Letter

Krylov complexity in conformal field theory

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Krylov complexity, or K-complexity for short, has recently emerged as a new probe of chaos in quantum systems. It is a measure of operator growth in Krylov space, which conjecturally bounds the operator growth measured by the out of time ordered correlator (OTOC). We study Krylov complexity in conformal field theories by considering arbitrary 2d CFTs, free field, and holographic models. We find that the bound on OTOC provided by Krylov complexity reduces to bound on chaos of Maldacena, Shenker, and Stanford. In all considered examples including free and rational CFTs Krylov complexity grows exponentially, in stark violation of the expectation that exponential growth signifies chaos.

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Quantum chaos and complexity play increasingly important role in understanding dynamical aspects of quantum field theory and quantum gravity. The notion of quantum chaos is difficult to define and there are different complementary approaches. The conventional approach in the context of quantum many-body systems is rooted in spectral statistics, eigenstate thermalization hypothesis (ETH), and absence of integrability [1]. In the context of field theory and large N models another well-studied signature of chaos is the behavior of the out of time ordered correlator (OTOC) [2]. These approaches focus on different aspects of quantum dynamics and usually apply to different systems. It is an outstanding problem to develop a uniform approach to chaos which would connect and unite them. Dynamics of quantum operators in Krylov space has been recently proposed as a potential bridge connecting dynamics of OTOC with the conventional signatures of many-body chaos [3].

Krylov space is defined as the linear span of nested commutators $[H\ldots,[H,\mathcal{O}]]$, where H is the system's Hamiltonian and $\mathcal{O}$ is an operator in question. Accordingly, time evolution $\mathcal{O}(t)$ can be described as dynamics in Krylov space. Krylov complexity $K_{\mathcal{O}}(t)$ defined below in (6) is a measure of operator size growth in *Krylov space*. For the chaotic systems it is expected to grow exponentially [3], $K_{\mathcal{O}}(t) \propto e^{\lambda_K t}$. For systems with finite-dimensional local Hilbert space, e.g., SYK model [4–6], it has been shown that at infinite temperature λ_K bounds Lypanunov exponent of OTOC

$$\lambda \leq \lambda_K. \tag{1}$$

This inequality conjecturally applies at finite temperature $\beta > 0$. From one side connection of Krylov complexity to OTOC is not that surprising given that the latter measures spatial operator growth [7]. From another side, dynamics in Krylov space is fully determined in terms of thermal 2pt function, see below. Hence, the bound on OTOC in terms of $K_{\mathcal{O}}(t)$ is the bound on thermal 4pt function in terms of thermal 2pt function. In this sense it is similar to the proposals of [8] and also [9], which derived the Maldacena-Shenker-Stanford (MSS) bound on chaos [2]

$$\lambda \leq 2\pi/\beta \tag{2}$$

from the ETH. From the effective field theory point of view the 4pt function is independent from the 2pt one, hence such a bound could only be very general and apply universally. One may not expect that a general theory would saturate the bound, casting doubt on the proposal that the exponent λ_K of Krylov complexity is indicative of the Lyapunov exponent λ. Indeed, we will see that in case of CFTs the conjectural bound (1) holds but reduces to MSS bound (2) such that λ_K would remain finite even when λ would approach zero or may not be well defined.

To conclude the introductory part, we remark that studying Krylov complexity should be seen in a broader context of relating it to holographic complexity [10–12] and studies of thermal 2pt function in holographic settings with the goal of elucidating quantum gravity in the bulk [13–20].

To remind the reader, we briefly introduce the main notions of Krylov space. More details can be found in [3,21]. Starting from an operator $\mathcal{O}$ one introduces iterative relation

$$\mathcal{O}_{n+1} = [H, \mathcal{O}_n] - b_{n-1}^2 \mathcal{O}_{n-1}, \tag{3}$$

where positive real Lanczos coefficients b_n are uniquely fixed by the requirement that $\mathcal{O}_n$ are mutually orthogonal with respect to scalar product $\mathrm{Tr}(e^{-\beta H/2}\mathcal{O}_n e^{-\beta H/2}\mathcal{O}_m) \propto \delta_{nm}$. Lanczos coefficients depend on the choice of the system Hamiltonian H, the operator $\mathcal{O}_0 = \mathcal{O}$, and inverse temperature β. Time evolution of the operator can be represented in terms of Krylov space,

$$\mathcal{O}(t) \equiv e^{iHt}\mathcal{O}e^{-iHt} = \sum_{n=0}^{\infty} \varphi(t)_n \mathcal{O}_n, \tag{4}$$

where normalized "wave function" $\varphi_n(t)$ satisfies discretized "Schrödinger" equation

$$-i\frac{d\varphi_n}{dt} = b_n\varphi_{n+1} + b_{n-1}\varphi_{n-1}, \tag{5}$$

with the initial condition $\varphi_n(0) = \delta_{n,0}$. It describes hopping of a quantum-mechanical "particle" on a one-dimensional chain. Krylov complexity is defined as the averaged value of an "operator" $\hat{n}$ measured in the "state" φ, where for convenience index n is shifted by 1,

$$K_{\mathcal{O}}(t) \equiv (\mathcal{O}|\hat{n}|\mathcal{O}) = 1 + \sum_{n=0}^{\infty} n|\varphi_n(t)|^2. \tag{6}$$

Lanczos coefficients, and hence $K_{\mathcal{O}}(t)$, are encoded in thermal Wightman 2pt function

$$\begin{aligned} C_0(\tau) &= \langle \mathcal{O}(-i(\tau+\beta/2))\mathcal{O}(0)\rangle_\beta \\ &\propto \mathrm{Tr}(e^{-(\frac{\beta}{2}-\tau)H}\mathcal{O}e^{-(\frac{\beta}{2}+\tau)H}\mathcal{O}). \end{aligned} \tag{7}$$

Precise relation evaluating b_n^2 in terms of C_0 and its derivatives is discussed in Supplemental Material [22]. We only note here that b_n^2 do not change under multiplication of C_0 by an overall constant.

In full generality for a physical system with local interactions $C_0(\tau)$ is analytic in the vicinity of $\tau = 0$. This implies that power spectrum

$$f^2(\omega) = \int dt\, e^{i\omega t} C_0(it) \tag{8}$$

decays at large ω at least exponentially,

$$f^2(\omega) \sim e^{-\tau^*\omega}, \qquad \omega \to \infty, \tag{9}$$

where $\tau^* > 0$ is the location of first singularity of $C_0(\tau)$ along the imaginary axis, if any. It was anticipated long ago that the high frequency behavior of $f^2(\omega)$ for a local operator in many-body system can be used as a signature of chaos. In particular exponential behavior (9) was proposed as a signature of chaos in *classical* systems in [23]. An equivalent formulation in terms of the singularity of $C_0(\tau)$ was proposed as a signature of chaos for *quantum* many-body systems in [24] based on the rigorous bounds constraining the magnitude of $C_0(\tau)$ in the complex plane. A further step had been taken in [3] who proposed the universal operator growth hypothesis: in generic, i.e., chaotic quantum many-body systems Lanczos coefficients b_n^2 associated with a local $\mathcal{O}$ exhibit maximal growth rate compatible with locality,

$$b_n \approx \left(\frac{\pi}{2\tau^*}\right)n + o(n), \quad n \gg 1. \tag{10}$$

This is stronger than the exponential behavior (9), i.e., it implies the latter, and reduces to it upon an additional assumption that the behavior of b_n^2 as a function of n is sufficiently smooth for $n \to \infty$. Modulo similar assumption of smoothness of b_n^2 Ref. [3] proved that in this case Krylov complexity grows exponentially as

$$K_{\mathcal{O}}(t) \propto e^{\lambda_K t}, \tag{11}$$

where $\lambda_K = \pi/\tau^*$.

In field theory Wightman thermal 2pt function of local operators $C_0(\tau)$ (7) necessarily has singularity at $\tau = \beta/2$, implying exponential decay of the power spectrum (9) with $\tau^* = \beta/2$. *Assuming* sufficient smoothness of b_n^2, one immediately arrives at (10) [25,26] (also see [3,24]), and exponential growth of Krylov complexity with $\lambda_K = 2\pi/\beta$. Hence the conjectural bound on OTOC (1) reduces to the MSS bound (2). This logic applies to *any* quantum field theory, including free, integrable or rational CFT models. Similarly, one can conclude that for field theories universal operator growth hypothesis (10) trivially holds, but the exponential behavior of Krylov complexity can not be regarded as an indication of chaos. We stress, these conclusions are premature as one needs to justify the smoothness assumption by e.g., evaluating b_n^2 explicitly. Without this assumption asymptotic behavior of b_n^2 is *not* determined by the high frequency tail of $f^2(\omega)$, or the singularity of $C_0(\tau)$, as is shown explicitly by a counterexample in [24]. We justify the smoothness assumption by considering several different CFT models placed on flat space $\mathbb{R}^{d-1}$ and evaluating corresponding Lanczos coefficients.

(i) In case of 2d CFTs thermal 2pt function of primary operators $\mathcal{O}$ is fixed by conformal invariance

$$C_0 = \frac{1}{\cos(\pi\tau/\beta)^{2\Delta}}, \tag{12}$$

where Δ is the dimension of $\mathcal{O}$. This functional form of C_0 has been thoroughly analyzed in [3] in the context of SYK model. In particular they found $b_n^2 = (n+1)(n+2\Delta)(\pi/\beta)^2$ and $K_\mathcal{O}(t) = 1 + 2\Delta\sinh^2(\pi t/\beta)$. In other words b_n^2 dependence on n is smooth and Krylov complexity grows exponentially with $\lambda_K = 2\pi/\beta$.

(ii) In case of free massless scalar in d dimensions, as well as generalized free field of conformal dimension Δ [18], thermal 2pt function is given by,

$$C_0 = c_d(\zeta(2\Delta, 1/2 + \tau/\beta) + \zeta(2\Delta, 1/2 - \tau/\beta)). \tag{13}$$

Coefficient c_d ensures canonical normalization in case of free massless scalar and is not important in what follows. In the latter case $\Delta = d/2 - 1$.

For (13) with general Δ explicit expression for Lanczos coefficients is not known. In the special case of $d = 4$, C_0 reduces to (12) with $\Delta = 1$, and the rest applies. For $d = 6$, $\Delta = 2$, and Lanczos coefficients can be evaluated in terms of special functions, see Supplemental Material [22]. In this case b_n^2 demonstrate "staggering" or "dimerization"—the sequences of b_n^2 for even and odd n can be combined into two families, each approximately described by smooth functions $b_n = h_n + (-1)^n \tilde{h}_n$, where $h_n \approx (\pi/2\tau^*)n + o(n)$ for $n \gg 1$. This is shown in Fig. 1. Such a behavior was analyzed in [27,28], where it was shown that for smooth functions $h_n, \tilde{h}_n$ in the large n region "Schrödinger equation" (5) reduces to continuous Dirac equation with the space-dependent mass. In the case when asymptotically $\tilde{h}_n \to 0$, mass eventually approaches zero for large x, describing propagation of a quantum "particle" with the speed of light $x(t) \sim t$ with respect to an auxiliary spatial continuous coordinate x which is related to n via $n \propto (e^{(2\pi/\beta)x} - 1)$ [28]. From this follows that for late times Krylov complexity will grow exponentially

$$K_\mathcal{O}(t) \approx e^{2\pi/\beta(t-t_0)} \tag{14}$$

where t_0 is the characteristic time "quantum particle" described by $\varphi_n(t)$ will spend near the edge of the Krylov space $n \sim O(1)$. From the analytic expression for $K_\mathcal{O}$ in case of 2d CFTs we conclude that t_0 is growing negative for large Δ, $t_0 \sim -\ln\Delta$. The only scenario to avoid exponential growth of $K_\mathcal{O}$ with t is for $\varphi_n(t)$ to be localized near the edge $n \sim O(1)$, which would presumably require erratic behavior of b_n for small n.

Numerical simulation of $K_\mathcal{O}$ for massless scalar in $d = 6$ shown in Fig. 2 confirms exponential behavior (14) with t_0 of order one. Thus, despite "staggering" Krylov complexity for free massless scalar in $d = 6$ behaves qualitatively similar to $d = 4$ case.

Next we numerically plot Lanczos coefficients for free scalar in $d = 5$ with $\Delta = 3/2$, see Fig. 1. Similarly to $d = 6$, b_n's exhibit staggering, which does not affect asymptotic exponential behavior of $K_\mathcal{O}$, see Fig. 2.

For large Δ we analyze (13) by employing $1/\Delta$ expansion to find at small n

$$\beta^2 b_n^2 = \begin{cases} 16\Delta^2 + 8(1+3n)\Delta + O(n^2) + \ldots & n\,\text{even}, \\ 16(1+n)\Delta + O(n^2) + \ldots & n\,\text{odd}. \end{cases} \tag{15}$$

Thus, staggering grows with Δ, but n dependence of b_n for odd and even n remain smooth.

For large n pole structure of C_0 suggests, see Supplemental Material [22],

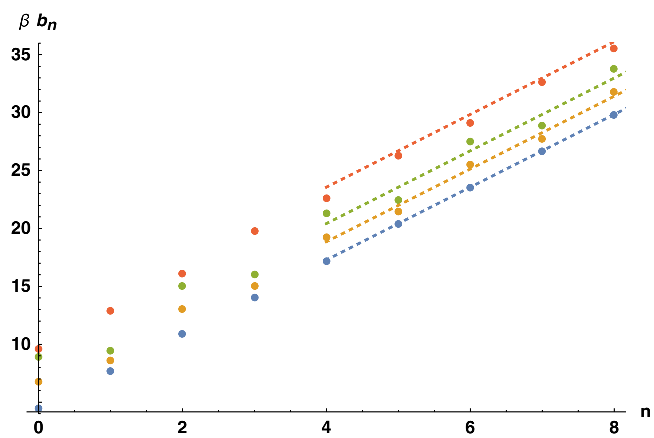

FIG. 1. Lanczos coefficients b_n for free massless scalar ϕ in $d = 4$ ($\Delta = 1$, blue), $d = 5$ ($\Delta = 3/2$, orange), $d = 6$ ($\Delta = 2$, green) dimensions, and for the composite operator ϕ^2 in $d = 5$ dimensions ($\Delta = 3$, red); dashed lines of the appropriate color show asymptotic behavior of b_n as given by (16).

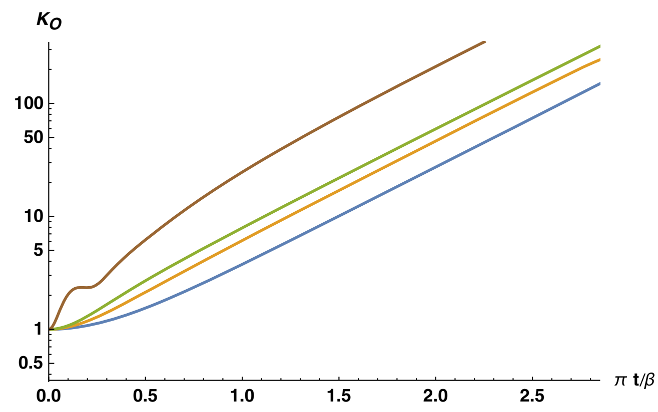

FIG. 2. Krylov complexity $K_\mathcal{O}$ shown in logarithmic scale for free scalar in $d = 4$ (blue), $d = 5$ (orange), $d = 6$ (green) dimensions and for generalized free field with $\Delta = 10$ (brown). Blue curve is known analytically, $\ln(1 + 2\sinh^2(\pi t/\beta))$. All four curves exhibit an apparent linear growth of $\ln K_\mathcal{O} \propto 2\pi t/\beta$ at late times.

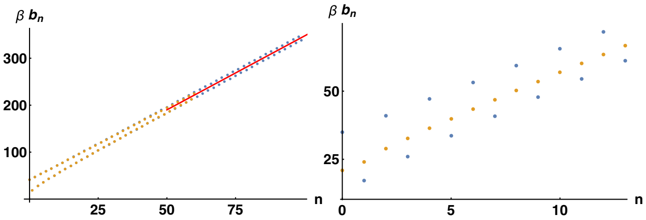

FIG. 3. Left panel. Lanczos coefficients b_n for generalized free field (13) with $\Delta = 10$ (blue) vs approximation for small n (15) (orange) and asymptotic behavior for large n (16) (red line). Right panel. Lanczos coefficients b_n for generalized free field (13) of dimension $\Delta = 8.5$ (blue) and for holographic operator $O = \int d^3x\mathcal{O}$ of effective dimension $\Delta = 8.5$, while $\mathcal{O}$ has dimension $\Delta = 10$ (orange). The same effective dimension means both sequences have the same asymptotic behavior $b_n \approx \pi(n+9)$.

$$\beta b_n \approx \pi(n + \Delta + 1/2). \tag{16}$$

These approximations accurately describe b_n for small and large n correspondingly, as is shown in the left panel of Fig. 3. Numerical simulation of $K_{\mathcal{O}}(t)$ for $\Delta = 10$ shown in Fig. 2 confirms exponential behavior with $\lambda_K = 2\pi/\beta$ and t_0 of order $-\ln\Delta$. In other words staggering, exhibited by b_n in case of free scalar field, which grows with Δ, is not affecting dynamics at late times: $K_{\mathcal{O}}$ grows exponentially with the exponent $\lambda_K = 2\pi/\beta$, although dynamics at early times becomes more complicated.

Finally, we discuss composite operators $\mathcal{O}^m$ for some integer m. By Wick theorem Wightman function simply becomes $C_0 \to C_0^m$ with an unimportant overall coefficient. In the case of 2d CFT or free massless scalar in $d = 4$ we again obtain C_0 of the form (12). In other cases Lanczos coefficients should be calculated numerically. We plot b_n for $\mathcal{O} = \phi^2$ in free massless scalar theory in $d = 5$ in Fig. 1.

(iii) In case of free fermions in d dimensions,

$$C_\psi(\tau) = r_d \sum_{n,k=0}^{1} (-1)^n \zeta\left(2\Delta, \frac{1+2n}{4} + (-1)^k \frac{\tau}{2\beta}\right), \tag{17}$$

where dimension of free fermion is $\Delta = (d-1)/2$. We notice that Lanczos coefficients for free fermion in dimension d are very close to those for the free boson of the same conformal dimension Δ, i.e., in dimension $d+1$. The same applies for b_n for the composite operators $\bar{\psi}\psi$ and ϕ^2. Corresponding comparison is delegated to Supplemental Material [22].

(iv) In case of holographic CFT thermal two-point function can be calculated by solving wave equation in the bulk [15,16]. We perform this numerically in Supplemental Material [22] to find that b_n smoothly depend on n. This is shown in the right panel of Fig. 3 where we superimposed b_n for the holographic model with Lanczos coefficients for the generalized free field of the same effective dimension, determined by the singularity of C_0 near $\tau \to \beta/2$. Smooth behavior perfectly matches the expectation that for holographic theories exhibiting maximal chaos, $\lambda = 2\pi/\beta$, growth of Krylov complexity also must be governed by the same exponent.

Conclusions. In this paper we studied Lanczos coefficients and operator growth in Krylov space for local operators in various CFT models. For some models b_n were calculated analytically, while for others we had to resort to numerical analysis. We also found asymptotic behavior of b_n for large n (16). One of the main goals was to study if Krylov complexity is sensitive to the underlying chaos. A general argument presented in the introduction dictates that so far asymptotic behavior of b_n as a function of n is sufficiently smooth, Lanczos coefficients exhibit universal operator growth hypothesis (10) and Krylov complexity grows exponentially (14). The only possible caveat is the possibility that for large n different subsequences of b_n would have different asymptotic, for example b_n for even and odd n would grow as n^a with different $a_{\mathrm{even}} \neq a_{\mathrm{odd}}$. Another hypothetical possibility, which will not affect (10) but may affect (14), is that erratic behavior of b_n for small n will cause approximate or complete localization of the operator "wave function" φ_n, leading to large or infinite t_0. We did not see any behavior of this sort in any model we considered, including arbitrary 2d CFTs, free bosons and fermions, composite operators, generalized free field of arbitrary dimension, and a holographic model in $d = 4$. On the contrary we observed linear growth of b_n at large n in full agreement with (16) and exponential growth of Krylov complexity with $\lambda_K = 2\pi/\beta$. In other words for considered models universal operator growth hypothesis of [3] trivially holds, and the conjectural bound (1) on of OTOC at finite temperature in terms of growth of Krylov complexity reduces to MSS bound [2]. At the same time exponential growth of $K_{\mathcal{O}}$ is not a signature of chaos as it grows with the same exponent $\lambda_K = 2\pi/\beta$ for maximally chaotic holographic CFTs as well as for rational 2d CFTs and free field theories, for which Lypanunov exponent may not be even properly defined [29–31]. It would be interesting to extend our analysis for massive an interacting models, especially those exhibiting nonmaximal chaos [32–35]. Nevertheless we expect a continuous deformation not to change the asymptotic behavior of b_n and our results to remain valid in the case of general interacting quantum field theory.

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