Letter

Shadowless rapidly rotating yet not ultraspinning Kerr-AdS$_4$ and Kerr-Newman-AdS$_4$ black holes

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We find that the Kerr-(Newman)-AdS$_4$ black hole will be shadowless if its rotation parameter is larger than a critical value and shadowless-ness may be related to the appearance of the null hypersurface caustics (NHC) both inside the Cauchy horizon and outside the event horizon for the black hole with the rotation parameter beyond the critical value. Our studies also further confirm that whether an ultraspinning black hole is superentropic or not is unrelated to the existence of the NHC outside the event horizon.

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The black hole, as one of the most remarkable and fascinating objects in nature, has always been a subject of extensive studies both theoretically and observationally. On the observational side, recent years have witnessed exciting achievements in "hearing" the gravitational waves from the coalescence of black holes [1–6] as well as "seeing" its shadow by the event horizon telescope [7–14]. On the theoretical side, the study of asymptotically anti–de Sitter (AdS) black holes has shed light on the nature of gravity through gauge-gravity dualities [15–17]. About ten years ago, it was conjectured that an AdS black hole should satisfy the reverse isoperimetric inequality (RII) [18,19]

$$\mathcal{R} = \left[\frac{(d-1)V}{\mathcal{A}_{d-2}}\right]^{1/(d-1)} \left(\frac{\mathcal{A}_{d-2}}{A}\right)^{1/(d-2)} \geq 1, \qquad (1)$$

where V is the thermodynamic volume of the black hole, $\mathcal{A}_{d-2} = 2\pi^{[(d-1)/2]}/\Gamma[(d-1)/2]$ the area of the unit $(d-2)$-sphere with d being the number of spacetime dimensions and A the area of the outer horizon. Equality is attained for the Schwarzschild-AdS black hole, which means that the Schwarzschild-AdS black hole has the maximum entropy or it occupies the least volume for a specified entropy.

The first counterexample of an AdS black hole, which does not satisfy the RII, is the ultraspinning black hole [20], which is constructed by boosting one of the angular velocities of the rotating AdS black hole to the speed of light [20]. Such constructed ultraspinning black hole has a finite horizon area but noncompact event horizon topology. Since the ultraspinning black hole exceeds the maximum entropy bound [21–23], it is dubbed "superentropic." Soon afterward, a lot of new ultraspinning AdS black hole solutions [24–30] have been generated successfully from the corresponding known rotating AdS black holes. And various aspects of the ultraspinning black holes, including thermodynamic properties [20,24–27,30–34], horizon geometry [21,24,25], Kerr/CFT correspondence [25,26,35] and geodesic motion [36], etc, have been explored widely. However, not all ultraspnning black holes are superentropic since a few new ultraspinning AdS black holes [24,28,29], i.e., the ultraspinning Kerr-Sen-AdS$_4$ (dyonic) black holes, violate the RII only in certain parameter spaces of the solutions. Therefore, ultraspinning is not a sufficient condition for a black hole to be superentropic.

So, a question arises naturally as to what the physical condition is for an ultraspinning black hole to be superentropic. Recently, it was suggested that the superentropicness may be related to the presence of the null hypersurface caustics (NHC) outside the event horizon of the black hole. This is because the NHC exists outside the event horizon for the (high dimensional) Kerr-(Newman)-AdS superentropic black hole [37,38]. The existence of the NHC is thought previously to mean that the causal structure of spacetime has some pathologies. However, it was recently found that the spacetime with the NHC is free of closed timelike curve [39].

For the ultraspinning Kerr-Sen-AdS$_4$ black hole, whether it is superentropic or not, the NHC always appears both out and inside of the horizon [39], which seems to indicate that the existence of the NHC outside the event horizon is related to ultraspinning rather than superentropicness. The ultraspinning means that the rotational angular velocity of

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the black hole is boosted to the speed of light. Then a further issue, which we are going to address, is whether the rotational angular velocity reaching the speed of light is a necessary requirement for existence of the NHC outside the event horizon.

Let us start with the four-dimensional Kerr-Newman-AdS black hole solution [40], whose metric has the form [41]

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(dt - \frac{a}{\Xi}\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2 \quad (2)$$

in the Boyer-Lindquist coordinates, where

$$\Delta_r = (r^2+a^2)(1+r^2/l^2) - 2mr + q^2, \qquad \Xi = 1 - a^2/l^2,$$
$$\Delta_\theta = 1 - a^2\cos^2\theta/l^2, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

a is the rotation parameter, m the mass parameter, q the electric charge parameter and l the AdS radius. The horizon is determined by equation $\Delta_r = 0$. When $q = 0$, the metric in Eq. (2) reduces to that of the Kerr-AdS$_4$ black hole. The RN-AdS$_4$ black hole is obtained if $a = 0$ in Eq. (2). In Fig. 1, we plot the rescaled horizon radius $\bar{r}$ as a function of the scaled mass $\bar{m}$. It is easy to see that if the black hole mass is larger than a critical value both the RN-AdS$_4$ black hole and the Kerr-(Newman)-AdS$_4$ black hole have the Cauchy and event horizons. When the black hole mass equals to the critical value, the Cauchy and event horizons coincide. While the Schwarzschild-AdS$_4$ black hole, which corresponds to the case with both $a = 0$ and $q = 0$, only has the event horizon.

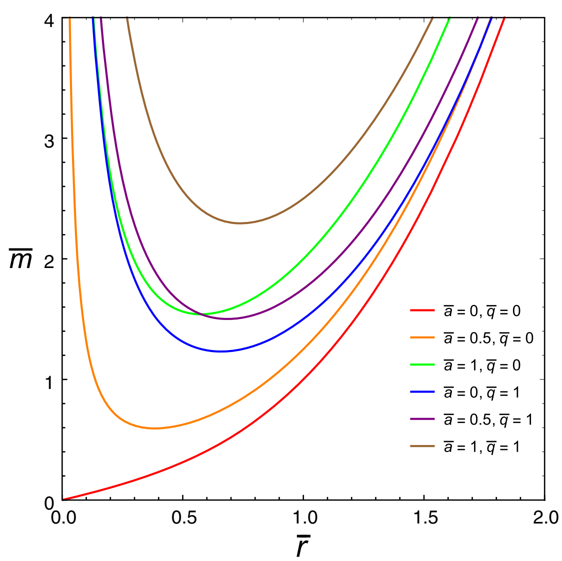

FIG. 1. The rescaled mass vs the rescaled horizon radius, where $\bar{m} = m/l, \bar{a} = a/l, \bar{q} = q/l, \bar{r} = r/l$. The red, orange, green, blue, purple and brown lines represent the Schwarzschild-AdS$_4$, Kerr-AdS$_4$, ultraspinnning Kerr-AdS$_4$, RN-AdS$_4$, Kerr-Newman-AdS$_4$ and ultraspinning Kerr-Newman-AdS$_4$ black holes, respectively.

To derive the condition for existence of the NHC for the Kerr-Newman-AdS$_4$ black hole, we use the method given in Refs. [37–39,42,43]. After introducing the outgoing and ingoing Eddington-Finkelstein coordinates defined in terms of the "generalized tortoise coordinate" $r_*(r,\theta)$: $u = t - r_*(r,\theta)$, $v = t + r_*(r,\theta)$, the null hypersurfaces can be described by $u =$ const, $v =$ const, which are dubbed the outgoing and ingoing null congruences of the hypersurfaces, respectively. It is easy to obtain that the null hypersurfaces defined by $u =$ const or $v =$ const satisfy the equation

$$g^{\mu\nu}\partial_\mu(t+r_*)\partial_\nu(t+r_*) = g^{tt} + g^{rr}(\partial_r r_*)^2 + g^{\theta\theta}(\partial_\theta r_*)^2 = 0. \quad (3)$$

By using the contravariant components g^{tt}, g^{rr}, and $g^{\theta\theta}$ of the metric (2), Eq. (3) can be re-expressed as

$$\Delta_r(\partial_r r_*)^2 - \Xi^2\frac{(r^2+a^2)^2}{\Delta_r} = -\Delta_\theta(\partial_\theta r_*)^2 - \Xi^2\frac{a^2\sin^2\theta}{\Delta_\theta}. \quad (4)$$

This equation can be solved by the method of separation of variables. Introducing the so-called "constant of separation" $a^2\lambda$ (hereinafter, λ is referred to as the separation constant) for Eq. (4), we have

$$(\partial_r r_*)^2 = \frac{Q^2(r)}{\Delta_r^2}, \qquad (\partial_\theta r_*)^2 = \frac{P^2(\theta)}{\Delta_\theta^2}. \quad (5)$$

where

$$Q^2(r) = \Xi^2[(r^2+a^2)^2 - a^2\lambda\Delta_r],$$
$$P^2(\theta) = \Xi^2 a^2(\lambda\Delta_\theta - \sin^2\theta). \quad (6)$$

In [37,43], λ is chosen as $\lambda < 1$ and it was then found that the NHC cannot exist outside the event horizon of the Kerr-(Newman)-AdS black hole. However, this choice is incorrect, since when $\theta = \pi/2$, $P^2(\theta) = \Xi^2 a^2(\lambda - 1)) \geq 0$ must be satisfied. Thus, the allowed region of λ should be $\lambda \geq 1$.

After some complicated computations, which are presented in detail in the Supplemental Material [44], we find that $Q(r) = 0$ is a sufficient condition for the existence of the NHC, which means that

$$(r^2+a^2)^2 - a^2\lambda\Delta_r = 0. \quad (7)$$

Figure 2 shows the numerical results of the NHC condition (7). In addition, the photon sphere condition, which is governed by the equation $(r^2+a^2)\Delta_r' - 4r\Delta_r = 0$ [45,46], and the horizon condition ($\Delta_r = 0$), are plotted, where a prime indicates the derivative with respect to r. In this figure, two intersection points between the dashed black line and the blue/purple line represent that the black hole has the Cauchy horizon and the event horizon. One can see

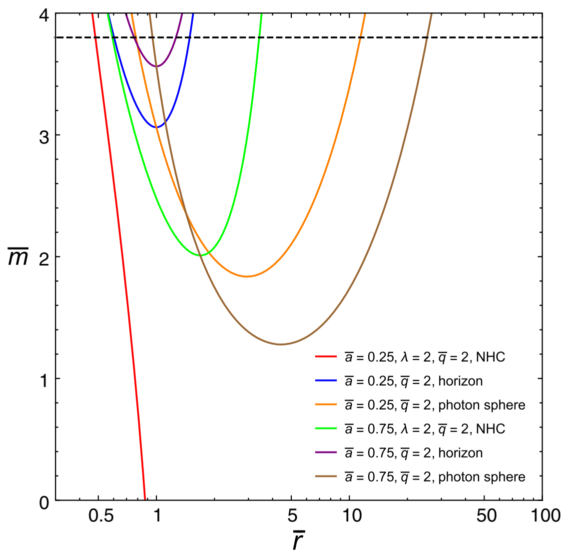

FIG. 2. Contours of the NHC, the horizon and the photon sphere in the ($\bar{r}$, $\bar{m}$) plane with a fixed $\bar{q}$ for the Kerr-Newman-AdS_4 black hole, where $\bar{m} = m/l$, $\bar{a} = a/l$, $\bar{q} = q/l$, $\bar{r} = r/l$.

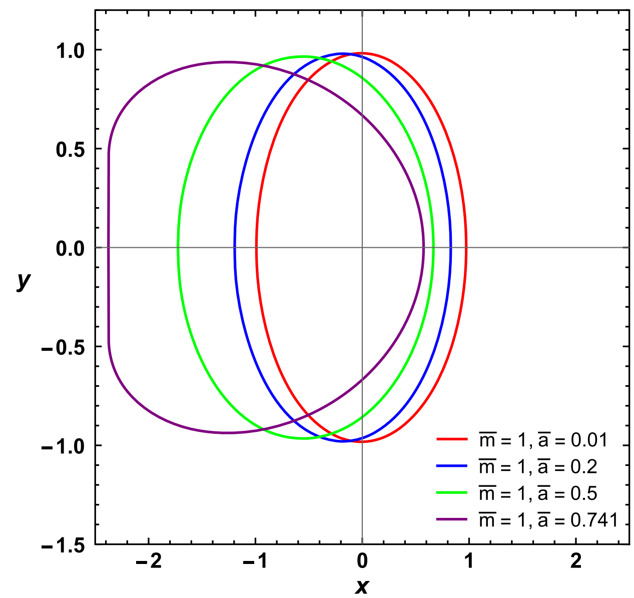

FIG. 3. The shadows for the Kerr-AdS_4 black hole for different rescaled rotation parameter $\bar{a}$ with $\bar{m} = 1$, where $\bar{m} = m/l$, $\bar{a} = a/l$.

that the NHC exists only inside the Cauchy horizon when $\bar{a} = 0.25$, and it appears both inside the Cauchy horizon and outside the event horizon of the black hole when $\bar{a} = 0.75$. This suggests that the NHC can appear outside the event horizon of the black hole for a large enough a which is not necessarily equal to l. The photon sphere line indicates that the NHC, which appears outside of the event horizon, locates between the event horizon and the photon sphere of the black hole.

How large is the value of the rotational angular velocity required for existence of the NHC outside the event horizon? After a detailed calculation, we find that when the rotation parameter a satisfies

$$a \geq \frac{l}{\sqrt{\lambda}}, \tag{8}$$

the NHC exists both outside the event horizon and inside the Cauchy horizon of the Kerr-Newman-AdS_4 black hole. Otherwise, there is the NHC only inside the Cauchy horizon. Thus, if the rotation parameter is larger than a critical value $a_c = l/\sqrt{\lambda}$, the NHC appears outside the event horizon. Apparently, this critical value is determined by l and λ. When λ equals unity, the NHC exists outside the event horizon only in the case of the ultraspinning Kerr-Newman-AdS_4 black hole ($a \to l$). If $\lambda > 1$, a large enough a, which is not necessarily equal to l, can lead to the existence of the NHC outside the event horizon of the black hole. Therefore, the value of λ determines how large a has to be to ensure the existence of the NHC outside the event horizon.

Now we demonstrate that when the rotation parameter is larger than a critical value, in addition to the existence of the NHC outside the event horizon, the Kerr-(Newman)-AdS black hole's shadow cannot be obtained and thus it should be shadowless. Here, we take the Kerr-AdS_4 black hole as an example and investigate its shadows. The results are expected to be valid for the Kerr-Newman-AdS_4 case. Figure 3 gives the numerical results in the equatorial plane $\theta = \pi/2$. With the increase of the rotational angular velocity, the black hole shadows change from an ellipse to a "D" shape. Once $a > 0.741l$, when $m = 1$ the shadow of the Kerr-AdS_4 black hole disappears. Clearly, both the shadowless-ness and the existence of the NHC outside the event horizon of the Kerr-(Newman)-AdS black hole require that the black hole rotates rapidly enough with the rotation parameter larger than a critical value. This seems to suggest that the existence of the HNC outside the event horizon might be the cause of the shadowless-ness of a rapidly rotating Kerr-(Newman)-AdS black hole. Therefore, we can obtain the value of λ from analyzing the shadow of the Kerr-(Newman)-AdS_4 black hole. From the inequality (8), we find that the value of λ should be chosen to be about $\lambda \sim 1.821$ when $\bar{m} = 1$.

It is easy to obtain that when the rotation parameter of the black hole is larger than the critical value a_c, i.e., $a_c \simeq 0.741l$ for the Kerr-AdS_4 black hole with $\bar{m} = 1$, the NHC will appear outside the event horizon of the black hole. Apparently, the ultraspinning limit ($a \to l$) of the black hole is not a necessary but sufficient condition for the appearance of the NHC outside the event horizon of black hole. Thus, we further confirm that the superentropicness of the black hole is unrelated to the presence of the NHC outside the event horizon of the rotating AdS black hole.

To summarize, by investigating the NHC of the Kerr-(Newman)-AdS_4 black hole, we find that there exists a critical value a_c for the rotation parameter a. When the rotation parameter of the black hole is smaller than this critical value, the NHC only exists inside the Cauchy horizon. Once a is larger than this critical value, the Kerr-(Newman)-AdS_4 black hole will have the NHC both inside the Cauchy horizon and outside the event horizon. Our results demonstrate that the NHC can exist outside the event horizon of the black hole which is not ultraspinning, in sharp contrast to the conclusion that the Kerr-(Newman)-AdS_4 black hole has the NHC outside the event horizon only when it is ultraspinning [37,43]. The fact that the shadows of the rotating AdS black holes cannot be obtained if the rotation parameter is larger than a critical value seems to suggest that the appearance of the NHC outside the event horizon leads to the disappearance of the black hole shadow. Thus, for the rapidly rotating yet not ultraspinning the Kerr-(Newman)-AdS_4 black holes, we will not see their shadows either. Our results confirm that the superentropicness of ultraspinning black holes is unrelated to the presence of the NHC outside the event horizon. Finally, we believe that our work can be generalized to other rotating AdS black holes, and for these black holes there is also a critical value for the rotation parameter.

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