Letter

Quasinormal modes and microscopic structure of the Schwarzschild black hole

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Maggiore observed that, in the high-damping regime, the quasinormal modes spectrum for the Schwarzschild black hole resembles that of a quantum harmonic oscillator. Motivated by this observation, we describe a black hole as a statistical ensemble of N quantum harmonic oscillators. By working in the canonical ensemble, we show that, in the large-mass black hole limit, the leading contribution to the Gibbs entropy is the Bekenstein-Hawking term, while the subleading one is a logarithmic correction, in agreement with several results in the literature. We also find that the number of oscillators scales holographically with the area of the event horizon.

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I. BACKGROUND AND MOTIVATIONS

Classically, a perturbed black hole reacts dynamically, producing characteristic oscillations, called quasi-normal modes (QNMs), which decay exponentially in time. At linear perturbation level, QNMs correspond to complex eigenfunctions of the system, namely modes characterized by complex frequencies, whose imaginary part describes the damping of the mode in time (see, e.g., Refs. [1–3]). Moreover, boundary conditions at infinity and at the horizon imply a discrete spectrum for the frequencies ω_n, with the imaginary part depending on an integer n, the overtone number.

In the high-damping regime (large-n limit), the spectrum of QNMs for a Schwarzschild black hole (SBH), with mass M, is independent of l, the angular momentum "quantum" number, and reads.[1] [4–7] (see Refs. [1–3,8] for reviews)

$$8\pi G M\omega_{\mathrm{n}} \sim \ln 3 + 2\pi \mathrm{i}\left(n+\frac{1}{2}\right) + \mathcal{O}(n^{-1/2}). \qquad (1)$$

By studying the system's response to external perturbations, in principle we could also have access to its internal microscopic structure. Therefore, despite being classical, QNMs could contain signatures of quantum gravity effects, encoding information about the quantum properties of black holes and their horizons [9]. This is particularly true in the large-n limit, which is expected to probe the black hole at short distances.

There are several indications supporting this perspective. On one hand, QNMs could be useful to understand the AdS/CFT conjecture. In fact, in the case of AdS black holes, the damping of QNMs can be mapped into the thermalization of the conformal field theory on the boundary [10]. On the other hand, the emergent and corpuscular gravity scenarios suggest that black holes could be characterized by long-range quantum gravity effects of N quanta building the black hole [11–16]. Similarly to what happens for the surface gravity of a black hole (see Ref. [15]), the quantum nature of Eq. (1) is obscured (the Planck constant $\hbar$ does not appear) by expressing ω_n in terms of the black-hole mass M, but becomes fully evident when we express it in terms of the black hole temperature T_{H}.

Thus, the QNMs spectrum could represent a coarse-grained description of the response of these N microscopic degrees of freedom to external perturbations, in the same spirit as the spectrum of the black body radiation is a manifestation of the collective behavior of a photon gas.

However, the no-hair theorem [17,18] makes a black hole drastically different from a black body. In the latter case the extensive thermodynamic parameters scale with the volume, while the number of photons, i.e., the number of microscopic degrees of freedom, is independent of the size of the system. For a black hole, on the other hand, the mass M and the entropy S are fixed by the temperature or, equivalently, by the horizon radius r_h. From a corpuscular gravity point of view, moreover, we can consider the black hole as a macroscopic quantum state that saturates a maximally packaging condition [16,19–21], which has

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[1]We adopt natural units, $c = \hbar = 1$.

been shown to be equivalent to the holographic scaling of N [15]. Essentially, in a black hole of a given mass, we can "pack" a maximum amount of degrees of freedom, which is constrained by the size of the system.

The proposal of using QNMs to capture some microscopic properties of black holes is not completely new, but was first proposed by Maggiore in [22]: the linear scaling of the QNMs frequencies with the overtone number n suggests that a SBH can be described, in the high-damping limit, as a harmonic oscillator, with proper frequency

$$\omega = \sqrt{\omega_{\mathrm{R}}^2 + \omega_{\mathrm{I}}^2}, \tag{2}$$

where ω_{R} and ω_{I} are the real and imaginary parts of the frequencies (1) respectively.[2]

Until now, the high-damped QNMs spectrum has been used in the quantum gravity context to explain and fix the area spectrum of the event horizon (see, e.g., Refs. [6,24–26]), whose quantization was first suggested by Bekenstein in [27,28]. This also allowed to fix the Barbero-Immirzi parameter [24], which is essential to correctly account for the Bekenstein-Hawking (BH) entropy in loop quantum gravity [29,30].

In this paper, we use Maggiore's result to model the SBH as a canonical ensemble of N harmonic oscillators and derive the black-hole entropy using the QNMs frequencies only, without assuming horizon-area quantization. Consistently with the no-hair theorem, for an asymptotic observer, the only physical observable is the black-hole mass M, which also determines the QNMs frequency spectrum. On the other hand, we assume that, quantum mechanically, the horizon area and the temperature can fluctuate independently from M. This will allow us to consistently define the canonical ensemble and to circumvent the no-hair theorem at quantum level.

II. THE MODEL

In the high-damping regime, $\omega_{\mathrm{I}} \gg \omega_{\mathrm{R}}$, QNMs probe the internal structure of the black hole, as the wavelength of each oscillator gets smaller and smaller as n grows. From Eqs. (1) and (2), we easily get the frequency spectrum

$$\omega_n \simeq |\omega_{\mathrm{I}}| = \frac{1}{4GM}\left(n + \frac{1}{2}\right) + \mathcal{O}(n^{-1/2}). \tag{3}$$

Following Maggiore's proposal, we model the black hole as a statistical ensemble of $N \gg 1$ indistinguishable noninteracting (at least in a first approximation) quantum harmonic oscillators with frequencies

$$\omega_n = \omega_0\left(n + \frac{1}{2}\right), \tag{4}$$

where $\omega_0 = 1/4GM$ is the proper frequency of each oscillator.

Our derivation relies entirely on equilibrium statistical mechanics, without resorting to usual black-hole thermodynamics. The black hole will be regarded as an ensemble in thermal equilibrium with its surroundings at temperature $T = 1/\beta$. We will therefore work entirely in the canonical ensemble and consider the number of oscillators N fixed. This is motivated by the no-hair theorem, which tells us that the chemical potential of a SBH is zero, being the mass M the only classical hair of the hole. Considering the SBH as a system at fixed temperature is also consistent with the fact that the QNMs spectrum is computed at fixed black-hole mass. For the asymptotic observer, the latter is related to the Hawking temperature, which is therefore fixed.

It is very important to stress that, in our statistical description, we treat β and ω_0 as *independent* variables, i.e., the temperature of the ensemble can change independently from the black-hole mass. At first sight, this may seem at odds with standard black-hole thermodynamics. However, we argue that quantum mechanically this is fully consistent.

In standard black-hole thermodynamics, the Hawking temperature T_{H} can be defined as the coefficient of proportionality between the entropy (the area of the black-hole event horizon, $\mathcal{A}_{\mathrm{H}}$) and its energy (the mass M). Classically, $\mathcal{A}_{\mathrm{H}}$ is a function of M, i.e., $\mathcal{A}_{\mathrm{H}} = 16\pi G^2 M^2$. The latter, however, should be considered as the mean value, measured at infinity, of the area of the event horizon, which can fluctuate around its expectation value, from a quantum mechanical point of view [27,28,31].[3] Only local measurements would allow us to probe these fluctuations [31]. An observer at infinity therefore would not have access to them, as the only degree of freedom he/she can measure is the classical hair of the black hole, i.e., its mass. This tells us that, at least quantum mechanically, the area of the event horizon can fluctuate *independently* from M. Hence, only the observer at infinity can make the identification $\beta = \beta_{\mathrm{H}} = 1/T_{\mathrm{H}} = 8\pi GM$.

Using the spectrum (3), the statistical Boltzmann weight of each harmonic oscillator black hole microstate is therefore

$$e^{-\beta\omega_n} = e^{-\beta\omega_0(n+\frac{1}{2})} e^{-\beta\frac{\kappa}{\sqrt{n}}}, \tag{5}$$

where κ is a dimensionful constant, proportional to ω_0 on dimensional grounds, parametrizing the low-n behavior of the damping modes. As long as we consider the limit of large-mass black holes $\omega_0 \to 0$ (or the high-temperature

[2] A similar proposal for the description of a black hole as a harmonic oscillator, from a corpuscular gravity perspective, can be found in [23].

[3] This is also supported by the fuzzball proposal for black holes in string theory [32].

limit, $\beta \to 0$), the factor $e^{-\beta\kappa n^{-1/2}}$ can be set equal to 1 and the partition function will be insensible to the subleading terms $\mathcal{O}(n^{-1/2})$.

Being the SBH effectively featureless, except from its mass, and having zero chemical potential, the probability of occupying a given energy level will be the same for all oscillators. The partition function for the composite system of N oscillators therefore reads

$$\mathcal{Z} = \left(\sum_{n=0}^{\infty} e^{-\beta\omega_0(n+\frac{1}{2})}\right)^N = \left(\frac{e^{\beta\omega_0/2}}{e^{\beta\omega_0}-1}\right)^N,$$
$$\ln\mathcal{Z} = \frac{N\omega_0}{2}\beta - N\ln\left(e^{\beta\omega_0}-1\right). \tag{6}$$

Using standard statistical mechanics relations, we compute the mean energy and the entropy

$$\langle E\rangle = -\partial_\beta \ln\mathcal{Z} = \frac{N\omega_0}{2}\,\mathrm{cotgh}\left(\frac{\beta\omega_0}{2}\right); \tag{7}$$

$$S = \ln\mathcal{Z} + \beta\langle E\rangle = -N\ln\left(e^{\beta\omega_0}-1\right) + \frac{N\beta\omega_0 e^{\beta\omega_0}}{e^{\beta\omega_0}-1}. \tag{8}$$

As expected for consistency, the expressions above satisfy the first law of thermodynamics, $d\langle E\rangle = TdS$.

Let us now focus on macroscopic black holes, by considering the large M limit, i.e., $\omega_0 \to 0$. By expanding the mean energy (7) and the entropy (8), we get:

$$\langle E\rangle = \frac{N}{\beta} + \frac{N}{12}\beta\omega_0^2 + \mathcal{O}(\omega_0^3); \tag{9}$$

$$S = N - N\ln(\beta\omega_0) + \frac{N}{24}\beta^2\omega_0^2 + \mathcal{O}(\omega_0^3). \tag{10}$$

We see that the leading terms in the expansions satisfy $\langle E\rangle = TS$, hence they capture only the purely thermal extensive contribution TS to the mean energy. However, for $\beta \to \infty$ (zero-temperature limit), the subleading terms in Eq. (9) diverge at fixed ω_0. The same problem appears in standard black-hole thermodynamics, where a zero Hawking temperature implies a divergence of the black-hole mass M. However, this is an artefact of the expansion. This divergence problem can be simply solved in our approach by first taking the $\beta \to \infty$ limit in the exact expression for $\langle E\rangle$ given by Eq. (7). We get $\langle E\rangle = N\omega_0/2$. This is a finite value, which cures the divergence appearing in Eq. (9) and has the simple physical interpretation of the zero-point energies $\omega_0/2$ of the N oscillators, representing therefore the contribution of the vacuum. Notice that this contribution cancels out when we perform first the $\omega_0 \to 0$ limit and keep the leading terms only.

The black-hole mass M measured by an observer at infinity can be seen as the sum of the purely extensive contribution of Eq. (9) and the contribution of the vacuum E_V. Our microscopic description of the SBH in terms of N noninteracting harmonic oscillators holds in the large-n limit. Thus, we may expect deviation of E_V from the naive value $N\omega_0/2$. Owing to the absence of an external scale different from the thermal one $\beta_H = 1/T_H$, we nevertheless expect E_V to get only order-one corrections, and M to take the form

$$M = \frac{N}{\beta} + c\,\frac{N\omega_0}{2}, \tag{11}$$

where c is a $\mathcal{O}(1)$ constant, which can be fixed using symmetry arguments. The expansions (9) and (10) can be also obtained by considering the limit $\beta \to 0$ instead of $\omega_0 \to 0$ in Eqs. (7) and (8). Despite being mathematically equivalent, these two limits have a very different physical meaning. While the latter corresponds to black holes with large masses, the former is related to small-mass SBHs. Treating ω_0 and β separately introduces some kind of duality between small and large black holes, which is again a consequence of the absence of an external scale different from β_H. This implies that in our microscopic description there is no difference between the thermodynamic properties of small- and large-mass black holes (a behavior very different from AdS black holes [33]).

Since the asymptotic observer can only measure the classical hair M, both the limits $\beta \to 0$ and $\beta \to \infty$ in Eq. (11) must lead to the same result, giving $c = 1/\pi$.

For the asymptotic observer, the black-hole equilibrium temperature is T_H, so that Eq. (11) gives

$$N = \frac{\beta_H M}{2} = 4\pi G M^2. \tag{12}$$

Seen by the distant observer, therefore, the number of oscillators scales holographically with the area of the event horizon. The leading term of Eq. (10), together with Eq. (12), yields

$$S = N = 4\pi G M^2 \tag{13}$$

which is exactly the BH entropy. The subleading term in Eq. (10) represents a logarithmic correction $N\ln T$, which is consistent with several results in the literature (see, e.g., Refs. [31,34–46]).

Our description of the black hole in terms of a canonical ensemble of harmonic oscillators, with frequency given by the fundamental QNMs frequency ω_0, is fully consistent with the corpuscular description [11,47]; the latter sees the black hole as a coherent state of particles with occupation numbers $n_j(p)$ sharply peaked around the characteristic momentum $p \sim 1/R_H$, with j labeling some internal microscopic degrees of freedom (DOF). The relation $\omega_0 \sim p$ is a highly nontrivial check of this consistency. Unfortunately, our thermodynamic treatment based on the QNMs spectrum does not give any information about the origin of these internal DOF and their occupation numbers $n_j(p)$. This is mainly due to the fact that the oscillators are treated as indistinguishable from the beginning, coherently with the no-hair theorem. Consequently, we do not have any chemical potential and the only observable is the total number of oscillators $N = \sum_j n_j$. It is quite obvious that, in order to gain information about n_j, we need some further insight, beyond the QNMs spectrum, on the "quantum hair" associated with the internal DOF.

III. CONCLUDING REMARKS

All the information a distant observer can achieve about a SBH is its mass and the response of the hole to perturbations, the QNMs spectrum. In this letter, we used this fact to build a microscopic description of the black hole in terms of a statistical ensemble of N harmonic oscillators. In this way, we were able to derive, microscopically, the BH entropy as the leading term in the large-mass expansion. We found a subleading logarithmic correction to the BH entropy, in agreement with several results in the literature. In our microscopic description, the holographic character of the BH formula is a natural consequence of the horizon-area scaling of N. An intriguing duality between black holes of small and large sizes also emerged in this approach.

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[1] E. Berti, V. Cardoso, and A. O. Starinets, Classical Quant. Grav. **26**, 163001 (2009).
[2] K. D. Kokkotas and B. G. Schmidt, Living Rev. Relativity **2**, 2 (1999).
[3] R. A. Konoplya and A. Zhidenko, Rev. Mod. Phys. **83**, 793 (2011).
[4] B. F. Schutz and C. M. Will, Astrophys. J. Lett. **291**, L33 (1985).
[5] H.-P. Nollert, Phys. Rev. D **47**, 5253 (1993).
[6] S. Hod, Phys. Rev. Lett. **81**, 4293 (1998).
[7] L. Motl, Adv. Theor. Math. Phys. **6**, 1135 (2002).
[8] H.-P. Nollert, Classical Quant. Grav. **16**, R159 (1999).
[9] I. Agullo, V. Cardoso, A. D. Rio, M. Maggiore, and J. Pullin, Phys. Rev. Lett. **126**, 041302 (2021).
[10] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D **62**, 024027 (2000).
[11] G. Dvali and C. Gomez, Fortschr. Phys. **61**, 742 (2013).
[12] G. Dvali and C. Gomez, J. Cosmol. Astropart. Phys. 01 (2014) 023.
[13] R. Casadio, A. Giugno, O. Micu, and A. Orlandi, Entropy **17**, 6893 (2015).
[14] R. Casadio, A. Giugno, and A. Giusti, Phys. Lett. B **763**, 337 (2016).
[15] M. Cadoni, M. Tuveri, and A. P. Sanna, Symmetry **12**, 1396 (2020).
[16] R. Casadio, arXiv:2103.14582.
[17] W. Israel, Commun. Math. Phys. **8**, 245 (1968).
[18] S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972).
[19] G. Dvali and C. Gomez, arXiv:1005.3497.
[20] G. Dvali, G. F. Giudice, C. Gomez, and A. Kehagias, J. High Energy Phys. 08 (2011) 108.
[21] G. Dvali, C. Gomez, and A. Kehagias, J. High Energy Phys. 11 (2011) 070.
[22] M. Maggiore, Phys. Rev. Lett. **100**, 141301 (2008).
[23] R. Casadio and A. Orlandi, J. High Energy Phys. 08 (2013) 025.
[24] O. Dreyer, Phys. Rev. Lett. **90**, 081301 (2003).
[25] G. Kunstatter, Phys. Rev. Lett. **90**, 161301 (2003).
[26] C. Corda, Adv. High Energy Phys. **2015**, 867601 (2015).
[27] J. D. Bekenstein, Lett. Nuovo Cimento **11**, 467 (1974).
[28] J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. B **360**, 7 (1995).
[29] G. Immirzi, Nucl. Phys. B, Proc. Suppl. **57**, 65 (1997).
[30] I. Agulló, J. F. Barbero G., J. Diaz-Polo, E. Fernandez-Borja, and E. J. S. Villasenor, Phys. Rev. Lett. **100**, 211301 (2008).
[31] G. Gour, Phys. Rev. D **61**, 021501 (1999).
[32] S. D. Mathur, Fortschr. Phys. **53**, 793 (2005).
[33] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
[34] R. B. Mann and S. N. Solodukhin, Nucl. Phys. **B523**, 293 (1998).
[35] M. Akbar and K. Saifullah, Eur. Phys. J. C **67**, 205 (2010).
[36] S. Carlip, Classical Quant. Grav. **17**, 4175 (2000).
[37] S. Mukherji and S. S. Pal, J. High Energy Phys. 05 (2002) 026.
[38] A. J. M. Medved, Classical Quant. Grav. **22**, 133 (2005).
[39] M. Domagala and J. Lewandowski, Classical Quant. Grav. **21**, 5233 (2004).
[40] D. Grumiller, in *Proceedings of the 8th International Conference on Path Integrals from Quantum Information to Cosmology*, http://www1.jinr.ru/Proceedings/Burdik-2005/index.html.
[41] D. V. Fursaev, Phys. Rev. D **51**, R5352 (1995).
[42] M. R. Setare, Int. J. Mod. Phys. A **21**, 3007 (2006).
[43] A. Ghosh and P. Mitra, Phys. Rev. Lett. **73**, 2521 (1994).
[44] R. K. Kaul and P. Majumdar, Phys. Rev. Lett. **84**, 5255 (2000).
[45] M. Cadoni, Phys. Lett. B **653**, 434 (2007).
[46] G. Gour and A. J. M. Medved, Classical Quant. Grav. **20**, 3307 (2003).
[47] G. Dvali, J. High Energy Phys. 03 (2021) 126.