - 2G(i,i;\beta)) + \mathcal{O}(\varepsilon^1), \tag{103}$$

$$\tilde{B}_9 = \frac{1}{189}i\varepsilon(\pi^2(G(-i;\beta) - G(i;\beta)) + 6(G(-i,-i,-i;\beta) - G(-i,-i,i;\beta) -G(-i,i,-i;\beta) + G(-i,i,i;\beta) - G(i,-i,-i;\beta) + G(i,-i,i;\beta) + G(i,i,-i;\beta) -G(i,i,i;\beta))) + \mathcal{O}(\varepsilon^0), \tag{104}$$

$$\tilde{B}_{10} = \frac{1}{441}(2G(-i,-i;\beta) - 2G(-i,i;\beta) - 2G(i,-i;\beta) + 2G(i,i;\beta) + \pi^2) + \frac{2}{441}\varepsilon(\pi^2G(-i;\beta) + \pi^2G(i;\beta) + 2G(-i,-i;\beta) - 2G(-i,i;\beta) -2G(i,-i;\beta) + 2G(i,i;\beta) - 4G(0,-i,-i;\beta) + 4G(0,-i,i;\beta) +4G(0,i,-i;\beta) - 4G(0,i,i;\beta) - 6G(-i,0,-i;\beta) + 6G(-i,0,i;\beta) +10G(-i,-i,-i;\beta) - 10G(-i,-i,i;\beta) + 2G(-i,i,-i;\beta) - 2G(-i,i,i;\beta) +6G(i,0,-i;\beta) - 6G(i,0,i;\beta) - 2G(i,-i,-i;\beta) + 2G(i,-i,i;\beta) -10G(i,i,-i;\beta) + 10G(i,i,i;\beta) + \pi^2 - 4\pi^2l_2) + \mathcal{O}(\varepsilon^2), \tag{105}$$

$$\begin{aligned}\tilde{B}_{11} = &-\frac{8}{9}(2G(-i,-i;\beta) - 2G(-i,i;\beta) - 2G(i,-i;\beta) + 2G(i,i;\beta) + \pi^2)\\
&-\frac{16}{9}\varepsilon(\pi^2 G(-i;\beta) + \pi^2 G(i;\beta) + 2G(-i,-i;\beta) - 2G(-i,i;\beta)\\
&-2G(i,-i;\beta) + 2G(i,i;\beta) - 4G(0,-i,-i;\beta) + 4G(0,-i,i;\beta)\\
&+4G(0,i,-i;\beta) - 4G(0,i,i;\beta) - 2G(-i,0,-i;\beta) + 2G(-i,0,i;\beta)\\
&+6G(-i,-i,-i;\beta) - 6G(-i,-i,i;\beta) - 2G(-i,i,-i;\beta) + 2G(-i,i,i;\beta)\\
&+2G(i,0,-i;\beta) - 2G(i,0,i;\beta) + 2G(i,-i,-i;\beta) - 2G(i,-i,i;\beta)\\
&-6G(i,i,-i;\beta) + 6G(i,i,i;\beta) + \pi^2 - 4\pi^2 l_2) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{106}$$

$$\begin{aligned}\tilde{B}_{12} = &-\frac{2}{11}(G(-i,-i;\beta) - G(-i,i;\beta) - G(i,-i;\beta) + G(i,i;\beta))\\
&-\frac{4}{11}\varepsilon(G(-i,-i;\beta) - G(-i,i;\beta) - G(i,-i;\beta) + G(i,i;\beta)\\
&-2G(0,-i,-i;\beta) + 2G(0,-i,i;\beta) + 2G(0,i,-i;\beta) - 2G(0,i,i;\beta)\\
&-3G(-i,0,-i;\beta) + 3G(-i,0,i;\beta) + 5G(-i,-i,-i;\beta) - 5G(-i,-i,i;\beta)\\
&+G(-i,i,-i;\beta) - G(-i,i,i;\beta) + 3G(i,0,-i;\beta) - 3G(i,0,i;\beta)\\
&-G(i,-i,-i;\beta) + G(i,-i,i;\beta) - 5G(i,i,-i;\beta) + 5G(i,i,i;\beta)) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{107}$$

$$\begin{aligned}\tilde{B}_{13} = &\, i\varepsilon^2(2G(-i,0,-i,-i;\beta) - 2(G(-i,0,-i,i;\beta) + G(-i,0,i,-i;\beta) - G(-i,0,i,i;\beta)\\
&+G(-i,-i,0,-i;\beta) - G(-i,-i,0,i;\beta) - 2G(-i,-i,i,-i;\beta) + 2G(-i,-i,i,i;\beta)\\
&-G(-i,i,0,-i;\beta) + G(-i,i,0,i;\beta) + 2G(-i,i,-i,-i;\beta) - 2G(-i,i,-i,i;\beta)\\
&+G(i,0,-i,-i;\beta) - G(i,0,-i,i;\beta) - G(i,0,i,-i;\beta) + G(i,0,i,i;\beta)\\
&-G(i,-i,0,-i;\beta) + G(i,-i,0,i;\beta) + 2G(i,-i,i,-i;\beta) - 2G(i,-i,i,i;\beta)\\
&+G(i,i,0,-i;\beta) - G(i,i,0,i;\beta) - 2G(i,i,-i,-i;\beta) + 2G(i,i,-i,i;\beta))\\
&+3\zeta_3(G(i;\beta) - G(-i;\beta))) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{108}$$

$$\begin{aligned}\tilde{B}_{14} = &\,\frac{1}{4}(G(-i,-i;\beta) - G(-i,i;\beta) - G(i,-i;\beta) + G(i,i;\beta))\\
&+\frac{1}{2}\varepsilon(G(-i,-i;\beta) - G(-i,i;\beta) - G(i,-i;\beta) + G(i,i;\beta)\\
&-2G(0,-i,-i;\beta) + 2G(0,-i,i;\beta) + 2G(0,i,-i;\beta) - 2G(0,i,i;\beta)\\
&-G(-i,0,-i;\beta) + G(-i,0,i;\beta) + 3G(-i,-i,-i;\beta) - 3G(-i,-i,i;\beta)\\
&-G(-i,i,-i;\beta) + G(-i,i,i;\beta) + G(i,0,-i;\beta) - G(i,0,i;\beta)\\
&+G(i,-i,-i;\beta) - G(i,-i,i;\beta) - 3G(i,i,-i;\beta) + 3G(i,i,i;\beta)) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{109}$$

$$\begin{aligned}\tilde{B}_{15} = &\,\frac{32}{9}(G(-i,-i;\beta) - G(-i,i;\beta) - G(i,-i;\beta) + G(i,i;\beta))\\
&+\frac{64}{9}\varepsilon(G(-i,-i;\beta) - G(-i,i;\beta) - G(i,-i;\beta) + G(i,i;\beta)\\
&-2G(0,-i,-i;\beta) + 2G(0,-i,i;\beta) + 2G(0,i,-i;\beta) - 2G(0,i,i;\beta)\\
&-G(-i,0,-i;\beta) + G(-i,0,i;\beta) + 3G(-i,-i,-i;\beta) - 3G(-i,-i,i;\beta)\\
&-G(-i,i,-i;\beta) + G(-i,i,i;\beta) + G(i,0,-i;\beta) - G(i,0,i;\beta)\\
&+G(i,-i,-i;\beta) - G(i,-i,i;\beta) - 3G(i,i,-i;\beta) + 3G(i,i,i;\beta)) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{110}$$

$$\tilde{B}_{16} = \frac{19}{60\varepsilon^2} + \frac{19}{30\varepsilon} + \frac{1}{360}(342 - 13\pi^2) + \frac{1}{180}\varepsilon(-374\zeta_3 + 228 + \pi^2(96l_2 - 13)) + \mathcal{O}(\varepsilon^2), \tag{111}$$

$$\begin{aligned}\tilde{B}_{17} = &\frac{1}{15}(-8G(-i,-i;\beta) + 8G(-i,i;\beta) + 8G(i,-i;\beta) - 8G(i,i;\beta) - \pi^2)\\
&+ \frac{1}{15}\varepsilon(4\pi^2 G(-1;\beta) - 8\pi^2 G(-i;\beta) - 8\pi^2 G(i;\beta) + 4\pi^2 G(1;\beta)\\
&-16G(-i,-i;\beta) + 16G(-i,i;\beta) + 16G(i,-i;\beta) - 16G(i,i;\beta)\\
&+32G(-1,-i,-i;\beta) - 32G(-1,-i,i;\beta) - 32G(-1,i,-i;\beta) + 32G(-1,i,i;\beta)\\
&+16G(0,-i,-i;\beta) - 16G(0,-i,i;\beta) - 16G(0,i,-i;\beta) + 16G(0,i,i;\beta)\\
&+16G(-i,0,-i;\beta) - 16G(-i,0,i;\beta) - 64G(-i,-i,-i;\beta) + 64G(-i,-i,i;\beta)\\
&+32G(-i,i,-i;\beta) - 32G(-i,i,i;\beta) - 16G(i,0,-i;\beta) + 16G(i,0,i;\beta)\\
&-32G(i,-i,-i;\beta) + 32G(i,-i,i;\beta) + 64G(i,i,-i;\beta) - 64G(i,i,i;\beta)\\
&+32G(1,-i,-i;\beta) - 32G(1,-i,i;\beta) - 32G(1,i,-i;\beta) + 32G(1,i,i;\beta)\\
&+21\zeta_3 - 2\pi^2 + 10\pi^2 l_2) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{112}$$

$$\begin{aligned}\tilde{B}_{18} = &\frac{2}{3}i\varepsilon(\pi^2(G(-i;\beta) - G(i;\beta)) + 6(G(-i,-i,-i;\beta) - G(-i,-i,i;\beta)\\
&- G(-i,i,-i;\beta) + G(-i,i,i;\beta) - G(i,-i,-i;\beta) + G(i,-i,i;\beta)\\
&+ G(i,i,-i;\beta) - G(i,i,i;\beta))) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{113}$$

3. Heavy propagator:

$$\begin{aligned}C_1 = &\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \frac{1}{3}(-2\zeta_3 + 12 + \pi^2)\varepsilon + \frac{1}{360}(-480\zeta_3 + 1800 + 180\pi^2\\
&+7\pi^4)\varepsilon^2 + \mathcal{O}(\varepsilon^3, w^2),\end{aligned} \tag{114}$$

$$C_2 = \frac{3}{2\varepsilon^2} + \frac{9}{2\varepsilon} + \frac{1}{4}\left(8\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) + 42 + \pi^2\right) + \varepsilon a_{1,1} + \varepsilon^2 a_{1,2} + \mathcal{O}(\varepsilon^3, w^2), \tag{115}$$

$$\begin{aligned}C_3 = &\frac{2w}{\varepsilon^2} + \frac{2w - 2\pi}{\varepsilon} + \left(\frac{1}{3}(6 + \pi^2)w + \pi(4l_2 - 6)\right) + \varepsilon\left(\frac{1}{3}w(-4\zeta_3 + 6 + \pi^2)\right.\\
&\left.-\frac{2}{3}\pi(21 + \pi^2 + 6l_2^2 - 18l_2)\right) + \varepsilon^2\left(\frac{1}{180}w(-240\zeta_3 + 360 + 60\pi^2 + 7\pi^4)\right.\\
&\left.+\frac{2}{3}\pi(8\zeta_3 - 45 + 4l_2^3 - 18l_2^2 + 42l_2 + \pi^2(2l_2 - 3))\right) + \mathcal{O}(\varepsilon^3, w^2),\end{aligned} \tag{116}$$

$$\begin{aligned}C_4 = &\frac{2w}{\varepsilon^2} + \frac{4w}{\varepsilon} + \frac{1}{3}(24 + \pi^2)w + \varepsilon\left(\frac{2}{3}w(-2\zeta_3 + 24 + \pi^2) + \frac{32\pi^2}{3}\right) + \varepsilon^2\left(\frac{1}{180}w(-480\zeta_3\right.\\
&\left.+5760 + 240\pi^2 + 7\pi^4) + \frac{64}{9}\pi^2(11 - 12l_2)\right) + \mathcal{O}(\varepsilon^3, w^2),\end{aligned} \tag{117}$$

$$C_5 = -\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon} + \left(-4\pi^2 w - \frac{\pi^2}{3} - 8\right) + \varepsilon\left(16\pi^2 w(2l_2 - 1) - \frac{2}{3}(-2\zeta_3 + 24 + \pi^2)\right) + \varepsilon^2\left(\frac{1}{180}(480\zeta_3 - 5760 - 240\pi^2 - 7\pi^4) - \frac{2}{3}w(7\pi^4 + 96\pi^2(1 + 2l_2^2 - 2l_2))\right) + \mathcal{O}(\varepsilon^3, w^2), \tag{118}$$

$$C_6 = \frac{w}{\varepsilon^2} + \frac{2w - 2\pi}{\varepsilon} + \left(-\frac{4}{3}\pi(6 + \sqrt{3}\pi - 3l_2) + \frac{1}{6}w\left(24\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) + 24 + \pi^2\right)\right) + \varepsilon\left(a_{2,1} - \frac{1}{6}w\left(-12a_{1,1} + 24\sqrt{3}\Im\left(\mathrm{Li}_2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) - 8\zeta_3 + 7\pi^2 + 222\right)\right) + \varepsilon^2\left(\frac{1}{180}w(-360a_{1,1} + 360a_{1,2} + 480\zeta_3 - 7\pi^4 - 240\pi^2 - 5760) + a_{2,2}\right) + \mathcal{O}(\varepsilon^3, w^2), \tag{119}$$

$$C_7 = 2\pi^2 + \varepsilon(16\pi^2 w - 8\pi^2(l_2 - 1)) + \varepsilon^2(32\pi^2 w(3 - 4l_2) + \pi^4 + 8\pi^2(3 + 2l_2^2 - 4l_2)) + \mathcal{O}(\varepsilon^3, w^2), \tag{120}$$

$$C_8 = -\frac{8(\pi w)}{\varepsilon} + (4\pi w(4l_2 - 4) + 4\pi^2) + \varepsilon\left(-\frac{4}{3}\pi w(24 + 2\pi^2 + 12l_2^2 - 24l_2) - 16\pi^2(l_2 - 1)\right) + \varepsilon^2\left(\frac{8}{3}w(4\pi(2\zeta_3 - 6 + l_2^3 - 3l_2^2 + 6l_2) + \pi^3(2l_2 - 2)) + 2(\pi^4 + 8\pi^2(3 + 2l_2^2 - 4l_2))\right) + \mathcal{O}(\varepsilon^3, w^2), \tag{121}$$

$$C_9 = -\frac{2\pi^2}{3\varepsilon} + \left(\frac{4}{3}\pi^2(2l_3 - 1) - \frac{16\pi^2 w}{3\sqrt{3}}\right) + \varepsilon\left(\frac{4}{9}w(3a_{2,1} + 2\pi^3 + 8(3 + \sqrt{3})\pi^2 + \pi(72 + 12l_2^2 - 48l_2)) + \frac{a_{3,1}}{2}\right) + \varepsilon^2\left(\frac{a_{3,2}}{2} - \frac{4}{9}w(6a_{2,1} - 3a_{2,2} + 8\pi(2\zeta_3 - 6 + l_2^3 - 3l_2^2 + 6l_2) + 48\pi^2(4l_2 - 3) + 2\pi^3(2l_2 - 2))\right) + \mathcal{O}(\varepsilon^3, w^2), \tag{122}$$

$$C_{10} = \frac{1}{2\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{7}{2} + \frac{\pi^2}{4} + \varepsilon\left(\frac{15}{2} + \frac{3\pi^2}{4} - \frac{4\zeta_3}{3}\right) + \varepsilon^2\left(\frac{31}{2} + \frac{7\pi^2}{4} + \frac{7\pi^4}{80} - 4\zeta_3\right) + \mathcal{O}(\varepsilon^3, w^2), \tag{123}$$

$$C_{11} = 0 + \mathcal{O}(\varepsilon^3, w^2), \tag{124}$$

$$C_{12} = \frac{w}{\varepsilon^2} + \frac{2w - 2\pi}{\varepsilon} + \left(\frac{1}{2}(8 + \pi^2)w + 4\pi(l_2 - 2)\right) + \varepsilon\left(w\left(-\frac{8\zeta_3}{3} + 8 + \pi^2\right) - \frac{2}{3}(5\pi^3 + 6\pi(6 + l_2^2 - 4l_2))\right) + \varepsilon^2\left(\frac{1}{120}w(-640\zeta_3 + 1920 + 240\pi^2 + 21\pi^4) + \frac{4}{3}\pi(7\zeta_3 - 48 + 2l_2^3 - 12l_2^2 + 5\pi^2(l_2 - 2) + 36l_2)\right) + \mathcal{O}(\varepsilon^3, w^2). \tag{125}$$

The integrals above are expressed in terms of Goncharov polylogarithms, which are generalizations of HPLs and defined recursively as [43,96,97]

$$G(b, a_1 \cdots a_n; x) = \int_0^x \frac{dy}{y - b} G(a_1 \cdots a_n; y), \qquad G(;x) = 1, \tag{126}$$

with

$$G(0...0;x) = \frac{1}{n!}\ln^n x \tag{127}$$

and the integration path being a straight line from 0 to $x \in \mathbb{C}$.

IV. CONCLUSION

We have employed the differential equations method to treat diagrams with heavy field insertions up to three-points two-loop order. Our analysis focused on vertex or form factor diagrams as these can be combined and used in studies beyond three-point order as they form the building blocks for a broad class of processes. By including a mass scale, our results are applicable for a wider range of theories and provide an IR structure for massless models studied. The treatment of the heavy-heavy and heavy-light vertex diagrams was achieved explicitly by reducing integrals to ε-form and expressing their results in terms of MPLs. On the other hand, the self-energy contributions needed for heavy field and residual mass renormalization were determined by a series expansion in the off-shell heavy field energy. Exact off-shell self-energies were shown to require treatment with elliptic polylogarithms. We instead employed the Frobenius method to obtain solutions as an expansion in small off-shell energies. Thus, we provide further proof-positive of the power of the differential equations and dimensional recurrence approaches, advocating for their use even when more exotic propagators are present.

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APPENDIX: ONE-LOOP MASTER INTEGRALS

Here, for completeness, we present a calculation of one-loop master integrals.

1. Heavy-heavy vertex

In the case of heavy-heavy vertices, the single integral family for prototype diagram shown in Fig. 1(a) is given by

$$I^{HH}_{\nu_1,\nu_2,\nu_3} = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l\cdot v_1)^{\nu_1}(l\cdot v_2)^{\nu_2}(l^2-M^2)^{\nu_3}}. \tag{A1}$$

Upon IBP reduction, all integrals in this family reduce to three master integrals $I^{HH}_{001}, I^{HH}_{011}, I^{HH}_{111}$. The first two master integrals are easy to calculate with Feynman parameters, and we have

$$I^{HH}_{001} = -\Gamma(\varepsilon-1)M^{2-2\varepsilon}, \qquad I^{HH}_{011} = \sqrt{\pi}\Gamma\left(-\frac{1}{2}+\varepsilon\right)M^{1-2\varepsilon} \tag{A2}$$

The third master integral can be conveniently calculated with the use of differential equation with respect to $w = v_1 \cdot v_2$

$$\frac{d}{dw} I^{HH}_{111}(w) = \frac{w}{1-w^2} I^{HH}_{111}(w) - \frac{2(\varepsilon-1)}{1-w^2}\frac{1}{M^2} I^{HH}_{001}. \tag{A3}$$

One can then easily solve Eq. (A3) by variation of parameters and boundary condition, $I^{HH}_{111}(1) = -2M^{-2\varepsilon}\Gamma(\varepsilon)$, to obtain,

$$I^{HH}_{111} = -\frac{2\Gamma(\varepsilon)M^{-2\varepsilon}}{\sqrt{1-w^2}} \arccos(w), \tag{A4}$$

which mimics the well-known result from Feynman parametrization and its modification for HQET-like propagators [68].

2. Heavy-light vertex

Here, the single integral family for prototype diagram shown in Fig. 1(b) is given by

$$I^{HL}_{\nu_1,\nu_2,\nu_3} = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l\cdot v_1)^{\nu_1}((l+p_2)^2-m^2)^{\nu_2}(l^2)^{\nu_3}}. \tag{A5}$$

Upon IBP reduction all integrals in this family reduce to two master integrals $I^{HL}_{010}, I^{HL}_{110}$. The first master we already have encountered in previous subsection

$$I^{HL}_{010} = -\Gamma(\varepsilon-1)m^{2-2\varepsilon} \tag{A6}$$

The second master integral can be again calculated with the use of differential equation, this time with respect to $w = v_1 \cdot p_2/m$

$$\frac{d}{dw} I^{HL}_{111}(w) = \frac{w(1-2\varepsilon)}{1-w^2} I^{HL}_{110}(w) + \frac{2(1-\varepsilon)}{1-w^2}\frac{1}{m} I^{HL}_{010} \tag{A7}$$

The Eq. (A7) is easily solved by variation of parameters and boundary condition,

$$I^{HL}_{110}(0) = \sqrt{\pi}\Gamma(-1/2+\varepsilon)m^{1-2\varepsilon}, \tag{A8}$$

to obtain,

$$I_{110}^{HL} = -2m^{1-2\varepsilon}\Gamma(\varepsilon)w_2F_1\left(1,\varepsilon;\frac{3}{2};w^2\right) + \sqrt{\pi}m^{1-2\varepsilon}\Gamma\left(-\frac{1}{2}+\varepsilon\right)(1-w^2)^{1/2-\varepsilon}. \quad \text{(A9)}$$

3. Heavy propagator

In this case, the single integral family for prototype diagram shown in Fig. 1(c) is given by

$$I_{\nu_1,\nu_2}^{SE} = M^{d-\nu_1-2\nu_2}\int\frac{d^dl}{i\pi^{d/2}}\frac{1}{(l\cdot v-w)^{\nu_1}(l^2-1)^{\nu_2}} \quad \text{(A10)}$$

Here we have the same master integrals as in previous subsection. These integrals are given by

$$I_{01}^{SE} = -\Gamma(\varepsilon-1)M^{2-2\varepsilon} \quad \text{(A11)}$$

and

$$I_{11}^{SE} = -2M^{1-2\varepsilon}\Gamma(\varepsilon)w_2F_1\left(1,\varepsilon;\frac{3}{2};w^2\right) + \sqrt{\pi}M^{1-2\varepsilon}\Gamma\left(-\frac{1}{2}+\varepsilon\right)(1-w^2)^{1/2-\varepsilon}. \quad \text{(A12)}$$

[1] A. Pineda and J. Soto, Effective field theory for ultrasoft momenta in NRQCD and NRQED, Nucl. Phys. B Proc. Suppl. **64**, 428 (1998).

[2] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Potential NRQCD: An effective theory for heavy quarkonium, Nucl. Phys. **B566**, 275 (2000).

[3] H. Georgi, An effective field theory for heavy quarks at low energies, Phys. Lett. B **240**, 447 (1990).

[4] B. Jantzen, J. H. Kühn, A. A. Penin, and V. A. Smirnov, Two-loop high-energy electroweak logarithmic corrections in a spontaneously broken SU(2) gauge model, Phys. Rev. D **72**, 051301 (2005).

[5] B. Jantzen and V. A. Smirnov, The two-loop vector form factor in the Sudakov limit, Eur. Phys. J. C **47**, 671 (2006).

[6] J.-y. Chiu, F. Golf, R. Kelley, and A. V. Manohar, Electroweak corrections to high energy processes using effective field theory, Phys. Rev. D **77**, 053004 (2008).

[7] J.-y. Chiu, F. Golf, R. Kelley, and A. V. Manohar, Electroweak Sudakov Corrections Using Effective Field Theory, Phys. Rev. Lett. **100**, 021802 (2008).

[8] J.-y. Chiu, R. Kelley, and A. V. Manohar, Electroweak corrections using effective field theory: Applications to the CERN LHC, Phys. Rev. D **78**, 073006 (2008).

[9] B. Assi and B. A. Kniehl, Matching the standard model to HQET and NRQCD, arXiv:2011.06447.

[10] B. Assi and B. A. Kniehl, Electroweak form factor in Sudakov and threshold regimes with effective field theories, arXiv:2011.14933.

[11] G. Ovanesyan, T. R. Slatyer, and I. W. Stewart, Heavy Dark Matter Annihilation from Effective Field Theory, Phys. Rev. Lett. **114**, 211302 (2015).

[12] M. Beneke, R. Szafron, and K. Urban, Wino potential and Sommerfeld effect at NLO, Phys. Lett. B **800**, 135112 (2020).

[13] M. Bauer, M. Neubert, S. Renner, M. Schnubel, and A. Thamm, The low-energy effective theory of axions and ALPs, J. High Energy Phys. 04 (2021) 063.

[14] B. Mecaj and M. Neubert, Effective field theory for leptoquarks, arXiv:2012.02186.

[15] P. H. Damgaard, K. Haddad, and A. Helset, Heavy black hole effective theory, J. High Energy Phys. 11 (2019) 070.

[16] R. Aoude, K. Haddad, and A. Helset, On-shell heavy particle effective theories, J. High Energy Phys. 05 (2020) 051.

[17] Z. Bern, D. Kosmopoulos, and A. Zhiboedov, Gravitational effective field theory islands, low-spin dominance, and the four-graviton amplitude, J. Phys. A **54**, 344002 (2021).

[18] A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, Three-Loop Static Potential, Phys. Rev. Lett. **104**, 112002 (2010).

[19] R. N. Lee and V. A. Smirnov, Evaluating the last missing ingredient for the three-loop quark static potential by differential equations, J. High Energy Phys. 10 (2016) 089.

[20] J. Ablinger, J. Blümlein, P. Marquard, N. Rana, and C. Schneider, Heavy quark form factors at three loops in the planar limit, Phys. Lett. B **782**, 528 (2018).

[21] J.-y. Chiu, A. Fuhrer, R. Kelley, and A. V. Manohar, Factorization structure of gauge theory amplitudes and application to hard scattering processes at the LHC, Phys. Rev. D **80**, 094013 (2009).

[22] M. Chiesa, G. Montagna, L. Barze, M. Moretti, O. Nicrosini, F. Piccinini, and F. Tramontano, Electroweak Sudakov Corrections to New Physics Searches at the LHC, Phys. Rev. Lett. **111**, 121801 (2013).

[23] W. Beenakker, S. Brensing, M. Krämer, A. Kulesza, E. Laenen, and I. Niessen, Supersymmetric top and bottom squark production at hadron colliders, J. High Energy Phys. 08 (2010) 098.

[24] P. Ciafaloni, D. Comelli, A. Riotto, F. Sala, A. Strumia, and A. Urbano, Weak corrections are relevant for dark matter indirect detection, J. Cosmol. Astropart. Phys. 03 (2011) 019.

[25] T. Gehrmann, E. Glover, T. Huber, N. Ikizlerli, and C. Studerus, Calculation of the quark and gluon form factors to three loops in QCD, J. High Energy Phys. 06 (2010) 094.

[26] A. von Manteuffel and R. M. Schabinger, Quark and gluon form factors to four-loop order in QCD: The n_f^3 contributions, Phys. Rev. D **95**, 034030 (2017).

[27] W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, and E. Remiddi, Two-loop QCD corrections to the heavy quark form factors: Anomaly contributions, Nucl. Phys. **B723**, 91 (2005).

[28] J. Blümlein, P. Marquard, N. Rana, and C. Schneider, The heavy fermion contributions to the massive three loop form factors, Nucl. Phys. **B949**, 114751 (2019)..

[29] P. Ciafaloni and D. Comelli, Electroweak Sudakov form factors and nonfactorizable soft QED effects at NLC energies, Phys. Lett. B **476**, 49 (2000).

[30] V. S. Fadin, L. Lipatov, A. D. Martin, and M. Melles, Resummation of double logarithms in electroweak high energy processes, Phys. Rev. D **61**, 094002 (2000).

[31] J. H. Kühn, A. A. Penin, and V. A. Smirnov, Summing up subleading Sudakov logarithms, Eur. Phys. J. C **17**, 97 (2000).

[32] B. Feucht, J. H. Kühn, A. A. Penin, and V. A. Smirnov, Two-Loop Sudakov form Factor in a Theory with a Mass Gap, Phys. Rev. Lett. **93**, 101802 (2004).

[33] A. Denner and S. Pozzorini, One-loop leading logarithms in electroweak radiative corrections, Eur. Phys. J. C **18**, 461 (2001).

[34] M. Hori, H. Kawamura, and J. Kodaira, Electroweak Sudakov at two loop level, Phys. Lett. B **491**, 275 (2000).

[35] A. V. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, Phys. Lett. B **254**, 158 (1991).

[36] A. V. Kotikov, New method of massive Feynman diagrams calculation, Mod. Phys. Lett. A **06**, 677 (1991).

[37] A. V. Kotikov, Differential equations method: The calculation of vertex type Feynman diagrams, Phys. Lett. B **259**, 314 (1991).

[38] A. V. Kotikov, Differential equation method: The calculation of N point Feynman diagrams, Phys. Lett. B **267**, 123 (1991); Erratum, Phys. Lett. B **295**, 409 (1992).

[39] E. Remiddi, Differential equations for Feynman graph amplitudes, Nuovo Cimento A **110**, 1435 (1997).

[40] J. M. Henn, Multiloop Integrals in Dimensional Regularization Made Simple, Phys. Rev. Lett. **110**, 251601 (2013).

[41] R. N. Lee, Reducing differential equations for multiloop master integrals, J. High Energy Phys. 04 (2015) 108.

[42] R. N. Lee and A. A. Pomeransky, Normalized Fuchsian form on Riemann sphere and differential equations for multiloop integrals, arXiv:1707.07856.

[43] A. B. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, Math. Res. Lett. **5**, 497 (1998).

[44] E. Remiddi and J. A. Vermaseren, Harmonic polylogarithms, Int. J. Mod. Phys. A **15**, 725 (2000).

[45] See for example [46] for nonalgebraic transformation to ε-form in the elliptic case and [47] for a notion of regular basis for nonpolylogarithmic integrals.

[46] L. Adams and S. Weinzierl, The ε-form of the differential equations for Feynman integrals in the elliptic case, Phys. Lett. B **781**, 270 (2018).

[47] R. N. Lee and A. I. Onishchenko, ϵ-regular basis for non-polylogarithmic multiloop integrals and total cross section of the process $e^+e^- \to 2(Q\bar{Q})$, J. High Energy Phys. 12 (2019) 084.

[48] A. Levin and A. Beilinson, Elliptic polylogarithms, in *Proceedings of Symposia in Pure Mathematics* (1994), Vol. 55, pp. 126–196.

[49] F. C. S. Brown and A. Levin, Multiple elliptic polylogarithms, arXiv:1110.6917.

[50] L. Adams, C. Bogner, and S. Weinzierl, The two-loop sunrise graph in two space-time dimensions with arbitrary masses in terms of elliptic dilogarithms, J. Math. Phys. **55**, 102301 (2014).

[51] S. Bloch and P. Vanhove, The elliptic dilogarithm for the sunset graph, J. Number Theory **148**, 328 (2015).

[52] L. Adams, C. Bogner, A. Schweitzer, and S. Weinzierl, The kite integral to all orders in terms of elliptic polylogarithms, J. Math. Phys. (N.Y.) **57**, 122302 (2016).

[53] E. Remiddi and L. Tancredi, An elliptic generalization of multiple polylogarithms, Nucl. Phys. **B925**, 212 (2017).

[54] J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: General formalism, J. High Energy Phys. 05 (2018) 093.

[55] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, Elliptic symbol calculus: From elliptic polylogarithms to iterated integrals of Eisenstein series, J. High Energy Phys. 08 (2018) 014.

[56] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, Elliptic Feynman integrals and pure functions, J. High Energy Phys. 01 (2019) 023.

[57] J. Broedel and A. Kaderli, Functional relations for elliptic polylogarithms, J. Phys. A **53**, 245201 (2020).

[58] S. Weinzierl, Modular transformations of elliptic Feynman integrals, Nucl. Phys. **B964**, 115309 (2021).

[59] M. Bezuglov, A. Onishchenko, and O. Veretin, Massive kite diagrams with elliptics, Nucl. Phys. **B963**, 115302 (2021).

[60] S. Bloch, M. Kerr, and P. Vanhove, A Feynman integral via higher normal functions, Compos. Math. **151**, 2329 (2015).

[61] A. Primo and L. Tancredi, Maximal cuts and differential equations for Feynman integrals. An application to the three-loop massive banana graph, Nucl. Phys. **B921**, 316 (2017).

[62] L. Adams, E. Chaubey, and S. Weinzierl, Planar Double Box Integral for Top Pair Production with a Closed Top Loop to All Orders in the Dimensional Regularization Parameter, Phys. Rev. Lett. **121**, 142001 (2018).

[63] L. Adams, E. Chaubey, and S. Weinzierl, Analytic results for the planar double box integral relevant to top-pair production with a closed top loop, J. High Energy Phys. 10 (2018) 206.

[64] J. L. Bourjaily, A. J. McLeod, M. Spradlin, M. von Hippel, and M. Wilhelm, Elliptic Double-Box Integrals: Massless Scattering Amplitudes beyond Polylogarithms, Phys. Rev. Lett. **120**, 121603 (2018).

[65] J. L. Bourjaily, Y.-H. He, A. J. Mcleod, M. Von Hippel, and M. Wilhelm, Traintracks through Calabi-Yau Manifolds: Scattering Amplitudes beyond Elliptic Polylogarithms, Phys. Rev. Lett. **121**, 071603 (2018).

[66] J. L. Bourjaily, A. J. McLeod, M. von Hippel, and M. Wilhelm, Bounded Collection of Feynman Integral Calabi-Yau Geometries, Phys. Rev. Lett. **122**, 031601 (2019).

[67] K. Bönisch, C. Duhr, F. Fischbach, A. Klemm, and C. Nega, Feynman integrals in dimensional regularization and extensions of Calabi-Yau motives, arXiv:2108.05310.

[68] A. V. Manohar and M. B. Wise, *Heavy Quark Physics* (Cambridge University Press, Cambridge, England, 2007), Vol. 10.

[69] F. V. Tkachov, A theorem on analytical calculability of four loop renormalization group functions, Phys. Lett. **100B**, 65 (1981).
[70] K. G. Chetyrkin and F. V. Tkachov, Integration by parts: The algorithm to calculate beta functions in 4 loops, Nucl. Phys. **B192**, 159 (1981).
[71] R. N. Lee, Libra: A package for transformation of differential systems for multiloop integrals, Comput. Phys. Commun. **267**, 108058 (2021).
[72] See also [42].
[73] See [41] for detail description of these steps.
[74] See [42] for the criterion of such reducibility.
[75] M. Barkatou, T. Cluzeau, and C. El Bacha, Frobenius methodfor computing powerseries solutions of linear higher-order differential systems, in *Proceedings of the 19th International Symposium on Mathematical Theory on Networks and Systems (MTNS), Budapest, Hungary* (2010), p. 1059–1066.
[76] B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, Three-loop massive tadpoles and polylogarithms through weight six, J. High Energy Phys. 08 (2017) 024.
[77] R. Mueller and D. G. Öztürk, On the computation of finite bottom-quark mass effects in Higgs boson production, J. High Energy Phys. 08 (2016) 055.
[78] K. Melnikov, L. Tancredi, and C. Wever, Two-loop $gg \to Hg$ amplitude mediated by a nearly massless quark, J. High Energy Phys. 11 (2016) 104.
[79] R. N. Lee, A. V. Smirnov, and V. A. Smirnov, Solving differential equations for Feynman integrals by expansions near singular points, J. High Energy Phys. 03 (2018) 008.
[80] R. N. Lee, A. V. Smirnov, and V. A. Smirnov, Evaluating 'elliptic' master integrals at special kinematic values: Using differential equations and their solutions via expansions near singular points, J. High Energy Phys. 07 (2018) 102.
[81] B. A. Kniehl, A. V. Kotikov, A. I. Onishchenko, and O. L. Veretin, Two-loop diagrams in non-relativistic QCD with elliptics, Nucl. Phys. **B948**, 114780 (2019).
[82] O. V. Tarasov, Connection between Feynman integrals having different values of the space-time dimension, Phys. Rev. D **54**, 6479 (1996).
[83] O. V. Tarasov, Hypergeometric representation of the two-loop equal mass sunrise diagram, Phys. Lett. B **638**, 195 (2006).
[84] R. Lee, Space-time dimensionality $\mathcal{D}$ as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to $\mathcal{D}$, Nucl. Phys. **B830**, 474 (2010).
[85] R. Lee, Calculating multiloop integrals using dimensional recurrence relation and d-analyticity, Nucl. Phys. B Proc. Suppl. **205–206**, 135 (2010).
[86] T. Binoth and G. Heinrich, An automatized algorithm to compute infrared divergent multiloop integrals, Nucl. Phys. **B585**, 741 (2000).
[87] T. Binoth and G. Heinrich, Numerical evaluation of multiloop integrals by sector decomposition, Nucl. Phys. **B680**, 375 (2004).
[88] T. Binoth and G. Heinrich, Numerical evaluation of phase space integrals by sector decomposition, Nucl. Phys. **B693**, 134 (2004).
[89] G. Heinrich, Sector decomposition, Int. J. Mod. Phys. A **23**, 1457 (2008).
[90] C. Bogner and S. Weinzierl, Resolution of singularities for multi-loop integrals, Comput. Phys. Commun. **178**, 596 (2008).
[91] C. Bogner and S. Weinzierl, Blowing up Feynman integrals, in *Proceedings of the 9th DESY Workshop on Elementary Particle Theory: Loops and Legs in Quantum Field Theory: Sondershausen, Germany*, (2008)[Nucl. Phys. B, Proc. Suppl. **183**, 256 (2008)].
[92] T. Kaneko and T. Ueda, A Geometric method of sector decomposition, Comput. Phys. Commun. **181**, 1352 (2010).
[93] A. V. Smirnov, FIESTA4: Optimized Feynman integral calculations with GPU support, Comput. Phys. Commun. **204**, 189 (2016).
[94] The differential equation system for heavy-heavy vertex was reduced to ε-form before. Still elliptics is present in its boundary conditions.
[95] R. N. Lee and K. T. Mingulov, Introducing SummerTime: A package for high-precision computation of sums appearing in DRA method, Comput. Phys. Commun. **203**, 255 (2016).
[96] H. Poincaré, Sur les groupes des équations linéaires, Acta Math. **4**, 201 (1884).
[97] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, Classical Polylogarithms for Amplitudes and Wilson Loops, Phys. Rev. Lett. **105**, 151605 (2010).