Worldline description of fractons

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We formulate the worldline approach of some field theories of fracton models and their symmetries. The distinction between the different models is based on their dispersion relations for the energy. In order to study the subsystem symmetries, we construct the Routhian functionals associated with the particle Lagrangians considered. We also build the pseudoclassical description of spinning fractons.

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I. INTRODUCTION

There has been recent interest in condensed matter physics to study lattice models with some peculiar properties, like the infinite degeneracy of the ground state and the existence of restricted motions along lines and planes. The excitations described by these models are known as "fractons" [1,2]. These states have zero energy but may have high momentum showing a mixing between the UV and the IR regions. Standard lore is that the long-distance properties of a lattice can be described in terms of field theories. It seems unlike this kind of lattices since, in a usual field theory, the ground state is unique, and low energy means suppressing the high momenta modes. In other terms, there is no UV/IR mixing.[1] However, the continuum limit of these lattice models can be formulated in terms of field theories.

In a series of papers [4–8] it has been shown that it is possible to consider field theories in $(d+1)$ dimensions with these peculiar properties. The main idea is to consider field theories that give up the invariance under the rotation group but only consider discrete subgroups like the lattice models. For instance, the dispersion relation in a scalar field theory in $d=2$, considered in [4], is given by

$$E^2 - \frac{p_x^2 p_y^2}{\mu^2} = 0, \tag{1}$$

where μ is a constant with dimension of momentum. This relation shows that there are infinite solutions with zero energy. For example, states with $p_x = 0$ and $p_y \neq 0$, and that these solutions have a restricted motion. In the previous example, the motion is along the spatial axis y. Therefore, this field theory reproduces the main features of the lattice models mentioned at the beginning.

The fracton models that we will consider here are classified as (m_E, n_p), where m_E is the exponent of the energy in the dispersion relation, and n_p is the overall power of the space momenta [9]. We will consider two classes of models: the first class is invariant under spatial rotations of 90° and depending quartically on the spatial momenta. In the second class, the models are invariant under rotations of 180° and depend quadratically on the spatial momenta. Therefore, one goes from the first class to the second class of models via the substitution $p_i^2 \to p_i$. These models will be identified as the $(m_E, n_p)'$ models.

As it is well known, Feynman was the first to consider the particle approach to quantum field theory [10,11], what today is known as the worldline approach. Here, we give the first steps towards constructing a worldline formulation of the field theory of some fracton models and their symmetries. One of the reasons to consider the construction of a worldline approach is that it brings us to a better understanding of the subsymmetries and interactions of fracton models. We will build particle models whose classical Lagrangian describe a point particle moving in space-time to carry out this study. As we shall see, using the dispersion relations of the field theory for the fractons will allow us to derive a Lagrangian describing the classical

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[1]In noncommutative field theories, this UV/IR mixing phenomena occurs, see, for instance, [3].

space-time motion of these particles, with no attempt of a quantization. The construction will be done in $(2+1)$ dimensions, but the extensions to other dimensions can be done using the same technique. An exciting aspect of the second class models is that, due to the quadratic character of the dispersion relation in the space momenta, it is possible to rewrite the kinetic term as a quadratic form in a Minkowski space with a particular nonstandard signature. The original coordinates are nothing but the light cone coordinates of this plane. Correspondingly, the fractons can be seen as particles moving along the light cone of this Minkowski space. We will also study all possible point symmetries of these models by constructing equations analogous to the Killing equations of a particle in a curved background. In [9] the subsystem symmetries of the fracton models were studied via a partial Fourier transform. In our approach, this study will be done using the Routhian functionals (see, for instance, [12]) associated with the particle Lagrangians, which is equivalent to taking a partial Legendre transform.

Finally, we will also consider the pseudoclassical worldline description of two fracton models by introducing a set of Grassman variables. There exist Dirac-like equations giving rise to dispersion relations quadratic in the energy (as it was done in the construction of the pseudoclassical description of the Dirac equation [13–15]).

The organization of the paper is as follows: in Sec. II, we will construct the particle Lagrangians associated with the (2,4) and (1,4) models, and we will study the related symmetries and subsymmetries. The same will be done in Sec. III, for the models $(2,2)'$ and $(1,2)'$. In Sec. IV, we will construct the pseudoclassical fractons. Finally, in Sec. V we will give some conclusions and outlook.

II. A CLASS OF LAGRANGIANS QUARTIC IN THE MOMENTA

In this section, we will construct two types of worldline Lagrangians with a mass-shell constraint quartic in the space momenta in $(2+1)$-space-time dimensions. We will consider the (2,4) and the (1,4) models with a quadratic and linear dependence on the energy, respectively.

A. The (2,4) model

The mass-shell condition that characterizes this model is given by

$$E^2 - \frac{p_x^2 p_y^2}{\mu^2} = 0, \tag{2}$$

which corresponds to the worldline version of the (2,4) model appearing in [16]. The canonical Lagrangian with the previous constraint is

$$L = -E\dot{t} + p_x\dot{x} + p_y\dot{y} + \pi_\lambda\dot{\lambda} + \frac{\lambda}{2}\left(E^2 - \frac{p_x^2 p_y^2}{\mu^2}\right) - \pi_\lambda\eta, \tag{3}$$

where the dot derivatives are done with respect to a parameter τ, which parametrizes the particle's worldline. The configuration space of the model has worldline coordinates (t, x, y, λ). The first four terms of (3) give the symplectic structure of the model, and the last two terms are the Dirac Hamiltonian, which contains the canonical Hamiltonian and the primary constraint π_λ multiplied by an arbitrary function $\eta(\tau)$. The equations of motion of this model are

$$\dot{t} = \lambda E, \qquad \dot{x} = \frac{\lambda p_x p_y^2}{\mu^2}, \qquad \dot{y} = \frac{\lambda p_x^2 p_y}{\mu^2}, \tag{4a}$$

$$\dot{p}_x = 0, \qquad \dot{p}_y = 0, \qquad \dot{E} = 0,$$
$$\dot{\lambda} = \eta, \qquad \dot{\pi}_\lambda = \frac{1}{2}\left(E^2 - \frac{p_x^2 p_y^2}{\mu^2}\right). \tag{4b}$$

Notice that λ is constrained by $\pi_\lambda = 0$, the primary constraint, which comes from the variation of η. The stability of the primary constraint gives the secondary one defined in Eq. (2). By eliminating the momenta, we get

$$L = -\frac{\dot{t}^2}{2\lambda} + \frac{3}{2}\sqrt[3]{\frac{\mu^2\dot{x}^2\dot{y}^2}{\lambda}}. \tag{5}$$

Varying with respect to λ on this last Lagrangian, we obtain its expression

$$\lambda = \sqrt{\frac{\dot{t}^6}{\mu^2\dot{x}^2\dot{y}^2}}. \tag{6}$$

Then, the Lagrangian in space-time variables is given by

$$L = \mu\sqrt{\frac{\dot{x}^2\dot{y}^2}{\dot{t}^2}}. \tag{7}$$

Since all the relevant quantities are one dimensional, the Lagrangian can also be expressed in terms of the absolute values

$$L = \mu\left|\frac{\dot{x}\,\dot{y}}{\dot{t}}\right|. \tag{8}$$

This Lagrangian is invariant under worldline diffeomorphisms. The associated first-class constraint is the mass-shell condition (2). The discrete global symmetries of this Lagrangian are spatial rotations of 90°, spatial inversions, and t inversion. In the following subsection, we will study the continuous symmetries.

1. Continuous symmetries of the (2,4) model

Now we study the point continuous symmetries of this particle model. In order to do that, we will consider the most general canonical point generator

$$G(t,x,y,\lambda,E,p_x,p_y,\pi_\lambda) = -E\xi^0(t,x,y) + p_x\xi^x(t,x,y) + p_y\xi^y(t,x,y) + \pi_\lambda\gamma(t,x,y) + F(t,x,y), \tag{9}$$

with ξ^0, ξ^x, ξ^y, and γ space-time functions to be determined. The presence of the function F is to take into account the possibility that the Lagrangian is invariant up to a total derivative [17]. The function G generates the following space-time transformations

$$\delta x^i = \{x^i, G\} = \xi^i(t,x,y), \qquad \delta t = \{t, G\} = \xi^0(t,x,y), \qquad i,j = 1,2. \tag{10}$$

The condition that G must satisfy to generate a symmetry is to be a constant of motion, namely $dG/d\tau = 0$, implying

$$-E(\dot{t}\partial_0\xi^0 + \dot{x}_i\partial^i\xi^0) + p_i(\dot{t}\partial_0\xi^i + \dot{x}_j\partial^j\xi^i) + \frac{\gamma}{2}\left(E^2 - \frac{p_x^2p_y^2}{\mu^2}\right) + \dot{t}\partial_0F + \dot{x}_j\partial^jF = 0, \tag{11}$$

where we have used the equations of motion (4). Notice that we have not made use of the secondary constraint $E^2 - p_x^2p_y^2/\mu^2 = 0$, whereas we have used the primary one $\pi_\lambda = 0$, since it vanishes identically at the Lagrangian level. Comparing the different orders in powers of the space-time momenta, we obtain the following system of partial differential equations, analogous to the Killing equations for a relativistic particle moving in a fixed curved space-time background

$$-\frac{\gamma}{2} + \lambda(\partial_x\xi_x + \partial_y\xi_y) = 0, \qquad -\lambda\partial_0\xi^0 + \frac{\gamma}{2} = 0, \tag{12a}$$

$$\partial_x\xi^0 = 0, \qquad \partial_y\xi^0 = 0, \tag{12b}$$

$$\partial_y\xi_x = 0, \qquad \partial_0\xi_x = 0, \tag{12c}$$

$$\partial_x\xi_y = 0, \qquad \partial_0\xi_y = 0, \tag{12d}$$

$$\partial_xF = 0, \qquad \partial_yF = 0, \qquad \partial_0F = 0, \tag{12e}$$

with $\partial_x \equiv \partial/\partial x$, $\partial_y \equiv \partial/\partial y$, and $\partial_0 \equiv \partial/\partial t$. Combining these equations, the system reduces to

$$\lambda(\tau)\left(\frac{\partial\xi_x(x)}{\partial x} + \frac{\partial\xi_y(y)}{\partial y}\right) - \frac{\gamma(\tau,t)}{2} = 0, \qquad \gamma(\tau,t) = 2\lambda(\tau)\frac{\partial\xi^0(t)}{\partial t}. \tag{13}$$

This system may be trivially solved, obtaining two kinds of symmetries: space-time dilations and translations only

$$\xi^0(t) = \chi_0 t + a_0, \qquad \xi_x(x) = \eta_0 x + a_1, \qquad \xi_y(y) = (\chi_0 - \eta_0)y + a_2, \qquad \gamma(\tau) = 2\lambda(\tau)\chi_0, \tag{14}$$

where χ_0, η_0, and as are integration constants. The generator (9) then reads

$$G(t,x,y,\lambda,E,p_x,p_y,\pi_\lambda) = -E(\chi_0 t + a_0) + p_x(\eta_0 x + a_1) + p_y((\chi_0 - \eta_0)y + a_2) + 2\pi_\lambda\lambda\chi_0. \tag{15}$$

Then, the space-time variable and momenta transformations read

$$\delta x = \eta_0 x + a_1, \qquad \delta y = -\eta_0 y + \chi_0 y + a_2, \qquad \delta t = \chi_0 t + a_0, \tag{16a}$$

$$\delta p_x = -\eta_0 p_x, \qquad \delta p_y = -\chi_0 p_y + \eta_0 p_y, \qquad \delta E = -\chi_0 E, \tag{16b}$$

$$\delta\lambda = 2\chi_0\lambda, \qquad \delta\pi_\lambda = 2\chi_0\pi_\lambda. \tag{16c}$$

The generators associated with the two dilatations, two space translations, and the time translation are given by

$$P_0 = -E, \qquad P_x = p_x, \qquad P_y = p_y, \tag{17}$$

$$D = -tE + yp_y + 2\lambda\pi_\lambda, \qquad \tilde{D} = xp_x - yp_y, \tag{18}$$

satisfying the nonvanishing commutation relations[2]

$$[D, P_0] = P_0, \qquad [D, P_x] = 0, \qquad [D, P_y] = P_y, \tag{19}$$

$$[\tilde{D}, P_0] = 0, \qquad [\tilde{D}, P_x] = P_x, \qquad [\tilde{D}, P_y] = -P_y. \tag{20}$$

Notice that there are no special conformal transformations. A fracton configuration like $E = 0$, $p_x = 0$ has restricted mobility since it only moves in the y spatial direction. The configuration preserves some of the symmetries of the Lagrangian; precisely, it maintains the two dilatations and the spatial translation along the y spatial direction.

[2] Using the conventions for the Poisson brackets $\{t, E\} = -1$, $\{x, p_x\} = +1$, and $\{y, p_y\} = +1$.

On the other hand, as we have seen, the number of continuous symmetries of the worldline Lagrangian is finite-dimensional in contrast with the ansatz of [9]. In this reference, the authors consider the symmetries of two $(1+1)$-dimensional subsystems obtained by doing the Fourier transform, one in p_x and the other in p_y, of the field theory Lagrangian. In any of these subsystems, there is an $(1+1)$-infinite-dimensional conformal symmetry. From the worldline approach point of view, this corresponds to considering the Routhian functionals, i.e., a partial Legendre transformation associated with the Lagrangian [12].

2. Routhian for the (2,4) model

Since the Lagrangian defined in Eq. (3) has two cyclic coordinates, x and y, the Routh's procedure can be carried out either along the x or y spatial direction. This approach is analogous to taking a partial Fourier transform of a field Lagrangian as was done in [9]. Let us consider the following partial Legendre transformation concerning the y coordinate given by

$$
\begin{aligned}
R_{(x)} &= L - p_x\dot{x} \\
&= -E\dot{t} + p_y\dot{y} + \pi_\lambda\dot{\lambda} + \frac{\lambda}{2}\left(E^2 - \frac{p_x^2 p_y^2}{\mu^2}\right) - \pi_\lambda\eta, \qquad (21)
\end{aligned}
$$

with the Hamilton's equations

$$
\dot{p}_y = 0, \qquad \dot{y} = \frac{\lambda p_x^2 p_y}{\mu^2}. \qquad (22)
$$

The Routhian functional (21) in configuration space is given by

$$
R_{(x)} = \frac{1}{2\lambda}\left(-\dot{t}^2 + \frac{\mu^2}{p_x^2}\dot{y}^2\right). \qquad (23)
$$

Because $p_x = \text{const}$, we can define $\tilde{y} \equiv \mu y/p_x$, then

$$
R_{(x)} = \frac{1}{2\lambda}(-\dot{t}^2 + \dot{\tilde{y}}^2), \qquad (24)
$$

which corresponds to the Lagrangian of a massless particle in $(1+1)$-space-time dimensions. Therefore, infinite-dimensional symmetries correspond to the conformal group in $(1+1)$-space-time dimensions. The same discussion holds if we perform the partial Legendre transform for the x coordinate. The authors of [9] argue that the full symmetry of the (2,4) model could be obtained by computing the closure of the infinite-dimensional conformal groups. In our case, the symmetries of the worldline Lagrangian for the fractons are not the closure of the symmetries of the two conformal groups. From our point of view, we notice that, in both cases, one of the momenta is kept fixed and should not be considered a canonical variable. Therefore, the symmetry of the entire model arises by considering only the transformations of the two conformal groups that do not depend on the value of the momenta that are kept fixed. These symmetries are precisely the translations and the dilations, agreeing with the study of the Killing equations made in the previous section.

B. The (1,4) model

The (1,4) model characterizes by the dispersion relation

$$
E = \frac{p_x^2 p_y^2}{\mu^3}, \qquad (25)
$$

and it can be described by the Lagrangian

$$
L = -E\dot{t} + p_x\dot{x} + p_y\dot{y} + \pi_\lambda\dot{\lambda} + \lambda\left(E - \frac{p_x^2 p_y^2}{\mu^3}\right) - \pi_\lambda\eta, \qquad (26)
$$

with equations of motion given by

$$
\dot{t} = \lambda, \qquad \dot{y} = \frac{2\lambda}{\mu^3}p_x^2 p_y, \qquad \dot{x} = \frac{2\lambda}{\mu^3}p_x p_y^2, \qquad (27)
$$

$$
\dot{p}_x = 0, \qquad \dot{p}_y = 0, \qquad \dot{\pi}_\lambda = E - \frac{p_x^2 p_y^2}{\mu^3}. \qquad (28)
$$

Notice that $\pi_\lambda = 0$ is a primary constraint, while the secondary constraint is given by Eq. (25). The configuration space Lagrangian is given by

$$
L = \frac{3\mu}{2}\sqrt[3]{\frac{\dot{x}^2\dot{y}^2}{2\dot{t}}}. \qquad (29)
$$

This Lagrangian is invariant under worldline diffeomorphisms. The associated second class constraint is the mass-shell condition (25). The discrete global symmetries of this Lagrangian are spatial rotations of 90°, spatial inversions, and $x \to -x$ and $y \to -y$.

1. Continuous symmetries of the (1,4) model

As in the previous section, we are interested in an analysis of the continuous space-time symmetries. To this end, let us consider the following point generator transformation

$$
\begin{aligned}
&G(t, x, y, \lambda, E, p_x, p_y, \pi_\lambda) \\
&\quad = -E\xi^0(t,x,y) + p_x\xi^x(t,x,y) + p_y\xi^y(t,x,y) \\
&\qquad + \pi_\lambda\gamma(\lambda, t, x, y). \qquad (30)
\end{aligned}
$$

Requiring that G generates a symmetry, we get the system for the unknown functions

$$\frac{\partial\xi_x(x)}{\partial x}+\frac{\partial\xi_y(y)}{\partial y}-\frac{1}{2}\frac{\partial\xi^0(t)}{\partial t}=0,\quad \gamma=\lambda(\tau)\frac{\partial\xi^0(t)}{\partial t}. \tag{31}$$

The solution is

$$\xi^0(t)=c_0+2(a_1+b_1)t,\qquad \xi_x(x)=b_0+b_1x,$$
$$\xi_y(y)=a_0+a_1y,\qquad \gamma=2\lambda(a_1+b_1), \tag{32}$$

with as, bs, and c_0 as arbitrary constants. As we see, the symmetries are the same of the (2,4) model at the Lagrangian level, but the rescaling in the dilations are different.

2. Routhian for the (1,4) model

In order to study the subdimensional symmetries of this model, let us perform a partial Legendre transformation along the y direction by considering the Routhian functional

$$R_{(x)}=L-p_x\dot{x}$$
$$=-E\dot{t}+p_y\dot{y}+\pi_\lambda\dot{\lambda}+\lambda\left(E-\frac{p_x^2p_y^2}{\mu^3}\right)-\pi_\lambda\eta, \tag{33}$$

with the equations of motion

$$\dot{t}=\lambda,\qquad \dot{y}=\lambda\frac{2p_x^2p_y}{\mu^3},\qquad p_x=\text{const},$$
$$\dot{E}=0,\qquad \dot{\pi}_\lambda=E-\frac{p_x^2p_y^2}{\mu^3}. \tag{34}$$

The Routhian functional in configuration space is

$$R_{(x)}=\frac{m}{2}\frac{\dot{y}^2}{\dot{t}}, \tag{35}$$

which is the Lagrangian for a one-dimensional nonrelativistic particle of mass $m\equiv\mu^3/2p_x^2$. For studying the symmetries of this model, we consider the following generator

$$G(t,y,E,p_y,\pi_\lambda)=-E\xi^0(t,y)+p_y\xi^y(t,y)$$
$$+\gamma(t,y)\pi_\lambda+F(t,y). \tag{36}$$

By requiring G be a generator of symmetries, we get the following ordinary differential equations

$$-\gamma+2\partial_y\xi^y=0,\qquad \gamma=\partial_0\xi^0, \tag{37a}$$

$$\partial_y\xi^0=0,\quad \partial_0\xi^y+\frac{2p_x^2}{\mu^3}\partial_yF=0,\quad \partial_0F=0, \tag{37b}$$

whose solution is given by

$$\xi^0(t)=b_0+2a_1t+a_3t^2,$$
$$\xi_y(t,y)=a_0+a_2t+a_1y+a_3ty,$$
$$F(y)=-m\left(a_2y+\frac{a_3}{2}y^2\right), \tag{38}$$

with as and bs as arbitrary constants. Then the generator reads

$$G(E,p_y,t,y)=-E(b_0+2a_1t+a_3t^2)$$
$$+p_y(a_0+a_2t+a_1y+a_3ty)$$
$$-m\left(a_2y+\frac{a_3}{2}y^2\right). \tag{39}$$

From here, we read the generators

$$H=-E,\qquad P=p_y,\qquad D=p_yy-2Et, \tag{40}$$

$$G=p_yt-my,\qquad C=-Et^2+p_yty-\frac{m}{2}y^2. \tag{41}$$

The nonvanishing commutators are given by

$$[D,H]=2H,\qquad [D,P]=P,$$
$$[C,P]=G,\qquad [D,G]=-G, \tag{42a}$$

$$[D,C]=-2C,\qquad [H,C]=-D,$$
$$[H,G]=-P,\qquad [G,P]=-m. \tag{42b}$$

This is the Schrödinger algebra in $(1+1)$-space-time dimensions. When we consider the Routhian functional associated with the coordinate y, we obtain the same symmetries. Notice the subsystem symmetries of the (1,4) model are different from the (2,4) model.

III. A CLASS OF LAGRANGIANS QUADRATIC IN THE SPACE MOMENTA

In this section, we will construct two types of worldline Lagrangians with a mass-shell constraint quadratic in the space momenta in $(2+1)$-space-time dimensions.

A. The $(2,2)'$ model

In this case, the model possesses the following dispersion relation

$$E^2=p_xp_y. \tag{43}$$

This dispersion relation gives rise to a sector with $E^2<0$. This could be a problem in the quantized version, but in the classical description we can restrict the analysis to the sector with $E^2\geq 0$. Notice that this model lives in a space with two times.

Now, introducing the space momenta

$$q^1 = \frac{1}{2}(p_x + p_y), \qquad q^2 = \frac{1}{2}(p_x - p_y), \tag{44}$$

the mass-shell constraint becomes

$$q^\mu \eta_{\mu\nu} q^\nu = (-q_0^2 + q_1^2 - q_2^2) = 0, \tag{45}$$

with μ, $\nu = 0$, 1, 2, where $q^\mu = (E, q^1, q^2)$, and $\eta_{\mu\nu} = (-1, 1, -1)$. Therefore, the mass-shell condition for this case can describe a massless particle in a Minkowski space with signature $(-2, 1)$, where -1 refers to the time coordinate and $+1$ to the space coordinate. The initial momenta are the light cone momenta in this space. From this point of view, a fracton is a particle moving along the light cone of a Minkowski space. The dispersion relation is invariant under the following transformation

$$\delta E = \frac{1}{2}\epsilon p_x, \qquad \delta p_x = 0, \qquad \delta p_y = \epsilon E. \tag{46}$$

A fracton configuration with $E = p_x = 0$ is also invariant. This describes a movement along the spatial direction y. The space-time configuration Lagrangian can be written either in terms of the original space-time variables as

$$L = \frac{1}{2\lambda}\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu = \frac{1}{2\lambda}(-\dot{t}^2 + \dot{x}\,\dot{y}), \tag{47}$$

where

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \tag{48}$$

or in terms of the new space coordinates

$$y_1 = \frac{1}{2}(x + y), \qquad y_2 = \frac{1}{2}(x - y), \tag{49}$$

with the Lagrangian taking now the form

$$L = \frac{1}{2\lambda}\dot{y}^\mu \eta_{\mu\nu}\dot{y}^\nu = \frac{1}{2\lambda}(-\dot{t}^2 + \dot{y}_1^2 - \dot{y}_2^2). \tag{50}$$

Therefore, this last Lagrangian describes a massless particle in $(1+2)$-space-time dimensions. It follows that L is invariant under the conformal group $O(3,2)$. The space-time symmetries, in this case, are finite dimensional. In fact, considering the generator of the point continuous symmetries to be

$$G(x^\mu, \lambda, p_\mu, \pi_\lambda) = p_\mu \xi^\mu(t, x, y) + \pi_\lambda \gamma(\tau, t, x), \qquad \mu = 0, 1, 2, \tag{51}$$

the Killing equations read

$$\partial^\mu \xi^\nu + \partial^\nu \xi^\mu = \eta(\tau) g^{\mu\nu}, \qquad \gamma = \lambda(\tau)\eta(\tau), \tag{52}$$

whose solutions give the conformal group in $(2+1)$-space-time dimensions.

1. Routhian for the $(2,2)'$ *model*

Let us consider the following Routhian functional along the y spatial direction

$$R_{(x)} = L - p_x \dot{x} = -E\dot{t} + p_y \dot{y} + \pi_\lambda \dot{\lambda} + \frac{\lambda}{2}(E^2 - p_x p_y) - \pi_\lambda \eta. \tag{53}$$

The Euler-Lagrange equations are

$$\dot{t} = \lambda E, \qquad \dot{y} = \frac{\lambda}{2} p_x, \qquad \dot{p}_y = 0, \qquad \dot{E} = 0, \qquad \dot{\pi}_\lambda = \frac{1}{2}(E^2 - p_x p_y). \tag{54}$$

The Routhian functional $R_{(x)}$ in configuration space is given by

$$R_{(x)} = -\frac{1}{4}\frac{\dot{t}^2}{\dot{y}} p_x. \tag{55}$$

This expression looks rather peculiar, but it can be understood by looking at the Routhian functional in coordinate space for the (1,4) model. Except for trivial factors, the two Routhian functionals can be obtained one from the other by exchanging time with the space coordinate y. This is since in the case of the Routhian functional for the (1,4) model, describing a nonrelativistic one-dimensional particle, the effective dispersion relation is of type (1,2), whereas in the actual case is of type (2,1), with an exchange of the energy with the spatial momentum.

By studying the Killing equations with the generator

$$G(t, y, E, p_y, \pi_\lambda) = -E\xi^0(t, y) + p_y \xi^y(t, y) + \pi_\lambda \gamma(t, y) + F(t, y), \tag{56}$$

and considering only primary first class constraints, we now find the system

$$\gamma + \partial_0 \xi^0 - \frac{1}{2}\partial_y \xi_y = 0, \qquad \partial_0 \xi^y = 0, \quad \gamma = 0, \tag{57a}$$

$$\partial_y \xi^0 - \frac{2}{p_x}\partial_0 F = 0, \qquad \partial_y F = 0, \tag{57b}$$

whose solution is given by

$$\xi^0(t,y) = b_0 + a_1 t + b_2 y + a_2 ty,$$
$$\xi_y(y) = a_0 + 2a_1 y + a_2 y^2,$$
$$F(t) = \frac{p_x}{2}\left(b_2 t + \frac{a_2}{2}t^2\right). \tag{58}$$

Then, the generators yields

$$G = -E(b_0 + a_1 t + b_2 y + a_2 ty) + p_y(a_0 + 2a_1 y + a_2 y^2) + \frac{p_x}{2}\left(b_2 t + \frac{a_2}{2}t^2\right), \tag{59}$$

where the as and bs are arbitrary constants. The generators of this symmetry (defining $p_x \equiv 2m$) read

$$P = p_y, \qquad D = 2p_y y - Et, \qquad H = -E, \tag{60a}$$

$$C = \frac{m}{2}t^2 - Ety + p_y y^2, \qquad G = mt - Ey. \tag{60b}$$

The nonvanishing commutators are given by

$$[D,H] = H, \qquad [D,P] = 2P,$$
$$[C,P] = D, \qquad [D,G] = -G, \tag{61a}$$

$$[D,C] = -2C, \qquad [C,H] = G,$$
$$[G,P] = H, \qquad [G,H] = m. \tag{61b}$$

We call this algebra the $(1+1)$-Schrödinger Carroll algebra. Notice that the last two commutators correspond to the ones appearing in Carroll algebra in two dimensions with a central charge (see, for instance, [18]).

B. The $(1,2)'$ model

The dispersion relation of this model is

$$E = \frac{1}{\mu}p_x p_y. \tag{62}$$

In this case, the energy is not bounded from below. This is not a problem at the classical level, as we are considering in this study. Various models in the literature present this problem, and different solutions are proposed for the specific cases.[3] In terms of the q variables defined in Eq. (44), it yields

$$E = \frac{1}{\mu}(q_1^2 - q_2^2), \tag{63}$$

which is the dispersion relation for a nonrelativistic particle of mass $\mu/2$ in a Minkowski space. This model can be treated as the previous ones. Starting from

$$L = -E\dot{t} + p_i\dot{x}^i + \pi_\lambda\dot{\lambda} + \lambda\left(E - \frac{1}{\mu}p_x p_y\right) - \pi_\lambda\eta, \; i, \; j = 1,2, \tag{64}$$

and proceeding as before we find

$$L = \mu\frac{\dot{x}\,\dot{y}}{\dot{t}}. \tag{65}$$

This Lagrangian is similar to the one of the (2,4) model of Sec. 2.1. The main difference is in the signs. In fact, $\sqrt{\dot{x}^2} = |\dot{x}|$ and not equal to $\dot{x}$. This implies different properties in the symmetries. The Lagrangian model is not invariant under rotations of 90°, but it is invariant under rotations of 180°.

C. Continuous symmetries of the $(1,2)'$ model

To study global symmetries of this model, let us consider the generator

$$G(t,x,y,E,p_x,p_y,\pi_\lambda) = -E\xi^0(t,x,y) + p_x\xi^x(t,x,y) + p_y\xi^y(t,x,y) + \pi_\lambda\gamma(t,x,y) + F(t,x,y). \tag{66}$$

For G to generate a symmetry, we find the system

$$\mu\frac{\partial\xi_x(t,x)}{\partial t} + \frac{\partial F(x,y)}{\partial y} = 0,$$
$$\mu\frac{\partial\xi_y(t,x)}{\partial t} + \frac{\partial F(x,y)}{\partial x} = 0, \tag{67a}$$

$$\frac{\partial\xi_x(t,x)}{\partial x} + \frac{\partial\xi_y(t,y)}{\partial y} - \frac{\partial\xi^0(t)}{\partial t} = 0,$$
$$\gamma(t) = \frac{\partial\xi^0(t)}{\partial t}. \tag{67b}$$

The solution of this system is

$$\xi^0(t) = a_0 + 2b_1 t + a_2 t^2, \tag{68}$$

$$\xi_x(t,x) = b_0 + b_2 t + b_1 x + a_2 tx, \tag{69}$$

$$\xi_y(t,y) = c_0 + c_2 t + b_1 y + a_2 ty, \tag{70}$$

$$F(x,y) = -\mu\left(b_2 x + c_2 y + \frac{a_2}{2}(x^2 + y^2)\right), \tag{71}$$

$$\gamma(t) = 2(b_1 + a_2 t), \tag{72}$$

[3]The typical example is the Dirac Lagrangian that has this problem at the classical level. Since, at the moment, we are not interested in the quantization of this model, we will not insist further on this point.

with as, bs, and cs again arbitrary constants. Then, the generator reads

$$\begin{aligned} G(t,x,y,E,p_x,p_y,\pi_\lambda) &= -E(a_0+2b_1t+a_2t^2)+p_x(b_0+b_2t+b_1x+a_2tx) \\ &\quad +p_y(c_0+c_2t+b_1y+a_2ty) \\ &\quad +2\pi_\lambda(b_1+a_2t)-\mu\left(b_2x+c_2y+\frac{a_2}{2}(x^2+y^2)\right). \end{aligned} \tag{73}$$

From here, we get the following generators

$$H=-E, \qquad P_i=p_i, \qquad G_i=p_it-\mu\delta_{ij}x^j, \tag{74a}$$

$$\begin{aligned} C &= -Et^2+tp_ix^i-\frac{\mu}{2}\delta_{ij}x^ix^j+2t\pi_\lambda, \\ D &= p_ix^i-2Et+2\pi_\lambda. \end{aligned} \tag{74b}$$

The generators satisfy the following nonvanishing commutators

$$\begin{aligned} &[D,H]=2H, \qquad [D,P_i]=P_i, \\ &[C,G_i]=G_i, \qquad [D,G_i]=-G_i, \\ &[D,C]=-2C, \qquad [C,H]=D, \\ &[G_i,P_j]=-\mu\delta_{ij}, \qquad [G_i,H]=P_i, \end{aligned} \tag{75}$$

with $i,\ j=1,\ 2$, which corresponds to the $(1+2)$-Schrödinger algebra.

1. The Routhian for the $(1,2)'$ model

The Routhian functional along the y spatial direction is given by

$$R_{(x)}=-E\dot{t}+p_y\dot{y}+\pi_\lambda\dot{\lambda}+\lambda\left(E-\frac{1}{\mu}p_xp_y\right)-\pi_\lambda\eta. \tag{76}$$

The relevant equations of motion are

$$\dot{t}=\lambda, \qquad \dot{y}=\lambda\frac{p_x}{\mu}. \tag{77}$$

As we see, it is not possible to solve all the velocities in terms of the momenta. Proceeding as we did for the Routhian functional of the (2,4) model, we define $\tilde{y}\equiv\mu/p_x$, from which

$$\dot{\tilde{y}}=\dot{t}. \tag{78}$$

This model is a branch of the Routhian functional of the (2,4) model. We got a one-dimensional massless particle that can move along the two branches of the light cone, whereas only one branch is allowed in this case. The symmetries of this case are translations and dilations.

IV. PSEUDOCLASSICAL SPINNING FRACTONS

In this section, we construct two pseudoclassical spinning fracton models associated with the previous (2,4) and $(2,2)'$ fracton models. Considering these particular models, we obtain Dirac-like equations giving rise to quadratic dispersion relations in the energy.

A. The spinning (2,4) model

The canonical (C) Lagrangian of the pseudoclassical spinning fracton is

$$\begin{aligned} L_C &= -E\dot{t}+p_x\dot{x}+p_y\dot{y}+\frac{i}{2}\lambda^0\dot{\lambda}^0-\frac{i}{2}\lambda^1\dot{\lambda}^1 \\ &\quad -\frac{e}{2}\Phi-i\chi\Phi_D, \end{aligned} \tag{79}$$

where the Grassmann variables λ^0 and λ^1 satisfy the nonvanishing Dirac brackets

$$\{\lambda^0,\lambda^0\}^*=-i, \qquad \{\lambda^1,\lambda^1\}^*=i, \tag{80}$$

where $\{\cdot,\cdot\}^*$ stands for the Dirac bracket associated to the second class constraints, and e and χ are Lagrange multipliers, with χ being a Grassmann variable (see, for instance, [13,14,19]). Here Φ denotes the mass-shell constraint

$$\Phi=E^2-\frac{p_x^2p_y^2}{\mu^2}, \tag{81}$$

while Φ_D stands for the odd constraint defined by

$$\Phi_D=-E\lambda^0+\frac{1}{\mu}\lambda^1p_xp_y, \tag{82}$$

which verifies

$$\{\Phi_D,\Phi_D\}^*=-i\Phi. \tag{83}$$

One can check that all of these brackets assumed for the Grassmann variables are the Dirac brackets that are obtained from the Lagrangian (79).

The continuous symmetries are the same as the bosonic counterpart except that, under dilations, the odd Lagrange multiplier should be scaled to compensate the overall scaling factor of the odd constraint, leaving invariant the Grassmann variables. The 90° rotation is still a symmetry in the Grassmann variables, which must transform as follows

$$\lambda^0\to\lambda^0, \qquad \lambda^1\to-\lambda^1. \tag{84}$$

At quantum level [20] we can realize the Grassmann in terms of the Pauli matrices

$$\hat{\lambda}^0=\frac{i}{\sqrt{2}}\sigma_3, \qquad \hat{\lambda}^1=\frac{i}{\sqrt{2}}\sigma_1, \tag{85}$$

where σ_1 and σ_2 are the standard Pauli matrices. Then, the fracton Dirac field equation for the fracton field $\Psi(t,x,y)$ is

$$\left(i\sigma_3\partial_t - \frac{1}{\mu}\sigma_1\partial_x\partial_y\right)\Psi(t,x,y) = 0. \tag{86}$$

The field $\Psi(t,x,y)$ also verifies the analog of the Klein-Gordon equation

$$\left(\partial_t^2 + \frac{1}{\mu^2}\partial_x^2\partial_y^2\right)\Psi(t,x,y) = 0. \tag{87}$$

Notice that the factorization of the quartic term is not unique, and it could also be done in terms of p_x^2 and p_y^2 instead of $p_x p_y$. In this case, we should have made use of three Grassmann variables instead of two, and the resulting Dirac equation should look different, more like the one that we have written in the following subsection for the $(2,2)'$ model.

B. The spinning $(2,2)'$ model

The spinning $(2,2)'$ model could be treated similarly to the (2,4) model. However, we will make use of the covariant form with a three-dimensional Minkowski space, where the dispersion relation assumes the form [see Eq. (45)]

$$\Phi = q^\mu \eta_{\mu\nu} q^\nu = 0,$$
$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,-1), \qquad \mu,\nu = 0,1,2. \tag{88}$$

Let us introduce a set of Grassmann variables, to say ξ^μ, with Dirac brackets given by

$$\{\xi^\mu, \xi^\nu\}^* = -i\eta^{\mu\nu}. \tag{89}$$

Then, defining $\Phi_D \equiv q^\mu \xi_\mu$, we get

$$\{\Phi_D, \Phi_D\}^* = -i\Phi. \tag{90}$$

We can easily write a Lagrangian in phase space by using the constraints Φ and Φ_D. This is given by

$$L = q^\mu \dot{y}_\mu + \frac{i}{2}\xi^\mu \dot{\xi}_\mu - \frac{e}{2}\Phi - i\chi\Phi_D, \tag{91}$$

where again e and χ are Lagrange multipliers, with χ a Grassmann variable. It is easy to show that this Lagrangian gives rise to the previous Dirac brackets [19]. The quantization of the Grassmann algebra leads to the Clifford algebra [20]

$$[\tilde{\gamma}_\mu, \tilde{\gamma}_\nu]_+ = \eta_{\mu\nu}. \tag{92}$$

The tilde gamma matrices $\tilde{\gamma}^\mu$ can be expressed in terms of the usual γ_μ as follows

$$\tilde{\gamma}^0 = \frac{1}{\sqrt{2}}\gamma^0, \qquad \tilde{\gamma}^1 = \frac{1}{\sqrt{2}}\gamma^1, \qquad \tilde{\gamma}^2 = \frac{i}{\sqrt{2}}\gamma^2. \tag{93}$$

Then the corresponding wave equation is

$$-i\tilde{\gamma}^\mu \partial_\mu \psi = 0, \tag{94}$$

where ψ corresponds to the "fracton" wave function.

V. CONCLUSIONS AND OUTLOOK

We have started the construction of the worldline approach to some field theories of fractons. We have analyzed particle Lagrangians whose mass-shell constraints give the dispersion relation of a set of models in $(2+1)$ dimensions. The extension to other models and directions follow the same lines. By writing the analog of Killing equations of a relativistic particle in a curved background, we have also studied the symmetries of the respective Lagrangians. The construction of the associated Routhian functionals to the Lagrangians allows finding the subsystem symmetries. The construction of pseudoclassical spinning fractons that lead in a natural way to the Dirac fracton equations was carried out.

A further study to consider is the analysis of three-dimensional worldline models. Here the number of possible models increases with the number of invariants. An interesting case would be the model $(2,2)'$ in three dimensions that, in analogy with the two-dimensional case, can be seen as a massless theory in a space-time with signature $(-1,1,-1,-1)$. We hope to return to this study in the future.

In the future, we would like to consider the self-interaction of fractons by using the geometrical interaction in the ordinary relativistic particle case, see for example [21]. The analysis will be done by using the path integral formulation.

Another interesting development will be to study the dynamics and symmetries of "fractonic strings." The canonical world sheet action for the (2,4) string with space-time coordinates $t(\tau,\sigma), x^i(\tau,\sigma)$, with $i = 1$, 2 and energy and momentum densities $\mathcal{E}(\sigma,\tau)$ and $\mathcal{P}_i(\sigma,\tau)$, is given by

$$S = \int d\tau d\sigma\Big(-\mathcal{E}\dot{t} + \mathcal{P}_i\dot{x}^i - \frac{\lambda}{2}\left(\mathcal{E}^2 - \frac{\mathcal{P}_x^2\mathcal{P}_y^2}{\mu^2}\right) - \rho(-\mathcal{E}t' + \mathcal{P}_i x'^i)\Big), \tag{95}$$

where the dot stands for partial derivative with respect to τ and the prime refers to differentiation with respect to σ with $0 \le \sigma \le \pi$. The Lagrange multipliers λ and ρ impose the first-class constraints of two-dimensional world sheet diffeomorphisms. Analogously, the $(2,2)'$-fractonic string has the following canonical action

$$S = \int d\tau d\sigma\Big(-\mathcal{E}\dot{t} + \mathcal{P}_i\dot{x}^i - \frac{\lambda}{2}(\mathcal{E}^2 - \mathcal{P}_x\mathcal{P}_y) - \rho(-\mathcal{E}t' + \mathcal{P}_i x'^i)\Big). \tag{96}$$

The study of these fractonic string models will be done in a subsequent work.

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