Some integrable deformations of the Wess-Zumino-Witten model

N. Mohammedi *

Institut Denis Poisson (CNRS - UMR 7013), Université de Tours,
Faculté des Sciences et Techniques, Parc de Grandmont, F-37200 Tours, France

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Lie algebra valued equations translating the integrability of a general two-dimensional Wess-Zumino-Witten model are given. We found a simple solution to these equations and identified a new integrable nonlinear sigma model. This is a two-parameter deformation of the Wess-Zumino-Witten model.

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I. INTRODUCTION

The search for integrable two dimensional nonlinear sigma model has known various developments. The early attempts dealt mostly with deformations of the principal chiral sigma model and examples based on the Lie algebra $SU(2)$ were found [1,2]. Later other integrable Wess-Zumino-Witten models, involving the Lie algebra $SU(2)$, were constructed [3–5]. The revival of the subject came after the work of Klimčík on the so-called Yang-Baxter deformation of the principal chiral model [6]. More recently Sfetsos presented a method for constructing integrable deformation of the Wess-Zumino-Witten model [7]. Various issues were treated later in the literature [8–31] and a nice account of these can be found in [32] and references within. Our interest in integrable nonlinear sigma models is motivated by their relation to string theories [33]. The hope is to find more solvable string theories and their spectrum in nontrivial backgrounds along the lines in [34–36].

In [37,38] we have given the conditions for the most general nonlinear sigma model to be integrable. These were specified in terms of the geometry and the structure of the target space manifold. A general two-dimensional nonlinear sigma model is given by the action[1]

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}[G_{ij}(\varphi) + B_{ij}(\varphi)]\partial\varphi^i\bar{\partial}\varphi^j. \qquad (1.1)$$

The invertible metric G_{ij} and the antisymmetric tensor B_{ij} are the backgrounds of the bosonic string theory. The equations of motion of this theory are

$$\bar{\partial}\partial\varphi^l + \Omega^l_{ij}\partial\varphi^i\bar{\partial}\varphi^j = 0, \qquad \Omega^k_{ij} = \Gamma^k_{ij} - H^k_{ij}, \qquad (1.2)$$

where Γ^k_{ij} and $H^k_{ij} = \frac{1}{2}G^{kl}(\partial_l B_{ij} + \partial_j B_{li} + \partial_i B_{jl})$ are, respectively, the Christoffel symbols and the torsion.

The equations of motion can be cast, for all values of the parameter μ, in the form of a zero curvature relation

$$\left[\partial + \frac{1}{1+\mu}(K_i - L_i)\partial\varphi^i, \bar{\partial} + \frac{1}{1-\mu}(K_j + L_j)\bar{\partial}\varphi^i\right] = 0 \quad (1.3)$$

if the space manifold is equipped with two sets of matrices $K_i(\varphi)$ and $L_i(\varphi)$ satisfying

$$\begin{aligned}
&\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij}K_l = 0,\\
&\partial_i L_j - \partial_j L_i + 2H^l_{ij}K_l = 0,\\
&\partial_i L_j + \partial_j L_i - 2\Gamma^l_{ij}L_l = [L_i, K_j] + [L_j, K_i],\\
&\partial_i K_j - \partial_j K_i + 2H^l_{ij}L_l = [L_i, L_j] - [K_i, K_j]. \qquad (1.4)
\end{aligned}$$

The last two equations determine the structure of the space manifold of the nonlinear sigma model. On the other hand, the first two relations indicate that the nonlinear sigma model is symmetric under a global isometry transformation [39,40] with $J = (K_i - L_i)\partial\varphi^i$ and $\bar{J} = (K_i + L_i)\bar{\partial}\varphi^i$ being the conserved currents. The zero curvature relation is then the same as the two equations $\partial\bar{J} + \bar{\partial}J = 0$ and $\partial\bar{J} - \bar{\partial}J + [J, \bar{J}] = 0$.

Although the conditions (1.4) specify the geometry of the manifold [38], their general solutions are not yet known. In this paper, we continue this program and consider simpler nonlinear sigma models. Namely, the most general integrable deformation of the Wess-Zumino-Witten (WZW) model. The conditions (1.4) are now more tractable. They are in the form of a Lie algebra valued relation which generalizes

*noureddine.mohammedi@univ-tours.fr

[1]The two-dimensional coordinates are (τ, σ) with $\partial_0 = \frac{\partial}{\partial\tau}$ and $\partial_1 = \frac{\partial}{\partial\sigma}$. In the rest of the paper, however, we will use the complex coordinates $(z = \tau + i\sigma, \bar{z} = \tau - i\sigma)$ together with $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial\bar{z}}$. Our conventions are such that the alternating tensor is $\epsilon^{z\bar{z}} = +1$.

the Yang-Baxter equation used in [6] and the integrable deformations of the principal chiral model [41]. We are able to find a solutions to this integrability condition. This leads to an integrable two-dimensional nonlinear sigma model in the form of a two-parameter family of integrable deformations of the Wess-Zumino-Witten model. Our result might be a generalization of the two-parameter integrable deformations of the WZW model found in [42]. Indeed, the two constructions coincide for a special case and we conjecture that our work contains more integrable models.

The paper is organized as follow: In the next section we give in details the steps leading to the equivalent relation to (1.4) for the case of the general Wess-Zumino-Witten model with a summary of the results at the end. For completeness, we show in Sec. III how the Yang-Baxter integrable sigma model is obtained as a particular case of our construction. In Sec. IV, we construct the solution to the integrability conditions and give, in Sec. V, the corresponding integrable nonlinear sigma models.

II. THE GENERAL CONSTRUCTION

We consider the two-dimensional nonlinear sigma model as defined by the action

$$S(g) = \int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z} \langle g^{-1}\partial g, (M+N) g^{-1}\bar{\partial} g\rangle_{\mathcal{G}} + \frac{\lambda}{6}\int_{\mathcal{M}} \mathrm{d}^3 x \epsilon^{\mu\nu\rho} \langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}}, \quad (2.1)$$

where $\mathcal{M}$ is a three-dimensional ball having x^μ, with $\mu = 1, 2, 3$, as coordinates and $\partial\mathcal{M}$ is the boundary of this ball with coordinates z and $\bar{z}$. The bilinear form $\langle , \rangle_{\mathcal{G}}$ is the Killing-Cartan form on the Lie algebra $\mathcal{G}$ and the field $g(z, \bar{z})$ is an element of the Lie group corresponding to $\mathcal{G}$. The Lie algebra is of dimension n. The Wess-Zumino-Witten term comes with a parameter λ.

The Lie algebra $\mathcal{G}$ is defined by the commutation relations $[T_a, T_b] = f^c_{ab} T_c$. For a semi-simple Lie algebra the Killing-Cartan form is $\eta_{ab} = f^d_{ac} f^c_{bd}$ and we have $\langle T_a, T_b\rangle_{\mathcal{G}} = \eta_{ab} = \mathrm{Tr}(T_a T_b)$. However, for a non semi-simple Lie algebra the bilinear form is such that $\langle T_a, T_b\rangle_{\mathcal{G}} = \eta_{ab}$ with η_{ab} an invertible matrix satisfying $\eta_{ab} f^b_{cd} + \eta_{cb} f^b_{ad} = 0$.

The two quantities M and N are linear operator acting on the generators of the Lie algebra $\mathcal{G}$. They are required to satisfy the relation

$$\langle X, (M+N)Y\rangle_{\mathcal{G}} = \langle (M-N)X, Y\rangle_{\mathcal{G}} \quad (2.2)$$

for any two elements X and Y in the Lie algebra $\mathcal{G}$. In other words, M is symmetric while N is antisymmetric with respect to $\langle , \rangle_{\mathcal{G}}$.

Putting indices, the action of M and N on the generators $\{T_a\}$ of the Lie algebra $\mathcal{G}$ is $MT_a = M^b_a T_b$ and $NT_a = N^b_a T_b$ and (2.2) is equivalent to

$$\eta_{ac} M^c_b = \eta_{bc} M^c_a, \quad \eta_{ac} N^c_b = -\eta_{bc} N^c_a, \quad (2.3)$$

where η_{ab} is the bilinear form corresponding to the Lie algebra $\mathcal{G}$ as stated above.

It is useful to introduce the two quantities

$$A = g^{-1}\partial g, \quad \bar{A} = g^{-1}\bar{\partial} g. \quad (2.4)$$

In terms of A and $\bar{A}$, the equations of motion of the model take the form

$$\partial[(M+N+\lambda I)\bar{A}] + \bar{\partial}[(M-N-\lambda I)A] + [A, (M+N)\bar{A}] + [\bar{A}, (M-N)A] = 0. \quad (2.5)$$

Multiplying this equation by g on the left and g^{-1} on the right, we get the conservation equation

$$\partial\bar{J} + \bar{\partial} J = 0, \quad (2.6)$$

where we have defined the two currents J and $\bar{J}$ as

$$J = g(P^{-1}A)g^{-1}, \quad \bar{J} = g(Q^{-1}\bar{A})g^{-1}. \quad (2.7)$$

Here the two linear operators P^{-1} and Q^{-1}, acting on A and $\bar{A}$ only, are defined as

$$P^{-1} = M - (N - \lambda I), \quad Q^{-1} = M + (N - \lambda I), \quad (2.8)$$

where I is the identity operator on the elements of the Lie algebra $\mathcal{G}$.

The conservation equation (2.6) is a result of to the global symmetry of the action (2.1) under the left multiplication

$$g \to hg, \quad (2.9)$$

where h is a constant group element.

It is, of course, assumed that the two linear operators P and Q are invertible. Hence, the inversion of (2.7) gives

$$A = P(g^{-1}Jg), \quad \bar{A} = Q(g^{-1}\bar{J}g). \quad (2.10)$$

However, the two currents A and $\bar{A}$ satisfy the Cartan-Maurer identity

$$\partial\bar{A} - \bar{\partial}A + [A, \bar{A}] = 0. \qquad (2.11)$$

In terms of the currents J and $\bar{J}$, after a use of (2.10) and (2.11), one finds the identity

$$\begin{aligned}
&\frac{1}{2}(Q-P)[g^{-1}(\partial\bar{J}+\bar{\partial}J)g] \\
&\quad+\frac{1}{2}(Q+P)[g^{-1}(\partial\bar{J}-\bar{\partial}J+\varepsilon[J,\bar{J}])g] \\
&\quad-\frac{\varepsilon}{2}(Q+P)[g^{-1}Jg, g^{-1}\bar{J}g] - Q[P(g^{-1}Jg), g^{-1}\bar{J}g] \\
&\quad+P[Q(g^{-1}\bar{J}g), g^{-1}Jg] + [P(g^{-1}Jg), Q(g^{-1}\bar{J}g)] \\
&=0. \qquad (2.12)
\end{aligned}$$

We have added and subtracted the term proportional to the constant ε. At this stage ε is just a bookkeeping device but will later join the constant λ to form one of the deformation parameters $\lambda\varepsilon$.

In order to have an identity that is suitable for the concept of integrability, we demand that the linear operators P and Q are such that the last four terms in (2.12) vanish. That is,

$$\begin{aligned}
&-\frac{\varepsilon}{2}(Q+P)[g^{-1}Jg, g^{-1}\bar{J}g] - Q[P(g^{-1}Jg), g^{-1}\bar{J}g] \\
&\quad+P[Q(g^{-1}\bar{J}g), g^{-1}Jg] + [P(g^{-1}Jg), Q(g^{-1}\bar{J}g)] = 0.
\end{aligned} \qquad (2.13)$$

Since the quantities $g^{-1}Jg$ and $g^{-1}\bar{J}g$ take values in the Lie algebra $\mathcal{G}$, this last equation is equivalent to requiring that

$$[PX, QY] - P[X, QY] - Q[PX, Y] = \frac{\varepsilon}{2}(P+Q)[X, Y] \qquad (2.14)$$

for any two Lie algebra elements X and Y. Notice that the constant ε can be absorbed by a rescaling of the two operators P and Q [which amounts to a rescaling of the two currents J and $\bar{J}$ in (2.7)].

When this last relation holds, the currents obey the identity

$$\begin{aligned}
&\frac{1}{2}(Q-P)[g^{-1}(\partial\bar{J}+\bar{\partial}J)g] \\
&\quad+\frac{1}{2}(Q+P)[g^{-1}(\partial\bar{J}-\bar{\partial}J+\varepsilon[J,\bar{J}])g] = 0. \qquad (2.15)
\end{aligned}$$

If in addition, the operator $(Q+P)$ is invertible then the two currents J and $\bar{J}$ obey the two relations

$$\begin{aligned}
&\partial\bar{J} + \bar{\partial}J = 0, \\
&\partial\bar{J} - \bar{\partial}J + \varepsilon[J, \bar{J}] = 0. \qquad (2.16)
\end{aligned}$$

Therefore, in addition of being on-shell conserved, the currents J and $\bar{J}$ have zero curvature.

These last two equations are the consistency conditions of the linear differential system

$$\begin{cases} \left(\partial + \frac{\varepsilon}{1+\mu}J\right)\Psi = 0 \\ \left(\bar{\partial} + \frac{\varepsilon}{1-\mu}\bar{J}\right)\Psi = 0 \end{cases}. \qquad (2.17)$$

Here $\Psi(z, \bar{z}, \mu)$ is a matrix valued field. The requirement that this linear differential system is consistent, for all values of the spectral parameter μ, leads to the equations of motion of the nonlinear sigma model (2.16). This is precisely the statement of the classical integrability of a two-dimensional nonlinear sigma model [43].

Finally, in terms of the linear operators P and Q, the relation (2.2) involving the bilinear form $\langle,\rangle_{\mathcal{G}}$ becomes upon using (2.8)

$$\langle X, Q^{-1}Y\rangle_{\mathcal{G}} = \langle P^{-1}X, Y\rangle_{\mathcal{G}} - 2\lambda\langle X, Y\rangle_{\mathcal{G}}. \qquad (2.18)$$

By writing $X = PZ$ and $Y = QW$, where X, Y, Z, and W are in the Lie algebra $\mathcal{G}$, this last relation becomes

$$\langle PZ, W\rangle_{\mathcal{G}} = \langle Z, QW\rangle_{\mathcal{G}} - 2\lambda\langle PZ, QW\rangle_{\mathcal{G}}. \qquad (2.19)$$

Summary: Given two linear operators P and Q (we assume that P, Q and $P+Q$ are invertible) on a Lie algebra $\mathcal{G}$ and satisfying, for any two elements X and Y in $\mathcal{G}$, the two relations

$$\langle PX, Y\rangle_{\mathcal{G}} = \langle X, QY\rangle_{\mathcal{G}} - 2\lambda\langle PX, QY\rangle_{\mathcal{G}}, \qquad (2.20)$$

$$\begin{aligned}
&[PX, QY] - P[X, QY] - Q[PX, Y] \\
&\quad= \frac{\varepsilon}{2}(P+Q)[X, Y] \qquad (2.21)
\end{aligned}$$

then the two-dimensional nonlinear sigma model defined by the action

$$\begin{aligned}
S(g) = \lambda &\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle_{\mathcal{G}} \\
&+\frac{\lambda}{6}\int_{\mathcal{M}} \mathrm{d}^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}} \\
&+\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\langle g^{-1}\partial g, Q^{-1}(g^{-1}\bar{\partial}g)\rangle_{\mathcal{G}} \qquad (2.22)
\end{aligned}$$

is classically integrable. We have used (2.8) to write $M + N = Q^{-1} + \lambda I$. The equations of motion stemming from this action are written in (2.16) in terms of two the currents J and $\bar{J}$

$$\begin{aligned}
&J = g[P^{-1}(g^{-1}\partial g)]g^{-1}, \\
&\bar{J} = g[Q^{-1}(g^{-1}\bar{\partial}g)]g^{-1} \qquad (2.23)
\end{aligned}$$

and are equivalent to the consistency conditions of the linear system (2.17).

III. THE YANG-BAXTER SIGMA MODEL

The so-called Yang-Baxter nonlinear sigma model is obtained as a special case of our construction. Indeed, let us first assume that the two linear operators are of the form

$$\begin{aligned} P &= \kappa I + \zeta R, \\ Q &= \kappa I - \zeta R, \end{aligned} \tag{3.1}$$

where R is a linear operator acting on the generators of the Lie algebra $\mathcal{G}$ and κ and $\zeta^2 = -\kappa(\kappa+\varepsilon) > 0$ are two constants. The parameters κ and ε are such ζ^2 is strictly positive. We also put the Wess-Zumino-Witten term in the action to zero. That is,

$$\lambda = 0. \tag{3.2}$$

When $\zeta^2 = -\kappa(\kappa+\varepsilon)$, the two relations in (2.20) and (2.21) become then respectively

$$\begin{aligned} \langle RX, Y\rangle_{\mathcal{G}} + \langle RY, X\rangle_{\mathcal{G}} &= 0, \\ [RX, RY] - R([RX, Y] + [X, RY]) &= [X, Y]. \end{aligned} \tag{3.3}$$

The last relation is known as the modified Yang-Baxter equation while the first equation says that the linear operator R is antisymmetric with respect to the bilinear form. A solution to these relations is given in [6,44] and is briefly recalled in the next section.

The corresponding action is obtained upon replacing Q^{-1} in (2.22) and is given by

$$\begin{aligned} S(g) &= \int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z} \langle g^{-1}\partial g, (\kappa I - \zeta R)^{-1}(g^{-1}\bar{\partial} g)\rangle_{\mathcal{G}}, \\ \zeta^2 &= -\kappa(\kappa+\varepsilon) > 0. \end{aligned} \tag{3.4}$$

This is precisely the action found in [6].

IV. CONSTRUCTING A SOLUTION

Our main concern now is to find solutions to (2.20) and (2.21). We start by recalling the commutation relations of a Lie algebra in the Cartan-Weyl basis

$$\begin{aligned} [H_i, H_j] &= 0, \quad i, j = 1 \ldots r, \\ [H_i, E_\alpha] &= \alpha_i E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= \alpha_i H_i, \\ [E_\alpha, E_\beta] &= \begin{cases} \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Sigma, \\ 0 & \text{if } \alpha+\beta \notin \Sigma. \end{cases} \end{aligned} \tag{4.1}$$

Here Σ is the set of roots.[2] The generators are normalized such that the Killing form (the bilinear form) is

$$\langle H_i, H_j\rangle = \delta_{ij}, \quad \langle H_i, E_\alpha\rangle = 0, \quad \langle E_\alpha, E_\beta\rangle = \delta_{\alpha+\beta,0}. \tag{4.2}$$

Since we will use the linear operator R, defined in (3.3), we start by giving its action on the generators of the Lie algebra in the Cartan-Weyl basis as found in [6,44]. This is

$$\begin{cases} RH_i = 0, \\ RE_\alpha = -iE_\alpha & \text{if } \alpha \in \Sigma^+, \\ RE_{-\alpha} = iE_{-\alpha} & \text{if } \alpha \in \Sigma^+, \end{cases} \tag{4.3}$$

where Σ^+ is the set of positive roots and $i^2 = -1$ (not to be confused with the index i used above). The action of the linear operator R on the generators of the Lie algebra in the basis $\{T_a\}$ is specified by

$$\begin{aligned} &RT_a = 0 \quad \text{if } T_a \in \mathcal{H}, \\ &\left.\begin{aligned} RT_a &= T_{a+1}, \\ RT_{a+1} &= -T_a, \end{aligned}\right\} \quad \text{with} \\ &E_{\alpha_a} = T_a + iT_{a+1} \text{ and such that } \alpha_a \in \Sigma^+. \end{aligned} \tag{4.4}$$

Here $\mathcal{H}$ is the Cartan subalgebra of the Lie algebra $\mathcal{G}$.

It is instructive to illustrate the action of the linear operator R on the generators of the Lie algebra $SU(3)$. The generalization to other Lie algebras can be figured out in a similar manner. The $SU(3)$ Cartan-Weyl basis is constituted as

$$\begin{aligned} &E_{\pm\alpha_{(1)}} = T_1 \pm iT_2, \qquad E_{\pm\alpha_{(2)}} = T_4 \pm iT_5, \\ &E_{\pm\alpha_{(3)}} = T_6 \pm iT_7, \qquad H_1 = T_3, \qquad H_2 = T_8. \end{aligned} \tag{4.5}$$

Using (4.4), one finds that the operator R acts on the $SU(3)$ generators $\{T_a\}$ as

$$R\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix}. \tag{4.6}$$

It is then clear that the matrix R^2 is diagonal with entries equal to either -1 or 0 (zero corresponds to the action of R^2

[2] We use the conventions and notations of Ref. [45].

on the elements of the Cartan subalgebra). The operator R^2 will be needed later.

Let us now return to the linear operators P and Q. We assume that they act on the generators of the Lie algebra in the Cartan-Weyl basis as

$$\begin{cases} PH_i = \sigma_i H_i, & \\ PE_\alpha = pE_\alpha & \text{if } \alpha \in \Sigma^+, \\ PE_{-\alpha} = p^* E_{-\alpha} & \text{if } \alpha \in \Sigma^+, \end{cases}$$
$$\begin{cases} QH_i = \xi_i H_i, & \\ QE_\alpha = qE_\alpha & \text{if } \alpha \in \Sigma^+, \\ QE_{-\alpha} = q^* E_{-\alpha} & \text{if } \alpha \in \Sigma^+, \end{cases} \tag{4.7}$$

where no summation over the repeated index i is implied. The constants σ_i and ξ_i are real while p and q are complex. In the basis $(H_i, E_\alpha, E_{-\alpha})$, the matrices associated to the operators P and Q are diagonal.

Using the commutation relations (4.1), the Killing form (4.2) and the action of the linear operators as in (4.7), the relations (2.20) and (2.21) are satisfied if

$$-pq = \frac{\varepsilon}{2}(p+q), \tag{4.8}$$

$$pq^* - q^*\sigma_i - p\xi_i = \frac{\varepsilon}{2}(\sigma_i + \xi_i), \tag{4.9}$$

$$\sigma_i = \xi_i - 2\lambda\sigma_i\xi_i, \tag{4.10}$$

$$p = q^* - 2\lambda p q^*. \tag{4.11}$$

The last two equations give simply σ_i in terms of ξ_i and p in terms of q

$$\sigma_i = \frac{\xi_i}{1+2\lambda\xi_i}, \tag{4.12}$$

$$p = \frac{q^*}{1+2\lambda q^*}. \tag{4.13}$$

Upon reporting (4.12) and (4.13) in (4.9) and (4.8) one finds

$$q^* = \tau_j \pm i\sqrt{-\tau_j\left(\tau_j + \frac{\varepsilon}{1+\lambda\varepsilon}\right)}, \quad j = 1\ldots r,$$
$$(1+\lambda\varepsilon)q^*q + \frac{\varepsilon}{2}(q^*+q) = 0. \tag{4.14}$$

Here $i^2 = -1$ and τ_j is defined as

$$\tau_j = \xi_j(1+\lambda\xi_j)(1+\lambda\varepsilon), \quad j = 1\ldots r. \tag{4.15}$$

We have therefore determined q in terms of ξ_j. The two equations in (4.14) are always compatible. Next, the parameter p is calculated from (4.13).

Now two paramaters ξ_i and ξ_j, say, must lead to the same value of q according to (4.14). This means that we must have also

$$\xi_i(1+\lambda\xi_i)(1+\lambda\varepsilon) = \xi_j(1+\lambda\xi_j)(1+\lambda\varepsilon),$$
$$i,j = 1\ldots r. \tag{4.16}$$

Therefore, the parameters ξ_i are such that

$$\xi_i = \xi_j \quad \text{or} \quad \xi_i = -\frac{1+\lambda\xi_j}{\lambda}, \quad i,j = 1\ldots r. \tag{4.17}$$

This means that they fall into two sets $\{\xi_1, \ldots, \xi_{r-l}\}$ and $\{\xi_{r-l-1}, \ldots, \xi_r\}$, $0 \le l \le r$, and the members of a set are identical. A set could be empty (if $l = 0$). The corresponding expressions for the parameters σ_i are found from (4.12).

These two choices for the constants ξ_i suggests the splitting of the Cartan subalgebra of $\mathcal{G}$ as

$$\mathcal{H} = \mathcal{H}_{r-l} \cup \mathcal{H}_l, \quad 0 \le l \le r, \tag{4.18}$$

where $\mathcal{H}_{r-l}$ contains the first $r-l$ elements of $\mathcal{H}$ and $\mathcal{H}_l$ the remaining l elements ($0 \le l \le r$).

We have now all the ingredients to put forward the full solution to the Eqs. (4.8)–(4.11). This is given by

$$\begin{aligned}
&\xi_1 = \xi_2 = \cdots = \xi_{r-l} = \xi, \qquad \xi_{r-l+1} = \xi_{r-l+2} = \cdots = \xi_r = -\frac{1+\lambda\xi}{\lambda}, \\
&\sigma_1 = \sigma_2 = \cdots = \sigma_{r-l} = \frac{\xi}{1+2\lambda\xi}, \qquad \sigma_{r-l+1} = \sigma_{r-l+2} = \cdots = \sigma_r = \frac{1+\lambda\xi}{\lambda(1+2\lambda\xi)}, \\
&q = \tau \mp i\omega, \\
&p = \tau' \pm i\omega',
\end{aligned} \tag{4.19}$$

where ξ is a free parameter. The constants τ, ω, τ', and ω' are given by

$$\begin{aligned} \tau &= \xi(1+\lambda\xi)(1+\lambda\varepsilon), \\ \omega &= \sqrt{-\tau\left(\tau+\frac{\varepsilon}{1+\lambda\varepsilon}\right)}, \\ \tau' &= \frac{\xi(1+\lambda\xi)(1-\lambda\varepsilon)}{(1+2\lambda\xi)^2}, \\ \omega' &= \frac{1}{(1+2\lambda\xi)^2}\sqrt{-\tau\left(\tau+\frac{\varepsilon}{1+\lambda\varepsilon}\right)}. \end{aligned} \tag{4.20}$$

The only restriction on the free parameter ξ is that the argument of the square root in the expression of ω is positive or zero. This is equivalent to demanding that

$$\tau = \xi(1+\lambda\xi)(1+\lambda\varepsilon) \in \left[-\frac{\varepsilon}{1+\lambda\varepsilon}, 0\right]. \tag{4.21}$$

The domain of parameters is therefore quite vast.

V. THE INTEGRABLE NONLINEAR SIGMA MODEL

The linear operators P and Q acting on the basis $\{T_a\}$ of the Lie algebra $\mathcal{G}$ are deduced from (4.7) and the solution (4.19). It might be helpful to work out their action on the Lie algebra $SU(3)$ first. For instance, $QE_{\alpha_{(1)}} = qE_{\alpha_{(1)}} = (\tau \mp i\omega)E_{\alpha_{(1)}}$ implies that $QT_1 = \tau T_1 \pm \omega T_2$ and $QT_2 = \tau T_2 \mp \omega T_1$, and so on. If we partition the $SU(3)$ Cartan subalgebra as $\mathcal{H} = \mathcal{H}_{r-l} \cup \mathcal{H}_l = T_3 \cup T_8$ then we have

$$Q\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} = \begin{pmatrix} \tau & \pm\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ \mp\omega & \tau & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau & \pm\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\omega & \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau & \pm\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \mp\omega & \tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1+\lambda\xi}{\lambda} \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix}. \tag{5.1}$$

The matrix corresponding to the operator P can be determined in a similar manner.

For the sake of condensing the expressions, we introduce the notation

$$\begin{aligned} \gamma &= \xi-\tau = -\lambda\xi[\varepsilon+\xi(1+\lambda\varepsilon)], \\ \rho &= -\frac{1+\lambda\xi}{\lambda}-\tau = -\frac{(1+\lambda\xi)}{\lambda}[1+\lambda\xi(1+\lambda\varepsilon)], \\ \gamma' &= \frac{\xi}{1+2\lambda\xi}-\tau' = \frac{\lambda\xi}{(1+2\lambda\xi)^2}[\varepsilon+\xi(1+\lambda\varepsilon)], \\ \rho' &= \frac{1+\lambda\xi}{\lambda(1+2\lambda\xi)}-\tau' = \frac{(1+\lambda\xi)}{\lambda(1+2\lambda\xi)^2}[1+\lambda\xi(1+\lambda\varepsilon)]. \end{aligned} \tag{5.2}$$

The operators P and Q are given by

$$\begin{aligned} P &= \tau' I \mp \omega' R + \gamma' \mathcal{Z}_{r-l} + \rho' \mathcal{Z}_l, \\ Q &= \tau I \pm \omega R + \gamma \mathcal{Z}_{r-l} + \rho \mathcal{Z}_l. \end{aligned} \tag{5.3}$$

The linear operator R is still that in (4.4), I is the identity operator and the action of the linear operators $\mathcal{Z}_{r-l}$ and $\mathcal{Z}_l$ on the basis $\{T_a\}$ is

$$\begin{cases} \mathcal{Z}_{r-l}T_a = T_a & \text{only if } T_a \in \mathcal{H}_{r-l}, 0 \le l \le r, \\ \mathcal{Z}_l T_a = T_a & \text{only if } T_a \in \mathcal{H}_l, 0 \le l \le r, \\ \mathcal{Z}_{r-l}T_a = \mathcal{Z}_l T_a = 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

The operators $\mathcal{Z}_{r-l}$ and $\mathcal{Z}_l$ act only on the elements of the Cartan subalgebra $\mathcal{H} = \mathcal{H}_{r-l} \cup \mathcal{H}_l$ with $0 \le l \le r$.

The next step in our construction is the computation of the inverses of the two operators P and Q. These are block diagonal matrices having either 2×2 or 1×1 matrices along the diagonal and are easily inverted. Indeed, we have

$$\begin{aligned} P^{-1} &= \frac{\tau'}{\tau'^2+\omega'^2} I \pm \frac{\omega'}{\tau'^2+\omega'^2} R + \left(\frac{1}{\tau'+\gamma'} - \frac{\tau'}{\tau'^2+\omega'^2}\right)\mathcal{Z}_{r-l} \\ &\quad + \left(\frac{1}{\tau'+\rho'} - \frac{\tau'}{\tau'^2+\omega'^2}\right)\mathcal{Z}_l, \\ Q^{-1} &= \frac{\tau}{\tau^2+\omega^2} I \mp \frac{\omega}{\tau^2+\omega^2} R + \left(\frac{1}{\tau+\gamma} - \frac{\tau}{\tau^2+\omega^2}\right)\mathcal{Z}_{r-l} \\ &\quad + \left(\frac{1}{\tau+\rho} - \frac{\tau}{\tau^2+\omega^2}\right)\mathcal{Z}_l. \end{aligned} \tag{5.5}$$

Explicitly, these expressions give

$$P^{-1} = -\frac{1}{\varepsilon}[(1-\lambda\varepsilon)I \pm \sqrt{-\alpha\beta}R + \alpha\mathcal{Z}_{r-l} + \beta\mathcal{Z}_l],$$
$$Q^{-1} = -\frac{1}{\varepsilon}[(1+\lambda\varepsilon)I \mp \sqrt{-\alpha\beta}R + \alpha\mathcal{Z}_{r-l} + \beta\mathcal{Z}_l]. \quad (5.6)$$

The two constants α and β are defined as

$$\alpha = -\frac{1}{\xi}[\varepsilon + \xi(1+\lambda\varepsilon)], \quad \beta = -\frac{[1+\lambda\xi(1+\lambda\varepsilon)]}{(1+\lambda\xi)}. \quad (5.7)$$

By eliminating the parameter ξ between α and β, we find that

$$\beta = -\left[1 - \frac{(\lambda\varepsilon)^2}{1+\alpha}\right]. \quad (5.8)$$

In terms of the parameters α and β, the operators P and Q are as given in (5.3) where

$$\tau = -\frac{\varepsilon(1+\alpha)(1+\lambda\varepsilon)}{(1+\alpha+\lambda\varepsilon)^2}, \quad \tau' = -\frac{\varepsilon(1+\alpha)(1-\lambda\varepsilon)}{(1+\alpha-\lambda\varepsilon)^2},$$
$$\omega = \frac{\varepsilon(1+\alpha)\sqrt{-\alpha\beta}}{(1+\alpha+\lambda\varepsilon)^2}, \quad \omega' = \frac{\varepsilon(1+\alpha)\sqrt{-\alpha\beta}}{(1+\alpha-\lambda\varepsilon)^2},$$
$$\gamma = \frac{\lambda\varepsilon^2\alpha}{(1+\alpha+\lambda\varepsilon)^2}, \quad \gamma' = -\frac{\lambda\varepsilon^2\alpha}{(1+\alpha-\lambda\varepsilon)^2},$$
$$\rho = -\frac{(1+\alpha)[1+\alpha-(\lambda\varepsilon)^2]}{\lambda(1+\alpha+\lambda\varepsilon)^2}, \quad \rho' = \frac{(1+\alpha)[1+\alpha-(\lambda\varepsilon)^2]}{\lambda(1+\alpha-\lambda\varepsilon)^2}. \quad (5.9)$$

We notice that the parameters $(\tau', \omega', \gamma', \rho')$ are obtained from $(\tau, \omega, \gamma, \rho)$ by the change $\lambda \to -\lambda$.

There is another way of writing the operators P^{-1} and Q^{-1}. Let $\mathcal{Z}_r$ be the operator that acts as

$$\begin{cases} \mathcal{Z}_r T_a = T_a & \text{only if } T_a \in \mathcal{H}, \\ \mathcal{Z}_r T_a = 0 & \text{otherwise.} \end{cases} \quad (5.10)$$

That is, $\mathcal{Z}_r$ acts on all the generator in the Cartan subalgebra $\mathcal{H}$. It satisfies the relation

$$\mathcal{Z}_r = I + R^2. \quad (5.11)$$

Furthermore, it can be seen that

$$\mathcal{Z}_{r-l} = \mathcal{Z}_r - \mathcal{Z}_l = (I + R^2) - \mathcal{Z}_l. \quad (5.12)$$

Using this last relation, we can write the operators P^{-1} and Q^{-1} in the form

$$P^{-1} = -\frac{1}{\varepsilon}[(1-\lambda\varepsilon+\alpha)I \pm \sqrt{-\alpha\beta}R + \alpha R^2 + (\beta-\alpha)\mathcal{Z}_l],$$
$$Q^{-1} = -\frac{1}{\varepsilon}[(1+\lambda\varepsilon+\alpha)I \mp \sqrt{-\alpha\beta}R + \alpha R^2 + (\beta-\alpha)\mathcal{Z}_l]. \quad (5.13)$$

A word of caution is necessary here. The operators P^{-1} and Q^{-1} are not invertible if either $(1-\lambda\varepsilon+\alpha) = -(\beta-\alpha)$ or $(1+\lambda\varepsilon+\alpha) = -(\beta-\alpha)$. In this case the operators $(1-\lambda\varepsilon+\alpha)I + \alpha R^2 + (\beta-\alpha)\mathcal{Z}_l$ or $(1+\lambda\varepsilon+\alpha)I + \alpha R^2 + (\beta-\alpha)\mathcal{Z}_l$ will have zeros as entries along the diagonal whenever acting on the generators in $\mathcal{H}_l$. The expression of β in (5.7) gives $(1+2\lambda\xi) = 0$ and $\lambda = 0$ as solutions to $(1-\lambda\varepsilon) = -\beta$ and $(1+\lambda\varepsilon) = -\beta$. These are precisely the two situations which are not allowed as can be seen from the solution (4.19).

Using the expression of Q^{-1} in (5.13), our action (2.22) takes then the form

$$S_l(g) = -\frac{1}{\varepsilon}\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\langle g^{-1}\partial g, [(1+\alpha)I \mp \sqrt{-\alpha\beta}R + \alpha R^2$$
$$+ (\beta-\alpha)\mathcal{Z}_l](g^{-1}\bar{\partial}g)\rangle_{\mathcal{G}}$$
$$+ \frac{\lambda}{6}\int_{\mathcal{M}} \mathrm{d}^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}}. \quad (5.14)$$

The parameters α and β are related by (5.8) and $\lambda\varepsilon$ is another free parameter ($\frac{1}{\varepsilon}$ is an overall factor). This is the main result of this paper. The above two dimensional nonlinear sigma model is integrable. The two current $J = g[P^{-1}(g^{-1}\partial g)]g^{-1}$ and $\bar{J} = g[Q^{-1}(g^{-1}\bar{\partial}g)]g^{-1}$, with P^{-1} and Q^{-1} as given in (5.13), are conserved and have a vanishing curvature on-shell.

At this stage a remark is due: In the case when $l = 0$, that is when the set $\mathcal{H}_l = \mathcal{H}_0$ is an empty set (consequently $\mathcal{Z}_0 T_a = 0$ for all T_a in the Lie algebra $\mathcal{G}$), the action $S_0(g)$ is precisely that constructed in ref. [42]. Their parameters, in this case, are related to ours as

$$\eta^2 = \alpha, \quad A = \mp\sqrt{-\alpha\beta}, \quad k^2 = (\lambda\varepsilon)^2, \quad K = \frac{1}{\varepsilon}. \quad (5.15)$$

With this identification, their relation $A = \eta\sqrt{1 - \frac{k^2}{1+\eta^2}}$ is exactly that written in (5.8).

In order to explore the novelty of our construction, we find it convenient to rewrite our final action as

$$
\begin{aligned}
S_l(g) = &-\frac{1}{\varepsilon}\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\langle g^{-1}\partial g, [I+\alpha\mathcal{Z}_{r-l}+\beta\mathcal{Z}_l \mp \sqrt{-\alpha\beta}R](g^{-1}\bar{\partial}g)\rangle_{\mathcal{G}} \\
&+\frac{\lambda}{6}\int_{\mathcal{M}} \mathrm{d}^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}}.
\end{aligned} \tag{5.16}
$$

In reaching this simplified version we have made use of (5.11) and (5.12) and the action of the linear operators $\mathcal{Z}_l$ and $\mathcal{Z}_{r-l}$ is as defined in (5.4). The Cartan subalgebra is split as $\mathcal{H} = \mathcal{H}_{r-l} \cup \mathcal{H}_l$ with $0 \le l \le r$ and $\mathcal{H}_0$ is the empty set.

As mentioned above, the case $l = 0$ is already treated in Ref. [42] and their integrable nonlinear sigma model is given by the action

$$
\begin{aligned}
S_0(g) = &-\frac{1}{\varepsilon}\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\langle g^{-1}\partial g, [I+\alpha\mathcal{Z}_{r} \mp \sqrt{-\alpha\beta}R](g^{-1}\bar{\partial}g)\rangle_{\mathcal{G}} \\
&+\frac{\lambda}{6}\int_{\mathcal{M}} \mathrm{d}^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}}.
\end{aligned} \tag{5.17}
$$

The linear operator $\mathcal{Z}_r = I + R^2$, given in (5.10), acts on all the generators in the Cartan subalgebra $\mathcal{H}$.

In the next section we will point out, by considering specific examples, that the action $S_l(g)$, for Lie algebras with rank $r \ge 2$, contains deformations of the Wess-Zumino-Witten model that are not accounted for by the action $S_0(g)$ (the nonlinear sigma model of Ref. [42]). Hence, this article is a generalization of the work of Ref. [42].

VI. THE DEFORMED $SU(2)$ WZW MODEL AND BEYOND

It is instructive to illustrate our construction by first considering the Lie algebras $SU(2)$. For this purpose, let us call

$$
\mathcal{D}_l = \alpha\mathcal{Z}_{r-l}+\beta\mathcal{Z}_l \mp \sqrt{-\alpha\beta}R \tag{6.1}
$$

the deformation operator. We will also consider α and β as our free parameters instead of α and $\lambda\varepsilon$. In terms of α and β, (5.8) gives

$$
(\lambda\varepsilon)^2 = (1+\alpha)(1+\beta). \tag{6.2}
$$

The deformed WZW action (5.16) is then written as

$$
\begin{aligned}
S_l(g) = &-\frac{1}{\varepsilon}\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\langle g^{-1}\partial g, [I+\mathcal{D}_l](g^{-1}\bar{\partial}g)\rangle_{\mathcal{G}} \\
&+\frac{1}{6}\frac{1}{\varepsilon}\sqrt{(1+\alpha)(1+\beta)}\int_{\mathcal{M}} \mathrm{d}^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}}.
\end{aligned} \tag{6.3}
$$

Notice that the coefficient of the WZW term is symmetric under the exchange $\alpha \leftrightarrow \beta$.

In the case of the $SU(2)$ Lie algebra, with generators $\{T_1, T_2, T_3\}$ and $\mathcal{H} = \{T_3\}$, there are two deformation operators and their action is given by

$$
\begin{aligned}
\mathcal{D}_0\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} &= \begin{pmatrix} 0 & \mp\sqrt{-\alpha\beta} & 0 \\ \pm\sqrt{-\alpha\beta} & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}. \\
\mathcal{D}_1\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} &= \begin{pmatrix} 0 & \mp\sqrt{-\alpha\beta} & 0 \\ \pm\sqrt{-\alpha\beta} & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}.
\end{aligned} \tag{6.4}
$$

These differ by their action on the generator T_3. However, by the parameter redefinition $\alpha \leftrightarrow \beta$, the deformation operators $\mathcal{D}_0$ and $\mathcal{D}_1$ are mapped to each other[3] and, therefore, lead to the same integrable nonlinear sigma model.

[3]I thank an anonymous referee for this remark.

Despite the fact that we have established that the deformations operator $\mathcal{D}_0$ and $\mathcal{D}_1$ are the same (up to a parameter redefinition), we will for completeness give the action for the deformed $SU(2)$ WZW model. The $SU(2)$ group element g is parametrized as

$$g = \begin{pmatrix} \cos(\varphi_1)e^{-i\varphi_2} & -\sin(\varphi_1)e^{-i\varphi_3} \\ \sin(\varphi_1)e^{i\varphi_3} & \cos(\varphi_1)e^{i\varphi_2} \end{pmatrix}. \tag{6.5}$$

For the bilinear form we take $\langle , \rangle = \mathrm{Tr}$. The nonlinear sigma model corresponding to the deformation operator $\mathcal{D}_0$ is given, up to a total derivative, by the action

$$\begin{aligned} S_0 = \frac{2}{\varepsilon}\int_{\partial\mathcal{M}} \mathrm{d}z\mathrm{d}\bar{z}\{&\partial\varphi_1\bar{\partial}\varphi_1 + [1+\alpha\cos^2(\varphi_1)]\cos^2(\varphi_1)\partial\varphi_2\bar{\partial}\varphi_2 \\ &+ [1+\alpha\sin^2(\varphi_1)]\sin^2(\varphi_1)\partial\varphi_3\bar{\partial}\varphi_3 - \alpha\cos^2(\varphi_1)\sin^2(\varphi_1)(\partial\varphi_2\bar{\partial}\varphi_3 + \partial\varphi_3\bar{\partial}\varphi_2) \\ &- 2\sqrt{(1+\alpha)(1+\beta)}\sin^2(\varphi_2-\varphi_3)\cos^2(\varphi_1)(\partial\varphi_2\bar{\partial}\varphi_3 - \partial\varphi_3\bar{\partial}\varphi_2)\}. \end{aligned} \tag{6.6}$$

The deformation operator $\mathcal{D}_1$ yields the same action with the replacement $\alpha \to \beta$.

Next, we consider the Lie algebra $SU(3)$. Its Cartan subalgebra is $\mathcal{H} = \{T_3, T_8\}$. The three deformation operators are 6

$$\begin{aligned} \mathcal{D}_0\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} &= \begin{pmatrix} 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix}, \\ \mathcal{D}_1\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} &= \begin{pmatrix} 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix}, \\ \mathcal{D}_2\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} &= \begin{pmatrix} 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix}, \end{aligned} \tag{6.7}$$

where $A = \mp\sqrt{-\alpha\beta}$ as in the dictionary (5.15). We see that $\mathcal{D}_2$ and $\mathcal{D}_0$ are related by the parameter redefinition $\alpha \leftrightarrow \beta$. However, $\mathcal{D}_1$ and $\mathcal{D}_0$ cannot be related by any parameter redefinition. It seems, therefore, that there are two independent deformations of the $SU(3)$ WZW model, namely $S_0(g)$ and $S_1(g)$. This remains though to be verified by an explicit calculation.

In general, one may decompose the Maurer-Cartan one-form along the Cartan-Weyl basis (4.1) as

$$g^{-1}dg = [e_a^{\gamma}E_{\gamma} + e_a^{-\gamma}E_{-\gamma} + e_a^{i_{(r-l)}}H_{i_{(r-l)}} + e_a^{i_{(l)}}H_{i_{(l)}}]d\varphi^a. \quad (6.8)$$

Here $\varphi^a(z,\bar{z})$ are the n local fields and the index γ runs over the positive roots Σ^+. The Cartan subalgebra is partitioned as $\mathcal{H} = \mathcal{H}_{r-l} \cup \mathcal{H}_l$ with $0 \le l \le r$. The indices $i_{(r-l)} = 0, \ldots, l$ and $i_{(l)} = l+1, \ldots, r$ are such that $H_{i_{(r-l)}} \in \mathcal{H}_{r-l}$ and $H_{i_{(l)}} \in \mathcal{H}_l$. The vielbiens are functions of $\varphi^a(z,\bar{z})$ and $e_a^{i_{(0)}} = 0$.

Using the bilinear form $\langle,\rangle$ as given in (4.2) and the action of the operator R in (4.3) together with the action of the operators $\mathcal{Z}_{r-l}$ and $\mathcal{Z}_l$ as defined in (5.4), we find that

$$\begin{aligned} S_l(g) = &-\frac{1}{\varepsilon}\int_{\partial\mathcal{M}} dz d\bar{z}[e_a^{\gamma}e_b^{-\gamma} + e_a^{-\gamma}e_b^{\gamma} + (1+\alpha)e_a^{i_{(r-l)}}e_b^{i_{(r-l)}} \\ &+ (1+\beta)e_a^{i_{(l)}}e_b^{i_{(l)}}]\partial\varphi^a\bar{\partial}\varphi^b \\ &\pm i\frac{\sqrt{-\alpha\beta}}{\varepsilon}\int_{\partial\mathcal{M}} dz d\bar{z}[e_a^{\gamma}e_b^{-\gamma} - e_a^{-\gamma}e_b^{\gamma}]\partial\varphi^a\bar{\partial}\varphi^b \\ &+ \frac{1}{6}\frac{1}{\varepsilon}\sqrt{(1+\alpha)(1+\beta)} \\ &\times \int_{\mathcal{M}} d^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}}. \end{aligned} \quad (6.9)$$

We see that the nonlinear sigma models defined by $S_l(g)$, $l = 0\ldots r$, share the same antisymmetric tensor field (coming from the last two terms) but differ by their target space metric (coming from the first term). It is clear that the two models $S_0(g)$ and $S_r(g)$ are related by the parameter redefinition $\alpha \leftrightarrow \beta$. Apart from this, we are inclined to conjecture that there are r different integrable models given by $S_l(g)$, $l = 0\ldots r-1$.

VII. CONCLUSIONS AND OUTLOOK

We have presented in this work an integrable two-dimensional nonlinear sigma model. It is a two-parameter deformation of the Wess-Zumino-Witten model. We have found a simple solution to the main integrability equations (2.20) and (2.21) of this article. It remains to see if these relations admit other solutions. The renormalizability of the sigma model studied here and its possible connection to string theories is another interesting subject to be explored.

There is a strong link between integrability and gauging as shown in [7,46]. This property is not very neat here. Indeed, the general WZW model (2.22) is related to another theory as follows: The nonlinear sigma model as defined by the action

$$\begin{aligned} S(g,h) = \lambda &\int_{\partial\mathcal{M}} dz d\bar{z}\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle_{\mathcal{G}} \\ &+ \frac{\lambda}{6}\int_{\mathcal{M}} d^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}} \\ &- \int_{\partial\mathcal{M}} dz d\bar{z}\langle h^{-1}\partial h, Q(h^{-1}\bar{\partial}h)\rangle_{\mathcal{G}} \\ &+ \int_{\partial\mathcal{M}} dz d\bar{z}(\langle h^{-1}\partial h, g^{-1}\bar{\partial}g\rangle_{\mathcal{G}} + \langle h^{-1}\bar{\partial}h, g^{-1}\partial g\rangle_{\mathcal{G}}) \end{aligned} \quad (7.1)$$

is invariant under the constant left multiplication $h \to lh$. This can be gauged by introducing a two components gauge field B_μ, with $\mu = z, \bar{z}$, transforming as $B_\mu \to lB_\mu l^{-1} - \partial_\mu l l^{-1}$. The gauging is carried out by replacing $h^{-1}\partial_\mu h$ with $h^{-1}(\partial_\mu + B_\mu)h$. The choice of the gauge $h = 1$ leads then, after the use of (2.18), to the action

$$\begin{aligned} S(g,B_\mu) = \lambda &\int_{\partial\mathcal{M}} dz d\bar{z}\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle_{\mathcal{G}} \\ &+ \frac{\lambda}{6}\int_{\mathcal{M}} d^3x\epsilon^{\mu\nu\rho}\langle g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\rho g]\rangle_{\mathcal{G}} \\ &+ \int_{\partial\mathcal{M}} dz d\bar{z}\langle g^{-1}\partial g, Q^{-1}(g^{-1}\bar{\partial}g)\rangle_{\mathcal{G}} \\ &- \int_{\partial\mathcal{M}} dz d\bar{z}\langle B - (P^{-1} - 2\lambda I)(g^{-1}\partial g), \\ &\times Q[\bar{B} - Q^{-1}(g^{-1}\bar{\partial}g)]\rangle_{\mathcal{G}}. \end{aligned} \quad (7.2)$$

The equations of motion of the nondynamical fields B and $\bar{B}$ are $B = (P^{-1} - 2\lambda I)(g^{-1}\partial g)$ and $\bar{B} = Q^{-1}(g^{-1}\bar{\partial}g)$. Substituting these into (7.2) we recover our general WZW action (2.22).

Now, the equations of motion corresponding to the original action (7.1) are

$$\begin{aligned} \partial[g\bar{a}g^{-1}] + \bar{\partial}[g(a + 2\lambda A)g^{-1}] = 0, \\ \partial[h(Q\bar{a} - \bar{A})h^{-1}] + \bar{\partial}[h((P^{-1} - 2\lambda I)^{-1}a - A)h^{-1}] = 0, \end{aligned} \quad (7.3)$$

where $a = h^{-1}\partial h$ and $\bar{a} = h^{-1}\bar{\partial}h$ and A and $\bar{A}$ are as defined in (2.4). The operator $(P^{-1} - 2\lambda I)$ is obtained from the expression of P^{-1} by simply changing λ to $-\lambda$ as can be seen from (2.8). These equations of motion, assuming that P and Q obey (2.20) and (2.21), do not seem to derive from some zero curvature conditions. Yet, the gauge fixed action

(7.2) leads to integrable nonlinear sigma model. This issue deserves to be investigated. As a matter of fact, this remark is true for most of the integrable sigma models found in the literature.

Note added.—After the completion of this work we became aware of the existence of Ref. [47] where (2.21) was also established. Their construction makes the formulations in [42,48] more compact and is inspired by the works of Klimčík [49–51]. Their assumption on the antisymmetric operator R is that it solves the homogeneous or inhomogeneous classical Yang-Baxter equation. In the case of the usual Drinfel'd-Jimbo solution, the R matrix satisfies the important relation $R^3 = -R$. They showed, in this particular case, that their integrable nonlinear sigma model is precisely that found in [42] (see their Sec. 3.2). Since our R matrix obeys also $R^3 = -R$, we conjecture that our models with $l = 1, \ldots, r-1$ are not covered by the construction of Ref. [47].

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