Quantum probing of null singularities

Ivo Sachs,[1,*] Marc Schneider,[2,†] and Maximilian Urban[1,‡]

[1]*Arnold Sommerfeld Center for Theoretical Physics, Theresienstraße 37, 80333 München, Germany*
[2]*Albert-Einstein Institute for Gravitational Physics, Am Mühlenberg 1, 14476 Potsdam, Germany*

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We adapt the dual-null foliation to the functional Schrödinger representation of quantum field theory and study the behavior of quantum probes in plane-wave space-times near the null singularity. A comparison between the Einstein-Rosen and the Brinkmann patch, where the latter extends beyond the first, shows a seeming tension that can be resolved by comparing the configuration spaces. Our analysis concludes that Einstein-Rosen space-times support exclusively configurations with nonempty gravitational memory that are focused to a set of measure zero in the focal plane with respect to a Brinkmann observer. To conclude, we provide a rough framework to estimate the qualitative influence of backreactions on these results.

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I. INTRODUCTION

Space-times featuring plane-fronted waves with parallel rays (pp-wave space-times) represent an important class of exact solutions to the Einstein equations, as they describe nonlinear gravitational waves in general relativity. The main applications of these space-times can be found in string theory, as these backgrounds support exact string solutions in scattering theory as an effective description of a nonperturbative scattering, or in terms of the gravitational memory effect [1], which will play a role later in this article. One special subclass thereof are plane-wave space-times which portray the even simpler situation where the profile of the wave front is constant along the transversal direction. Penrose extensively studied these space-times from a geometric perspective in a series of seminal articles [2,3]. Those space-times' most distinguished property is the focusing of null rays that have crossed the wave. Since the weak-energy condition holds, the plane wave will act like a converging lens such that collimated light rays meet in a focal plane. The space-time then develops two focal planes with respect to future- and past-directed null rays. This intersection of null geodesics indicates that the manifold itself is null-geodesically incomplete. A further consequence is that this class does not admit a Cauchy surface and therefore fails to be globally hyperbolic. Albeit curvature invariants are vanishing globally, null and time-like geodesic incompleteness is the minimal criterion for a space-time to be called singular [4].

Owing to their peculiar causal structure, plane-wave space-times can be described by so-called Einstein-Rosen patches—i.e., coordinate systems ranging from null infinity to the focal plane, where the metric degenerates. Since focusing singularities are classified as weak, one can still find a consistent extension of the Einstein-Rosen patches beyond the singularity, subject to certain integrability conditions [5]. This extension, the Brinkmann patch, fully covers the manifold from past to future null infinity without degenerating.

In recent analyses [6–11], classical singularities have been probed by quantum fields within the functional Schrödinger formalism with intriguing results—e.g., Schwarzschild black holes admit a consistent quantum field theory. Plane-fronted waves provide an excellent setting to further investigate the concepts developed therein, as they are distinctly different from previous examples. As the singularity is null, and therefore of a different type from previous studies, it may shed some light on how the nature of the singular surface, as well as the bordering space-time, impacts previously seen effects. Furthermore, these space-times have zero curvature everywhere away from the wave front. In fact, the regions flanking the wave are Minkowski patches, which allows us to use properties of flat space-time. In particular, one does not expect to observe dissipative effects, such as those occurring in Schwarzschild space-time. Hence, a focused massless quantum field theory is predicted to encounter the null singularity, providing a consistency check for the previous interpretation of quantum completeness.

The nonexistence of Cauchy hypersurfaces imposes an obstacle to the definition of a sensible evolution problem,

*ivo.sachs@physik.uni-muenchen.de
†marc.schneider@aei.mpg.de
‡maximilian.urban@physik.uni-muenchen.de

as this strongly depends on a well-defined initial-value problem and Hamiltonian flow. A remedy was found by Hayward [12–14], making use of the fact that well-defined initial-value problems prohibit only timelike separations between two points on the initial hypersurface. The resulting dual-null foliation allows for the construction of Lagrangian and Hamiltonian dynamics. These then support well-defined initial-value problems, as well as Hamiltonian flows, such that the Cauchy problem can be generalized to some nonglobally hyperbolic space-times. The construction also admits a consistent path integral quantization, making it particularly useful.

In this article, we utilize the dual-null foliation to derive the functional Schrödinger representation of quantum field theory in these cases. While our construction is completely general and applicable to all space-times, the particular geometry of plane-wave space-times makes the application especially simple and transparent.

In the following section, we review some geometrical features of plane-wave space-times with the emphasis on the different coordinate systems. Section III briefly discusses Hayward's dual-null foliation for Hamiltonian developments in general before it is explicitly applied to the functional Schrödinger analysis. After constructing the Hamilton operator and the corresponding states, we will perform a null reduction that reflects our initial conditions. In Sec. IV, the special case of a delta-distribution-shaped shockwave is examined explicitly in both coordinate patches. We show that there appears to be a tension between the two results which can be resolved by carefully studying the underlying configuration spaces. The last section discusses backreactions on the system. Similarly to Ref. [15], we chose a heuristic approach that yields a corrected metric around the focal plane and estimates its influence on the Schrödinger wave functional. Afterwards, we set our results in context to the previous results in dynamical space-times and discuss further directions. Note that we will work in the units $c = \hbar = G_N = 1$ throughout the article.

II. PLANE-WAVE SPACE-TIME

The general four-dimensional space-time $(\mathbb{W}^4, g)$ describing a gravitational-electromagnetic pp wave is given by a semi-Riemannian manifold $\mathbb{W}^4$ and the corresponding metric in Brinkmann or harmonic coordinates [2]

$$g = -2\mathrm{d}u \otimes \mathrm{d}v + H(u,x)\mathrm{d}u \otimes \mathrm{d}u + \delta_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b, \quad (1)$$

where the function $H(u,x)$ describes the profile of an outgoing plane wave that propagates through Minkowski space. Here, u is the ingoing and v the outgoing null direction, while the x^a's are spatial coordinates, with δ_{ab} being the identity matrix. This space-time can be thought of as two Minkowski patches linked by a plane wave of compact support in the u direction (sandwich wave). The pp-wave metric enjoys a high degree of symmetry and is therefore characterized almost entirely by H. For the special case of plane waves, the function $H(u,x)$ is quadratic in x^a—that is, $H(u,x) \to H_{ab}(u)x^a x^b$. In particular, gravitational waves are described by profile functions that fulfill $\mathrm{tr}(H) = 0$—i.e., that have a vanishing Ricci tensor—while electromagnetic waves, $H_{ab}(u) = H(u)\delta_{ab}$, are characterized by vanishing Weyl tensors. The Ricci tensor in this space-time is then nontrivial at all points through which the wave propagates—i.e., $R_{uu} = -\mathrm{tr}(H)(u)$—while other components are zero. Hence, Eq. (1) is the solution to the Einstein equation with Einstein tensor $G_{\mu\nu} = 0$, except for $G_{uu} = 8\pi T_{uu} = -\mathrm{tr}(H)(u)$. Focusing properties of plane waves are implied, since the weak energy condition $\mathrm{tr}(H) \leq 0$ holds. It follows from the fact that the profile function $H(u,x)$ is independent of v that there are no self-focusing effects on the wave front itself.

The first extensive study of these space-times was undertaken by Penrose [2,3] and found that, although they are strongly causal, they do not admit Cauchy hypersurfaces due to a focusing singularity. More specifically, every light cone will degenerate in at least one direction at some distance after crossing the wave front. Thus, all geodesics are destined to meet at said focal point u_f, clearly excluding the possibility of a Cauchy surface as defined in Ref. [2].

While the Brinkmann coordinates are useful as they cover the entire manifold, the symmetries of the space-time are more transparent in the Einstein-Rosen or group coordinates:

$$g = -2\mathrm{d}U \otimes \mathrm{d}V + \gamma_{ij}\mathrm{d}y^i \otimes \mathrm{d}y^j. \quad (2)$$

The transition functions between Eqs. (1) and (2) are given through the vierbein $E^a_i(u)$, which in this case is only a function of u and satisfies the Einstein equation $\ddot{E}_{ai} = H_{ab}(u)E^b_i$, together with the relation $g_{ij} = E^a_i E^b_j \delta_{ab}$ and the symmetry condition $\dot{E}_{ai}E^i_b = \dot{E}_{bi}E^i_a$ stemming from the Wronski determinant. The asymptotic symmetries of Minkowski space-time must be reflected by the vierbeins such that $\lim_{u\to\pm\infty} E^{\pm}_{ia} = \delta_{ia}$ [16]. Explicitly, the coordinate transformations that lead to Eq. (2) are given by $u \to U$, $v \to V + \frac{1}{2}\dot{\gamma}_{ij}(U)y^i y^j$, and $x^a \to E^a_i y^i$. The metric of the two-dimensional spatial submanifold can be constructed from the vierbeins via $\gamma_{ij}(U) = \frac{1}{2}(E^a_i E_{aj} + E^a_j E_{ai})$. The overdot unambiguously denotes a derivative with respect to the ingoing direction u or U coordinate, as the two are identical. In Eq. (2), we can immediately read off the Killing vectors $\frac{\partial}{\partial V}$ and $\frac{\partial}{\partial y^i}$. The remaining ones are given by the combination $\mathcal{D}^i = y^i \frac{\partial}{\partial V} + F^{ij}(U)\frac{\partial}{\partial y^j}$ with $F^{ij}(U) = \int^U \mathrm{d}v \gamma^{ij}(v)$. The Killing vectors D in Eq. (1) can be constructed directly from the vierbeins E and are $D^i = E^{ia}\frac{\partial}{\partial x^a} - \dot{E}^i_a x^a \frac{\partial}{\partial v}$. Geometrically, these can be

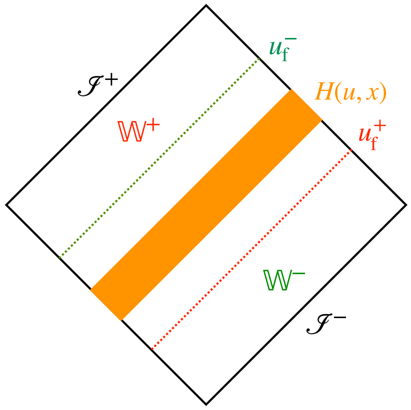

FIG. 1. The Carter-Penrose diagram of the pp-wave space-time shows the full Brinkmann chart $(\mathbb{W}^4, g)$, with embedding $\iota\colon \mathcal{U}\times\mathcal{V}\times\mathcal{S}\to\mathcal{M}$, ranging from $\mathscr{I}^-$ to $\mathscr{I}^+$. The outgoing wave $H(u,x)$ is depicted by the orange bar that divides the manifold into two parts, where all white areas describe Minkowski space-time. Additionally, we see the two Einstein-Rosen patches, $\mathbb{W}^- = (-\infty, u_f^-)\times\mathcal{V}\times\mathcal{S}$ and $\mathbb{W}^- = (\infty, u_f^+)\times\mathcal{V}\times\mathcal{S}$, that end at the null singularities at the focusing points $u_f^\pm$ displayed by the dashed lines.

thought of as connecting vectors between neighboring geodesics [17].

Although most calculations are easier in Einstein-Rosen coordinates, the drawback is that they only cover a part of $\mathbb{W}^4$, either from past null infinity $\mathscr{I}^-$ to the focusing plane in the future of the wave, or from future null infinity $\mathscr{I}^+$ to the focusing singularity in the past of the wave. We will call the chart with the asymptotic boundary in the future $\mathbb{W}^+$, and the chart with the asymptotic boundary in the past $\mathbb{W}^-$. The geometry is illustrated in the Penrose diagram in Fig. 1. Here, the Brinkmann patch describes the entire diamond, while the Einstein-Rosen charts $(\mathbb{W}^\pm, g^\pm)$ range from $\mathscr{I}^\pm$ to the null singularity at $u_f^\pm$ and together cover the whole space-time with overlap between the null singularities. It should be mentioned that the distance in u, or U, between the focal point and the wave front is inversely proportional to the amplitude of the wave. The Killing vectors alone already encode many of the interesting features of plane-wave space-times, such as, for example, the gravitational memory effect. Null geodesics starting at $\mathscr{I}^-$ experience a Minkowski evolution—that is to say, $E_{ia}^- = \delta_{ia}$—before they have crossed the wavefront. It is therefore clear that we recover the spatial translation invariance of Minkowski space-time by finding $D^a \to \frac{\partial}{\partial x^a}$. However, once a geodesic has crossed the wave, the vierbein develops a u dependence that triggers the deviation of ingoing geodesics towards the v direction, resulting in a focusing. This u dependence reflects the gravitational memory as it persists throughout the entire future development. In the patches $\mathbb{W}^-$ and $\mathbb{W}^+$, the spatial metric accounts for the gravitational memory as $\gamma_{ij}^\pm \to \delta_{ij}$ at the asymptotic boundary $\mathscr{I}^\pm$; however, it will develop a U dependence that leads to a degeneracy at some $U_f^\pm$ beyond the pp wave. In other words, the spatial hypersurfaces in Einstein-Rosen coordinates will degenerate based on the nature of the wave itself: While for electromagnetic waves, the leading order in the mode expansion is a dipole that supports an isotropic contraction, gravitational waves are governed by quadrupole modes that contract in one direction and simultaneously expand in the perpendicular direction, resulting in an astigmatic focusing (caustic).

From the transformation between the spatial coordinates x_a and y_i, we see that the spatial volume of the Einstein-Rosen patch collapses, while remaining constant for Brinkmann coordinates. The latter is hence an extension of space-time through the singularity with a nondegenerating spatial part, as can be seen in Eq. (1). This is possible because the null-singularity occurring here is weak according to the definition of Ref. [5], such that a geometrical extension is possible. The gravitational memory is seen in the u dependence of the E_{ia}, which strongly influences the Killing vectors.

III. DUAL-NULL FOLIATION

The global hyperbolicity of space-time is considered essential for most analyses in Hamiltonian systems; however, generically this requirement is not necessary for a well-defined initial-value problem. Following Ref. [4], the necessary and sufficient condition is the existence of an achronal set (a surface without two events that are separated by a timelike curve) and a global flow which respects this property of the set. Hayward constructed in a series of seminal articles [12–14] the dual-null formulation of Hamiltonian dynamics, showing that the initial-value problem of the Cauchy evolution from the initial Cauchy surface is replaced by a combined boundary and initial-value problem in the null evolution. More precisely, the initial data is prescribed by two intersecting null surfaces in terms of a boundary-value problem on the null infinities and a set of initial values on the intersection. This introduces a new set of canonical momenta; consistency is enforced through additional integrability conditions.

The first step is to create a suitable embedding: Let $\mathcal{M}$ be a d-dimensional, globally time-orientable manifold; then we construct a $(d-2)$-dimensional, compact, orientable, and time-orientable spacelike submanifold $\mathcal{S}$ [18]. Additionally, we take two half-open intervals $\mathcal{V} = [0, v)$ and $\mathcal{U} = [0, u)$ such that we can define a smooth embedding $\iota\colon \mathcal{U}\times\mathcal{V}\times\mathcal{S}\to\mathcal{M}$. Using ι, we can build the phase space of the elementary observables: Consider a vector bundle Q over the base space B; then the configuration variable will be $q \in CQ$, where C denotes the space of smooth sections. We can now define the two directions given by vectors in $\mathcal{U}$ and $\mathcal{V}$ such that they are aligned with the null congruences. Hence, the space-time is foliated by two sets of three-surfaces: Σ_- along the u direction, and Σ_+

along the v direction. The manifold is thus covered by two stacks of hypersurfaces along the two corresponding null directions.

A. Hamilton density

Consider the tangent space TCQ to CQ; we can define the velocity fields $(q, q^+, q^-) \in TCQ \oplus TCQ$. From here, it is clear that the evolution space is given by $\mathcal{V} \times \mathcal{U}$, that q^+ is the velocity field tangent to the outgoing v direction, and q^- is tangent to the ingoing u direction. The Hamilton density can be constructed by a Legendre transformation [14] of the Lagrange density $\mathcal{L}$:

$$\mathcal{H}(q, p^+, p^-) = ((0, p^+, p^-) - \mathcal{L})\Lambda^{-1}(q, p^+, p^-). \quad (3)$$

Here the conjugate fields are $p^\pm \in T^*CQ$ and $\Lambda\colon TCQ \oplus TCQ \to T^*CQ \oplus T^*CQ$, where $\Lambda\colon (q, q^+, q^-) \mapsto (q, \frac{\delta\mathcal{L}}{\delta q^+}, \frac{\delta\mathcal{L}}{\delta q^-})$ denotes the invertible Laplace transformation. The identification of the conjugate momenta $p^\pm$ with the functional derivative of $\mathcal{L}$ with respect to the velocity fields $q^\pm$ follows from the Hamilton equations $p^+ = \frac{\delta\mathcal{L}}{\delta q^+}$, $p^- = \frac{\delta\mathcal{L}}{\delta q^-}$, and $\frac{\partial q}{\partial u} = \frac{\delta\mathcal{H}}{\delta p^+}$, $\frac{\partial q}{\partial v} = \frac{\delta\mathcal{H}}{\delta p^-}$, $\frac{\partial p^+}{\partial u} + \frac{\partial p^-}{\partial v} = -\frac{\delta\mathcal{H}}{\delta q}$, with the additional integrability condition $\frac{\partial}{\partial u}(\frac{\delta\mathcal{H}}{\delta p^-}) = \frac{\partial}{\partial v}(\frac{\delta\mathcal{H}}{\delta p^+})$.

In the double-null case, it was shown by Hayward that these can always be satisfied [18].

The above construction can be applied directly to the quantization of fields in curved spaces, where it is especially useful to study transmissions through null surfaces. Plane-wave space-times particularly lend themselves to this setup, as there is a natural choice of null surfaces dictated by the wave front. Hence, we consider a free massless scalar field theory $S = -\frac{1}{2}\int d^4x\sqrt{-g}\ (g(\nabla\phi, \nabla\phi) + \zeta\mathcal{R}\phi^2)$ with covariant derivative ∇, Ricci scalar $\mathcal{R}$, and coupling constant ζ. Rewriting this action in Brinkmann coordinates, using the embedding $\imath$ given by the dual-null foliation, yields

$$S = -\frac{1}{2}\int_{\mathbb{W}^4} d^4x\sqrt{-g}\bigg(-2\phi^+\phi^- + H(u,x)\phi^+\phi^+ + \delta(\nabla\phi, \nabla\phi) + \zeta\mathcal{R}\phi^2\bigg), \quad (4)$$

where δ is the induced Euclidean flat metric on $\mathcal{S}$, and the ∇'s are understood to be the spatial derivatives. Considering $\imath$, the boundary and initial data for the velocity fields with respect to the u and v directions are given by $\phi^+ \restriction \Sigma_- = \{0\} \times \mathcal{V} \times \mathcal{S}$, and $\phi^- \restriction \Sigma_+ = \mathcal{U} \times \{0\} \times \mathcal{S}$, while the spatial data are given by $\phi \restriction \{0\} \times \{0\} \times \mathcal{S}$. An illustration of the dual-null foliation can be found in Fig. 2, where the velocity fields and the hypersurfaces are shown explicitly. From here, it is obvious why the evolution in this foliation is an initial-value problem on $\mathcal{S}$ combined with a boundary condition at $\mathscr{I}^+$ or $\mathscr{I}^-$.

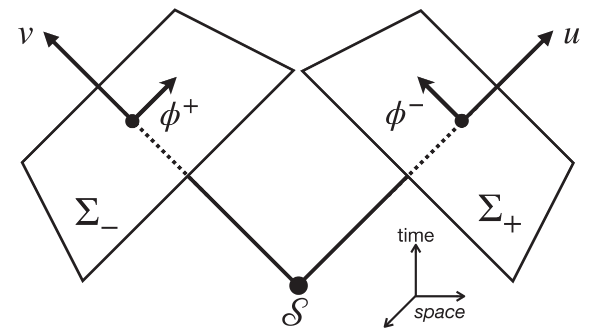

FIG. 2. Sketch of the dual-null foliation (spatial dimensions partially suppressed). Attached to the two-dimensional spacelike surface $\mathcal{S}$ on which the spatial fields live are the ingoing and outgoing null vectors along $v \in \mathcal{V}$ and $u \in \mathcal{U}$, respectively. Perpendicular to the null directions are the three-surfaces Σ_- and Σ_+ and the velocity fields $\phi^\pm \restriction \Sigma_\mp$. Points in the two-dimensional spatial submanifold are represented by intersections of the three-submanifolds Σ_+ and Σ_- at certain values of the light-cone coordinates.

Applying Eq. (3) to Eq. (4), we find the Hamilton density in Brinkmann coordinates to be

$$\mathcal{H} = \frac{1}{2}(2\pi^+\pi^- + H(u,x)\pi^-\pi^- + \delta(\nabla\phi, \nabla\phi)). \quad (5)$$

This is the general form for a Hamilton density in Brinkmann coordinates; however, for an outgoing wave, the only nonzero component of the Ricci tensor is $R_{uu} = -\mathrm{tr}(H)$, which is zero for gravitational waves, as we have seen before. Hence, we have safely set $\mathcal{R} \equiv 0$ in the above equation. Furthermore, to replace the mixing terms in the Legendre transformation, we use the Hamiltonian equations

$$\pi^- = \frac{\delta\mathcal{L}}{\delta\phi^-} = \phi^+, \quad (6)$$

$$\pi^+ = \frac{\delta\mathcal{L}}{\delta\phi^+} = \phi^- - H(u,x)\phi^+. \quad (7)$$

We see that the outgoing momentum is not affected by the wave, since it propagates parallelly, while the ingoing momentum that crosses the wave experiences a distortion which will lead to the inevitable focusing. Hence, the first term on the right-hand side of Eq. (3) becomes $\pi^-\phi^- = \phi^+\phi^-$ and $\pi^+\phi^+ = \phi^-\phi^+ - H(u,x)\phi^+\phi^+$, such that we can complete the Legendre transformation using Eqs. (6) and (7) to get Eq. (5). Note that the integrability condition in this space-time can be satisfied trivially because it simply reduces to Schwartz's theorem for the exchange of partial derivatives. Although the Hamilton density in Einstein-Rosen coordinates misses the term proportional to $H(u,x)$ and therefore speciously looks simpler, the spatial metric will be given by γ and has a nontrivial U dependence.

B. Functional Schrödinger states

In this part, we demonstrate how to construct the functional Schrödinger representation in the dual-null foliation. For a Cauchy problem with time direction t, the differential operator describing the Schrödinger equation $\hat{P}\psi = 0$ with Hamilton operator $\hat{H}(t)$ is given by $\hat{P} = i\partial_t - \hat{H}(t)$. To apply the above formalism to the Schrödinger representation, let us recall the definition of the light-cone coordinates $u = t - z$ and $v = t + z$, with z being a spatial direction. Hence, $\partial_t \to \frac{1}{2}(\partial_u + \partial_v)$ in this coordinate chart, and the Schrödinger equation becomes $\hat{P} = i(\partial_u + \partial_v) - 2\hat{H}(u, v)$ accordingly. We see that the evolution equation in this foliation looks involved due to the combination of evolution directions. In the functional Schrödinger representation, the differential operator will be constructed using Eq. (5). The difference from a quantum-mechanical state ψ is that in this representation, the states are wave functionals Ψ that read in the instantaneous fields ϕ as a configuration variable.

Consider the infinite-dimensional space of instantaneous field configurations within the embedding $\imath$ to be $\mathcal{C}(\Sigma_\pm) \ni \phi_\mp$ depending on the three-surface they are defined on. We can therefore define a generalized $\mathcal{L}^2$ space for the wave functionals. The corresponding formal measure space is given by $\mathfrak{M}_\pm = (\mathcal{C}(\Sigma_\pm), \mathcal{D}\phi^\mp)$, with infinite-dimensional uniform measure $\mathcal{D}\phi^\mp$. Let $\mathcal{L}^2(\mathfrak{M}_\pm)$ denote the space of square-integrable, $\mathcal{D}\phi^\mp$-measurable wave functionals Ψ: $\mathcal{C}(\Sigma_\pm) \to \mathbb{C}$ with the seminorm $\|\Psi\|_2 = (\int_{\mathcal{C}(\Sigma_\pm)} \mathcal{D}\phi^\mp |\Psi|^2)^{1/2} < \infty$. To define a proper norm, we need to divide out the wave functionals, yielding $\Psi[\phi] = 0$ almost everywhere with respect to the functional measure [9]. We note that the functional measure is elected to be a uniform measure. As a basis for the wave functional, we choose the eigenbasis of the field operator, such that all ϕ's become multiplicative operators that yield the classical field φ as an eigenvalue.

To represent the momenta π conjugate to ϕ in the configuration space, we must impose a consistent quantization prescription. We will impose the quantization prescription in the Minkowski region, where we also formulate our initial conditions. For the dual-null formulation, there exists a commonly used quantization that holds on every three-surface $\Sigma_\pm$, as well as on $\mathcal{S}$ [19]:

$$[\pi^\pm, \phi^\pm]_{\Sigma_\mp} = -i\delta^{(2)}(x, x')\delta(\xi_\pm, \xi'_\pm), \tag{8}$$

$$[\phi^+, \phi^-]_{\mathcal{S}} = -i\delta^{(2)}(x, x'), \tag{9}$$

where $\xi_+ = v$ and $\xi_- = u$. Relations (8) and (9) suggest the representation $\pi^\pm \to -i\frac{\delta}{\delta\phi^\pm}$. The resulting wave functional for the free-field theory can be constructed by the following ansatz:

$$\Psi[f](u, v) = N(u, v) \exp\left(-\frac{1}{2}[f]\mathcal{K}(u, v)[f]\right), \tag{10}$$

with $\mathcal{C}(\Sigma_+) \times \mathcal{C}(\Sigma_-) \ni f(x) = (\phi^+(x), \phi^-(x))^T$ as the field vector and the kernel matrix $\mathcal{K}_{AB}(u, v)$ where $A, B \in \{+, -\}$. The entries of $\mathcal{K}$ appearing in Eq. (10) are bilocal functionals $\mathcal{K}\colon \mathcal{C}(\Sigma_\pm) \times \mathcal{C}(\Sigma_\pm) \to \mathbb{C}$, $(f_1, f_2) \mapsto [f_1]\mathcal{K}[f_2]$ of the form

$$[f]\mathcal{K}[f] = \iint_{\Sigma_{A,B'}} \mathrm{dvol}_{A,B'} f^A(x) K_{AB}(x, x') f^B(x'). \tag{11}$$

It should be noted that the f^A's are only defined on the corresponding Σ_A, while the x-dependence in K_{AB} has to be interpreted with respect to $\mathbb{W}^4$. The primed index B' signals that the three-surface or volume integration, respectively, is associated with the primed coordinate. From here, we see that $\mathcal{K}_{AB}$ develops a dependence on the u and v coordinates. The field-independent part of Eq. (10) also depends on K_{AB}:

$$\frac{N(u, v)}{N(u_0, v_0)} = \exp\left(-\frac{i}{2}\int_{u_0}^{u}\int_{v_0}^{v}\int_{\mathcal{S}} \mathrm{d}^4x\sqrt{-g}\sum_{A,B} K_{AB}(x, x)\right). \tag{12}$$

A defining equation for the evolution kernel can be derived by plugging Eq. (10) into the functional Schrödinger equation and solving the resulting equation for the components K_{AB}.

In full generality, the solution to these equations seems daunting; however, in a plane-wave background, the high degree of symmetry simplifies them significantly. Extensive studies [15–17,20] of these space-times have shown that there occurs no mixing between outgoing and ingoing velocity fields. This leads to a decoupling of evolution directions and allows us to focus on the direction that crosses the plane wave—i.e., the ingoing fields ϕ^- traveling along the u direction. Physically, this makes sense, as we do not expect the ϕ^+ fields evolving parallel to the wavefront to be affected by it. For the fields ϕ^-, we can formulate a well-defined initial-value problem, since we can use Σ_- as an achronal set. Consequently, we perform a null reduction, such that we set the outgoing fields $\phi^+(v) \equiv 0$ per default; then Eq. (5) as well as Eq. (10) will depend on u alone, and the Schrödinger equation will reduce to $P = i\partial_u - 2H(u)$ [21]. As we can see, Eq. (10) simplifies such that we can use a Gaussian ansatz in ϕ^-, and the kernel matrix $\mathcal{K}_{AB}$ becomes a scalar bilocal function, because $K_{--} \coloneqq K$ will be the only nonvanishing contribution. In this specific situation, the resulting equation for K is a Riccati equation that can be transformed to the Klein-Gordon equation for the scalar modes φ^- by inserting [9,22]

$$K_k(u) = \frac{-i}{\sqrt{-g}} \partial_u \ln\left(\frac{\varphi^-(u;k)}{\varphi^-(u_0;k)}\right). \quad (13)$$

Fortunately, the exact solution to the mode equation on plane-wave space-times is known [23]. For purely ingoing modes, the solutions can be constructed analytically using Huygens's principle [16,17] to be of the form $\varphi_k^-(u,x) = \Omega(u)e^{i\phi_k}$, where $\Omega = |\det(\gamma)|^{-1/4}$ and

$$\phi_k = \frac{k_0}{2}\Xi_{ab}x^a x^b + k_i E^i_a x^a + k_0 v + \frac{F^{ij}k_i k_j}{2k_0}. \quad (14)$$

Here, the shear-expansion tensor defined by $\Xi_{ab} = \dot{E}^i_a E_{ib}$ describes the distortion along the u direction after crossing the wave, and the k's come from a spatial Fourier transformation. Employing the total differential $\mathrm{d}\phi_k$ [16], we can deduce the u derivative that appears in Eq. (13) to find

$$K_k(u) = \frac{1}{\sqrt{-g}}\left[\frac{k_0}{2}\dot{\Xi}_{ab}x^a x^b + k_i \dot{E}^i_a x^a + \frac{\gamma^{ij}k_i k_j}{2k_0} - i\frac{\dot{\Omega}}{\Omega}\right]. \quad (15)$$

Note that K develops a real and an imaginary part. With this in hand, we are equipped to calculate the full wave functional in order to study its behavior near the focusing singularity.

IV. SHOCKWAVE

Since we are interested in how the focusing singularity affects ingoing modes after passing through the wave, we consider the example of a shockwave $H_{ab}(u) = \lambda^{(a)}\delta(u)\delta_{ab}$ at $u = 0$, described by a Dirac delta distribution. Here, $\lambda^{(a)}$ is the eigenvalue that denotes the physical amplitude of the wave in the x^a direction.

A. Brinkmann coordinates

For this analysis, we will start with Brinkmann coordinates because of the wave's explicit appearance in the metric. The focusing singularity can be shown to occur at a distance u_f from the wave front that is inversely proportional to $\lambda^{(a)}$. To see this, we first calculate the explicit form of the vierbeins by solving $\ddot{E}_{ia} = H_{ab}E^b_i = \lambda^{(a)}\delta(u)E_{ia}$ with boundary conditions at $\mathscr{I}^\pm$: $\lim_{u\to\pm\infty} E^{\pm a}_{\ i} = \delta^a{}_i$. We find for the shockwave with the corresponding boundary condition

$$E^\pm_{ia} = \delta_{ia}(1 \mp \lambda^{(a)}u\Theta(\mp u)), \quad (16)$$

with $\Theta(u)$ being the Heaviside theta distribution. Insertion of Eq. (16) into Eq. (2) shows that the metric becomes degenerate at the value $u^\pm_\mathrm{f} = \mp 1/\lambda^{(a)}$, as stated before. The pair of focusing points is due to the past or future evolution. In the following, we will restrict ourselves to only the future development of the ingoing fields and suppress the superscript of E^-_{ia} (we choose to start at $\mathscr{I}^-$, which fixes the boundary condition there). Because we have chosen a symmetric setup by setting the shockwave to $u = 0$, we could easily carry out the calculation for the past development by choosing the opposite sign. In order to calculate the explicit form of $K(x,x')$, we need the metric $\gamma_{ij} = \delta_{ij}(1 + \lambda^{(i)}u\Theta(u))(1 + \lambda^{(j)}u\Theta(u))$ and its determinant $\gamma = (1 + \lambda^{(1)}u\Theta(u))^2(1 + \lambda^{(2)}u\Theta(u))^2$. The last relevant contribution in Eq. (14) is the shear-expansion tensor

$$\Xi_{ab} = \delta_{ab}\frac{-\lambda^{(a)}\Theta(u)(1 + \lambda^{(b)}u\Theta(u))}{(1 + \lambda^{(a)}u\Theta(u))^2}. \quad (17)$$

Without loss of generality, we omit the second term appearing from the differentiation, which is proportional to $u\delta(u)$, and hence vanishes identically for every value of u. The tensor in Eq. (17) describes the distortion of geodesics, and consequently the focusing at u_f for $\lambda^{(a)} < 0$. Such an eigenvalue will definitely exist for gravitational waves, since $H(u,x)$ is traceless. The shear-expansion tensor manifests a memory effect—that is, it is zero unless the field has encountered the plane wave. In other words, by looking at this tensor, we can determine immediately whether a scalar field has met a plane wave or not. As we can see, γ^{ij} diverges for $u \to u_\mathrm{f}$, as well as Ξ_{ab}, Ω, and E^i_a, while Eq. (16) and γ_{ij} approach zero. The u derivative of Ξ_{ab} is then given by

$$\dot{\Xi}_{ab} = \delta_{ab}\left[\frac{(\lambda^{(a)}(\lambda^{(b)}u+2) - \lambda^{(b)})\lambda^{(a)}\Theta(u)}{(1+\lambda^{(a)}u\Theta(u))^3} + h(u)\right]. \quad (18)$$

Besides the singular contribution at the focusing point, Eq. (18) will also develop a divergence at $u = 0$, because the second term $h(u)$ is just proportional to the shockwave $H_{ab}(u) \propto \delta(u)$ for all $u \in \mathbb{R}$. This divergence is not physical, however, as it simply stems from the choice of $H(u,x)$ as delta distribution. As we are not interested in the wave itself, we will set this term to zero due to the vanishing support at $u \neq 0$. For more realistic smooth profile functions, it remains finite in any case.

The functional Schrödinger representation is particularly useful to studying the influence of space-time singularities on quantum field theory. Formerly, the concept of quantum completeness [6–9] has been used to investigate the Schwarzschild and Kasner singularity. Those space-times admit a globally hyperbolic slicing in which the singularity is not a part of the manifold. In fact, we could see it as a geodesic border towards which quantum fields could leak into the classically singular configuration. Structurally, this type of singularity is different from the null singularity in plane-wave space-times. The former space-times admit a spacelike singularity where geodesics (and so space-time itself) end abruptly. In our case, the space-time itself is perfectly regular at the focusing point if one considers the fully extended space-time in Brinkmann coordinates. However, $(\mathbb{W}^4, g)$ is null-geodesically incomplete because of the focusing on Σ_{f-}. Due to the intersection of geodesics, we lose predictability, but we could still cross Σ_F in Brinkmann coordinates. In Einstein-Rosen coordinates, we would instead find that at the focusing point the Einstein-Rosen patches $(\mathbb{W}^{\pm}, g^{\pm})$ end abruptly. While there is no analytic extension in Schwarzschild and Kasner space-times, we must adapt the scientific objective to the situation where space-time continues beyond the singular hypersurface. Hence, for the presented setup, the research question should be whether or not quantum field theory develops pathologies while crossing the focusing hyperplane. In other words, we investigate whether the wave functionals remain normalizable throughout the whole evolution such that $\|\Psi\|_2^2 < \infty,\ \forall\, u$ on $(\mathbb{W}^4, g)$.

Taking Eq. (16), we can derive all relevant contributions that arise in Eq. (15) where we can identify the problematic contribution to be the first term in Eq. (18). We choose a linearly polarized plane-wave [15] with $\lambda^{(1)} = 1$ and $\lambda^{(2)} = -1$, or equivalently, $H_{ab}(u) = \sigma^z_{ab}\delta(u)$, where $\sigma^z_{ab} = \mathrm{diag}(1, -1)$. In this system, the focusing occurs at $u_f = 1$; the problematic value is the case of the negative eigenvalue. This choice greatly simplifies Eq. (18), since we are only interested in the region close to the singularity—i.e., $u \approx 1$, where $\Theta(u_f \pm \Delta u) = 1$ for $0 < \Delta u < 1$—such that the second term in Eq. (18) stays zero. These assumptions simplify Eq. (18) further, such that the leading singularity is given by $\dot{\Xi}_{22} \to 1/(1-u)^2$, which is clearly divergent at the focusing point. In fact, all other contributions in the real part of Eq. (15) scale similarly. We use our knowledge to determine the behavior of $\Psi[\phi^-](u)$, first for the field-independent part. It is worth pointing out that the field-independent part of the wave functional $N(u)$ is usually referred to as the normalization due to its resemblance to a normalization constant in quantum mechanics. Due to its time dependence, however, it can not be chosen to compensate the norm of the field-dependent part at all times so as to "normalize" the functional. The resulting state would clearly fail to solve the Schrödinger equation for later times. We are interested in the norm of Eq. (10) in close proximity to u_f, and therefore only consider the most divergent terms. The normalization $|N|^2(u)$ is determined by the imaginary part of Eq. (15):

$$\frac{\dot{\Omega}}{\Omega} = \frac{-(\lambda^{(1)} + \lambda^{(2)} + 2\lambda^{(1)}\lambda^{(2)}u)\Theta(u)}{2((1+\lambda^{(1)}u\Theta(u))(1+\lambda^{(2)}u\Theta(u)))}, \tag{19}$$

where we use $\Omega = |(1+\lambda^{(1)}u\Theta(u))(1+\lambda^{(2)}u\Theta(u))|^{-1/2}$. The expression can be simplified by assuming $\lambda^{(1)} = 1$, $\lambda^{(2)} = -1$, and $u > 0$ to be $\dot{\Omega}/\Omega = u/(1-u^2)$. As we see, this term is divergent in the limit $u \to 1$, even after performing the u integration in Eq. (20) which yields $\ln(1-u^2)$ in the exponent. We then find for $0 < u_0 < 1$

$$\frac{|N(u)|^2}{|N(u_0)|^2} = |1-u^2|^{\mathrm{vol}(\Sigma_-)}, \tag{20}$$

where the volume factor $\mathrm{vol}(\Sigma_-)$ comes from the integration over the hypersurface Σ_- in Eq. (20) and can be interpreted as an infrared regulator. Due to the Heaviside distribution, $N(u)$ is trivially equal to 1 for all $u < 0$, as in this case space is just a Minkowski patch.

Similarly, we can address the field-dependent part. First, we note that only the real part of Eq. (15) contributes. The determinant g in the Brinkmann chart is just given by -1, and is hence not u dependent. Altogether, we find that

$$\|\exp(-[\phi^-]\mathcal{K}[\phi^-])\|_2^2 = \sqrt{\left|\frac{C}{\mathrm{Det}(-g\mathrm{Re}(K))}\right|} \propto |1-u|^{\frac{\Lambda}{2}}, \tag{21}$$

where the last expression is the leading behavior near the focusing singularity $u_f = 1$. We have absorbed all constants into C and introduced the ultraviolet cutoff Λ that regulates the infinite dimensionality of the field space. We will comment on this in more detail later. It should be noted that because the space-time itself does not terminate at the focusing singularity, we need to comment on the evolution beyond u_f. As can be checked by the explicit form of Eq. (15), the kernel function resembles a Minkowski limit for $u \to -\infty$ such that $\Psi[\phi^-](u) \to \Psi_{\mathbb{M}^4}[\phi^-]$, with $\Psi_{\mathbb{M}^4}[\phi^-] = N_0 \exp(-[\phi^-]\mathcal{K}_{\mathbb{M}^4}[\phi^-])$ being the Minkowski wave functional and $\mathcal{K}_{\mathbb{M}^4}$ being the square root of the inverse Minkowski propagator in the dual-null foliation. In the limit $u \to \infty$, we see that the terms dominating at u_f are subdominant, and Eqs. (20) as well as (21) approach constant values. Although the terms tend asymptotically to a Minkowski solution, the contributions from Ξ_{ab} will not vanish, which accounts for the gravitational memory of those solutions that have crossed the gravitational wave.

B. Einstein-Rosen coordinates

From the perspective of Einstein-Rosen coordinates, the setup is similar to the Kasner or Schwarzschild case, where the singularity marks the geodesic border. However, as we see in Fig. 3, the degenerate hypersurface Σ_{u^-} with the focusing singularity on the patch $(\mathbb{W}^-, g^-)$ is just a Minkowski hypersurface for $(\mathbb{W}^+, g^+)$, the other Einstein-Rosen patch. Performing the diffeomorphism $\{u, v, x^a\} \to \{U, V, y^i\}$, we can deduce the kernel in $(\mathbb{W}^-, g^-)$ to be

$$K(U) = \frac{1}{\sqrt{-\gamma}}\left[\frac{\gamma^{ij}k_i k_j}{2k_0} - i\frac{\dot{\Omega}}{\Omega}\right]. \tag{22}$$

In the limit of large U values, it reduces precisely to the kernel for a Minkowski functional [24], and while

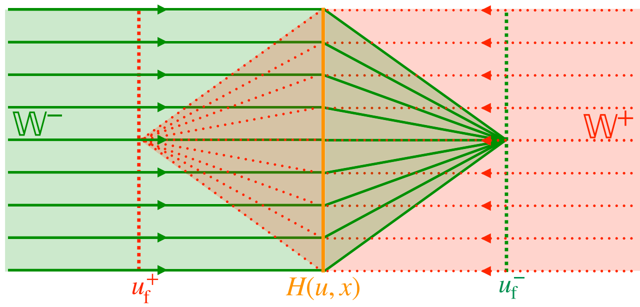

FIG. 3. Focusing that occurs in the different Einstein-Rosen patches $\mathbb{W}^-$ and $\mathbb{W}^+$ is described by the green and red lines. Einstein-Rosen coordinates do not extend beyond the null singularities, such that $\mathbb{W}^-$ is given by the green shaded and $\mathbb{W}^+$ by the red shaded area. The diamond-shaped overlap belongs to both patches. Brinkmann patches $\mathbb{W}^4$ have nondegenerate focal planes, and $\mathrm{vol}(\Sigma_{f^-})$ is nonzero, in contrast to $\mathrm{vol}(\Sigma_{f^-})$ in Einstein-Rosen coordinates at the null singularities.

approaching the focal point, it generates an imaginary part, as well as a nontrivial U dependence. Calculating $\|\Psi\|_2^2(U)$ in this coordinate neighborhood yields the same result as Eq. (20) for the normalization. However, the volume of the singular hypersurface $\mathrm{vol}(\Sigma_-)$ is in principle infinite in Brinkmann coordinates. Since none of the x^a directions degenerate, the volume of a Brinkmann submanifold is equal to a submanifold of a dual-null foliated Minkowski space-time. However, for the Einstein-Rosen metric, the volume $\mathrm{vol}(\Sigma_-)$ tends to 0 because the submanifold degenerates in at least one of the y^i coordinates; this can be seen from the form of the vierbeins at the focal point. Hence, in Einstein-Rosen coordinates, $|N(U)|^2 \to 1$.

The field-dependent part again requires the regularization of the functional determinant, similar to Eq. (21). For $\mathrm{Det}(-\gamma \mathrm{Re}(K))$ and Eq. (22), we find by explicitly writing out the eigenvalues that

$$\mathrm{Det}(-\gamma \mathrm{Re}(K)) = \prod_{k_1,k_2} \left(\frac{1+U}{1-U} k_1^2 + \frac{1-U}{1+U} k_2^2 \right). \tag{23}$$

Again, this bears great resemblance to the Minkowski result. We can now employ a zeta function regularization in order to obtain a finite result and extract from it the relevant U behavior. A very similar calculation was already carried out in the context of the generating functional of Thirring models [25] and may be used with slight alterations. To do so, we rewrite the infinite product over the eigenvalues λ_k in the form

$$\prod_k \lambda_k = \prod_{n\in\mathbb{Z}^2} \left(\frac{2\pi}{L}\right)^2 \rho^{\mu\nu} \left(\frac{1}{2}\mathbb{1}_\mu + c_\mu + n_\mu\right)\left(\frac{1}{2}\mathbb{1}_\nu + c_\nu + n_\nu\right) \tag{24}$$

by choosing toroidal compactification of length L to regulate the infinite volume for now. Here, n_μ's are reference coordinates on the torus, ρ is the reference metric, c_μ is a constant shift, and $\mathbb{1}_\mu = (1,1)^T$ is the one-vector. Equation (24) can be reexpressed in terms of a generalized ζ function:

$$\zeta(s) \coloneqq \sum_k (\lambda_k)^{-s}. \tag{25}$$

The desired determinant will then be given by the value of the derivative of the zeta function at $s=0$. Writing the zeta function in terms of a Mellin transform and using the Poisson resummation formula, we arrive at

$$\zeta(s) = \frac{\Gamma(s-1)}{\Gamma(s)} \pi^{2s-1} \sqrt{\rho} \sum_n (\rho_{\mu\nu} n^\mu n^\nu)^{\frac{s-1}{2}} \times \exp\left[-2\pi n^\mu \left(c_\mu + \frac{1}{2}\mathbb{1}_\mu\right)\right]. \tag{26}$$

Fortunately, due to the form of Eq. (23), we read off $c_\mu = -\frac{1}{2}\mathbb{1}_\mu$ such that the phase factor is equal to unity, $\rho = \mathrm{diag}(\frac{1+U}{1-U}, \frac{1-U}{1+U})$, and the derivative at zero is given by

$$\zeta'(0) = \pi^{-1}\sqrt{\rho}\sum_n (\rho_{\mu\nu} n^\mu n^\nu)^{-\frac{1}{2}}. \tag{27}$$

The sum is well defined in terms of the Epstein zeta function and gives a result independent of U. The technical reason for this is that the determinant of the metric in Eq. (23) is constant. In Ref. [26], Hawking gives a slightly different argument, noting that the integrals depending on the infrared regulator must vanish, confirming that the determinant will become constant. In the Appendix, we show that the zeta-regularized determinant in Eq. (23) will just become a constant that can be set to 1. In Einstein-Rosen coordinates, the volume element $\mathrm{vol}(\Sigma_-)$ of the transversal directions appearing in the expression for the normalization factor [Eq. (20)] degenerates with respect to one spatial direction, as seen from Eq. (16) explicitly. While for Brinkmann the transversal submanifold $\Sigma_- = \mathcal{V} \times \mathcal{S}$ is a flat three-manifold like in a dual-null foliated Minkowski space-time, the spatial part $\mathcal{S}$ degenerates in Einstein-Rosen coordinates. Suppose we start with a spatially limited bundle of rays with a quadratic cross section at $\mathscr{I}^-$. After traversing the wave, the cross section will elongate in the x_1 direction and contract in the x_2 direction. Albeit the elongation is finite (here it is doubled), the contraction will be total, such that the cross section degenerates. Hence, the volume $\mathrm{vol}(\Sigma_-)$ essentially captures a subtle dependence on U such that one y coordinate degenerates and the resulting volume shrinks to zero when Σ_- becomes a null two-manifold:

$$\lim_{U\to 1} |1-U^2|^{\mathrm{vol}(\Sigma_-)} = 1 \quad \text{with} \quad \lim_{U\to 1} \mathrm{vol}(\Sigma_-) = 0. \tag{28}$$

Therefore, the contribution to the overall probability stemming from the field-independent part of the wave functional will simply equal 1 at the focal point. As we have seen prior, the functional determinant will approach a constant too, that may be set to 1 by an appropriate choice of the constant C. Thus, we see that the wave functional remains normalizable throughout the entire evolution, and its norm $\|\Psi\|_2^2(U) \to 1$ at the focal point.

The results in Secs. IV A and IV B seem to be in tension, as the probabilities do not agree. This effect is due to the gravitational memory. Unlike the examples studied in Refs. [6,9], the geometry does not trivialize the theory, and the singular configuration is populated, as the wave functional enjoys probabilistic support. Moreover, it will be reached inevitably by the essentially Minkowski-like evolution, resulting in a probability of exactly 1 at the geodesic border. Due to the fact that Brinkmann coordinates are a nontrivial extension of the Einstein-Rosen metric, a direct comparison is intricate. This is due to the difference in the functional spaces over which the functional integration is understood in Eq. (21). The solution space to the Brinkmann d'Alembert operator P_B is a unification of the solution spaces associated with both Einstein-Rosen patches. In other words, as the Brinkmann patch is an extension of the Einstein-Rosen patch, we are also integrating over modes that lack an interaction with the plane wave in the past, as illustrated by Fig. 3. As can be checked explicitly, the solution space to $P_B\phi = 0$ consists of solutions that show a memory effect, since they have passed the wave and solutions that are plane waves. Considering the explicit form of the mode solutions for $u > 0$, Eq. (14) describes two different kinds of modes in Brinkmann space-time: One kind corresponds to the (green) $\mathbb{W}^-$ patch and show a gravitational memory because of the nontrivial vierbeins $E^-_{ia} = \delta_{ia}(1 + \lambda^{(a)}U)$, while the other modes belong to the (red) $\mathbb{W}^+$ patch and have $E^+_{ia} \equiv \delta_{ia}$—that is to say, they have an empty gravitational memory in the future development.

The inclusion or omission of these modes will naturally affect the induced probability measure. For a direct comparison, consider a finite region in the patch where both coordinates overlap—i.e., the right part of the diamond. The functional measure in Brinkmann space-time will not distinguish between the $\mathbb{W}^-$ modes and the $\mathbb{W}^+$ modes, and an integration on a hypersurface will therefore incorporate both modes. To make it comparable to an Einstein-Rosen development, we need to project out modes with empty gravitational measurement in order to match both results. The remaining modes with a nonempty gravitational memory will be confined within a region (right side of the diamond) that, to a Brinkmann observer, will continuously shrink until it ends at a caustic, thereby rendering the evolution the same as for the local Einstein-Rosen observer.

V. BACKREACTION

Ending inevitably in the coincidence limit, it is clear that the evolution requires some form of completion to remain physically viable. Classically, the energy-momentum tensor will diverge at the focal point, and it is therefore to be expected that backreactions may regularize the singular behavior, at least to some degree. While exact statements are difficult to come by, owing to the technical difficulty of the resulting expressions, some general observations can be made. It should be mentioned that we assume we will stay in the class of plane-wave space-times—that is, all perturbations of the wave front itself are ignored throughout this analysis. To get an estimate for the leading contribution, we focus on the strongest diverging component of the energy-momentum tensor [15]

$$T_{uu} = (\partial_u \varphi^-)^2 \approx (\partial_u \sqrt{|E|}^{-1})^2 \sim \frac{\dot{\mathcal{E}}^2}{\mathcal{E}^3}, \tag{29}$$

where we define $\mathcal{E} \coloneqq |\det(E)|$. Since our aim is to calculate a backreacted metric from the Einstein equation, we use our knowledge that the only nonvanishing contribution to the Ricci tensor will come from the uu component, and that in our case the Ricci scalar is zero, to find the Einstein tensor's uu component:

$$G_{uu} = R_{uu} = \frac{\ddot{\mathcal{E}}}{\mathcal{E}}. \tag{30}$$

Note that our ansatz for the backreacted metric is of Einstein-Rosen form, such that our objective is to construct an improved γ_{ij}. We may attempt to find a self-consistent solution to $G_{uu} = T_{uu}$ in terms of the tetrad E. In this case, the solution for the square root of the determinant is given by

$$\mathcal{E} = \frac{1}{\ln(1-u)} \tag{31}$$

to leading order as u approaches 1. We can construct a new γ as a diagonal matrix by treating Eq. (31) as the tetrad that corresponds to the degenerating entry while we set the remaining entry to be 1 for simplicity, as it would tend toward a constant anyway. This yields the backreacted metric

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\ln^2(1-u)} \end{pmatrix}. \tag{32}$$

In this approximate case, the expression for the shear tensor becomes very straightforward—in fact, it reduces to just one component, as there is only one nonvanishing component of $\dot{E}$,

$$\dot{\Xi}_{22} = \frac{1 - \ln(1-u)}{(1-u)^2 \ln^2(1-u)}. \quad (33)$$

This is still divergent as u approaches 1, but weakened through the logarithmic factor in the denominator. Approximating $K(u) \sim \dot{\Xi}_{22}$ shows that $\mathrm{Det}(-g\mathrm{Re}(K))^{-1/2}$ decreases more slowly than Eq. (21) but eventually approaches zero as well. The same holds for the imaginary part of the kernel,

$$\frac{\dot{\Omega}}{\Omega} = -\frac{1}{2(1-u)\ln(1-u)}. \quad (34)$$

The qualitative behavior is thus unchanged; however, the strength of the contraction near the focal point is attenuated. This feature seems to be systemic unless a disturbance in v direction of the wave front is considered. In this case, we leave the class of plane-wave space-times, giving $H(u, v, x)$ an explicit v dependence that induces a mode mixing between ϕ^+ and ϕ^- which intuitively leads to a defocusing. However, a singularity avoidance might only result from a much less idealized system, since a naive addition of a v dependence might result in a Khan-Penrose space-time [27], where the singularity becomes spacelike.

VI. CONCLUSION

This article investigates the evolution and consistency of massless quantum fields in a plane-wave space-time. We constructed the functional Schrödinger formalism in the dual-null embedding such that we can formulate a well-defined initial condition and boundary value problem along the Hamiltonian flow. On its own, this formalism has various applications—e.g., it can be used to study systems where evolution on null paths is favored, such as gravitational tunneling across horizons [28]. One objective of this article was to test the hypothesis of incompleteness by using quantum fields as probing devices. We found indeed that the null singularity will be populated by quanta that have passed the wave. This is not surprising, since the focusing of the plane wave is not fronted by any effect that can prevent the caustic, since here the space-time itself is Minkowski, and the field theory is free. In other words, we saw that unitarity of the dual-null evolution forces quantum fields into the caustic while keeping normalizability intact.

An interesting feature of these results comes from the comparison between the two different patches: while in Einstein-Rosen coordinates, we recovered that within one patch, there exists a unitary evolution for all initial conditions, and the probability is conserved throughout the evolution. Brinkmann coordinates show a steady decrease of the probability amplitude. Brinkmann patches are nontrivial extensions of Einstein-Rosen patches with respect to field configurations. Geometrically, the Brinkmann patch consists of Minkowski hypersurfaces that cover both Einstein-Rosen patches. This, in turn, implies for the field configurations that the Brinkmann configuration space $\mathcal{C}^{\mathbb{W}^4} \triangleq \mathcal{C}(\Sigma_u)$ is the combination of the configuration spaces of both Einstein-Rosen patches $\mathcal{C}^{\mathbb{W}^\pm} \triangleq \mathcal{C}(\Sigma_U^\pm)$ with hypersurfaces Σ_u and $\Sigma_U^\pm$ in Brinkmann geometry, and past/future geodesically incomplete Einstein-Rosen geometries, respectively. Hence, the functional measure space for Brinkmann $\mathfrak{M}^{\mathbb{W}^4}$ equals $(\mathcal{C}(\Sigma_u), \mathcal{D}\phi_{\mathbb{W}^4})$, which in turn equals $\mathfrak{M}^{\mathbb{W}^+} \cup \mathfrak{M}^{\mathbb{W}^-}$, where $\phi_{\mathbb{W}^4}$'s are the instantaneous field configurations on a Brinkmann null hypersurface. At the null singularity, $\mathfrak{M}^{\mathbb{W}^-}$ degenerates to a null set with respect to $\mathfrak{M}^{\mathbb{W}^4}$, which inevitably entails the vanishing of the probability amplitude for these configurations, as they have measure zero according to $\mathcal{D}\phi_{\mathbb{W}^4}$. In other words, the Brinkmann observer would describe a focusing of the fields into a caustic for gravitational waves. Albeit the results for the norm of the wave functionals seem to develop a tension at first glance, they are in agreement when properly considering the subtleties of the configuration spaces. As we saw, various subtleties arise from treating such infinite-dimensional measures that need due care. Situations like a quantum probing involve free fields described by Gaussian ground states which are tractable and lead to a well-defined measure. However, non-Gaussian deformations can be treated through a (time-dependent) Rayleigh-Schrödinger perturbation theory [7]. Since the functional Schrödinger equation describes the evolution of the field configurations itself, there is limited information we can extract concerning the individual degrees of freedom.

It is always difficult to reflect on the physical nature of singular solutions in general relativity. One of the more general statements known from globally hyperbolic space-times, such as Kasner universes, is the Belinskii-Khalatnikov-Lifshitz (BKL) conjecture, claiming that in the approach to the singular point, velocity terms will dominate over the potential terms, $\dot{\varphi} \ll \varphi'$ [29,30]. Although the original formulation is tied to spacelike singularities, a similar BKL behavior has been observed for the null singularity at inner horizons in Kerr black holes [31]. In the present example, we consider the derivative of the scalar field $d\varphi = (d\Omega + \Omega d\phi_k)e^{i\phi_k}$, where the $d\Omega$ term only arises for the u derivative. To estimate the behavior close to the focal point, we use Eq. (14), the explicit form of $\dot{\Omega}$, and $\partial_a \phi_k = k_i E^i_a + k_0 \Xi_{ab} x^b$, the spatial derivative of the phase. The scaling shows that the kinetic term will go as $\dot{\varphi} \sim \dot{\Omega} + \Omega\dot{\Xi} \sim 1/(1-u)^{5/2}$, while the spatial derivative will be $\varphi' \sim \Omega\Xi \sim 1/(1-u)^{3/2}$ in leading order.

Our analysis shows explicitly that, although the equations of motions might suggest a BKL-type behavior, the wave functional in plane-wave space-times shows major differences from the wave functional in a Kasner universe [9] and that of a Schwarzschild space-time [6]. Most notably, the probability density in these previously examined cases decreases to zero towards singularity.

The explanation of this difference in qualitative behavior lies in the physics of the underlying space-time: plane-wave space-times provide no curvature that can act as a source for dissipation such as in Kasner or Schwarzschild cases—i.e., they provide a unitary evolution in the Schrödinger picture, while this cannot hold on a time-dependent, curved manifold like Kasner or Schwarzschild [32,33]. In this case, information from the quantum sector gets transferred to the classical background [8], leading to the question of how one can read all pieces of information in this representation.

Another distinctive feature of both of these examples is the pronounced anisotropy developed by the singular hypersurface. In both cases, at least one of the spatial directions grows without bound as the others contract, and thus complete coincidence is avoided. In the present example, this is not the case, as all spatial directions are either contracting or bounded, and therefore nothing prevents a coincidence at the caustic.

In conclusion, we see that the caustic singularity may only be avoided when the strict symmetry requirements on the space-time are lifted. This may be achieved by the possibility of mixing the u and v dependency of the modes, either by allowing for the field to have self-interactions or alternatively, by considering backreactions on the space-time, leading to less restrictive geometries.

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APPENDIX: ZETA FUNCTION REGULARIZATION

Integration over the space of field configurations as performed in Eq. (21) leads to functional determinants that have to be regularized due to the infinite dimensionality of the configuration space $\mathcal{C}(\Sigma_-)$. A standard technique (cf. Refs. [26,34–37]) involves a generalized local zeta function of some operator $\mathcal{O}$ [37]:

$$\zeta\left(s;x,y\middle|\frac{\mathcal{O}}{\nu^2}\right) = \int_0^\infty \mathrm{d}\tau \frac{\nu^{2s}\tau^{s-2}}{\Gamma(s)}[L_\tau(x,y|\mathcal{O}) - P_0(x,y|\mathcal{O})] = \sum_j \left(\frac{\lambda_j}{\nu^2}\right)^{-s}\phi_j(x)\phi_j^*(x), \tag{A1}$$

where $L_\tau(x,x)$ represents the kernel of the operator, $P_0(x,x)$ are the zero modes, and ν is an arbitrary scale which we set to 1. The second equation clarifies the relation to functional determinants by introducing the eigenvalues λ_j corresponding to the eigenvectors $\phi_j(x)$ that are normalized as $\int \mathrm{d}^4x\sqrt{-g}\phi_a(x)\phi_b(x) = \delta_{ab}$ with respect to the background [26]. The convergence of Eq. (A1) in two dimensions is ensured whenever $\mathrm{Re}(s) > 1$. The functional determinant is then equivalent to

$$\mathrm{Det}(\mathcal{O}) = \prod_j^N \lambda_j = e^{-\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s;x,y|\mathcal{O})}|_{s=0}. \tag{A2}$$

Following Refs. [26,34], we use the explicit form of Eq. (15) in Einstein-Rosen coordinates and reformulate the determinant using Eq. (A2) to define the zeta function

$$\zeta(s;U|K) = \frac{\mathrm{Vol}}{4\pi^2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left(\frac{1+U}{1-U}k_1^2 + \frac{1-U}{1+U}k_2^2\right)^{-s}, \tag{A3}$$

where k_i's are the momenta in the directions y^i and Vol is a volume regulator. The above integral can be simplified by a transformation of the measures $\mathrm{d}k_1 \to f(U)\mathrm{d}\ell_1$ and $\mathrm{d}k_2 \to f^{-1}(U)\mathrm{d}\ell_2$, where we introduce $f(U) = \sqrt{(1+U)/(1-U)}$. It can be seen that the prefactors from the Jacobi determinants cancel each other, and the U dependence vanishes. After shifting to polar coordinates $\{\ell, \alpha\}$, the angular α integration just yields a constant, and we are left with an integral that can be analytically continued such that [35,38]

$$\int_0^\infty \mathrm{d}\ell\,\ell^z = 0, \tag{A4}$$

from which we get that $\zeta(s;U|K)|_{s=0}$ is identically zero, and so is the derivative. For the functional determinant in Einstein-Rosen coordinates, it follows immediately that $\mathrm{Det}(-\gamma\,\mathrm{Re}(K)) = \exp(-\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s;U|K))|_{s=0} = 1$ for all $U \in (0,1]$. In Ref. [26], Hawking gave a heuristic argument by introducing an infrared regulator ε and found that when replacing the lower integration bound with ε, the result yields ε^{2-2s} (for our dimensions), which goes to zero in the infrared limit.

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Correction: The inline equations in the second sentence after Eq. (3) and Eqs. (4)–(7) contained errors and have been fixed.