Parameter symmetries of neutrino oscillations in vacuum, matter, and approximation schemes

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Expressions for neutrino oscillations contain a high degree of symmetry, but typical forms for the oscillation probabilities mask these symmetries of the oscillation parameters. We elucidate the 2^7 parameter symmetries of the vacuum parameters and draw connections to the choice of definitions of the parameters as well as interesting degeneracies. We also show that in the presence of matter an *additional* set of 2^7 parameter symmetries of the matter parameters exists. Due to the complexity of the exact expressions for neutrino oscillations in matter, numerous approximations have been developed; we show that under certain assumptions approximate expressions have at most 2^6 additional parameter symmetries of the matter parameters. We also include one parameter symmetry related to the large mixing angle (LMA)-dark degeneracy that holds under the assumption of CPT invariance; this adds one additional factor of 2 to all of the above cases. Explicit, nontrivial examples are given of how physical observables in neutrino oscillations, such as the probabilities, CP violation, the position of the solar and atmospheric resonance, and the effective Δm^2's for disappearance probabilities, are invariant under all of the above symmetries. We investigate which of these parameter symmetries apply to numerous approximate expressions in the literature and show that a more careful consideration of symmetries improves the precision of approximations.

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I. INTRODUCTION

The propagation of neutrinos is described by the eigenvalues and eigenvectors of the Hamiltonian. The eigenvectors form up into a unitary matrix, which, after rephasing of the charged leptons and the neutrinos,[1] results in 4 degrees of freedom. There is a considerable number of options for parametrizing the matrix; see e.g., Ref. [1] for a recent overview. Even within one parametrization scheme, there may still be a large number of choices to be made, and that is the focus of this paper.

The lepton[2] mixing matrix [4,5] is usually described [6] as the product of three rotations: (23), (13), (12), with a complex phase associated with the (13)-rotation resulting in four different physical degrees of freedom: θ_{23}, θ_{13}, θ_{12}, and δ. Distilling the 18 degrees of freedom of a complex 3×3 matrix down to four parameters is due to the fact that the mixing matrix is unitary,[3] but any choice of parametrization as a sequence of rotations leaves a number of discrete symmetries of the parameters including the usual Particle Data Group (PDG) choice. Once the order of rotations is chosen and the parametrization is made, there are still many different configurations that result in the same physics. The parameter symmetries that connect these different configurations are directly related to the definition of the range of the parameters; see e.g., Ref. [7]. Typically, the initial definition made is to restrict the ranges of the three mixing angles from $[0, 2\pi)$ down to $[0, \pi/2)$—a factor of 4 reduction for each angle. That implies that each quadrant of each angle can be related to the other quadrants,

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[1]Neutrinos can be rephased if they have only Dirac mass terms or are in the ultrarelativisitic $p \gg m$ regime; since we are focused on oscillations, the latter condition always holds.

[2]Many of the points made here apply to the quark mixing matrix [2,3] as well; however, there is no connection to the Majorana phases, and the matter effect is not relevant.

[3]We assume that any deviations from unitarity due to sterile neutrino hints or neutrino mass generation are negligible.

implying $4^3 = 64$ discrete parameter symmetries. This means that for any given configuration of the mixing angles there are 63 others that lead to exactly equivalent physics. There is also one discrete parameter symmetry which again doubles the number of parameter symmetries related to the choice of how one defines the mass eigenstates [8]. This leads to a total of $2^7 = 128$ discrete parameter symmetries. Reference [7] concentrates on the allowed range for each parameter, while in this paper, we concentrate on the nontrivial parameter symmetries of the physical oscillation observables when written in terms of mixing angles and a phase. These parameter symmetries restrict the allowed combinations of such parameters which we elucidate with important examples. In addition, Ref. [9] assumed CP conservation and vanishing Δm^2_{21} and examined a subset of the symmetries presented in this work to determine the allowed ranges of the parameters. In addition, if one generalizes the mixing matrix to include a phase on each rotation, there are two additional continuous degrees of freedom related to rephasing of the neutrino states or the Majorana phases. In this paper, we show exactly how all of the parameter symmetries arise.

Going beyond the symmetries of the vacuum parameters which can all be addressed via a choice of definition of the mixing angles and mass eigenstates, these parameter symmetries also naturally extend to the mixing angles and eigenvalues in matter as well, providing another $2^7 = 128$ discrete parameter symmetries. Since both of these sets of symmetries apply simultaneously in matter, there are in total $2^{14} = 16,384$ discrete parameter symmetries of the oscillation probabilities in matter for both neutrinos and antineutrinos.

The above symmetries all leave the Hamiltonian unchanged up to rephasing. There is an additional parameter symmetry which changes the Hamiltonian but leaves all physical observables unchanged known as the CPT parameter symmetry. This is due to the fact that, under the assumption that CPT is a good symmetry, all of the oscillation physics is unchanged by the transformation $H \to -H^*$. This symmetry, when combined with some of the others discussed above, is equivalent to the so-called LMA-light/LMA-dark symmetry often discussed in the literature [10–14]. This leads to an additional power of 2 for the total number of symmetries for each of the vacuum case, the matter case, and approximate case.

It is well known that the exact solutions to neutrino oscillations in constant matter density are quite intractable. To gain insights into the physics of neutrino oscillations in matter, a large number of approximate expressions have been developed. We also examine these parameter symmetries in the scenario of a perturbative Hamiltonian. The symmetries of both the vacuum parameters and the new approximate parameters also apply subject to certain conditions, one of which restricts the number of parameter symmetries of the perturbative parameters by 2 to bring the total number of parameter symmetries for a perturbative scheme to $2^{13} = 8,192$. Finally, we examine which of these parameter symmetries are satisfied by various approximate expressions in the literature.

II. PARAMETER SYMMETRIES

Since the Hamiltonian in the flavor basis uniquely and exactly determines the neutrino oscillation probabilities, any parameter symmetry of this Hamiltonian is necessarily a parameter symmetry of the probabilities. We first focus on the vacuum case for simplicity and will later show that the presence of matter, handled either exactly or perturbatively, behaves in a similar way. We define the Hamiltonian[4] in the usual PDG fashion [6] except with a complex phase on each rotation,

$$H_{\rm flav} = \frac{1}{2E} U_{23} U_{13} U_{12} M^2 U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger, \tag{1}$$

where the nontrivial part of the 2×2 submatrix of the 3×3 complex rotation matrices is defined as[5]

$$U_{ij}(\theta_{ij}, \delta_{ij}) = \begin{pmatrix} c_{ij} & s_{ij} e^{i\delta_{ij}} \\ -s_{ij} e^{-i\delta_{ij}} & c_{ij} \end{pmatrix}, \tag{2}$$

and the mass-squared matrix is $M^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$. That is, we describe the mixing matrix with three rotation angles θ_{ij} and three associated complex phases δ_{ij}. To understand the parameter symmetries, it is useful to work with this slightly more general mixing matrix with three distinct complex phases, one for each rotation; however, ultimately oscillation probabilities only depend on the sum of these three complex phases.

A. Discrete parameter symmetries

We now list the various parameter symmetries of the Hamiltonian. First, there are 128 discrete parameter symmetries of the vacuum Hamiltonian up to rephasing. To describe these, we introduce the parameters m_{ij} and n_{ij}, each of which are $\in \mathbb{Z}_2$ (that is, $\{0, 1\}$). A sign flip of a cosine term is given by the m_{ij} parameters, $c_{ij} \to (-1)^{m_{ij}} c_{ij}$, and a sign flip of a sine term is given by the n_{ij} parameters, $s_{ij} \to (-1)^{n_{ij}} s_{ij}$. This allows us to write down "angle reflection" (n_{ij}) and "angle shift and reflection" (m_{ij}) parameter symmetries as follows:

(i) $(i, j) = (2, 3)$ or (1,2):

$$\begin{aligned} &c_{ij} \to (-1)^{m_{ij}} c_{ij}, \qquad s_{ij} \to (-1)^{n_{ij}} s_{ij} \\ &\text{so long as } \delta_{ij} \to \delta_{ij} + (m_{ij} + n_{ij})\pi. \end{aligned} \tag{3}$$

[4]We only focus on the flavor basis in this paper.

[5]We use the usual $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$, $\Delta m^2_{ij} = m_i^2 - m_j^2$ convention.

(ii) $(i,j) = (1,3)$:

$$c_{13} \to (-1)^{m_{13}} c_{13}, \qquad s_{13} \to (-1)^{n_{13}} s_{13}$$
$$\text{so long as } \delta_{13} \to \delta_{13} + n_{13}\pi. \tag{4}$$

In each case, if one of the angles is changed via the m_{ij} or n_{ij} parameters, then the corresponding δ_{ij} must accumulate a factor of π with the exception of changes to c_{13}. That is, the six m_{ij} and n_{ij} are all independent and can each be either 0 or 1, leading to $2^6 = 64$ combinations.

The remaining discrete parameter symmetry is the 1-2 interchange [15] for which all of the following simultaneously happen:

(i) $(i,j) = (1,2)$:

$$c_{12} \to s_{12} \quad \text{and} \quad s_{12} \to c_{12}$$
$$\text{so long as } \delta_{12} \to \delta_{12} + \pi \quad \text{and} \quad m_1^2 \leftrightarrow m_2^2. \tag{5}$$

The final part of the interchange is equivalent to $\Delta m^2_{21} \to \Delta m^2_{12} = -\Delta m^2_{21}$ and $\Delta m^2_{31} \leftrightarrow \Delta m^2_{32}$. Note that the 1-2 interchange can be done simultaneously with the shift and reflection described above.[6] The parameter symmetry related to this interchange provides one more factor of 2, leading to 128 total parameter symmetries.

After performing any combination of these parameter symmetries including the 1-2 interchange, the Hamiltonian is exactly the same up to a possible overall phase matrix,

$$H_{\rm flav} \to {\rm diag}((-1)^{m_{13}+m_{23}}, 1, 1) H_{\rm flav} {\rm diag}((-1)^{m_{13}+m_{23}}, 1, 1); \tag{6}$$

see Appendix A for an explicit derivation of this. This (-1) phase coming from m_{13} and/or m_{23} terms can then be absorbed into the definition of $|\nu_e\rangle$, and nothing has changed. The presence of m_{13} and m_{23} and not m_{12} is due to the fact that the sign from sending $c_{12} \to -c_{12}$ can go through the middle of the Hamiltonian and commutes with the M^2 matrix, while this is not true for the m_{13} or m_{23} parameter symmetries. The n_{ij} parameter symmetries operate on the rotation matrix level and thus do not affect the total Hamiltonian.

The various interchanges can be thought of equally in terms of sines and cosines (e.g., $s \to -s$) or in terms of angles (e.g., $\theta \to -\theta$). For convenience, we include the relationship between these in a table:

Sines/cosines	Angles
$s \to -s$, $c \to c$	$\theta \to -\theta$
$s \to s$, $c \to -c$	$\theta \to \pi - \theta$
$s \to -s$, $c \to -c$	$\theta \to \pi + \theta$
$c \to s$, $s \to c$	$\theta \to \pi/2 - \theta$
$c \to -s$, $s \to c$	$\theta \to \pi/2 + \theta$
$c \to s$, $s \to -c$	$\theta \to -\pi/2 + \theta$
$c \to -s$, $s \to -c$	$\theta \to -\pi/2 - \theta$

B. Continuous parameter symmetries

In addition to the above discrete parameter symmetries, there are two continuous degrees of freedom as well, dubbed the "delta shuffle."[7] We define the three rotations and complex phases by[8]

$$U_{23}(\theta_{23}, \delta_{23}) U_{13}(\theta_{13}, -\delta_{13}) U_{12}(\theta_{12}, \delta_{12}). \tag{7}$$

In fact, in the context of neutrino oscillations, these three phases are related by

$$\delta_{23} + \delta_{13} + \delta_{12} = \delta \mod 2\pi, \tag{8}$$

where δ is a new fixed parameter and is equal to the usual single complex phase. That is, there are two additional degrees of freedom which, it turns out, are equivalent to the Majorana phases; see Sec. IV. For example, the three following scenarios, each with a single complex phase, are all equivalent:

$$U_{23}(\theta_{23}, \delta) U_{13}(\theta_{13}, 0) U_{12}(\theta_{12}, 0), \tag{9}$$

$$U_{23}(\theta_{23}, 0) U_{13}(\theta_{13}, -\delta) U_{12}(\theta_{12}, 0), \tag{10}$$

$$U_{23}(\theta_{23}, 0) U_{13}(\theta_{13}, 0) U_{12}(\theta_{12}, \delta). \tag{11}$$

See Appendix B for the explicit rephasing expression relating Eq. (7) to Eqs. (9)–(11). This can also be expressed from the point of view of rephasing the flavor states as well as the mass eigenstates,

$$D_f U_{23} U_{13} U_{12} D_m M^2 D_m^\dagger U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger D_f^\dagger, \tag{12}$$

where D_f and D_m are diagonal rephasing matrices, ${\rm diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma})$. Note that, since D_m commutes with M^2, it trivially cancels, while the D_f rephasing can be absorbed into the definitions of the neutrino flavor states. The D_f rephasing matrix requires some care if the Hamiltonian is split into two parts for a perturbative description; see Sec. III B below.

[6]While the m_{ij} and n_{ij} notation makes it appear as though the 1-2 interchange does not commute with a parameter symmetry related to either m_{12} or n_{12}, note that m_{12} and n_{12} always appear as a sum, so swapping one for the other leads to no change.

[7]These were pointed out in Ref. [16].

[8]The choice of a minus sign on δ_{13} is arbitrary and is chosen for consistency with the PDG convention.

C. *CPT* parameter symmetry

All of the above parameter symmetries are symmetries of the vacuum Hamiltonian up to rephasing, $H \to D_f H D_f^\dagger$ where D_f is an arbitrary rephasing matrix. There is an additional parameter symmetry of all oscillation observables (e.g., the probabilities) that is not of the above form that depends on the assumption of CPT invariance [10,17]. This parameter symmetry can be written as

$$m_i^2 \to -m_i^2 \quad \text{and} \quad \delta \to -\delta. \tag{13}$$

Note that this requires the sum of the three phases, $\sum \delta_{ij}$, to change signs. This is a parameter symmetry of the vacuum Hamiltonian because it is equivalent to sending $H \to -H^*$. The minus sign applies a time reversal to the vacuum Hamiltonian, and the complex conjugate applies a charge-parity reversal to the vacuum Hamiltonian; under the reasonable assumption that CPT is a good symmetry at scales relevant for neutrino oscillations, all physical observables remain the same.

D. Summary of parameter symmetries

These parameter symmetries can be rewritten relatively compactly. For $m_{ij}, n_{ij} \in \{0, 1\}$, the following are exact parameter symmetries of the Hamiltonian:

(i) (13): $c_{13} \to (-1)^{m_{13}} c_{13}$, $s_{13} \to (-1)^{n_{13}} s_{13}$, and $\delta_{13} \to \delta_{13} \pm n_{13}\pi$,

(ii) (23): $c_{23} \to (-1)^{m_{23}} c_{23}$, $s_{23} \to (-1)^{n_{23}} s_{23}$, and $\delta_{23} \to \delta_{23} \pm (m_{23} + n_{23})\pi$,

(iii) (12): $c_{12} \to (-1)^{m_{12}} c_{12}$, $s_{12} \to (-1)^{n_{12}} s_{12}$, and $\delta_{12} \to \delta_{12} \pm (m_{12} + n_{12})\pi$,

(iv) (12): $c_{12} \to (-1)^{m_{12}} s_{12}$, $s_{12} \to (-1)^{n_{12}} c_{12}$, and $\delta_{12} \to \delta_{12} \pm (m_{12} + n_{12} + 1)\pi$ plus $m_1^2 \leftrightarrow m_2^2$,

(v) For three phases defined $U_{23}(\theta_{23}, \delta_{23}) \times U_{13}(\theta_{13}, -\delta_{13}) \times U_{12}(\theta_{12}, \delta_{12})$, there are two free degrees of freedom after applying the constraint: $\delta_{23} + \delta_{13} + \delta_{12} = \delta$.

(vi) CPT: $m_i^2 \to -m_i^2$ and $\delta \to -\delta$,

See, for example, Ref. [18] for neutrino oscillation amplitudes in vacuum which explicitly satisfy all of these discrete parameter symmetries without additional manipulation, up to an overall unphysical phase.

III. PARAMETER SYMMETRIES IN DIFFERENT HAMILTONIANS

While the previous discussion was focused on the vacuum Hamiltonian, we now show that it extends to the matter Hamiltonian for constant (e.g., long-baseline accelerator), smoothly varying (day-time solar), or sharply varying (atmospheric and nighttime solar) density profiles. In addition, it also applies to both exact and perturbative scenarios, provided that the perturbative approach satisfies certain properties discussed in Sec. III B below.

The three forms of the Hamiltonian considered in this paper are

$$H_{\rm flav} = \frac{1}{2E}[U_{23}U_{13}U_{12}M^2U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger + A], \tag{14}$$

$$= \frac{1}{2E} W_{23}W_{13}W_{12}\Omega W_{12}^\dagger W_{13}^\dagger W_{23}^\dagger, \tag{15}$$

$$\begin{aligned} = \frac{1}{2E}[&V_{23}V_{13}V_{12}\Lambda V_{12}^\dagger V_{13}^\dagger V_{23}^\dagger \\ &+ (U_{23}U_{13}U_{12}M^2U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger + A \\ &- V_{23}V_{13}V_{12}\Lambda V_{12}^\dagger V_{13}^\dagger V_{23}^\dagger)], \end{aligned} \tag{16}$$

each of which are exactly equivalent, and all terms have identical mathematical structure of the form $UM^2U^\dagger$ apart from matter term. In general, the rotation matrices have complex phases associated with them as in the vacuum case; see Eq. (7). The rotation and diagonal matrices are as follows: (U, M^2) in Eq. (14) are the vacuum parameters, and (W, Ω) in Eq. (15) are the diagonalized exact eigenvectors and eigenvalues in matter [19]. For Eq. (16), (V, Λ) are any approximation[9] for the Hamiltonian in matter that is diagonalized by rotations of the order: (23), (13), (12) of any angles and phases, e.g., Denton, Minakata, and Parke (DMP) [15]. That is, some of the V could be vacuum rotations (U), exact rotations (V), or anything else. We use hats to denote the parameters that exactly diagonalize the Hamiltonian as shown in Eq. (15), e.g., $\hat\theta_{ij}$ and $\hat\delta_{ij}$. We use tildes to denote the parameters of the first line of the perturbative Hamiltonian in Eq. (16), e.g., $\hat\theta_{ij}$ and $\tilde\delta_{ij}$, and the indices for the parameter symmetries are similarly expressed as $\tilde m_{ij}$ and $\tilde n_{ij}$. Typically, the first line of Eq. (16) is considered H_0, and the second line is considered H_1.

Since we are working at the Hamiltonian level, any symmetry of the Hamiltonian is necessarily a symmetry of the oscillation probability regardless of whether the matter density profile is constant or not. For example, if the density profile is a complicated function such as for atmospheric or solar neutrinos, it may be difficult to solve the Schrödinger equation (analytically or numerically), but the symmetries are still valid since all of the information required for propagation is in the Hamiltonian. In this case, the effective oscillation parameters in matter, $\hat\theta_{ij}$, $\hat\delta_{ij}$, and $\widehat{\Delta m^2}_{ij}$, all evolve as well, but since the symmetries discussed below do not depend on the value of the matter potential, a, they all apply for the entire probability for any matter density profile.[10]

[9]These approximations need not be good approximations for these parameter symmetries to hold.

[10]Note that these symmetries for supernova neutrinos, where neutrino-neutrino interactions are relevant, require additional care.

A. Parameter symmetries in matter

Since $A \equiv \mathrm{diag}(a, 0, 0)$, the matter effect matrix[11] respects all of the parameter symmetries in question apart from the CPT parameter symmetry, adding A to the vacuum Hamiltonian in Eq. (14) results in a matrix that also respects the parameter symmetries when described in terms of the vacuum parameters. Therefore, the diagonalization of the Hamiltonian in matter also makes no difference since the Hamiltonians in Eqs. (14) and (15) are equal. This means that Eq. (15) also respects the parameter symmetries in terms of the vacuum parameters.

Moreover, since it is now written in the same form as that of the vacuum Hamiltonian, the new diagonalized parameters (the eigenvalues, the mixing angles, and complex phase of the eigenvectors in matter) also obey an additional set of parameter symmetries summarized here. For $\hat{m}_{ij}, \hat{n}_{ij} \in \{0, 1\}$:

- (i) (13): $\hat{c}_{13} \to (-1)^{\hat{m}_{13}}\hat{c}_{13}$, $\hat{s}_{13} \to (-1)^{\hat{n}_{13}}\hat{s}_{13}$, and $\hat{\delta}_{13} \to \hat{\delta}_{13} \pm \hat{n}_{13}\pi$,
- (ii) (23): $\hat{c}_{23} \to (-1)^{\hat{m}_{23}}\hat{c}_{23}$, $\hat{s}_{23} \to (-1)^{\hat{n}_{23}}\hat{s}_{23}$, and $\hat{\delta}_{23} \to \hat{\delta}_{23} \pm (\hat{m}_{23} + \hat{n}_{23})\pi$,
- (iii) (12): $\hat{c}_{12} \to (-1)^{\hat{m}_{12}}\hat{c}_{12}$, $\hat{s}_{12} \to (-1)^{\hat{n}_{12}}\hat{s}_{12}$, and $\hat{\delta}_{12} \to \hat{\delta}_{12} \pm (\hat{m}_{12} + \hat{n}_{12})\pi$,
- (iv) (12): $\hat{c}_{12} \to (-1)^{\hat{m}_{12}}\hat{s}_{12}$, $\hat{s}_{12} \to (-1)^{\hat{n}_{12}}\hat{c}_{12}$, and $\hat{\delta}_{12} \to \hat{\delta}_{12} \pm (\hat{m}_{12} + \hat{n}_{12} + 1)\pi$ plus $\widehat{m^2}_1 \leftrightarrow \widehat{m^2}_2$,
- (v) For three phases defined $W_{23}(\hat{\theta}_{23}, \hat{\delta}_{23}) \times W_{13}(\hat{\theta}_{13}, -\hat{\delta}_{13}) W_{12}(\hat{\theta}_{12}, \hat{\delta}_{12})$, there are two free degrees of freedom after applying the constraint: $\hat{\delta}_{23} + \hat{\delta}_{13} + \hat{\delta}_{12} = \hat{\delta}$.

Therefore, the exact expressions for neutrino oscillations in matter [19,21,22] respect the 128 symmetries of the vacuum parameters as described in the previous section except for the CPT parameter symmetry. They also respect 128 symmetries in terms of the matter parameters described above, for a total of $2^{14} = 16,384$ possible vacuum plus matter parameter symmetries. This includes flipping $|\nu_1\rangle \leftrightarrow |\nu_2\rangle$ and/or $|\hat{\nu}_1\rangle \leftrightarrow |\hat{\nu}_2\rangle$ where the $|\hat{\nu}_i\rangle$ are the exact eigenstates of the Hamiltonian in matter. Adding the CPT parameter symmetry, given by[12]

$$CPT: \widehat{m_i^2} \to -\widehat{m_i^2} \text{ and } \hat{\delta} \to -\hat{\delta}\ ,$$

doubles the total number of symmetries. All of these symmetries impose constraints on the analytic form of the oscillation probabilities matter, as will be discussed in Sec. IV.

Each of these parameter symmetries can be applied at each point in propagation in either a constant or varying matter potential, and the physics at the step remains invariant. In principle, one could choose a different parameter symmetry at each step; however, for continuity of the mixing angles and CP phase, applying the same parameter symmetries along the entire route is certainly the simplest option, and the physics cannot depend on the choice made.

B. Parameter symmetries of a perturbative Hamiltonian

Since the exact expressions for neutrino oscillations in matter for constant or sharply varying density profiles tend to be fairly opaque, numerous approximation schemes have been considered in the literature; for an overview, see Ref. [23]. One technique is that of splitting the Hamiltonian into a large part and a small part; $H = H_0 + H_1$. H_0 is then the diagonal part of H after successive diagonalizing with two component rotations until the off-diagonal elements are sufficiently small. Then, H_1 is just the remaining off-diagonal part; see Refs. [15,24–27]. While different perturbative schemes have different benefits,[13] we focus on that described in Ref. [15] by DMP as a concrete example.

First, we note that the above parameter symmetries still apply to the vacuum parameters, as they must, since the addition of the matter potential matrix is invariant under the parameter symmetries. Second, we find that there are four key conditions for a perturbative Hamiltonian to satisfy in order for the new approximate eigenvalues and approximate angles (denoted with tildes) to be independent under the parameter symmetries:

1. (1) The order of rotations must match that of the vacuum rotations.
2. (2) The approximate eigenvalues must respect all the vacuum parameter symmetries.
3. (3) The phases of the new rotations must match the corresponding vacuum ones mod π,

$$\tilde{\delta}_{ij} = \delta_{ij} \mod \pi. \qquad (17)$$

4. (4) Certain vacuum and perturbative parameter symmetries must match:

$$m_{13} + m_{23} = \tilde{m}_{13} + \tilde{m}_{23} \mod 2. \qquad (18)$$

We now explain in more detail exactly what these conditions are.

[11] The matter effect is given by $a = 2\sqrt{2}G_F E N_e \rho$ [20].

[12] The CPT parameter symmetry of the matter parameters can be achieved by the simultaneous transformations $m_i^2 \to -m_i^2$, $\delta \to -\delta$, and $a \to -a$.

[13] If in the approximation scheme one is considering, $V_{12} = \mathbb{I}$, such as Ref. [25], then a $|\hat{\nu_1}\rangle \leftrightarrow |\hat{\nu_3}\rangle$ interchange symmetry is possible, which can be made exact by appropriate choices for $\hat{\theta}$ and $\hat{\delta}$ in V_{13}, similar to what was performed for the $|\hat{\nu_1}\rangle \leftrightarrow |\hat{\nu_2}\rangle$ interchange symmetry in V_{12}. Using the methods of this paper, one can work out all the parameter symmetries for such approximation schemes; see Appendix C.

First, it is necessary to follow the same sequence of rotations as in vacuum. That is, for the PDG parametrization of the lepton mixing matrix [6], the zeorth order part of the Hamiltonian must be diagonalized by a (23)-rotation, then a (13)-rotation, followed by a (12)-rotation. If the lepton mixing matrix is parametrized in a different way [1], then a different sequence of rotations would need to be used, which may or may not be advantageous depending on the exact region of interest. This condition is satisfied in the work by DMP [15] (as well as in Ref. [25]), where it was a noted convenient benefit that happened by chance due to focusing on the large off-diagonal elements and removing the level crossings. Note that, since the sequence of rotations used in Ref. [24] is a different order, the approximate matter parameters used there do not satisfy these parameter symmetries.

Second, in general, the approximate eigenvalues (these make up the Λ matrix) need to respect the symmetries of the vacuum parameters in order for the parameter symmetries to be satisfied by expressions derived from such an approximation scheme. While the full Hamiltonian in Eq. (16) does respect these parameter symmetries, if the split between H_0 and H_1 is not done in a way that respects these parameter symmetries, then expressions derived from only part of the Hamiltonian will not respect these vacuum parameter symmetries. We explicitly show in Appendix D that each of the eigenvalues in the exact solution respects the vacuum parameter symmetries as they must and in Appendix E that each of the approximate eigenvalues in the DMP scheme do respect the vacuum parameter symmetries.

Third, assuming the vacuum mixing matrix is parametrized with three phases as in Eq. (7), the phases in the diagonalizing matrices V_{ij} must be given by $\tilde{\delta}_{ij} = \delta_{ij} \bmod \pi$. This condition guarantees that no net phase appears between the exact and approximate expressions. Therefore, there are no parameter symmetry degrees of freedom related to the delta shuffle for the perturbative complex phases, $\tilde{\delta}_{ij}$. Note that in DMP this was satisfied as the vacuum matrix was parametrized with the phase on the (23)-rotation as in Eq. (9) the (23) diagonalization matrix was the same as the vacuum parameters,

$$\sin 2\tilde{\theta}_{23} e^{i\tilde{\delta}_{23}} = \sin 2\theta_{23} e^{i\delta_{23}}. \tag{19}$$

This form of the definition is motivated as the imaginary part of Eq. (19) is known to be exactly satisfied for the exact versions of the matter variables [28].[14]

Fourth, and most interestingly, a certain combination of parameter symmetries of the vacuum and perturbative parameters is not parameter symmetries of the Hamiltonian. Equation (6) shows how the Hamiltonian accumulates an overall phase if m_{13} or m_{23} parameter symmetries are applied. As this phase can be absorbed into the definition of $|\nu_e\rangle$, it is not a problem, but if different phases appear on H_0 and H_1, then the phase cannot be simply absorbed into the definition of $|\nu_e\rangle$ anymore. Since the impact of such a phase on H_0 comes only from the symmetries of the approximate parameters denoted $\tilde{m}_{13}$ and $\tilde{m}_{23}$ and the impact of such a phase on H_1 comes from not only m_{13} and m_{23} but also $\tilde{m}_{13}$ and $\tilde{m}_{23}$ as can be seen from the second line of Eq. (16), the only way to ensure that these phases can be absorbed into $|\nu_e\rangle$ is if the impact of both are the same. This only happens when the conditions of Eq. (18) are satisfied. This extra condition implies that the number of symmetries in the perturbative parameters is a factor of 2 lower; one can think of it as once choices are made about m_{23}, m_{13}, and $\tilde{m}_{23}$, then $\tilde{m}_{13}$ is fixed. Therefore, there are an additional $128/2$ parameter symmetries for the perturbative parameters for a total of $2^{13} = 8,192$ parameter symmetries for a perturbative Hamiltonian that meets the above requirements.

IV. DISCUSSION

A. Definition of the parameters

The $2^6 = 64$ discrete parameter symmetries of the vacuum parameters not including the 1–2 interchange are exactly equivalent to the fact that one can define the range of each of the three mixing angles to be in only one quadrant. That is, one could choose that each of the mixing angles θ_{ij} exists in $[\eta_{ij}\pi/2, (\eta_{ij}+1)\pi/2)$ where the $\eta_{ij} \in \mathbb{Z}$ need not be the same for each angle.[15] For convenience, the standard convention is that $\eta_{ij} = 0$ for each of the mixing angles. This is equivalent to requiring that $s_{ij} \geq 0$ and $c_{ij} \geq 0$.

The remaining parameter symmetry for the vacuum parameters is the 1-2 interchange symmetry. Given the above range of the mixing angles and complete freedom in identifying the mass states, the 1-2 interchange symmetry exists. As with the mixing angles, this implies that one should make a definition restricting this; there are two typical approaches to proceed (for a comprehensive examination of how one labels the mass states, see e.g., Ref. [8]). First, one could fix the order of the mass states such that the first mass state is smaller than the second. Second, one could fix θ_{12} to be contained within one octant, typically

[14]The Toshev identity, $\sin 2\tilde{\theta}_{23} \sin\tilde{\delta} = \sin 2\theta_{23} \sin\delta$ [28], gains a sign under certain changes in the definitions; this is consistent with the rest of the results of this paper since the Toshev identity is not a physically measurable quantity like $\Delta P_{\rm CP}$, given in Eq. (26). With the correct signs, the Toshev identity reads $(-1)^{\tilde{m}_{12}+\tilde{n}_{12}+\tilde{n}_{13}} \sin 2\tilde{\theta}_{23} \sin\tilde{\delta} = (-1)^{m_{12}+n_{12}+n_{13}} \sin 2\theta_{23} \sin\delta$ without $1 \leftrightarrow 2$ interchanges. With such interchanges in matter/vacuum an additional factor of (-1) is needed on the matter/vacuum side of this generalized Toshev identity.

[15]In fact, this generalizes somewhat. The angles can be defined to be within any region given by $\theta \in \cup_i [x_i, y_i)$ for any x_i and y_i possibly different for each angle such that $\{|\cos\theta|\}$ are all unique as are $\{|\sin\theta|\}$ across the allowed range of θ. For example, one could define θ_{12} to be in the range $(-0.5\pi, -0.2\pi] \cup [0, 0.2\pi)$.

$\theta_{12} \in [0, \pi/4)$. Each of these are equivalent as described by the interchange symmetry (up to a factor of π on δ_{12}). Thus, the fact that we can define $\Delta m^2_{21} > 0$ *or* $\theta_{12} \in [0, \pi/4)$ is exactly due to the 1-2 interchange symmetry.

We also note that for the same reason that there is a parameter symmetry of mass states $|\nu_1\rangle$ and $|\nu_2\rangle$ related to θ_{12} there is also a different discrete parameter symmetry related to flavor states $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ and θ_{23}, also subject to appropriate modifications if the lepton mixing matrix is parametrized in a different way. This parameter symmetry is not a parameter symmetry like the others since we can differentiate between $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ by measuring the properties of their associated charged leptons.

B. LMA-dark degeneracy

The *CPT* parameter symmetry discussed in Sec. II C has appeared in the literature typically accompanied with the 1-2 interchange parameter symmetry stated in Eq. (5) and is often written as

$$c_{12} \leftrightarrow s_{12}, \quad \Delta m^2_{31} \leftrightarrow -\Delta m^2_{32}, \quad \text{and} \quad \delta \to \pi - \delta. \tag{20}$$

This form, or variations thereof, is often referred to as the LMA-light/LMA-dark degeneracy or the Generalized Mass Ordering Degeneracy; see e.g., Refs. [9,12–14]. This particular combination of parameter symmetries is of interest due to interesting phenomenological implications, in particular when the matter effect is included. In the presence of matter, the vacuum *CPT* parameter symmetry changes the Hamiltonian as follows,

$$H_{\text{vac}} + A \to -H^*_{\text{vac}} + A, \tag{21}$$

where $A \equiv \text{diag}(a, 0, 0)$ and a is the matter potential which is unchanged by this parameter symmetry. This is the LMA-light to LMA-dark interchange. The fact that this is the only parameter symmetry of the vacuum Hamiltonian but not of the matter Hamiltonian is exactly why measuring the matter effect (as has already been done by combining solar data [29] with KamLAND data [30]) is a necessary condition for measuring both mass orderings.

In the presence of new physics, it is possible that the matter effect could take the opposite sign of the expectation in the SM (i.e., $A \to -A$) where the new physics is described in the neutrino nonstandard interactions (NSI) framework [20,31,32]. This also means that, in the presence of NSIs, it is not possible to determine the atmospheric mass ordering since then even the matter Hamiltonian is invariant under $\Delta m^2_{31} \leftrightarrow -\Delta m^2_{32}$, although the details of the NSI model may allow one to break this degeneracy in most cases [9,13,14].

This parameter symmetry adds a factor of 2 to the number of symmetries in vacuum. In matter, it also adds a factor of 2 because, while the probabilities are no longer invariant under this parameter symmetry of the vacuum parameters as shown in Eq. (21), they are invariant under the same symmetry but for the equivalent parameters in matter,

$$\widehat{m^2_i} \to -\widehat{m^2_i} \quad \text{and} \quad \hat{\delta} \to -\hat{\delta}. \tag{22}$$

Similarly, for any approximate diagonalization scheme, so long as all three eigenvalues change sign along with the complex phase, the *CPT* parameter symmetry holds.

C. Some technical details

In this section, we aim to understand these parameter symmetries conceptually. We begin with the delta shuffle since it is distinct from the others. The main notable difference from the other parameter symmetries is that this represents a continuous parameter symmetry. That is, it represents two additional continuous parameters in the matrix. These two extra parameters are physical if neutrinos have a Majorana mass term [16,33]. If one writes the mixing matrix as a product of the usual PDG [6] form and the Majorana phase matrix $P = \text{diag}(e^{-i\alpha}, e^{-i\beta}, 1)$ as $U_{\text{PDG}}P$, then our parametrization in Eq. (7) with δ_{ij} on each rotation is equivalent to $P^\dagger U_{\text{PDG}} P$, which is equal to $U_{\text{PDG}}P$ after rephasing the charged leptons. The Majorana phases α and β are related to the phases in our notation by

$$\delta = \delta_{23} + \delta_{13} + \delta_{12}, \quad \alpha = \delta_{12} + \delta_{23}, \quad \beta = \delta_{23}. \tag{23}$$

See Appendix B for more on rephasing.

Note that, while the discrete parameter symmetries are related to the definition of the ranges of the parameters (see Sec. IV A), they are still parameter symmetries of the probabilities. This means that every probability expression must respect these parameter symmetries and they can be used as a valuable and quick cross-check for any expression.

If the Hamiltonian is written as a series of three rotations in a different order, see Ref. [1], the same parameter symmetries apply, but care is required with regard to the inner vs outer rotations for the delta shuffle and the angle shift (m_{ij}). It is the sign change on the cosine in the middle rotation that does not result in a factor of π on the associated complex phase, and the 1-2 interchange applies only to the third rotation. For example, if the lepton matrix is parametrized by a sequence of rotations in the order (23), (12), (13), then the 1-2 interchange would become a 1-3 interchange involving θ_{13} instead of θ_{12} and $m^2_1 \leftrightarrow m^2_3$ instead of $m^2_1 \leftrightarrow m^2_2$. It would then be θ_{12} for which a $c_{12} \to -c_{12}$ change would not include a change to δ, and it would be $m_{12} + m_{23}$ that would need to be equivalent between the vacuum and perturbative parameters.

While the parameter symmetries derived in the Hamiltonian framework presented in Sec. II are all exact, some care is required. The Hamiltonian framework benefits from a high level of generality: any parameter symmetries of the Hamiltonian are automatically parameter symmetries of the physical observables—the probabilities. It is possible, however, to artificially break the parameter symmetries of the Hamiltonian while the parameter symmetries of the probabilities persist. For example, it is known that the probabilities are invariant under the addition of any matrix proportional to the identity matrix as this is equivalent to adding an overall phase which is not detectable in oscillations. But if one adds a matrix that does not respect the parameter symmetries, then the resultant Hamiltonian will also no longer respect some of the parameter symmetries, but the probabilities still will, of course. For example, if one writes the Hamiltonian with $M^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$, then all the parameter symmetries are respected in the Hamiltonian. If, however, one subtracts $m_1^2\mathbb{I}$ so that

$$M^2 \to \mathrm{diag}(0, \Delta m_{21}^2, \Delta m_{31}^2), \tag{24}$$

the 1-2 interchange symmetry is not respected since $m_1^2\mathbb{I}$ is not invariant under the 1-2 interchange symmetry. One can, however, subtract a term that is known to be invariant under the parameter symmetries, see the previous paragraph, and the parameter symmetries will still remain valid. For example, one could subtract $(c_{12}^2 m_1^2 + s_{12}^2 m_2^2)\mathbb{I}$, which sends

$$\begin{aligned} M^2 &= \mathrm{diag}(m_1^2, m_2^2, m_3^2) \\ &\to \mathrm{diag}(-s_{12}^2 \Delta m_{21}^2, c_{12}^2 \Delta m_{21}^2, \Delta m_{ee}^2), \end{aligned} \tag{25}$$

which still respects all the parameter symmetries since the initial Hamiltonian does, as does the subtracted identity matrix. It is interesting to note then that these parameter symmetries (particularly the 1-2 interchange symmetry) applies to the physical observables so long as there is at least one Hamiltonian which can be shifted to by a multiple of the identity matrix such that that parameter symmetry applies; the fact that this is true for the physically motivated definition of $M^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$ is one of convenience.

We also note that the form of the parameter symmetries as shown in Sec. II D and elsewhere through this paper is not fully symmetric under the different parameters. This is because the specific description of parameter symmetries discussed depend on the parametrization of the mixing matrix. A different parametrization in terms of a different sequence of rotations will result in similar parametrization symmetries where one needs to pay attention to which rotation is next to the diagonal mass matrix for the 1-2 interchange (which could become the 1-3 interchange or the 2-3 interchange accordingly) and which rotation is the final one to get to the flavor basis.

At least some of the discrete parameter symmetries can also be embedded in a group structure [34].

While we do not know for sure if the list presented here contains all of the relevant parameter symmetries of this form, we do believe that it is complete.

V. PHYSICAL CONSEQUENCES

All physical oscillation observables must satisfy these parameter symmetries. In this section, we discuss a few important examples of how these parameter symmetries appear in physical observable variables and constrain the dependence on the parameters.

The *CP* violating term in appearance experiments proportional to the Jarlskog invariant [35] is given by

$$\begin{aligned} \Delta P_{\mathrm{CP}} \propto\, & s_{23}c_{23}s_{13}c_{13}^2 s_{12}c_{12} \sin(\delta_{23} + \delta_{13} + \delta_{12}) \\ & \times \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32}, \end{aligned} \tag{26}$$

which is unchanged under all of the above parameter symmetries including the *CPT* symmetry. We note that the fact that m_{13} (the change on c_{13}) is treated differently from the other m_{ij}, n_{ij} is the same reason that c_{13}^2 appears in Eq. (26); if $m_{13} = 1$, then $c_{13}^2 \to c_{13}^2$, and thus we must have $\sin\delta \to \sin\delta$ to remain invariant, while for the remaining s_{ij} and c_{ij} terms as each picks up a minus sign, it is exactly offset by the minus sign in $\sin(\delta_{23} + \delta_{13} + \delta_{12})$. This is a useful nontrivial cross-check of the symmetries discussed in this paper.

The Hamiltonians are invariant under these parameter symmetries as described above, but certainly individual elements that appear in the Hamiltonian are not, such as s_{13} or Δm_{21}^2. That said, there are a number of interesting nontrivial terms that appear regularly in exact and various approximate oscillation probabilities that are invariants of all of the parameter symmetries. We list the parameter symmetry invariant factors here:

(i) $s_{13}e^{i\delta_{13}}$, s_{13}^2 and c_{13}^2,
(ii) $s_{23}c_{23}e^{i\delta_{23}}$, s_{23}^2 and c_{23}^2,
(iii) $s_{12}c_{12}e^{i\delta_{12}}\Delta m_{21}^2$,
(iv) $\cos 2\theta_{12}\Delta m_{21}^2 = (c_{12}^2 - s_{12}^2)\Delta m_{21}^2$,
(v) $c_{12}^2\Delta m_{31}^2 + s_{12}^2\Delta m_{32}^2 \equiv \Delta m_{ee}^2$ (see ref. [36]).

In addition, combinations of these expressions appear in the probabilities. For example, the usual $\cos\delta$ or $\sin\delta$ terms must appear in the combination, $s_{13}s_{23}c_{23}s_{12}c_{12}e^{i(\delta_{23}+\delta_{12}+\delta_{13})}\Delta m_{21}^2$ possibly with additional parameter symmetry invariant factors such as $c_{13}^2, s_{23}^2, \ldots$. See, for example, Table 1 of Ref. [15]. Also, these parameter symmetries exclude some combinations of parameters in physical observables, such as, odd powers of c_{13}.

VI. PARAMETER SYMMETRIES OF APPROXIMATIONS IN THE LITERATURE

A. Simple probability approximations

The DMP approximation [15] has many useful features including the fact that it automatically respects the maximum number of parameter symmetries for an approximation scheme. There are numerous other interesting approximate expression in the literature; see Ref. [23] for a review. While many of these approximate expressions do not follow the form of Eq. (16), we nonetheless investigate which of these parameter symmetries they respect.

We first note that all other expressions only consider the single complex phase $\delta \equiv \delta_{23}+\delta_{13}+\delta_{12}$; thus, the delta shuffle parameter symmetry is not relevant. Next, we investigate the behavior of these approximations under the discrete parameter symmetries.

We begin with the expressions for the probabilities written as simple functions of the vacuum parameters and the matter potential. As such, there are no approximate matter parameter symmetries. This includes seven expressions[16] [37–42]. All of these expressions respect the $2^6=64$ parameter symmetries of the mixing angles represented by the m_{ij}, n_{ij} except for that of Ref. [38], which only respects the parameter symmetry associated with m_{13}. On the other hand, *none* satisfies the 1-2 interchange, although many could with a simple change of $\Delta m^2_{31}\to\Delta m^2_{ee}$ (a change which, in some cases, is known to increase the precision of approximations [43,45]). In addition, since these are all expressions as functions of the vacuum parameters only but include the matter effect, none satisfies the CPT symmetry of the vacuum parameters either.

One interesting approximate expression is that from Ref. [39], which we reproduce here,[17]

$$\begin{aligned}P_{\mu e} &= 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{\Delta m^2_{31}-a}\right)^2\sin^2\left(\frac{(\Delta m^2_{31}-a)L}{4E}\right)\\ &\quad+4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2\sin^2\left(\frac{aL}{4E}\right)\\ &\quad+8J_r\left(\frac{\Delta m^2_{21}}{a}\right)\sin\left(\frac{aL}{4E}\right)\left(\frac{\Delta m^2_{31}}{\Delta m^2_{31}-a}\right)\\ &\quad\times\sin\left(\frac{(\Delta m^2_{31}-a)L}{4E}\right)\cos\left(\delta+\frac{\Delta m^2_{31}L}{4E}\right),\end{aligned}\tag{27}$$

where $J_r=s_{23}c_{23}s_{13}c^2_{13}s_{12}c_{12}$ is the reduced Jarlskog invariant [15,35]. We now rewrite this expression in such a way that it respects the 1-2 interchange symmetry, and thus all the vacuum parameter symmetries,

$$\begin{aligned}P_{\mu e} &= 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta}{m^2_{ee}}\Delta m^2_{ee}-a\right)^2\sin^2\left(\frac{(\Delta m^2_{ee}-a)L}{4E}\right)\\ &\quad+4c^2_{23}c^2_{13}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2\sin^2\left(\frac{aL}{4E}\right)\\ &\quad+8J_r\left(\frac{\Delta m^2_{21}}{a}\right)\sin\left(\frac{aL}{4E}\right)\left(\frac{\Delta m^2_{ee}}{\Delta m^2_{ee}-a}\right)\\ &\quad\times\sin\left(\frac{(\Delta m^2_{ee}-a)L}{4E}\right)\cos\left(\delta+\frac{\Delta m^2_{ee}L}{4E}\right).\end{aligned}\tag{28}$$

[16] We consider Eqs. (31) and (48) of Ref. [37] and Eq. 36 of Ref. [38].

[17] The notation is slightly modified, and the absolute value signs are removed as they are not relevant, for convenience.

Note that we have changed $\Delta m^2_{31}\to\Delta m^2_{ee}$ and have also added in a factor of c^2_{13} to the second term compared to the expression in Ref. [39]. The impact of this c^2_{13} term is small, 2% on an already small term, but there is a slight improvement in the precision of the expression. More importantly, it allows one to easily write Eq. (28) as the sum of two amplitudes squared $P_{\mu e}=|\mathcal{A}_{\mu e}|^2$, where

$$\begin{aligned}\mathcal{A}_{\mu e} &= 2c_{13}\left\{s_{23}s_{13}\left(\frac{\Delta}{m^2_{ee}}\Delta m^2_{ee}-a\right)\sin\left(\frac{(\Delta m^2_{ee}-a)L}{4E}\right)\right.\\ &\quad+c_{23}s_{12}c_{12}\left(\frac{\Delta m^2_{21}}{a}\right)\sin\left(\frac{aL}{4E}\right)\\ &\quad\left.\times\exp\left[i\left(\delta+\frac{\Delta m^2_{ee}L}{4E}\right)\right]\right\}.\end{aligned}\tag{29}$$

B. Probability approximations with rotations

The remaining approximate expressions [15,24,25] use perturbative techniques of diagonalizing a part of the Hamiltonian. In Ref. [24], we note that the approximate eigenvalues denoted $\lambda'_\pm$ and $\lambda''_\pm$ do not respect the 1-2 interchange symmetry of the vacuum parameters but do respect the m_{ij}, n_{ij} discrete parameter symmetries of the vacuum angles. With regard to the approximate matter variables, they work in the vacuum mass basis so the equivalent form of Eq. (16) becomes

$$\begin{aligned}H_{\rm flav} &= U_{23}U_{13}U_{12}V_{12}V_{23}\Lambda V^\dagger_{23}V^\dagger_{12}U^\dagger_{12}U^\dagger_{13}U^\dagger_{23}\\ &\quad+(U_{23}U_{13}U_{12}M^2U^\dagger_{12}U^\dagger_{13}U^\dagger_{23}+A\\ &\quad-U_{23}U_{13}U_{12}V_{12}V_{23}\Lambda V^\dagger_{23}V^\dagger_{12}U^\dagger_{12}U^\dagger_{13}U^\dagger_{23}),\end{aligned}\tag{30}$$

for some V_{ij} and Λ. Note that in Ref. [24] the $U_{12}V_{12}$ rotations are subsequently combined into a single rotation. In addition, the sequence of rotations is different for antineutrinos than for neutrinos. Since the order of the diagonalization of the V_{ij} matrices is not the same as the lepton mixing matrix, the discrete parameter symmetries of the perturbative parameters in general do not remain.

Next, Refs. [15,25] both start in the mass basis and perform a (23)-rotation first with $\tilde\theta_{23}=\theta_{23}$. They then perform a (13)-rotation for some angle $\tilde\theta_{13}$. After this, Ref. [15] also performed a (12)-rotation for some angle $\tilde\theta_{12}$, while Ref. [25] was done and can be thought of as setting $\tilde\theta_{12}=0$ and is thus of the same form for the context of the parameter symmetries discussed here. The remaining point to confirm is that the eigenvalues all satisfy the symmetries of the vacuum parameters, which is shown in Appendix E. Thus, the expressions in Refs. [15,25] respect the 2^7 discrete parameter symmetries of the vacuum parameters and the additional 2^6 discrete parameter symmetries of the perturbative parameters provided that one considers $\tilde\theta_{23}$ and

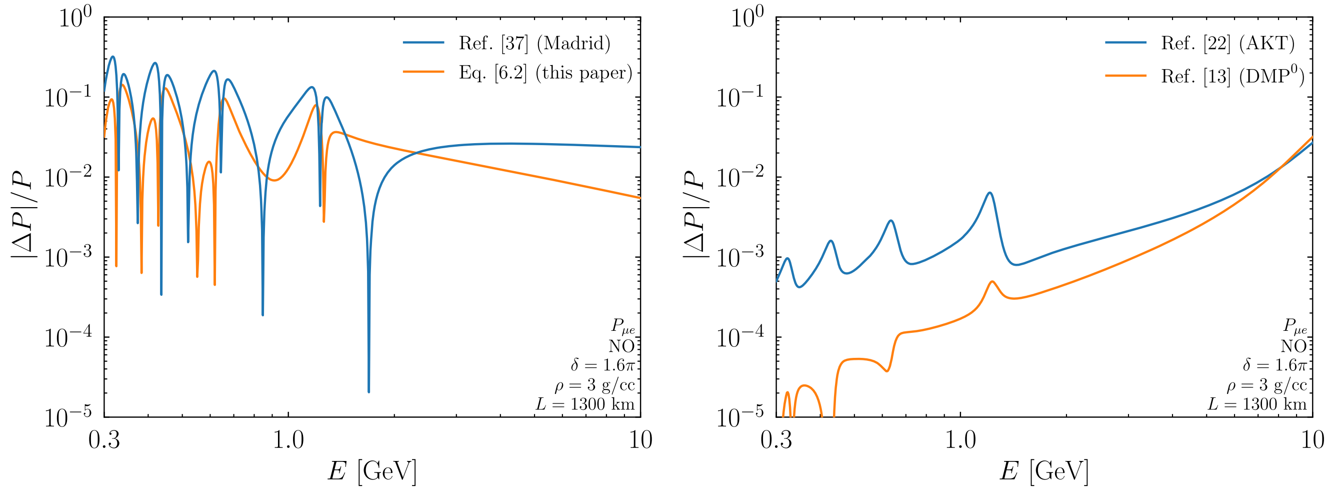

FIG. 1. Left panel: we show the fractional precision for the appearance probability in Ref. [39] compared to the exact expression in blue. In orange, we change $\Delta m^2_{31} \to \Delta m^2_{ee}$, which makes the expression in Ref. [39] respect all $2^7 = 128$ discrete parameter symmetries of the vacuum parameters including the 1-2 interchange and include a c^2_{13} factor; see Eq. (28). Right panel: we show the precision of Agarwalla-Kao-Takeuchi, [24], which does not respect the parameter symmetries as well as DMP, [15], which does respect the parameter symmetries. In both of these panels, expressions that respect the parameter symmetries have better precision. We use the best fit oscillation parameters from Ref. [44] and $\delta = 1.6\pi$ to avoid any accidental CP phase cancellations.

$\tilde{\delta}$ as separate parameters from θ_{23} and δ that are allowed to transform differently even though they take the same values in the standard ranges. In addition, the CPT parameter symmetry of the matter parameters is also respected. Finally, we note that as pointed out in Ref. [23] the approximation scheme in Ref. [15] which respects all the possible parameter symmetries of a perturbative expression is numerically more precise than the similar approach in Ref. [24] which respects fewer parameter symmetries.

In fact, we found that there is generally an increase in precision[18] when using a form that respects all the available parameter symmetries for both the simpler expressions discussed at the beginning of the section and those based on rotations in the previous papers. In Fig. 1, we show the precision for the expressions from Refs. [15,24,39] and Eq. (28). We also verified that in Refs. [37,40], which are similar to Ref. [39], the original approximate expressions have a constant error ~2% as $E \to \infty$, but a change from $\Delta m^2_{3i} \to \Delta m^2_{ee}$ results in convergence at high energies for the expressions from Refs. [39,40] and generally improved precision.

C. Approximations for other oscillation parameters

Another example of the impact of these parameter symmetries on the physics of neutrino oscillations is the size of the matter potential at the solar and atmospheric resonances given by

$$a^R_{\rm sol} \approx \cos 2\theta_{12} \Delta m^2_{21}/\cos^2\theta_{13}$$
$$\text{and} \quad a^R_{\rm atm} \approx \cos 2\theta_{13} \Delta m^2_{ee}, \tag{31}$$

respectively. These expressions are both extremely accurate as fractional corrections to these expressions are of order [47]

$$\mathcal{O}(s^4_{13}, s^2_{13}(\Delta m^2_{21}/\Delta m^2_{ee}), (\Delta m^2_{21}/\Delta m^2_{ee})^2). \tag{32}$$

The above expressions for the matter potential at the resonances satisfy all the parameter symmetries including the 1-2 symmetries. Note for the atmospheric resonance it is Δm^2_{ee}, not Δm^2_{31} nor Δm^2_{32} that appears, as the approximation is best when the 1-2 interchange symmetry is respected.

The Jarlskog invariant in matter is the coefficient of the $\mathcal{O}(L^3)$ term in the appearance probability and can also be simply approximated as [45]

$$\hat{J} \approx \frac{J}{\mathcal{S}_\odot \mathcal{S}_{\rm atm}}, \tag{33}$$

$$\mathcal{S}_\odot \equiv \sqrt{(\cos 2\theta_{12} - c^2_{13} a/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}}, \tag{34}$$

$$\mathcal{S}_{\rm atm} \equiv \sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}. \tag{35}$$

Each of these terms respects all the symmetries, and if Δm^2_{ee} is changed to Δm^2_{31} or Δm^2_{32} which do not respect these parameter symmetries, the precision of the approximation is a factor $\gtrsim 10$ times worse.

[18]The improved precision using Δm^2_{ee} instead of Δm^2_{3i} broadly applies, except for a small range of δ values near $\delta = \pi/2$, a region that is disfavored by current T2K data [46].

The effective Δm^2's for disappearance experiments in vacuum are given by

$$\Delta m^2_{ee} = \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{31}, \qquad (36)$$

$$\Delta m^2_{\mu\mu} \approx \sin^2\theta_{12}\Delta m^2_{31} + \cos^2\theta_{12}\Delta m^2_{31} + 2\sin\theta_{23}\cos\theta_{23}\sin\theta_{13}\sin\theta_{12}\cos\theta_{12} \times \Delta m^2_{21}\cos(\delta_{12}+\delta_{13}+\delta_{23})/\cos^2\theta_{23}, \qquad (37)$$

$$\Delta m^2_{\tau\tau} \approx \sin^2\theta_{12}\Delta m^2_{31} + \cos^2\theta_{12}\Delta m^2_{31} - 2\sin\theta_{23}\cos\theta_{23}\sin\theta_{13}\sin\theta_{12}\cos\theta_{12} \times \Delta m^2_{21}\cos(\delta_{12}+\delta_{13}+\delta_{23})/\sin^2\theta_{23}, \qquad (38)$$

for ν_e, ν_μ, and ν_τ disappearance, respectively [36]. Each of these satisfies the symmetries given in Sec. II for (12), (13), and (23) sectors independently. Similarly, Ref. [48] found that the eigenvalues in matter can be better approximated by using an atmospheric mass splitting of either Δm^2_{ee} or $\frac{1}{2}(\Delta m^2_{31} + \Delta m^2_{32})$, both of which respect the above parameter symmetries, over other possible definitions which may not respect these parameter symmetries.

Finally, we note that the relevant exact and approximate two-flavor Δm^2 and mixing angle for ν_e disappearance in matter [49],

$$\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2) - [m^2_3 - (m^2_1 + m^2_2)] + \Delta m^2_{ee}, \qquad (39)$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\mathcal{S}_{\mathrm{atm}}, \quad \text{and} \quad \sin 2\tilde{\theta}_{13} = \frac{\sin 2\theta_{13}}{\mathcal{S}_{\mathrm{atm}}}, \qquad (40)$$

also respect all of the above symmetries and that other possible forms that do not respect these symmetries are less precise.

VII. CONCLUSIONS

The PDG [6] parametrization of the lepton mixing matrix of three rotations has been the de facto standard parametrization for neutrino oscillations for many years now. This parametrization has many favorable phenomenological properties [1], but it also leaves open many parameter symmetries. These parameter symmetries relate two seemingly different sets of parameters to the same underlying physics, such as $\theta_{13} = 8.5°$ and $\theta_{13} = 171.5°$. In this paper, we elucidate what these parameter symmetries are in the context of changing signs of $\sin\theta_{ij}$ or $\cos\theta_{ij}$ and adjusting the appropriate complex phase adjustment. These parameter symmetries then allow one to define the mixing angles as within a range spanning $\pi/2$ rad subject to certain restrictions, typically taken to be $\theta_{ij} \in [0, \pi/2)$. In addition, these parameter symmetries allow one to either fix $\Delta m^2_{21} > 0$ or $\theta_{12} \in [0, \pi/4)$ depending on one's preference.

While the allowed ranges of the vacuum parameters have been previously identified, the framework presented here makes the connection to the parameter symmetries manifest, which makes it clear that the same parameter symmetries separately apply to the parameters in the presence of matter. Thus, in matter, one has $2^{14} = 16,384$ parameter symmetries including both vacuum and matter parameters. In addition, if one uses an approximate perturbative scheme such as that in DMP [15], one has nearly all of the parameter symmetries to any order in perturbation theory as well, $2^{13} = 8,192$, subject to key matching conditions between the vacuum parameters and the approximate matter parameters. Finally, we note that CPT invariance leads to another factor of 2 in the number of symmetries for each vacuum, matter, and approximate matter expression. All combined, this paper highlights 49,408 discrete parameter symmetries which apply to both neutrino and antineutrino oscillations across the three different frameworks. The CPT parameter symmetry combined with one of the other discrete parameter symmetries gives rise to the well-known LMA-light/LMA-dark degeneracy.

These parameter symmetries not only make it clear where the restricted ranges on the parameters come from, but they also provide a powerful tool when working with approximations for various physical quantities. While such approximate expressions need not satisfy any of the parameter symmetries mentioned here to be useful, these symmetries do provide an important cross-check and generally lead to improved precision.

We have focused on the standard PDG parametrization of the lepton mixing matrix, but the results presented here apply to different parametrizations containing a different sequence of rotations after straightforward modifications. It may be interesting to explore connections to other parametrizations involving generators of SU(3) [50–52], four complex phases [53], five rotations and a complex phase [54], and the exponential of a complex matrix [55].

Recently, Ref. [56] appeared on a related topic. We note that the various vacuum, matter, and perturbative parameter symmetries mentioned there are all covered in this paper. We explicitly show the connection to our work for a representative example in Appendix F.

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APPENDIX A: TRANSFORMATIONS OF THE ROTATION MATRICES

In this Appendix, we give the effects of the transformations given in Sec. II on the rotation matrices U_{23}, U_{13}, and U_{12}, respectively.

The application of the parameter symmetries to the (23)-rotation,

$$\begin{pmatrix} 1 & & \\ & (-1)^m c & (-1)^n s e^{i(\delta+(m+n)\pi)} \\ & -(-1)^n s e^{-i(\delta+(m+n)\pi)} & (-1)^m c \end{pmatrix} = \begin{pmatrix} (-1)^m & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & c & s e^{i\delta} \\ & -s e^{-i\delta} & c \end{pmatrix} (-1)^m. \tag{A1}$$

In this equation, $c = c_{23}, s = s_{23}, \delta = \delta_{23}$ with $m = m_{23}, n = n_{23}$.

The application of the parameter symmetries to the (13)-rotation is

$$\begin{pmatrix} (-1)^m c & & (-1)^n s e^{-i(\delta+n\pi)} \\ & 1 & \\ -(-1)^n s e^{i(\delta+n\pi)} & & (-1)^m c \end{pmatrix} = \begin{pmatrix} (-1)^m & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} c & & s e^{-i\delta} \\ & 1 & \\ -s e^{i\delta} & & c \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & (-1)^m \end{pmatrix}. \tag{A2}$$

In this equation, $c = c_{13}, s = s_{13}, \delta = \delta_{13}$ with $m = m_{13}, n = n_{13}$. The left (right) diagonal matrix commutes with U_{23} (U_{12} and also M^2).

The application of the parameter symmetries to the (12)-rotation without the 1-2 interchange is

$$\begin{pmatrix} (-1)^m c & (-1)^n s e^{i(\delta+(m+n)\pi)} & \\ -(-1)^n s e^{-i(\delta+(m+n)\pi)} & (-1)^m c & \\ & & 1 \end{pmatrix} = \begin{pmatrix} c & s e^{i\delta} & \\ -s e^{-i\delta} & c & \\ & & 1 \end{pmatrix} \begin{pmatrix} (-1)^m & & \\ & (-1)^m & \\ & & 1 \end{pmatrix}. \tag{A3}$$

In this equation, $c = c_{12}, s = s_{12}, \delta = \delta_{12}$ with $m = m_{12}, n = n_{12}$.

The application of the symmetries to the (12)-rotation with 1-2 interchange is

$$\begin{pmatrix} (-1)^m s & (-1)^n c e^{i(\delta+(m+n+1)\pi)} & \\ -(-1)^n c e^{-i(\delta+(m+n+1)\pi)} & (-1)^m s & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} c & s e^{i\delta} & \\ -s e^{-i\delta} & c & \\ & & 1 \end{pmatrix} \begin{pmatrix} -(-1)^m e^{i\delta} & & \\ & (-1)^m e^{-i\delta} & \\ & & 1 \end{pmatrix}. \tag{A4}$$

In this equation, $c = c_{12}, s = s_{12}, \delta = \delta_{12}$ with $m = m_{12}, n = n_{12}$. The second matrix in Eq. (A4) performs the $m_1^2 \leftrightarrow m_2^2$ interchange.

Then, under all the parameter symmetries given in Sec. II, we have the symmetry transformation equation for the Hamiltonian in the flavor basis:

$$U_{23}U_{13}U_{12}M^2U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger \Rightarrow \mathrm{diag}((-1)^{(m_{23}+m_{13})}, 1, 1)U_{23}U_{13}U_{12}M^2U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger \mathrm{diag}((-1)^{(m_{23}+m_{13})}, 1, 1). \tag{A5}$$

Only the diagonal matrices immediately after the equal side in Eqs. (A1) and (A2) appear in this transformation of the vacuum Hamiltonian in the flavor basis. This can also be seen directly by calculating the Hamiltonian in the flavor basis and then applying the parameter symmetries. Only the first row and column get modified by multiplication by $(-1)^{(m_{23}+m_{13})}$. An identical result to Eq. (A5) exists for DMP at zero order, i.e., for $V_{23}V_{13}V_{12}\Lambda V_{12}^\dagger V_{13}^\dagger V_{23}^\dagger$ with $(m_{23}+m_{13})$ replaced with $(\widetilde{m_{23}}+\widetilde{m_{13}})$.

Adding the matter potential to both sides of Eq. (A5) does not effect this result as

$$\mathrm{diag}((-1)^{(m_{23}+m_{13})}, 1, 1)A\mathrm{diag}((-1)^{(m_{23}+m_{13})}, 1, 1) = A. \tag{A6}$$

Therefore, Eq. (6) is valid in both vacuum and matter.

APPENDIX B: DELTA SHUFFLE EXAMPLES

In this Appendix, we show exactly how one relates Eq. (7) to Eqs. (9)–(11) via diagonal rephasing matrices. Recall that we drew attention to the delta shuffle, which allows us to write the following mixing matrices in the same way,

$$\begin{aligned} U_{23}(\theta_{23},\delta_{23})U_{13}(\theta_{13},-\delta_{13})U_{12}(\theta_{12},\delta_{12}) &= D_{23}U_{23}(\theta_{23},\delta)U_{13}(\theta_{13},0)U_{12}(\theta_{12},0)D_{23}^\dagger, \\ &= D_{13}U_{23}(\theta_{23},0)U_{13}(\theta_{13},-\delta)U_{12}(\theta_{12},0)D_{13}^\dagger, \\ &= D_{12}U_{23}(\theta_{23},0)U_{13}(\theta_{13},0)U_{12}(\theta_{12},\delta)D_{12}^\dagger, \end{aligned} \tag{B1}$$

where[19]

$$D_{23}=\mathrm{diag}(1,e^{-i\delta_{12}},e^{+i\delta_{13}}), \qquad D_{13}=\mathrm{diag}(e^{+i\delta_{12}},1,e^{-i\delta_{23}}), \qquad D_{12}=\mathrm{diag}(e^{-i\delta_{13}},e^{+i\delta_{23}},1) \tag{B2}$$

given that $\delta=\delta_{23}+\delta_{13}+\delta_{12} \mod 2\pi$. Note the D_{jk}'s are not unique.

APPENDIX C: PARAMETER SYMMETRIES IN MP

In the approximation scheme of Minakata and Parke (MP) [25], where $V_{12}=1$, Eq. (16) becomes

$$H_{\rm flav}=\frac{1}{2E}[V_{23}V_{13}\Lambda V_{13}^\dagger V_{23}^\dagger+(U_{23}U_{13}U_{12}M^2U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger+A-V_{23}V_{13}\Lambda V_{13}^\dagger V_{23}^\dagger)]. \tag{C1}$$

Applying the 2^7 vacuum symmetries to $\frac{1}{2E}[U_{23}U_{13}U_{12}M^2U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger+A]$ results in the phase matrix $\mathrm{diag}((-1)^{m_{23}+m_{13}},1,1)$ multiplying the both from the left and the right, whereas $\frac{1}{2E}[V_{23}V_{13}\Lambda V_{13}^\dagger V_{23}^\dagger]$ is invariant.

Applying the following symmetries for the matter variables, we find, for $\tilde m_{ij},\tilde n_{ij}\in\{0,1\}$:

(i) (23): $\tilde c_{23}\to(-1)^{\tilde m_{23}}\tilde c_{23}$, $\tilde s_{23}\to(-1)^{\tilde n_{23}}\tilde s_{23}$, and $\tilde\delta_{23}\to\tilde\delta_{23}\pm(\tilde m_{23}+\tilde n_{23})\pi$,

(ii) (13): $\tilde c_{13}\to(-1)^{\tilde m_{13}}\tilde c_{13}$, $\tilde s_{13}\to(-1)^{\tilde n_{13}}\tilde s_{13}$, and $\tilde\delta_{13}\to\tilde\delta_{13}\pm(\tilde m_{13}+\tilde n_{13})\pi$,

(iii) (13): $\tilde c_{13}\to(-1)^{\tilde m_{13}}\tilde s_{13}$, $\tilde s_{13}\to(-1)^{\tilde n_{13}}\tilde c_{13}$, and $\tilde\delta_{13}\to\tilde\delta_{13}\pm(\tilde m_{13}+\tilde n_{13}+1)\pi$, plus $\lambda_-\leftrightarrow\lambda_+$,

(iv) The two phases given in $V_{23}(\tilde\theta_{23},\tilde\delta_{23})V_{13}(\tilde\theta_{13},-\tilde\delta_{13})$ are determined from the cancellations that must occur for the MP perturbation theory; these are

$$\begin{aligned} \tilde s_{23}\tilde c_{23}e^{i\tilde\delta_{23}} &= s_{23}c_{23}e^{i\delta_{23}} \\ \mathrm{sign}(\tilde c_{23})\tilde c_{13}\tilde s_{13}e^{i\tilde\delta_{13}}\Delta\lambda_{+-} &= \mathrm{sign}(c_{23})c_{13}s_{13}e^{i\delta_{13}}\Delta m_{ee}^2. \end{aligned} \tag{C2}$$

Not surprisingly, CP violation is determined by the sum of the vacuum δ's: $\delta_{23}+\delta_{13}+\delta_{12}=\delta$ as in the work by DMP.

The above 2^5 matter parameter symmetries give

$$\frac{1}{2E}[V_{23}V_{13}\Lambda V_{13}^\dagger V_{23}^\dagger]\to\mathrm{diag}((-1)^{\hat m_{23}},1,1)\frac{1}{2E}[V_{23}V_{13}\Lambda V_{13}^\dagger V_{23}^\dagger]\mathrm{diag}((-1)^{\hat m_{23}},1,1). \tag{C3}$$

Matching the phase of ν_e between the two terms leads to the constraint that $\hat m_{23}=m_{23}+m_{13}$, so the total number of matter and vacuum parameter symmetries for MP perturbation theory is 2^{11}.

APPENDIX D: SYMMETRIES OF THE EXACT HAMILTONIAN IN MATTER

The Hamiltonian in matter was first explicitly solved by Zaglauer and Schwarzer (ZS) [19], but the eigenvalues as written do not respect the parameter symmetries, yet the probabilities must. If we adjust the diagonal matrix by

$$M^2=(c_{12}^2m_1^2+s_{12}^2m_2^2)\mathbb{I}+\mathrm{diag}(-s_{12}^2\Delta m_{21}^2,c_{12}^2\Delta m_{21}^2,\Delta m_{ee}^2), \tag{D1}$$

then coefficients of the characteristic equation,

[19] Note that the D_{ij} matrices may also have an additional overall phase which cancels out.

$$(\widehat{m^2}_i)^3 - A(\widehat{m^2}_i)^2 + B\widehat{m^2}_i - C = 0, \tag{D2}$$

where $\widehat{m^2}_i$ are the three eigenvalues, are[20]

$$A = \Delta m^2_{ee} + (c^2_{12} - s^2_{12})\Delta m^2_{21} + a, \tag{D3}$$

$$B = (c^2_{12} - s^2_{12})\Delta m^2_{21}\Delta m^2_{ee} - (s_{12}c_{12}\Delta m^2_{21})^2 + a[c^2_{13}\Delta m^2_{ee} + (c^2_{12} - s^2_{12})\Delta m^2_{21}], \tag{D4}$$

$$C = -(s_{12}c_{12}\Delta m^2_{21})^2\Delta m^2_{ee} + a[c^2_{13}(c^2_{12} - s^2_{12})\Delta m^2_{21}\Delta m^2_{ee} - s^2_{13}(s_{12}c_{12}\Delta m^2_{21})^2]. \tag{D5}$$

From Eq. (17) of ZS [19], the shift must be accounted for in order to see that the parameter symmetries are respected. Note that Δm^2_{ee}, $(c^2_{12} - s^2_{12})\Delta m^2_{21}$, and $(s_{12}c_{12}\Delta m^2_{21})^2$ are all unchanged under the 1-2 interchange. Under this parameter symmetry, the mass-squared matrix changes

$$M^2 \Rightarrow \mathrm{diag}(-c^2_{12}\Delta m^2_{12}, s^2_{12}\Delta m^2_{12}, \Delta m^2_{ee}) = \mathrm{diag}(c^2_{12}\Delta m^2_{21}, -s^2_{12}\Delta m^2_{21}, \Delta m^2_{ee}), \tag{D6}$$

which leads to an identical characteristic equation; see Eq. (25).

Recall that once the eigenvalues of the Hamiltonian and the eigenvalues of the principal minors of the Hamiltonian are determined the norm of the elements of the eigenvectors (or the norm of the elements of the diagonalizing matrix, $|W_{\alpha j}|^2$) are determined [22,57] and only the relative phases between the elements of the diagonalizing matrix are to be determined. There are multiple ways to choose these relative phases, and this leads to most, but not all, of the parameter symmetries discussed in this paper.

Alternatively, without using an explicit parametrization of the Pontecorvo-Maki-Nakagawa-Sakata matrix, U, and using

$$M^2 = \mathrm{diag}(m^2_1, m^2_2, m^2_3), \tag{D7}$$

then the coefficients of the characteristic equation [Eq. (D2)] are given by

$$A = m^2_1 + m^2_2 + m^2_3 + a \tag{D8}$$

$$B = m^2_1 m^2_2 + m^2_2 m^2_3 + m^2_3 m^2_1 + a\sum_i m^2_i(1 - |U_{ei}|^2) \tag{D9}$$

$$C = m^2_1 m^2_2 m^2_3 + a\sum_{i,j,k} m^2_i m^2_j |U_{ek}|^2, \qquad ((\mathrm{i,j,k})\ \text{all different}), \tag{D10}$$

where $|U_{\alpha i}|^2$'s are the fraction of the α-flavor in the ith mass eigenstate in vacuum, and thus invariant under all the vacuum parameter symmetries of this paper. Therefore, the eigenvalues in matter, $\widehat{m^2}_i$, are also invariant under the vacuum parameter symmetries.

In matter, the fraction of the α-flavor in the ith matter mass eigenstate is given by, see Refs. [22,57],

$$|W_{\alpha i}|^2 = \frac{((\widehat{m^2}_i)^2 - (\xi + \chi)_\alpha \widehat{m^2}_i + (\xi\chi)_\alpha)}{(\widehat{m^2}_i - \widehat{m^2}_j)(\widehat{m^2}_i - \widehat{m^2}_k)}, \tag{D11}$$

with (i,j,k) all different. The sum and product of the eigenvalues of the principal minors are given by

$$(\xi + \chi)_\alpha = \sum_i m^2_i(1 - |U_{\alpha i}|^2) + \begin{cases} 0 & \alpha = e \\ a & \alpha = \mu, \tau \end{cases} \quad \text{with} \quad \sum_\alpha (\xi + \chi)_\alpha = 2A, \tag{D12}$$

[20] A and C (B) are odd (even) under the CPT symmetry, so that the $\widehat{m^2_j}$'s are odd under this symmetry, as expected.

$$(\xi\chi)_\alpha = \sum_{i,j,k} m_i^2 m_j^2 |U_{\alpha k}|^2 + \begin{cases} 0 & \alpha = e \\ a\sum_i m_i^2 |U_{\tau i}|^2 & \alpha = \mu \\ a\sum_i m_i^2 |U_{\mu i}|^2 & \alpha = \tau \end{cases} \quad \text{with} \quad \sum_\alpha (\xi\chi)_\alpha = B. \tag{D13}$$

Thus, all vacuum parameter symmetries are respected by the matter physical observables as well. Only the unitarity of U has been used to derive these expressions.[21] The variables $\widehat{m^2}_i$'s and $|W_{ei}|^2$'s are only dependent on m_i^2's and $|U_{ei}|^2$'s and are explicitly independent of $|U_{\mu i}|^2$'s and $|U_{\tau i}|^2$'s (θ_{23} and δ in the PDG parametrization).

For oscillation physics, the only other quantity needed is the Jarlskog in matter, $\hat{J}$, which is related to Jarlskog in vacuum, J, by [58,59]

$$\hat{J}\Pi_{i>j}(\widehat{m^2}_i - \widehat{m^2}_j) = J\Pi_{i>j}(m_i^2 - m_j^2). \tag{D14}$$

As discussed earlier the lhs (rhs) of this equation is invariant under all matter (vacuum) parameter symmetries of this paper. Thus, all physical observables of the exact solution in uniform matter are invariant under all parameter symmetries of both the matter variables and the vacuum variables.

APPENDIX E: CONFIRMATION OF THE PERTURBATIVE CONDITIONS FOR DMP

Reference [15] (DMP) presented an approximation scheme for neutrino oscillations in matter based on diagonalizing "most" of the Hamiltonian via three rotations defined as a rotation of the vacuum parameters $[U_{23}(\theta_{23}, \delta)]$ and then diagonalizing a 2×2 submatrix twice in succession: 13 and then 12. We confirm here that the requirements for the parameter symmetries to hold are all satisfied by the details of this scheme.

For definiteness we consider NO $\Rightarrow \lambda_1 < \lambda_2 < \lambda_3$. We start with M^2 shifted as in Eq. (25). Then, we define the initial eigenvalues after the (23)-rotation before either of the diagonalizations,

$$\lambda_a = a + s_{13}^2 \Delta m_{ee}^2, \qquad \lambda_b = (c_{12}^2 - s_{12}^2)\Delta m_{21}^2,$$
$$\lambda_c = c_{13}^2 \Delta m_{ee}^2; \tag{E1}$$

thus, all parameter symmetries are satisfied for $(\lambda_a, \lambda_b, \lambda_c)$.

Next, after the (13)-rotation, we obtain the eigenvalues,

$$\lambda_\pm = \frac{1}{2}(a + \Delta m_{ee}^2 \pm \Delta\lambda_{+-}),$$
$$\Delta\lambda_{+-} = +\sqrt{(\Delta m_{ee}^2 \cos 2\theta_{13} - a)^2 + (\sin 2\theta_{13}\Delta m_{ee}^2)^2},$$
$$\lambda_0 = \lambda_b = (c_{12}^2 - s_{12}^2)\Delta m_{21}^2; \tag{E2}$$

thus, all parameter symmetries are also satisfied for the $(\lambda_-, \lambda_0, \lambda_+)$ eigenvalues. Therefore, the diagonalizing angle, $\phi \equiv \tilde{\theta}_{13}$, given by

$$\sin^2\phi = \frac{\lambda_+ - \lambda_c}{\lambda_+ - \lambda_-}, \tag{E3}$$

also satisfies all parameter symmetries.[22] This is the point at which Ref. [25] stops; thus, the eigenvalues in that paper all respect the parameter symmetries of the vacuum parameters.

Continuing on with the framework of Ref. [15], after the (12)-rotation,

$$\lambda_3 = \frac{1}{2}\left(a + \Delta m_{ee}^2 + \sqrt{(\Delta m_{ee}^2 \cos 2\theta_{13} - a)^2 + (\sin 2\theta_{13}\Delta m_{ee}^2)^2}\right),$$
$$\lambda_{2,1} = \frac{1}{2}(\lambda_- + \lambda_0 \pm \Delta\lambda_{21}),$$
$$\Delta\lambda_{21} = +\sqrt{(\Delta m_{21}^2 \cos 2\theta_{12} - a_{12})^2 + (\cos^2\xi)(\sin 2\theta_{12}\Delta m_{21}^2)^2}, \tag{E4}$$

where parameter symmetry invariant quantities a_{12} and $\cos^2\xi$ are given by[23]

$$a_{12} = \frac{1}{2}(a + \Delta m_{ee}^2 - \Delta\lambda_{+-}), \tag{E5}$$

[21] In particular, use of the identity $|U_{\alpha i}|^2 = |U_{\beta j}U_{\gamma k} - U_{\beta k}U_{\gamma j}|^2$, with (i, j, k) and (α, β, γ) all different, is needed. This relation follows from $U^\dagger = U^{-1} = \mathrm{adj}(U)/\det(U)$.

[22] The choice of sign when taking the square root gives rise to the parameter symmetries.

[23] In the work by DMP, $\cos^2\xi$ is related to ϕ and θ_{13} via $\xi = \phi - \theta_{13}$, which may experience various reflections or shifts.

$$\cos^2\xi = \frac{1}{2}\frac{\Delta m^2_{ee} + \Delta\lambda_{+-} - a\cos 2\theta_{13}}{\Delta\lambda_{+-}}. \tag{E6}$$

All parameter symmetries are satisfied for $(\lambda_1, \lambda_2, \lambda_3)$.

In the Hamiltonian, whether we chose $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ or $\Lambda = \mathrm{diag}(\lambda_2, \lambda_1, \lambda_3)$ determines whether the second diagonalizing angle, $\psi \equiv \tilde{\theta}_{12}$, is given by

$$\sin^2\psi \quad \text{or} \quad \cos^2\psi = \frac{\lambda_2 - \lambda_0}{\lambda_2 - \lambda_1}. \tag{E7}$$

Again, we have the sign choices when the square root is taken.

Nothing here depends on what sign choices one has made in vacuum. However, whether $\delta_\phi = \delta_{13}$ or $\delta_\phi = \delta_{13} + \pi$ and similarly for δ_ψ, does depend on both sets of choices. This is related to Eq. (16) where in the work of DMP V_{23} is a rotation of a new angle and phase that are taken to be the same as θ_{23} and δ in vacuum, V_{13} is a rotation of the angle ϕ, and V_{12} is a rotation of the angle ψ. This section confirms that the approximate eigenvalues that make up the Λ matrix are invariant under all the parameter symmetries as required by the second condition.

Here, we precisely quantify how the procedure carried out in the work by DMP [15] is properly generalized to take advantage of all available parameter symmetries. First, we note that it makes sense to treat $\tilde{\theta}_{23}$ as a separate parameter from θ_{23}, just like how $\tilde{\theta}_{13}$ and $\psi = \tilde{\theta}_{12}$ are treated as separate variables from θ_{13} and θ_{12}, similarly for the $\tilde{\delta}_{ij}$. Thus, the conditions that need to be satisfied to follow the DMP procedure are

$$\begin{aligned} s_{\tilde{23}}c_{\tilde{23}}e^{i\tilde{\delta}_{23}} &= s_{23}c_{23}e^{i\delta_{23}} \\ \mathrm{sign}(c_{\tilde{23}})c_\phi s_\phi e^{i\delta_\phi}\Delta\lambda_{ee} &= \mathrm{sign}(c_{23})c_{13}s_{13}e^{i\delta_{13}}\Delta m^2_{ee} \\ s_\psi c_\psi e^{i\delta_\psi}\Delta\lambda_{21} &= (|c_\phi c_{13}| + |s_\phi s_{13}|)s_{12}c_{12}e^{i\delta_{12}}\Delta m^2_{21}. \end{aligned} \tag{E8}$$

From these equations, its clear that the new phases must satisfy

$$\tilde{\delta}_{jk} = \delta_{jk} \mod \pi. \tag{E9}$$

In addition, the final requirement from Sec. III B,

$$m_{13} + m_{23} = m_\phi + \tilde{m}_{23} \mod 2, \tag{E10}$$

follows from the middle condition of Eq. (E8) when combined with the other requirements.

APPENDIX F: RELATIONSHIP TO HM21

Recently, Ref. [56] (HM21) appeared on a related topic. All of the parameter symmetries presented in HM21 fit within our framework. To illustrate the relationship, we pick one representative example containing all relevant features, Symmetry-IVB (last line of Table 1 in HM21) for which the parameter symmetry is proven through first order in perturbation theory (our result in Sec. III B is correct to all orders).

Symmetry-IVB is written as the following interchanges:

$$\begin{aligned} &\theta_{23} \to -\theta_{23}, \quad \theta_{13} \to -\theta_{13}, \quad \theta_{12} \to -\theta_{12}, \quad \delta \to \delta + \pi, \\ &\phi \to -\phi, \quad \lambda_1 \leftrightarrow \lambda_2, \quad c_\psi \leftrightarrow \pm s_\psi, \quad s_\psi \leftrightarrow \pm c_\psi. \end{aligned} \tag{F1}$$

We note that in the notation of HM21 (which matches that by DMP [15]; see also Appendix E) λ_i are the approximate matter eigenvalues of the perturbative matrix and ϕ (ψ) are $\tilde{\theta}_{13}$ ($\tilde{\theta}_{12}$). Care is required in translating the statement of the parameter symmetry in HM21 to that of this paper since in HM21 it is implicitly assumed that $\tilde{\theta}_{23}$ is the same (and thus transforms the same) as θ_{23}; $\tilde{\delta}$ and δ are similarly linked. In fact, each of the $\widetilde{\delta_{ij}}$ are taken to transform the same as the corresponding δ_{ij} (although this can be written as just a single complex phase for each side given the delta shuffle).

Then, this parameter symmetry with the upper signs is equivalent to our framework with the 1-2 interchange on the approximate variables, all the $m_{ij} = \tilde{m}_{ij} = 0$, and all the $n_{ij} = \tilde{n}_{ij} = 0$ except those listed here:

(i) for $\theta_{23} \to -\theta_{23}$, therefore $n_{23} = 1$, and this requires a π be added to δ_{23},
(ii) for $\theta_{13} \to -\theta_{13}$, therefore $n_{13} = 1$, and this requires a π be added to δ_{13},
(iii) for $\theta_{12} \to -\theta_{12}$, therefore $n_{12} = 1$, and this requires a π be added to δ_{12},

Thus, $\delta \to \delta + \pi$ and $m_{23} + m_{13} = 0$. For the approximate matter parameters, we recall that we must also modify $\tilde{\theta}_{23}$ and that the resultant modifications to $\tilde{\delta}$ must match the factor of π gained for δ. We have:

(i) for $\tilde{\theta}_{23} \to -\tilde{\theta}_{23}$, therefore $\tilde{n}_{23} = 1$, and this requires π be added to $\tilde{\delta}_{23}$,
(ii) for $\tilde{\theta}_{13} \to -\tilde{\theta}_{13}$, therefore $\tilde{n}_{13} = 1$, and this requires π be added to $\tilde{\delta}_{13}$,
(iii) $m_1^2 \leftrightarrow m_2^2$ with $c_{\tilde{12}} \to \pm s_{\tilde{12}}$ and $s_{\tilde{12}} \to \pm c_{\tilde{12}}$, and this requires π be added to $\tilde{\delta}_{12}$,

Thus, $\tilde{\delta} \to \tilde{\delta} + \pi$ and $\tilde{m}_{23} + \tilde{m}_{13} = 0$. So, δ and $\tilde{\delta}$ match as they must *by the choice of definition for DMP*, and this example satisfies our parameter symmetry requirements including the constraint given by Eq. (18). For the lower signs, one additionally sets $\tilde{m}_{12} = \tilde{n}_{12} = 1$, each of which results in a factor of π added to $\tilde{\delta}_{12}$ and thus no change to the complex phase. We have also verified that our scheme contains all of the parameter symmetries presented in Table 1 of HM21.

Since HM21 uses the same symbol for δ in vacuum and in matter, it is important that they agree. However, using the same symbol in vacuum and matter for θ_{23} and δ limits the number of possible discrete parameter symmetries by $2^3 = 8$ (4 for θ_{23} and 2 for δ). Note that in a general perturbative framework $\tilde{\delta}$ and δ can differ by π. Even accounting for these parameter symmetries, there are still additional parameter symmetries unexplored in HM21 such as those including a 1-2 interchange of both vacuum and matter parameters, those without the 1-2 interchange of the matter parameters, or those which send $c_{ij} \to -c_{ij}$ (and $c_{\widetilde{ij}} \to -c_{\widetilde{ij}}$). Using a different symbol for θ_{23} in matter with its associated δ_{23} in matter allows additional parameter symmetries. To address the maximum possible parameter symmetries, one must use different symbols for matter and vacuum for all variables and associate a separate δ for each angle, even though the oscillation probabilities only depend on the sum of these δ's.

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