Wick rotation for spin foam quantum gravity

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Models of spin foam quantum gravity come in different spacetime signatures, either Euclidean or Lorentzian. The choice is reflected in the use of $Spin(4)$ or $SL(2,\mathbb{C})$ as gauge groups for the holonomies of the connection. In this work we show that a rotation of the Immirzi parameter to purely imaginary values maps the Euclidean Engle-Pereira-Rovelli-Livine spin foam model to its Lorentzian version and vice versa. Our methods provide a general recipe for relating spin foam models of different signature through analytic continuation of the gauge groups and their unitary irreducible representations.

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I. INTRODUCTION

The importance of working with theories defined on Euclidean spacetime in modern physics can hardly be overstated. In quantum field theory, one can recover the Wightman distributions from the analytic continuation of Schwinger functions in imaginary time. The Feynman path integral quantization is much more under precise mathematical control when working with Euclidean signature, as oscillatory integrals turn into Gaussian integrals. This opens the door to, among others, analytical methods from statistical mechanics and numerical methods such as Monte Carlo integration that are routinely employed for gauge theories on lattices. In gravitation, the Hartle-Hawking or Euclidean vacuum provides a route to understand the thermodynamical properties of spacetime and the initial conditions of the universe.

Yet one wants, in the end, to recover physical laws in Lorentzian spacetime. In quantum field theories, where the background metric is constant and flat, the Wick rotation is the prescription for switching signature by "rotating" the physical time variable t to a purely imaginary time variable $\tau = it$. This is not limited to flat spacetimes, as for example the same rotation works for extending the Schwarzschild solution to its Euclidean sector. It is not clear, however, how to generalize this procedure to nonflat or nonstationary metrics.

When working with generally curved spacetime, the choice of signature is enforced from the start. The Ashtekar formulation [1] of general relativity as a gauge theory of spacetime connections comes in two versions, Euclidean and Lorentzian. This formulation is the starting point for the quantization program of loop quantum gravity (LQG), which then necessarily picks one of the two signatures. This is especially the case for spin foam models [2], which define the dynamics of $SU(2)$ spin networks by regularizing the gravitational path integral over simplicial complexes. Spin foam models come in two versions, Euclidean and Lorentzian, which differ by the gauge groups of the holonomies on the spin foam edges. Euclidean models are defined with the compact group $Spin(4)$ while the noncompact group $SL(2,\mathbb{C})$ is used in Lorentzian ones. This is true also for the Engle-Pereira-Rovelli-Livine (EPRL) model [3,4], which is currently the standard model employed in calculations. Most of the results in the literature have been obtained in the Euclidean model, with a few exceptions [5–7]. The calculations in the Euclidean version are generally much simpler, but it is not clear if and how the results can be applied to the Lorentzian version.

In this work we show that the Euclidean and Lorentzian EPRL models are related through the "rotation" of the Immirzi parameter, a fundamental constant of LQG quantization, to purely imaginary values. To do so, we use two analog decompositions of the gauge groups $Spin(4)$ and $SL(2,\mathbb{C})$ to rotate their "boost" parts in their complexification. Our methods can be used to build a relation between these two groups that are fundamental to quantum gravity theories. In the spin foam setting, this relation provides an analytic continuation between Euclidean and Lorentzian models in close analogy with the analytic continuation in complex time of n-point correlation functions of ordinary quantum field theories.

The paper is organized as follows. We start by defining the decomposition of the two gauge groups using similar

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notation to highlight the analogies. Then we show how to relate their algebras, group elements and unitary irreducible representation through analytic continuation. After a brief presentation of the EPRL model, we prove our result about the Euclidean and Lorentzian vertex amplitudes. We conclude discussing the applications of our findings to quantum gravity and spin foam theory.

II. MATHEMATICAL PRELIMINARIES

We start by discussing how to decompose the gauge groups to isolate a one-dimensional subgroup of boost transformations.

A. Decomposition of $SL(2,\mathbb{C})$

The algebra of $SL(2,\mathbb{C})$ is generated by $\vec{L}$, the generators of the spatial rotation subgroup, and $\vec{K}$, the generators of the corresponding boosts. The two Casimir operators are $K^2 - L^2$ and $\vec{K}\cdot\vec{L}$. The unitary irreducible representations in the principal series are labeled by (ρ, k) where ρ is a real number and k is a half-integer [8]. The Casimirs in this representation take the values

$$\begin{aligned}(K^2 - L^2)|\rho,k\rangle &= (\rho^2 - k^2 + 1)|\rho,k\rangle,\\ \vec{K}\cdot\vec{L}|\rho,k\rangle &= \rho k|\rho,k\rangle.\end{aligned} \tag{1}$$

Since the representation $(-\rho,-k)$ is unitarily equivalent to the representation (ρ,k) we restrict ourselves to only positive values of ρ and k. The group $SL(2,\mathbb{C})$ is noncompact so the generic unitary representation (ρ,k) is infinite dimensional. The canonical basis for the representation (ρ,k), given by $|\rho,k;jm\rangle$ with $j\geq k$ and $m=-j,\ldots,j$, diagonalizes L^2 and L_3. The Cartan decomposition of the group $SL(2,\mathbb{C})$ is

$$\begin{aligned}SU(2)\times A^+ \times SU(2) &\to SL(2,\mathbb{C})\\ (u, e^{\frac{r}{2}\sigma_3}, v) &\mapsto u e^{\frac{r}{2}\sigma_3} v^\dagger,\end{aligned} \tag{2}$$

where A^+ is the diagonal subgroup $\{e^{\frac{r}{2}\sigma_3}|r\geq 0\}$. The Haar measure with respect to this decomposition is

$$\mathrm{d}\mu_{SL(2,\mathbb{C})} = \frac{1}{\pi}\sinh^2 r\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v. \tag{3}$$

Note we choose a normalization factor that slightly differs from the one used in the literature [9] by a factor 4 to uniform our notation with the $Spin(4)$ case.

B. Decomposition of $Spin(4)$

The algebra of $Spin(4)\simeq SU(2)\times SU(2)$ is generated by two commuting $SU(2)$ algebras with generators $\vec{J}_L$ and $\vec{J}_R$. For our purposes it is convenient to parametrize the algebra in terms of L_i, the generators of the spatial rotation subgroup, and A_i, the generators of time rotations or (Euclidean) boosts, as we will call them with a slight abuse of language. We define the rotations and boost generators as

$$\vec{L} = \vec{J}_L + \vec{J}_R, \qquad \vec{A} = \vec{J}_L - \vec{J}_R. \tag{4}$$

From the standard Casimirs J_L^2 and J_R^2 we obtain an equivalent set of two $Spin(4)$ invariant operators:

$$A^2 + L^2 = 2(J_L^2 + J_R^2), \qquad \vec{L}\cdot\vec{A} = J_L^2 - J_R^2.$$

We parametrize the representation (j_L, j_R) in terms of two other half-integer quantum numbers $p\equiv j_L + j_R + 1$ and $k\equiv j_L - j_R$ [10]. Without loss of generality we will assume that $j_L\geq j_R$ such that $p>k\geq 0$. The canonical basis for the representation (p,k), given by $|p,k;jm\rangle$ with $p-1\geq j\geq k$ and $m=-j,\ldots,j$, diagonalizes L^2 and L_3. In this basis, the Casimirs assume the values

$$\begin{aligned}(A^2 + L^2)|p,k\rangle &= (p^2 + k^2 - 1)|p,k\rangle,\\ \vec{L}\cdot\vec{A}|p,k\rangle &= pk|p,k\rangle.\end{aligned} \tag{5}$$

There exists a decomposition analog to (2) for the group $Spin(4)$. We parametrize an arbitrary element $(g_L, g_R)\in Spin(4)$ using two copies of the diagonal subgroup $(a,a) = D\simeq SU(2)$. We define the map

$$\begin{aligned}SU(2)\times T^+ \times SU(2) &\to Spin(4)\\ (u, e^{-i\frac{t}{2}\sigma_3}, v) &\to (u e^{-i\frac{t}{2}\sigma_3} v^\dagger, u e^{i\frac{t}{2}\sigma_3} v^\dagger),\end{aligned} \tag{6}$$

where $T^+ = \{\exp(-i\frac{t}{2}\sigma_3)|t\in[0,2\pi)\}$. Let also E^+ be the subgroup $\{(g,g^\dagger)|g\in T^+\}$ of $Spin(4)$. We show that the map (6) is surjective. Let (g_L, g_R) be a generic element of $Spin(4)$. The equations

$$\begin{aligned}g_L &= u e^{-i\frac{t}{2}\sigma_3} v^\dagger\\ g_R &= u e^{i\frac{t}{2}\sigma_3} v^\dagger\end{aligned}$$

imply

$$\begin{aligned}g_L g_R^\dagger &= u e^{-it\sigma_3} u^\dagger\\ g_R^\dagger g_L &= v e^{-it\sigma_3} v^\dagger.\end{aligned}$$

The elements $g_L g_R^\dagger$ and $g_R^\dagger g_L$ are conjugate, and every element of $SU(2)$ is conjugate to a diagonal matrix of the form $\exp(it\sigma_3)$. Hence we can solve the last equations for u, v. Notice, importantly, that it is enough to require $t\in[0,2\pi)$ to get a unique solution. We have thus shown that (6) is the sought for Cartan decomposition for $Spin(4)$. The Haar measure with respect to this decomposition is

$$\mathrm{d}\mu_{Spin(4)} = \frac{1}{\pi}\sin^2 t\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v. \tag{7}$$

III. ANALYTIC CONTINUATION

We use the decompositions (2) and (6) to relate the groups $SL(2,\mathbb{C})$ and $Spin(4)$ through analytic continuation.

A. Algebras and group elements

To start, we map the Lie algebras and group elements. From the (real) algebra of $Spin(4)$ we get the (realification of the) algebra of $SL(2,\mathbb{C})$ by rotating half of the algebra to purely imaginary generators:

$$\mathfrak{spin}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{su}(2) \oplus i\mathfrak{su}(2) \simeq \mathfrak{sl}(2,\mathbb{C}).$$

In terms of the previously defined generators, the rotation maps the generators of Euclidean boosts to the generators of Lorentzian boosts and vice versa:

$$(\vec{L}, i\vec{K}) \simeq \mathfrak{spin}(4) \quad \text{and} \quad (\vec{L}, -i\vec{A}) \simeq \mathfrak{sl}(2,\mathbb{C}). \tag{8}$$

We write these isomorphisms of (real) Lie algebras as $\vec{A} \leftrightarrow i\vec{K}$ and $\vec{K} \leftrightarrow -i\vec{A}$. From the decompositions (2) and (6) we see also that (8) induces a map between the compact subgroup E^+ and the noncompact subgroup A^+ seen as subgroups of the complexified groups $Spin(4)_\mathbb{C} \simeq SL(2,\mathbb{C})_\mathbb{C}$. For example, the map from E^+ to A^+ can be achieved by sending $t \to ir$ where $t \in [0, 2\pi)$ parametrizes E^+ and $r \in \mathbb{R}^+$ parametrizes A^+.

This relation between the E^+ and A^+ subgroups has an interesting geometrical interpretation. Since the action of $Spin(4)$ is transitive on $SU(2) \simeq S^3 \subset \mathbb{H}$, where $\mathbb{H}$ is the group of quaternions, and the diagonal subgroup D stabilizes the identity, the 3-sphere S^3 is a homogeneous space for $Spin(4)$. We can thus identify the quotient subgroup $Spin(4)/D$ with the 3-sphere. There is a similar result for $SL(2,\mathbb{C})$ (a well-known construction of geometric analysis [11]): the quotient group $SL(2,\mathbb{C})/SU(2)$ can be identified with hyperbolic 3-space H^3. In light of the decompositions (2) and (6), write an element of the quotient $Spin(4)/D$ as $de(t)D$, $e(t) \in E^+, d \in D$, and an element of $SL(2,\mathbb{C})/SU(2)$ as $ka(r)K$, $a(r) \in A^+, k \in K = SU(2)$. The parameters r, t act as radial coordinates in the corresponding 3-manifolds. Hence the inverse map $r \to -it$ from A^+ to E^+ can be interpreted geometrically as mapping hyperbolic 3-space to spherical 3-space, similarly to the usual rotation $t \to i\tau$ from physical time to Euclidean time that transforms Lorentzian metrics into Riemannian ones (and in particular Minkowski space to Euclidean space). This becomes manifest if we consider the metric of hyperbolic 3-space in radial coordinates,

$$\mathrm{d}H^2 = \mathrm{d}r^2 + \sinh^2 r \mathrm{d}\Omega_2^2,$$

where $\mathrm{d}\Omega_2$ is the metric on the 2-sphere. The map $r \to -it$ maps this metric to (up to a minus sign)

$$\mathrm{d}S^2 = \mathrm{d}t^2 + \sin^2 t \mathrm{d}\Omega_2^2$$

which is exactly the metric of the 3-sphere. From these metrics we can also read the Jacobian factors that enter the Haar measures (3) and (7).

B. Representations

The isomorphisms (8) can be used to find a correspondence between the unitary irreducible representations of $Spin(4)$ and $SL(2,\mathbb{C})$. We compute the action of the Casimirs (1) and (5) in the complexified algebras and find the map between representations looking at their eigenvalues on the respective canonical bases. From $\vec{A} \leftrightarrow i\vec{K}$ we get

$$\begin{aligned} p^2 + k^2 - 1 &\leftrightarrow -\rho^2 + k^2 - 1, \\ pk &\leftrightarrow i\rho k. \end{aligned} \tag{9}$$

Looking at the second Casimir we read the maps between $SL(2,\mathbb{C})$ and $Spin(4)$ representations

$$\begin{aligned} (\rho \to -ip, k) &\simeq (p, k), \\ (p \to i\rho, k) &\simeq (\rho, k). \end{aligned} \tag{10}$$

In the following we write $(p, k) \leftrightarrow (i\rho, k)$ to denote both maps. The maps (10) can be realized explicitly in terms of matrix elements. Using these maps between representations, we can define a generalized reduced matrix element that we denote $d^{(a,k)}_{jlm}(z)$, $z \in \mathbb{C}$, which represents both the A^+ and E^+ subgroups. For $a = i\rho$, $\rho \in \mathbb{R}$ and $z = e^{-r}$ it reduces to

$$d^{(\rho,k)}_{jlm}(r) \equiv \langle \rho, k; j, m | e^{\frac{r}{2}\sigma_3} | \rho, k; l, m \rangle \tag{11}$$

in the $SL(2,\mathbb{C})$ case, and for $a = p$, $p > k \in \mathbb{Z}/2$ and $z = e^{it}$ to

$$d^{(p,k)}_{jlm}(t) \equiv \langle p, k; j, m | e^{-i\frac{t}{2}\sigma_3} | p, k; l, m \rangle \tag{12}$$

in the $Spin(4)$ case.

We outline a derivation of the expression for the generalized matrix elements $d^{(a,k)}_{jlm}(z)$ using only elementary algebra and properties of the hypergeometric functions. The complete proof can be found in [12] by one of the authors. Similar results appear in [13–16] but using different conventions.

The boost matrix elements for $SL(2,\mathbb{C})$ can be written as analytically continued $SU(2)$ Clebsch-Gordan coefficients

$$
\begin{aligned}
d_{jlm}^{(\rho,k)}(r) = &\sum_n e^{-(i\rho-k-1+m-2n)r}\left\langle \frac{i\rho+k-1}{2},\frac{i\rho-k-1}{2}+m-n;\frac{i\rho-k-1}{2},n-\frac{i\rho-k-1}{2}\middle| j,m\right\rangle \\
&\times\left\langle \frac{i\rho+k-1}{2},\frac{i\rho-k-1}{2}+m-n;\frac{i\rho-k-1}{2},n-\frac{i\rho-k-1}{2}\middle| l,m\right\rangle \\
&+[\rho\to-\rho,k\to-k]
\end{aligned}
\tag{13}
$$

using the expression of the Clebsch-Gordan coefficients in terms of ${}_3F_2$ hypergeometric functions and factorials (continued to Gamma functions) (pages 240 and 241 of [17]). The last line corresponds to the same expression as in the first sum $\sum_n \ldots$ but with the signs of ρ and k flipped. Sending $\rho\to -ip$, equivalently $i\rho\to p$, and $r\to -it$, equivalently $ir\to t$, we obtain

$$
\begin{aligned}
d_{jlm}^{(-ip,k)}(it) = &\sum_n e^{(p-k-1+m-2n)it}\left\langle \left(\frac{p+k-1}{2},\frac{p-k-1}{2}+m-n\right),\left(\frac{p-k-1}{2},n-\frac{p-k-1}{2}\right)\middle| j,m\right\rangle \\
&\times\left\langle \left(\frac{p+k-1}{2},\frac{p-k-1}{2}+m-n\right),\left(\frac{p-k-1}{2},n-\frac{p-k-1}{2}\right)\middle| l,m\right\rangle \\
&+[p\to-p,k\to-k].
\end{aligned}
\tag{14}
$$

The second term vanishes identically since $k\le j,l\le p-1$ while the Clebsch-Gordan coefficients vanish if $j,l\le -p-1$. If we shift the first summation $n\to n'+(p-k-1)/2$ we obtain the expression for the reduced matrix elements of $Spin(4)$. The complete expression for $d_{jlm}^{(a,k)}(z)$, which covers both the previous cases, is

$$
\begin{aligned}
d_{jlm}^{(a,k)}(z) = &(-1)^{j-l}\sqrt{\frac{(a-j-1)!(j+a)!}{(a-l-1)!(l+a)!}}\frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!}z^{-(a-k-m-1)} \\
&\times\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!} \\
&\times\sum_{s,t}(-1)^{s+t}z^{2t}\frac{(k+s+m+t)!(j+l-k-m-s-t)!}{t!s!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!} \\
&\times {}_2F_1[\{l-a+1,k+m+s+t+1\},\{j+l+2\};1-z^2],
\end{aligned}
\tag{15}
$$

where $z\in\mathbb{C}$, $k\in\mathbb{Z}/2$ and either $a=p\in\mathbb{Z}/2, p>1$ or $a=i\rho,\rho\in\mathbb{R}^+$. If $a=i\rho$ and $z=e^{-r}$ then (15) turns into the $SL(2,\mathbb{C})$ reduced matrix elements (11). If $a=p$ and $z=e^{it}$ then (15) turns into the $Spin(4)$ reduced matrix elements (12). The formula for the minimal case $l=j=k$ is much simpler:

$$
\begin{aligned}
d_{kkm}^{(\rho,k)}(z) = &z^{k-a+m+1} \\
&\times {}_2F_1[k-a+1,k+m+1,2k+2;1-z^2],
\end{aligned}
\tag{16}
$$

where ${}_2F_1$ is the Gauss hypergeometric function. This is the formula that will be used later in the proof of the main result.

IV. THE EPRL MODEL

The harmonic analysis of $Spin(4)$ and $SL(2,\mathbb{C})$ is central to spin foam models of quantum gravity. The EPRL model is built upon the spin foam quantization of topological BF theory. Topological invariance is broken implementing at the quantum level the simplicity constraints of the Plebanski formulation of general relativity via the so-called Y_γ map [2].

In the Lorentzian model, the linear simplicity constraints impose the linear relation $\vec{K}=\gamma\vec{L}$ between the boost and rotation generators of the $SL(2,\mathbb{C})$ algebra. In terms of the representations (ρ,k) on the canonical basis, these are implemented as the map Y_γ from the $\mathfrak{su}(2)$ representation of spin j to a subspace of the $\mathfrak{sl}(2,\mathbb{C})$ representation $(\rho,k)=(\gamma j,j)$:

$$
Y_\gamma : |j,m\rangle \to |\gamma j,j;j,m\rangle. \tag{17}
$$

In the Euclidean model, similarly, the constraints $\vec{A}=\gamma\vec{L}$ between the (Euclidean) boost and rotation generators are implemented through the same map (17) where, in this case, the embedding is in the $(p,k)=(\gamma j,j)$ representation on the canonical basis for $\mathfrak{spin}(4)$. Requiring this map to be valid for all $j\in\mathbb{Z}/2$ imposes $\gamma\in\mathbb{Z}$. To be consistent with the convention of taking $p>k$ we restrict to the case of $\gamma>1$.

The decompositions (2) and (6) allow to recast the EPRL vertex amplitude as a sum of $SU(2)$ $15j$-symbols weighted by the product of four *booster functions* B_4 [18]:

$$A_v(j_f, i_e) = \sum_{l_f, i'_e} \left(\prod_e d_{i'_e} B_4(j_f, l_f, i_e, i'_e) \right) \{15j\}(l_f, i'_e),$$

where $d_{i'_e} = 2i'_e + 1$ and j_f, i_e are spin and intertwiner variables on the vertex boundary. The dependence on the Immirzi parameter γ is in the functions B_4. The formula holds both for the Lorentzian and Euclidean models, with the following differences. First, the l_f spins are unbounded in the Lorentzian case while they are bounded in the Euclidean case. Second, the Lorentzian booster functions contain the integral over A^+ of the product of $SL(2,\mathbb{C})$ reduced matrix elements while the Euclidean ones contain the integral over E^+ of the product of $Spin(4)$ matrix elements (with the corresponding Haar measures). We denote B_4^L the former and B_4^E the latter, and similarly A_v^L and A_v^E for the vertex amplitudes.

V. ANALYTIC CONTINUATION OF VERTEX AMPLITUDES

We now prove that the Lorentzian and Euclidean vertex amplitudes can be related through the rotation $\gamma \leftrightarrow i\gamma$ of the Immirzi parameter. We do so by showing how to relate the booster functions B_4^L and B_4^E. For simplicity of notation, we focus on the minimal case $B_4(j_f, j_f, i_e, i'_e)$ but the proof applies also to the case $l_f > j_f$ with minimal modifications. For further ease of notation, we set in the following $k_i \equiv j_f$ and we drop the indices i_e, i'_e.

Using the generalized matrix element (16), the booster function B_4 can be written as

$$B_4(k_i, k_i) = \int_\Omega \mathrm{d}\mu_\Omega(z) f(a_i, k_i; z), \tag{18}$$

where Ω can be either $A^+ \simeq (0, 1]$ or $E^+ \simeq S^1$ embedded in the complex plane $\mathbb{C}$, $\mathrm{d}\mu_\Omega(z)$ is the Haar measure (3) or (7) in the $z = e^{-r}$ or $z = e^{it}$ variable and

$$f(a_i, k_i; z) = \sum_{m_i} \begin{pmatrix} k_i \\ m_i \end{pmatrix} \prod_{i=1}^4 d^{(a_i,k_i)}_{k_i k_i m_i}(z) \begin{pmatrix} k_i \\ m_i \end{pmatrix}$$

with $4jm$-symbols in round parentheses. The value of a_i depends on the signature as discussed previously. It is easily seen that the Haar measures on A^+ and E^+ are the same up to a factor of i when embedded into the complex plane. In the Euclidean case, where $a_i = p_i$, let us denote $I_E(p_i, k_i)$ the integral (18), considering it as a function of the representation labels. In the following formulas we also drop the $4jm$-symbols which come from the double $SU(2)$ integration in the Haar measures, and assume $M = \sum_i m_i = 0$ (since otherwise the $4jm$-symbols vanish and the result is zero). After these simplifications, we have

$$\begin{aligned} I_E(p_i, k_i) &= \frac{i}{4\pi} \oint_{S^1} \mathrm{d}z (1 - z^2)^2 z^{K-A+1} \\ &\times \sum_{m_i} \prod_{i=1}^4 {}_2F_1[k_i - p_i + 1, k_i + m_i + 1, 2k_i + 2; 1 - z^2] \end{aligned} \tag{19}$$

with $K = \sum_i k_i$ and $A = \sum_i a_i = \sum_i p_i$. The first argument of all the hypergeometric functions is a strictly negative integer since $k_i < p_i$, therefore they reduce to polynomials in $1 - z^2$. The prefactor z^{K-A+1} introduces a pole singularity in $z = 0$ and the complete integrand is meromorphic, or analytic in the punctured plane. Therefore, it is irrelevant which contour one uses, as long as it contains $z = 0$. Let us consider the contour $\mathcal{C}_\varepsilon$ represented in red in Fig. 1. The horizontal segments have small distance ε from the real axis. The semicircles around 0 and 1 have small radius ε. The integral I_E can be equivalently evaluated on $\mathcal{C}_\varepsilon$, in the limit $\varepsilon \to 0$.

Let us now consider (18) in the Lorentzian case, when $a_i = i\rho_i$. As in the previous case, we denote the integral with $M = 0$ and dropping the $4jm$-symbols as

$$\begin{aligned} I_L(\rho_i, k_i) &= \frac{1}{4\pi} \int_{(0\,1]} \mathrm{d}z (1 - z^2)^2 z^{K-A+1} \\ &\times \sum_{m_i} \prod_{i=1}^4 {}_2F_1[k_i - i\rho_i + 1, k_i + m_i + 1, 2k_i + 2; 1 - z^2], \end{aligned} \tag{20}$$

where now $A = \sum_i a_i = i \sum_i \rho_i$.

A. Rotation $I_E \to I_L$

The substitution $p_i \leftrightarrow i\rho_i$ maps the integrand of I_E into the integrand of I_L and vice versa, up to a factor of i. We now show that this provides a rotation of the Euclidean integral I_E to the Lorentzian integral I_L, up to a prefactor.

FIG. 1. The contour $\mathcal{C}_\varepsilon$. The point $z = 0$ is a pole singularity in the Euclidean case and a branch point singularity in the Lorentzian case.

In the Lorentzian case the integrand is not meromorphic because both the hypergeometric functions and the prefactor z^{K-A+1} develop a branch point singularity in $z = 0$. Consider first the hypergeometric part. The analytic continuation of the hypergeometric series outside of the unit disk using Euler's formula has a branch cut discontinuity along real numbers $x \geq 1$ (Chapter 15 of [19]). When computed in $1 - z^2$, the branch cut discontinuity is along the whole imaginary axis, represented as a dot-dashed line in Fig. 1, and there are two disconnected domains of analyticity. We assign the principal branch $|\arg(z)| < \pi/2$ on both sides of the imaginary axis and we define the value on the imaginary axis minus the origin by continuity from the left.

At the origin each one of the hypergeometric functions in (20) is in general divergent. This happens when $\mathrm{Re}(\delta_i - \alpha_i - \beta_i) < 0$, where α_i, β_i, δ_i are the three parameters of ${}_2F_1[\{\alpha_i, \beta_i\}, \{\delta_i\}; 1 - z^2]$ (Sec. 15.4(ii) of [19]). In our case $\mathrm{Re}(\delta_i - \alpha_i - \beta_i) = -m_i$, therefore for some m_i the hypergeometric function is divergent at most of order $d_i = 2\max(0, m_i)$, i.e.

$$\lim_{z\to 0^-} z^{d_i+\sigma}{}_2F_1[\{k_i - a_i + 1, k_i + m_i + 1\}, \{2k_i + 2\}; 1 - z^2] = 0$$

for any real $\sigma > 0$. The product of four hypergeometric functions in (20) is divergent at most logarithmically in the origin, i.e. $\sum_i d_i = 2\max(0, \sum_i m_i) = 2\max(0, M) = 0$, since we can always take $M = 0$. This implies that the prefactor z^{K-A+1} cures any potential divergence in $z = 0$. This is not peculiar for the minimal case $j_i = l_i = k_i$, as can be verified using (15).

The prefactor z^{K-A+1} also has branch point singularities in 0 and ∞ since A is purely imaginary. For this term we consider the branch $|\arg(-z)| < \pi$ so that the discontinuity is along the positive real axis (represented in Fig. 1 as a dot-dashed line). We note, however, that the hypergeometric functions are continuous across the positive real axis.

We split the contour $\mathcal{C}_\varepsilon$ in four pieces: let $\mathcal{C}_\varepsilon{}^0$ be the small semicircle around 0, $\mathcal{C}_\varepsilon{}^1$ be the small semicircle around 1, $\mathcal{C}_\varepsilon{}^+$ be the straight line above the real axis and $\mathcal{C}_\varepsilon{}^-$ be the straight line below the real axis. The integral along $\mathcal{C}_\varepsilon{}^1$ vanishes since $\mathrm{Re}(K - A + 1) > 0$ and the hypergeometric function is regular there. The integral along $\mathcal{C}_\varepsilon{}^0$ vanishes for the same reason, since the eventual logarithmic divergence at $z = 0$ of the product of the four hypergeometric functions is more than canceled by the prefactor z^{K-A+1}. The integrals along $\mathcal{C}_\varepsilon{}^+$, $\mathcal{C}_\varepsilon{}^-$ differ by a factor $\exp 2\pi i(K - A + 1)$ due to the presence of the branch cut of z^{K-A+1} while the hypergeometric function is continuous in the right half-plane $\mathrm{Re}\, z > 0$. In the limit $\varepsilon \to 0$ we obtain

$$I_E(p_i, k_i) \overset{p_i \to i\rho_i}{\longrightarrow} i(e^{2\pi i \sum_i (k_i - i\rho_i)} - 1) I_L(\rho_i, k_i), \tag{21}$$

concluding the proof of the rotation $I_E \to I_L$. The generalization of (21) to any $j_i, l_i \geq k_i$, and in particular to the case $k_i = j_i$ and $l_i \geq j_i$, relevant for the booster functions, is straightforward. All the previous arguments are immediately extended, since the prefactor z^{K-A+1} remains the same and all the considerations about the product of the hypergeometric functions with minimal arguments apply also to the more complicated sum over s_i, t_i of products of hypergeometric functions.

B. Rotation $I_L \to I_E$

The converse direction $I_L \overset{i\rho_i \to p_i}{\longrightarrow} I_E$ holds too after regularizing the divergent prefactor with a limit. Write the rotation $i\rho_i \to p_i$ as

$$I_E(p_i, k_i) = i \lim_{q_i \to p_i} (e^{2\pi i \sum_i (k_i - q_i)} - 1) I_L(-iq_i, k_i), \tag{22}$$

where first we do the analytic continuation $\rho_i \to -iq_i$ of $I_L(\rho_i, k_i)$ with $q_i \in \mathbb{R}\backslash\mathbb{Z}$ and then we take the limit $q_i \to p_i = k_i + n_i$ and $n_i \in \mathbb{N}^+$ to regularize the product of the vanishing prefactor with the divergent function $I_L(-iq_i, k_i)$. In fact, the defining integral representation (20) of $I_L(\rho_i, k_i)$ is divergent if we perform the substitution $i\rho_i \to p_i$, for any half-integer $p_i > k_i \geq 0$. However, we can overcome this difficulty noticing that the same apparent obstruction appears for example in the Euler's integral representation of the standard Gamma and Beta functions. After the substitution $i\rho_i \to p_i$ the integral in (20) reduces to

$$\int_0^1 \mathrm{d}z (1 - z^2)^2 z^{1 + \sum_i (k_i - p_i)} P[1 - z^2], \tag{23}$$

where $P[1 - z^2]$ stands for a generic polynomial in the variable $1 - z^2$. Changing variables $z^2 \to w$ we can write this integral as a finite sum of Beta functions

$$\int_0^1 \mathrm{d}w w^{-2 + \frac{1}{2}\sum_i (k_i - p_i + 1)} P'[1 - w] \sim \sum_j B\left(-2 + \frac{1}{2}\sum_i (k_i - p_i + 1), n_j\right) \tag{24}$$

with the first argument always a negative integer and the second argument a positive integer. The Beta function can be analytically continued to complex values of its arguments using for example the Pochammer contour, possibly with simple poles at the negative integers. Since $I_L(-iq_i, k_i)$ tends to (24) continuously for $q_i \to p_i$, it is possible to analytically extend $I_L(\rho_i, k_i)$ to generic complex values of the first parameter, again possibly with simple poles at the negative integers, i.e. at the values $k_i - p_i + 1$ relevant for our case. The simple form (24) holds however only in a small neighborhood of the poles since in general the hypergeometric functions will not be expressible as simple polynomials. Remarkably, the vanishing prefactor in (22) exactly cancels the divergence of the analytically continued $I_L(-iq_i, k_i)$ at its simple poles.

Since each ρ_i spans an open subset of $\mathbb{C}$, the previous considerations support strongly the conjecture that the two functions $I_E(p_i, k_i)$ and $I_L(\rho_i, k_i)$ are particular integral representations of a unique function $I(a_i, k_i)$ defined on the whole space $\mathbb{C}^4 \times \mathbb{Z}^4$, which agrees with $I_E(p_i, k_i)$ for $a_i = p_i$ and with $I_L(\rho_i, k_i)$ for $a_i = i\rho_i$. Hence, we can speak unambiguously of *the* analytic continuation of I_E and I_L. We do not provide a rigorous proof of this claim here, which would require a more careful treatment of the interplay between the analytic continuation of the hypergeometric functions, the prefactor z^{K-A+1} and the integration on the unit interval.

C. Rotation of vertex amplitudes

Imposing the Y_γ map to both I_E and I_L integrals we find the desired relation between the Lorentzian and Euclidean booster functions. The analytic continuation $p_i \to i\rho_i$ reads $\gamma j_i \to i\gamma j_i$ and corresponds to the rotation of the Immirzi parameter $\gamma \to i\gamma$. In terms of the vertex amplitudes we get

$$A_v^L(j_f, i_e) = \left(\prod_e \frac{i}{1 - e^{2\pi\gamma \sum_{i\in e} j_i}}\right) A_v^E(j_f, i_e)^{(\gamma\to i\gamma)}, \qquad (25)$$

where the superscript $(\gamma \to i\gamma)$ means that equality holds after performing the rotation, or analytic continuation, of the expression. The converse rotation $i\gamma \to \gamma$ gives the other direction

$$A_v^E(j_f, i_e) = i\left[\prod_e \left(e^{2\pi\gamma \sum_{i\in e} j_i} - 1\right) A_v^L(j_f, i_e)\right]^{(i\gamma\to\gamma)}, \qquad (26)$$

where the prescribed regularization is implicit. This completes the proof of the relation between the analytic continuations of the vertex amplitudes.

VI. CONCLUSIONS

Our result explains the many similarities between results obtained in the two versions of the EPRL model. It is important to remark that formulas (25) and (26) are *exact*, while all the known results in the literature use the saddle-point approximation to estimate the amplitudes when the spin labels are large. Care is needed when analytically continuing the saddle-point expressions, since the relevant terms for the rotation may have been discarded in the approximation. The Euclidean asymptotic formula is [20]

$$A_v^E \approx \frac{1}{\lambda^{12}} \left(N_1 \cos(\lambda\gamma S_R) + N_2 e^{i\lambda S_R} + N_3 e^{-i\lambda S_R}\right),$$

where S_R is the Regge action of the reconstructed 4-simplex and λ gives the scaling. In the rotation $\gamma \to i\gamma$, the cosine term vanishes since the corresponding saddle-point disappears, while the two exponentials survive the rotation and contribute to the "degenerate" sector of the Lorentzian model. The converse rotation holds too. We show the details in the Appendix.

The map between the saddle-point expressions implies a map between derived results. An example is given by the "flatness problem" of spin foam models [21–24]. Various analyses suggested that the curvature of spin foam models, measured by interpreting the holonomies around internal faces as Regge deficit angles, might be accidentally suppressed in the large-spin limit. Most of the analyses have been done in the Euclidean model, with the last work [25] considering a particular configuration with three vertices. Our result implies that the flatness constraint $\theta = 0 \mod 4\pi/\gamma$ holds in both models, with the rotation $\gamma \to i\gamma$ mapping Euclidean to Lorentzian angles. The presence of the flatness constraint within Lorentzian EPRL has recently been verified numerically [26]. Another compatible result pertains to the divergence of the self-energy graph, i.e. the one-loop correction to the spin foam propagator. The Lorentzian analysis [7] seems to match the Euclidean analysis [27]. This is implied by our result since the leading order of divergence does not depend on γ.

In the context of computer simulations, we expect our result to contribute to existing numerical codes [26,28,29]. A complicated simulation can be performed with the computationally simpler Euclidean vertex and then general results can be inferred to be valid also in the Lorentzian case, similarly to what is done in lattice gauge theories. Our result can also be compared to other proposed analytic continuations of the EPRL model [30–32] or to effective spin foam models [33].

It is interesting to relate our findings to the early formulation of loop quantum gravity. The original canonical formulation of LQG was based on complex (self-dual) Ashtekar variables that require the imposition of "reality conditions" to recover real Lorentzian general relativity. Since the quantization of these reality conditions is problematic, the focus of the LQG community soon shifted to the use of real Barbero-Immirzi variables [34,35]. The price to pay is a more complicated form of the constraints and a less clear geometric interpretation of the real connection [36]. In the Euclidean signature these problems do not arise since the self-dual connection is real, and a "Wick transform" has been proposed for mapping the Euclidean constraints of general relativity to the Lorentzian ones [37–39]. Our result implements this classical construction into the quantum theory.

The applications of our methods go beyond spin foam models. For example, the booster functions are related to the Clebsch-Gordan coefficients of the respective groups [18]. Our results can be used to relate the Clebsch-Gordan coefficients of $SL(2,\mathbb{C})$ in the principal series to the analytic continuation of the ones of $Spin(4)$ (given by a $9j$-symbol) [13,16]. In addition, our construction strongly suggests that the integral forms of the booster functions, hence of the Clebsch-Gordan coefficients, are special cases of a general meromorphic function defined in the whole complex plane, a point already raised in the group theory literature [40].

In general, our prescription for mapping through analytic continuation the algebras, group elements and unitary irreducible representations of $Spin(4)$ and $SL(2,\mathbb{C})$ can be adapted to any physical model based on each of the two groups. Unitary irreducible representations of the Lorentz group are surely relevant to quantum gravity, but the possibility of finding other physical realizations of these representations is certainly worth considering, and indeed may be suggested by this relation.

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APPENDIX: ON THE LARGE-SPIN APPROXIMATION

Here we show how to apply the rotation $\gamma \to i\gamma$ to the large-spin asymptotic approximation of the Euclidean EPRL model. The Euclidean EPRL amplitude for $\gamma > 1$ with coherent boundary data represented by the set of spinors $|\zeta_{ab}\rangle$ can be written as the exponential of the action

$$\begin{aligned} S = \sum_{a<b} &(\gamma+1) j_{ab} \log\langle\zeta_{Lba}|g_{Lba}|\zeta_{Lab}\rangle \\ &+ (\gamma-1) j_{ab} \log \overline{\langle\zeta_{Lba}|g_{Rba}|\zeta_{Lab}\rangle} \\ &+ 2 j_{ab} \log \langle\zeta_{ba}|\zeta_{Lba}\rangle\langle\zeta_{Lab}|\zeta_{ab}\rangle, \end{aligned} \tag{A1}$$

where $g_{ba} = (g_{Lba}, g_{Rba}) \in Spin(4)$, $g_{ba} = g_b^{-1} g_a$ and ζ_{Lba}, ζ_{Lab} are dummy spinors part of the integration variables.

The critical point equations for (A1) are given by alignment conditions of the dummy spinors, closure conditions on the boundary data and orientation conditions. The solutions of these are straightforward and we find that, for suitable boundary data representing a Euclidean 4-simplex, $\theta^L_{ba} = \pm\theta_{ba}$ and $\theta^R_{ba} = \pm\theta_{ba}$ where θ_{ba} is the dihedral angle between the tetrahedron a and b in the 4-simplex [20]. The plus or minus signs in θ^L_{ba} and θ^R_{ba} are independent, therefore we have four critical points:

$$\langle\zeta_{ba}|g_{ba}|\zeta_{ab}\rangle = e^{\pm\frac{i}{2}\theta_{ba}}.$$

Two of these satisfy the condition $g_{Lba} = g_{Rba}$ while the other two satisfy $g_{Lba} = g^\dagger_{Rba}$. The critical points satisfying $g_{Lba} = g_{Rba}$ belong to the diagonal $SU(2)$ subgroup of $Spin(4)$ and has zero (Euclidean) boost parameter $t = 0$. We refer to them as "topological" critical points since they reduce to the BF $SU(2)$ ones if restricting the amplitude to the diagonal $SU(2)$ subgroup. Moreover, the dihedral angles are always defined as positive and by definition both angles $\theta^{L,R}_{ba} \in [0, 2\pi)$, therefore $-\theta_{ba}$ must be taken as $2\pi - \theta_{ba}$. The contribution to the amplitude from the topological critical points is given by

$$A \approx \frac{1}{\lambda^{12}} e^{-i\gamma 4\pi \sum_{a<b} j_{ab}} \left(N_1 e^{i\sum_{a<b} 2 j_{ab}\theta_{ba}} + N_2 e^{-i\sum_{a<b} 2 j_{ab}\theta_{ba}} \right).$$

When performing the rotation, the prefactor $e^{-i\gamma 4\pi \sum_{a<b} j_{ab}}$ cancels the one of (25) in the large-spin limit.

The contributions of the critical points are rotated from the Euclidean to the Lorentzian asymptotic formula only if they are also critical points of the Lorentzian action, otherwise their contributions vanish. The topological critical points are present both in the Euclidean and Lorentzian model, and in fact their contribution is correctly rotated since they do not depend on the γ.

On the contrary, it is readily seen that in general the nontopological points are not critical points of the Lorentzian action. Using the decomposition (6), the rotation from $Spin(4)$ to $SL(2,\mathbb{C})$ acts on $g_{ba} \in Spin(4)$ as

$$(g_{Lba}, g_{Rba}) \to (h_{ba}, (h^{-1}_{ba})^\dagger).$$

The property $g_{Lba} = g^\dagger_{Rba}$ that defines the nontopological points is mapped into $h_{ba} = h^{-1}_{ba}$, which in general is not satisfied by critical points of the Lorentzian action. These latter however satisfy $h_{ba} = (h^{-1}_{ba})^\dagger$, which is compatible with the topological points as we have just argued. This shows that the rotation (25) maps also the asymptotic Euclidean formula into the (degenerate sector of the) Lorentzian one, for boundary data corresponding to a Euclidean 4-simplex and after all the details are taken into account. The converse rotation (26) and the case of Lorentzian boundary data require a similar analysis.

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