Separability of Klein-Gordon equation on near horizon extremal Myers-Perry black hole

Saeedeh Sadeghian [*]
Department of Theoretical Physics, Faculty of Basic Sciences, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran
and ICRANet-Mazandaran, University of Mazandaran, P. O. Box 47415-416, Babolsar, Iran

Hovhannes Demirchian [†]
Yerevan Physics Institute, 2 Alikhanian Brothers St., 0036, Yerevan, Armenia

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We investigate the separability of the Klein-Gordon equation on near horizon of d-dimensional rotating Myers-Perry black hole in two limits: (i) generic extremal case and (ii) extremal vanishing horizon case. In the first case, there is a relation between the mass and rotation parameters so that black hole temperature vanishes. In the latter case, one of the rotation parameters is restricted to zero on top of the extremality condition. We show that the Klein-Gordon equation is separable in both cases. Also, we solved the radial part of that equation and discuss its behavior in small- and large-r regions.

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I. INTRODUCTION

The four-dimensional Kerr black hole has been widely studied from various aspects. From the geometric point of view, Carter showed that it has integrable geodesics [1]. The extension to higher dimensions is known as Myers-Perry black hole. In d dimensions, it is described by one mass parameter and $N(=[\frac{d}{2}])$ number of rotation parameters. It has integrable geodesic equations just like its four-dimensional equivalent. Other extensions such as including the cosmological constant and Newman-Unti-Tamburino (NUT) charge does not change this behavior [2–4]. The integrability of the Klein-Gordon, Maxwell field and gravitational perturbation equations have been also studied on the d-dimensional Kerr-(A)dS-NUT space-time [2,3,5–9].

One can construct another solution to Einstein equations in the near horizon extremal limit [10–13] of Myers-Perry (NHEMP) black hole [14] (see [15–20] for recent studies). The extremal limit of the parameters describes a black hole (BH) with the biggest allowed angular momentum for a given BH mass. When one of the rotation parameters of the BH vanishes, we arrive at yet another solution of Einstein equations referred to as extremal vanishing horizon (EVH) geometry [21,22]. Although the generic extremal limit of Myers-Perry black hole exists in both even and odd dimensions, the special EVH limit exists only in odd dimensions.

A set of the Killing vectors of NHEMP obeys the structural relation of $SL(2,\mathrm{R})$ algebra corresponding to AdS_2 subspace. It has been demonstrated (e.g., [23–26]) that the Casimir element of this $SO(2,1)$ algebra gives rise to a reduced Hamiltonian system called spherical or angular mechanics, which contains all the necessary information about the near horizon geometry. In other words, a massive particle moving in the near horizon geometry of an extremal rotating black hole possesses a dynamical conformal symmetry, i.e., defines "conformal mechanics" [15,17–19,25–32], whose Casimir element can be viewed as a reduced Hamiltonian, which contains all the necessary information about the whole system.

An eye-catching difference between NHEMP and its vanishing horizon limit is that the latter has a larger isometry group [33–35]: it includes two copies of SL(2,R) corresponding to the AdS_3 subspace instead of one copy of SL(2,R) for the AdS_2 factor of the metric. This symmetry enhancement does not add to the number of independent constants of motion though [16].

The near horizon geometry of Myers-Perry black holes contains integrable and superintegrable systems like Rosochatius and Pöschl-Teller which are interestingly related to the Klein-Gordon equation through a geometrization procedure [36]. The angular part of the near horizon limit of a fully isotropic Myers-Perry black hole is a superintegrable mechanics called Rosochatius system. This is a

[*]s.sadeghian@umz.ac.ir
[†]demhov@gmail.com

direct generalization of the Higgs oscillator. Separation of variables in Rosochatius system results in a recursive family of one-dimensional Pöschl-Teller system. The quantum equivalents of Higgs oscillator, Rosochatius, and Pöschl-Teller systems can be associated with a Klein-Gordon equation on a static space-time.

In this work, we examine the mentioned near horizon extremal geometries of odd-dimensional Myers-Perry black hole by studying a probe scalar field which satisfies the Klein-Gordon equation. First, we take the near horizon geometry in the generic extremal case and use the elliptical coordinates in which the geodesic equation is separable. The $\mathrm{SL}(2,\mathrm{R})\times \mathrm{U}(1)^N$ isometry group of the background metric helps us to simply separate the part of the Klein-Gorodon equation related to AdS_2 subspace and azimuthal angles. We observe that the Klein-Gordon equation is separable in the elliptical coordinates. The total number of independent separation constants is as much as the number required for the integrability of the Klein-Gordon equation. This integrability is inherited from the hidden symmetries associated with the Killing tensors of background metric (see [37] for review). Interestingly, we find that the radial part of the Klein-Gordon equation is solved by Whittaker functions which are related to the confluent hypergeometric functions. By requiring smoothness in the large- and small-r regions, we fix the solution. Then, we study the near horizon EVH space-time and show the separability of the Klein-Gordon equation on that metric. In this case, the radial part is solved by Bessel's functions which are restricted to Bessel's function J_ν by demanding the smoothness in the large- regions.

II. KLEIN-GORDON EQUATION ON NEAR HORIZON EXTREMAL GEOMETRY

The near horizon extremal metric of odd-dimensional $(d = 2N + 1)$ NHEMP in Gaussian null coordinates was given in [14] and can be written in the Boyer-Lindquist coordinates as

$$ds^2 = \frac{F_H}{b(r_H)}\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + \sum_{i=1}^{N}(r_H^2 + a_i^2)d\mu_i^2 + \sum_{i,j=1}^{N}\gamma_{ij}D\varphi^i D\varphi^j, \quad (1)$$

where

$$D\varphi^i \equiv d\varphi^i + \frac{B^i}{b} r d\tau, \qquad B^i = \frac{2}{r_H^2}\frac{\sqrt{m_i - 1}}{m_i^2}. \quad (2)$$

The metric functions are

$$F_H = \sum_{i=1}^{N}\frac{\mu_i^2}{m_i}, \qquad b(r_H) = \frac{4}{r_H^2}\sum_{i<j}^{N}\frac{1}{m_i}\frac{1}{m_j},$$
$$\gamma_{ij} = (r_H^2 + a_i^2)\mu_i^2\delta_{ij} + \frac{1}{F_H}a_i\mu_i^2 a_j\mu_j^2. \quad (3)$$

Here, m_i's are some constant parameters related to the horizon radius (r_H) and N the number of rotation parameters (a_i's) corresponding to azimuthal coordinates φ^i defined by

$$m_i = \frac{r_H^2 + a_i^2}{r_H^2} \geq 1, \quad (4)$$

and μ_i are the latitudinal coordinates which satisfy the following relation:

$$\sum_{i=1}^{N}\mu_i^2 = 1. \quad (5)$$

The location of the horizon r_H is determined by the largest positive solution of

$$\Pi(r_H) = m r_H^2, \quad (6)$$

where $\Pi(r) \equiv \prod_{i=1}^{N}(r^2 + a_i^2)$ and m is a constant related to the mass of the Myers-Perry black hole.

The extremal limit is given by

$$\Pi'(r_H) = 2 m r_H. \quad (7)$$

By combining these two relations one finds that m_i's are restricted by

$$\sum_{i=1}^{N}\frac{1}{m_i} = 1. \quad (8)$$

By substituting $\mu_i = x_i/\sqrt{m_i}$ for $i = 1\ldots N$, the new form of NHEMP metric in an arbitrary odd dimension becomes

$$\frac{ds^2}{r_H^2} = A(x)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{i=1}^{N}dx_i dx_i + \sum_{i,j=1}^{N}\tilde{\gamma}_{ij}x_i x_j D\varphi^i D\varphi^j, \quad (9)$$

where

$$D\varphi^i \equiv d\varphi^i + k^i r d\tau, \qquad k^i = \frac{B^i}{b},$$
$$A(x) = \frac{\sum_{i=1}^{N}x_i^2/m_i^2}{r_H^2 b}, \qquad \sum_{i=1}^{N}\frac{x_i^2}{m_i} = 1,$$
$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{\sum_l^N x_l^2/m_l^2}\frac{\sqrt{m_i - 1}x_i}{m_i}\frac{\sqrt{m_j - 1}x_j}{m_j}. \quad (10)$$

The behavior of a massive scalar field Φ in the gravitational background is governed by the Klein-Gordon equation:

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi) = M^2\Phi, \quad (11)$$

where M is the mass of the scalar field and g is the determinant of the metric. Separation of variables of the Klein-Gordon equation in the background of odd-dimensional

NHEMP can be carried out in elliptic coordinates λ_a which are related to x_i with the following relation:

$$x_i^2 = (m_i - \lambda_i) \prod_{j=1, j\neq i}^{N} \frac{m_i - \lambda_j}{m_i - m_j}, \tag{12}$$

where $\lambda_N < m_N < \ldots < \lambda_2 < m_2 < \lambda_1 < m_1$. To resolve the last relation in (10) one should choose $\lambda_N = 0$. In these coordinates λ_a

$$\sum_{i=1}^{N} dx_i^2 = \sum_{a=1}^{N-1} h_a(\lambda) d\lambda_a^2, \tag{13}$$

with

$$h_a = -\frac{\lambda_a \prod_{b\neq a}^{N-1}(\lambda_b - \lambda_a)}{4\prod_{i=1}^{N}(m_i - \lambda_a)}, \tag{14}$$

the NHEMP metric (9) becomes

$$\frac{ds^2}{r_H^2} = A(\lambda)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{a=1}^{N-1} h_a(\lambda) d\lambda_a^2 + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i(\lambda) x_j(\lambda) D\varphi^i D\varphi^j, \tag{15}$$

where

$$A(\lambda) = \frac{1}{b} \frac{\prod_{a=1}^{N-1} \lambda_a}{\prod_{i=1}^{N} m_i}. \tag{16}$$

To analysis the Klein-Gordon equation (11), we need to find the determinant and the inverse metric of (15). The inverse of NHEMP metric is

$$r_H^2 \left(\frac{\partial}{\partial s}\right)^2 = -\frac{1}{A(\lambda) r^2}\left(\frac{\partial}{\partial \tau} - \sum_{i=1}^{N} r k^i \frac{\partial}{\partial \varphi^i}\right)^2 + \frac{r^2}{A(\lambda)}\left(\frac{\partial}{\partial r}\right)^2 + \sum_{a=1}^{N-1} h^a(\lambda)\left(\frac{\partial}{\partial \lambda_a}\right)^2 + \sum_{i,j=1}^{N} \tilde{\gamma}^{ij} \frac{1}{x_i(\lambda)} \frac{1}{x_j(\lambda)} \frac{\partial}{\partial \varphi^i} \frac{\partial}{\partial \varphi^j},$$

where $h^a(\lambda)$ and $\tilde{\gamma}^{ij}$ are the inverses of $h_a(\lambda)$ and $\tilde{\gamma}_{ij}$, respectively,

$$h^a(\lambda) = h_a^{-1}(\lambda), \qquad \tilde{\gamma}^{ij} = \delta^{ij} - x_i \frac{\sqrt{m_i - 1}}{m_i} x_j \frac{\sqrt{m_j - 1}}{m_j}. \tag{17}$$

The determinant of metric (15) has the following form:

$$-\det g = A(\lambda) r^2 \cdot \frac{A(\lambda)}{r^2} \cdot \prod_{a=1}^{N-1} h_a \cdot \det(\tilde{\gamma}_{ij} x_i x_j). \tag{18}$$

Taking into account the definition of h_a (13) and the relation (12), $\prod_a h_a$ simplifies to

$$\prod_{a=1}^{N-1} h_a = c \cdot \frac{A(\lambda)(\prod_{a<b}^{N-1}(\lambda_b - \lambda_a))^2}{\prod_{i=1}^{N} x_i^2}, \quad c = \frac{(-1)^{\frac{N(N-1)}{2}} b (\prod_{i=1}^{N} m_i)^2}{4^{N-1} \prod_{i,j=1, i\neq j}^{N}(m_i - m_j)}. \tag{19}$$

Using matrix determinant lemma (A5), the determinant of $\tilde{\gamma}_{ij}$ will take a simple form:

$$\det(\tilde{\gamma}_{ij} x_i x_j) = \frac{1}{bA(\lambda)} \prod_{i=1}^{N} x_i^2. \tag{20}$$

Finally inserting (19) and (20) in (18), we find

$$-\det g = c'\left(\prod_{a=1}^{N-1} \lambda_a\right)^2 \left(\prod_{a<b}^{N-1}(\lambda_b - \lambda_a)\right)^2, \tag{21}$$

with

$$c' = 4^{1-N}\left(b \prod_{i<j}^{N}(m_i - m_j)\right)^{-2}. \tag{22}$$

Equipped with (17) and (21), we can rewrite the Klein-Gordon equation (11), and noting two important relations

$$\partial_{\lambda_c} \sqrt{-\det g} = \left(\frac{1}{\lambda_c} + \sum_{b=1, b\neq c}^{N-1} \frac{1}{\lambda_c - \lambda_b}\right) \sqrt{-\det g}, \tag{23}$$

$$\partial_{\lambda_a} h^a = \left(-\frac{1}{\lambda_a} - \sum_{i}^{N} \frac{1}{m_i - \lambda_a} - \sum_{b=1, b\neq a}^{N-1} \frac{1}{\lambda_a - \lambda_b}\right) h^a, \tag{24}$$

we rewrite the Klein-Gordon equation on NHEMP metric:

$$+\frac{1}{A(\lambda)}\left(-\frac{1}{r^2}\left[\frac{\partial}{\partial \tau} - \sum_{i=1}^{N} r k^i \frac{\partial}{\partial \varphi^i}\right]^2 \Phi + r^2 \partial_r^2 \Phi + 2r\partial_r \Phi\right) + \sum_{a=1}^{N-1} h^a \partial_{\lambda_a}^2 \Phi - \sum_{a=1}^{N-1} \sum_{i=1}^{N} \frac{h^a}{m_i - \lambda_a} \partial_{\lambda_a} \Phi + \sum_{i=1}^{N} \frac{1}{x_i^2} \partial_{\varphi_i}^2 \Phi - \sum_{i,j=1}^{N} \frac{\sqrt{m_i - 1}}{m_i} \frac{\sqrt{m_j - 1}}{m_j} \partial_{\varphi_i} \partial_{\varphi_j} \Phi = M^2 \Phi. \tag{25}$$

To separate the variables in this equation, we apply the following ansatz:

$$\Phi = R_r(r) \cdot \prod_{a=1}^{N-1} R_{\lambda_a}(\lambda_a) \cdot e^{i\omega\tau} \cdot \prod_{b=1}^{N} e^{iL_b \varphi_b}, \tag{26}$$

where ω and L_i are arbitrary constants. In this form λ_a and r derivatives of the scalar Φ will be

$$\partial_r\Phi = \frac{R'_r}{R_r}\Phi, \qquad \partial_r^2\Phi = \frac{R''_r}{R_r}\Phi,$$
$$\partial_{\lambda_a}\Phi = \frac{R'_{\lambda_a}}{R_{\lambda_a}}\Phi, \qquad \partial^2_{\lambda_a}\Phi = \frac{R''_{\lambda_a}}{R_{\lambda_a}}\Phi, \tag{27}$$

and the Klein-Gordon equation transforms into

$$\left[\frac{1}{A(\lambda)}\left(r^2\frac{R''_r}{R_r} + 2r\frac{R'_r}{R_r} + \frac{1}{r^2}\left(\omega - r\sum_{i=1}^{N}k^iL_i\right)^2\right)\right.$$
$$+\sum_{a=1}^{N-1}h^a\frac{R''_{\lambda_a}}{R_{\lambda_a}} - \sum_{a=1}^{N-1}\sum_{i=1}^{N}\frac{h^a}{m_i-\lambda_a}\frac{R'_{\lambda_a}}{R_{\lambda_a}} \tag{28}$$

$$\left.-\sum_{i=1}^{N}\frac{L_i^2}{x_i^2} + \sum_{i,j=1}^{N}\frac{\sqrt{m_i-1}}{m_i}\frac{\sqrt{m_j-1}}{m_j}L_iL_j\right]\Phi = M^2\Phi. \tag{29}$$

One can see that the first term in the above equation only depends on r. The separability requirement forces it to be a constant,

$$r^2\frac{R''_r}{R_r} + 2r\frac{R'_r}{R_r} + \frac{1}{r^2}\left(\omega - r\sum_{i=1}^{N}k^iL_i\right)^2 = \mathscr{C}_2. \tag{30}$$

To write the radial equation (30) as a known differential equation, we change the radial variable r to z by

$$z = \frac{2i\omega}{r}, \tag{31}$$

which brings Eq. (30) into the familiar form of Whittaker's equation:

$$\frac{d^2R_r}{dz^2} + \left(-\frac{1}{4} + \frac{K}{z} + \frac{(1/4-\mu^2)}{z^2}\right)R_r = 0, \tag{32}$$

with

$$K = i\sum_{j=1}^{N}k^jL_j, \quad \text{and} \quad \mu^2 = \frac{1}{4} - \left(\sum_{i=1}^{N}k^iL_i\right)^2 + \mathscr{C}_2. \tag{33}$$

The general solutions to Eq. (32) are Whittaker's functions: $\mathscr{M}_{K,\mu}(z)$ and $\mathscr{W}_{K,\mu}(z)$. These are related to the confluent hypergeometric functions. It is interesting to study the behavior of these functions in small- and large-z region. (we keep K, μ parameter fixed and generic.) For the small-z region, which is related to the $r \to \infty$ region in the space-time, we have

$$\mathscr{M}_{K,\mu}(z) \sim z^{\mu+1/2}, \quad \text{as } z \to 0$$
$$\mathscr{W}_{K,\mu}(z) \sim \alpha_1 z^{\mu+1/2} + \alpha_2 z^{-\mu+1/2}, \quad \text{if } \mathrm{Re}(\mu) < \frac{1}{2}$$
$$\mathscr{W}_{K,\mu}(z) \sim z^{-\mu+1/2} \quad \text{if } \mathrm{Re}(\mu) \geq \frac{1}{2}. \tag{34}$$

Here, $\alpha_{1,2}$ are some constants related to μ, K. The asymptotic behavior ($z \to \infty$) of $\mathscr{M}_{K,\mu}(z)$ and $\mathscr{W}_{K,\mu}(z)$ which is corresponding to the $r \to 0$ region of space-time is given as

$$\mathscr{M}_{K,\mu}(z) \sim z^{-K}e^{z/2}$$
$$\mathscr{W}_{K,\mu}(z) \sim z^{K}e^{-z/2}, \quad \text{as z} \to \infty. \tag{35}$$

Depending on the values of μ and K in (33), the behavior of the solution is different: $\mathscr{M}_{K,\mu}$ is blowing up in the $r \to 0$ ($z \to \infty$) limit, so we discard it. In the small z, *i.e.*, ($r \to \infty$), $\mathscr{W}_{K,\mu}$ is vanishing if $\mathrm{Re}(\mu) < 1/2$ which is guaranteed by e.g., $\mathscr{C}_2 < 0$.

The rest of the variables can be separated after applying the relations (A2), (A3), and (A4) to the Klein-Gordon equation, which will be transformed to

$$\sum_{a=1}^{N-1}\frac{P_{\lambda_a}(\lambda_a) - M^2(-\lambda_a)^{N-2}}{\prod_{b=1;a\neq b}^{N-1}(\lambda_b-\lambda_a)} = 0, \tag{36}$$

where $P_{\lambda_a}(\lambda_a)$ are defined by

$$P_{\lambda_a}(\lambda_a) \equiv -\frac{4}{\lambda_a}\left(\frac{R''_{\lambda_a}}{R_{\lambda_a}} - \frac{R'_{\lambda_a}}{R_{\lambda_a}}\sum_{i}^{N}\frac{1}{m_i-\lambda_a}\right)\prod_{j=1}^{N}(m_j-\lambda_a)$$
$$+\frac{b}{\lambda_a}\mathscr{C}_2\prod_{i=1}^{N}m_i + (-1)^{N-1}\sum_{i=1}^{N}\frac{g_{\varphi_i}}{m_i-\lambda_a}$$
$$+g_0(-\lambda_a)^{N-2}. \tag{37}$$

Here g_{φ_i} and g_0 are some constants, defined by

$$g_{\varphi_i} \equiv \frac{k^2_{\varphi_i}}{m_i}\prod_{\substack{j=1\\ j\neq i}}^{N}(m_i-m_j), \qquad g_0 \equiv \left(\sum_{i=1}^{N}\frac{\sqrt{m_i-1}}{m_i}L_i\right)^2. \tag{38}$$

Equation (36) is only satisfied when

$$P_{\lambda_a}(\lambda_a) = \sum_{\alpha=1}^{N-1}k_\alpha\lambda_a^{\alpha-1}, \tag{39}$$

where $k_\alpha(\alpha = 1, \ldots, N-2)$ are arbitrary constants and $k_{N-1} = (-1)^{N-2}M^2$.

Accordingly, the R_{λ_a} satisfies

$$R''_{\lambda_a} + \left(\sum_i^N \frac{1}{m_i - \lambda_a}\right) R'_{\lambda_a} - \frac{R_{\lambda_a}}{4\prod_{j=1}^N (m_j - \lambda_a)} \left[g_0 \lambda_a^{N-1} + b\mathscr{C}_2 \prod_{i=1}^N m_i + \lambda_a \sum_{i=1}^N \frac{(-1)^{N-1} g_i}{(m_i - \lambda_a)} - \sum_{\alpha=1}^{N-1} k_\alpha \lambda_a^{\alpha-1} \right] = 0, \tag{40}$$

which is an ordinary differential equation and depends on only the λ_a coordinate.

III. KLEIN-GORDON EQUATION ON NEAR HORIZON EVH GEOMETRY

The metric of near horizon EVH Myers-Perry in the elliptical coordinates parametrized with $(\tau, \rho, \psi, \lambda_a, \varphi_i)$ in $d = 2N + 1$ dimensions is of the form

$$ds^2 = F(\lambda)\left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2 d\psi^2\right) + \sum_{a=1}^{N-1} \hat{h}_a d\lambda_a^2 + \sum_{a,b=1}^{N-1} \hat{\gamma}_{ab} \hat{x}_a(\lambda) \hat{x}_b(\lambda) d\varphi_a d\varphi_b, \tag{41}$$

where the metric functions are

$$F(\lambda) = \prod_{a=1}^{N-1} \frac{\lambda_a}{m_a}, \qquad \hat{h}_a = \frac{1}{4} \frac{\prod_{b\neq a}^{N-1}(\lambda_b - \lambda_a)}{\prod_{c=1}^{N-1}(m_c - \lambda_a)},$$
$$\hat{\gamma}_{ab} = \delta_{ab} + \frac{1}{F(\lambda)} \frac{\hat{x}_a(\lambda)}{\sqrt{m_a}} \frac{\hat{x}_b(\lambda)}{\sqrt{m_b}}, \tag{42}$$

with

$$\hat{x}_a^2(\lambda) = \frac{\prod_{b=1}^{N-1}(m_a - \lambda_b)}{\prod_{l\neq a}^{N-1}(m_a - m_l)}, \qquad \sum_{a=1}^{N-1} \frac{1}{m_a} = 1. \tag{43}$$

(We start this section in elliptical coordinates to avoid writing multiple metrics. For more details, we refer the reader to [15,16,21].)

The inverse of metric (41) is

$$\left(\frac{\partial}{\partial s}\right)^2 = \frac{1}{F(\lambda)}\left(-\frac{1}{\rho^2}\left(\frac{\partial}{\partial \tau}\right)^2 + \rho^2 \left(\frac{\partial}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial}{\partial \psi}\right)^2\right) + \sum_{a=1}^{N-1} \hat{h}^a \left(\frac{\partial}{\partial \lambda_a}\right)^2 + \sum_{a,b=1}^{N-1} \frac{\hat{\gamma}^{ab}}{\hat{x}_a \hat{x}_b} \frac{\partial}{\partial \varphi^a} \frac{\partial}{\partial \varphi^b}, \tag{44}$$

where

$$\hat{\gamma}^{ab} = \delta_{ab} - \frac{\hat{x}_a(\lambda)}{\sqrt{m_a}} \frac{\hat{x}_b(\lambda)}{\sqrt{m_b}}, \qquad \hat{h}^a = (\hat{h}_a)^{-1}. \tag{45}$$

To study the Klein-Gordon equation on this metric,

$$\Box \Phi = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta \Phi) = M^2 \Phi, \tag{46}$$

we need to compute the determinant of the metric,

$$-\det g = \rho^2 F_0(\lambda)^3 \left(\prod_{a=1}^{N-1} \hat{h}_a\right) \det(\hat{\gamma}_{ab} \hat{x}_a \hat{x}_b). \tag{47}$$

It is easy to show that

$$\prod_{l=1}^{N-1} \hat{h}_l = \hat{c} \cdot \frac{\prod_{a,b=1, a\neq b}^{N-1}(\lambda_b - \lambda_a)}{\prod_{a=1}^{N-1} \hat{x}_a^2}, \qquad \hat{c} = 4^{1-N} \prod_{a,b=1, a\neq b}^{N-1} (m_a - m_b)^{-1}. \tag{48}$$

Using matrix determinant lemma (A5), we have

$$\det(\hat{\gamma}_{ab} \hat{x}_a \hat{x}_b) = \frac{1}{F(\lambda)} \prod_{a=1}^{N-1} \hat{x}_a^2. \tag{49}$$

Finally, the determinant of metric (41) simplifies to

$$-\det g = c'' \rho^2 \left(\prod_{l=1}^{N-1} \lambda_l\right)^2 \left(\prod_{a,b=1, b<a}^{N-1} (\lambda_b - \lambda_a)\right)^2, \tag{50}$$

with

$$c'' = \frac{1}{4^{N-1}} \left(\prod_{l=1}^{N-1} m_l\right)^{-2} \left(\prod_{a,b=1, b<a}^{N-1} (m_b - m_a)\right)^{-2}. \tag{51}$$

The key relations for the separability of the Klein-Gordon equation are as follows:

$$\partial_{\lambda_a} \sqrt{-\det g} = \left(\frac{1}{\lambda_a} - \sum_{b=1, b\neq a}^{N-1} \frac{1}{\lambda_b - \lambda_a}\right) \sqrt{-\det g},$$
$$\partial_{\lambda_a} \hat{h}^a = \left(-\sum_{b=1}^{N-1} \frac{1}{m_b - \lambda_a} + \sum_{b=1, b\neq a}^{N-1} \frac{1}{\lambda_b - \lambda_a}\right) \hat{h}^a. \tag{52}$$

The Klein-Gordon equation becomes

$$\frac{1}{F(\lambda)}\left(-\frac{\partial_\tau^2 \Phi - \partial_\psi^2 \Phi}{\rho^2} + \rho^2 \partial_\rho^2 \Phi + 3\rho \partial_\rho \Phi\right) + \sum_{a=1}^{N-1} \frac{1}{\hat{x}_a^2} \partial_{\varphi_a}^2 \Phi - \sum_{a,b}^{N-1} \frac{1}{\sqrt{m_a}} \frac{1}{\sqrt{m_b}} \partial_{\varphi_a} \partial_{\varphi_b} \Phi + \sum_{a=1}^{N-1} \hat{h}^a \partial_{\lambda_a}^2 \Phi + \sum_{a=1}^{N-1} \frac{\hat{h}^a}{\lambda_a} \partial_{\lambda_a} \Phi - \sum_{a,b=1}^{N-1} \frac{\hat{h}^a}{m_b - \lambda_a} \partial_{\lambda_a} \Phi = M^2 \Phi. \tag{53}$$

where $\hat{x}^a$ is defined in (43). To separate the variables, we use the following ansatz for the scalar field:

$$\Phi = R_\rho(\rho) \cdot \prod_{a=1}^{N-1} R_a(\lambda_a) \cdot e^{i(-k_\tau \tau + m_\psi \psi)} \cdot \prod_{b=1}^{N-1} e^{iL_b \varphi_b}, \quad (54)$$

into the Klein-Gordon equation (53). We observe that the first parentheses include only ρ-dependent term. The separability requires it to be a constant:

$$\left(\frac{k_\tau^2 - m_\psi^2}{\rho^2} + \rho^2 \partial_\rho^2 + 3\rho\partial_\rho\right) R_\rho(\rho) = -4\hat{\mathscr{C}}_2 R_\rho(\rho). \quad (55)$$

Replacing

$$R_\rho(\rho) = \frac{u(\rho)}{\rho} \quad \text{and} \quad \rho = \frac{\sqrt{k_\tau^2 - m_\psi^2}}{z} \quad (56)$$

into the radial equation (55), we get Bessel's equation:

$$z^2 \frac{d^2}{dz^2} u + z \frac{d}{dz} u + (z^2 - \nu^2) u = 0, \quad (57)$$

with

$$\nu^2 = 1 - 4\hat{\mathscr{C}}_2, \quad (58)$$

whose solutions are Bessel functions $J_\nu(z)$, $Y_\nu(z)$. The small- and large-r behavior of the solution is dictated by the asymptotic behavior of Bessel functions. In the small-r region of space-time which is related to $z \to \infty$, they give

$$J_\nu(z) \sim \frac{1}{\sqrt{z}} \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right),$$
$$Y_\nu(z) \sim \frac{1}{\sqrt{z}} \sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \quad \text{as } z \to \infty, \quad (59)$$

which is vanishing. Also, in the large-r region of space-time, corresponding to $z \to 0$, they behave as

$$J_\nu(z) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu,$$
$$Y_\nu(z) \sim -\frac{\Gamma(\nu)}{\pi} \left(\frac{z}{2}\right)^{-\nu}. \quad (60)$$

In the case of $\hat{\mathscr{C}}_2 < 0$ which leads to $\nu > 1$, Y_ν blows up while J_ν falls off in the large-r region and is an acceptable solution.

For the rest of the Klein-Gordon equation, we have

$$\left(-\frac{4\hat{\mathscr{C}}_2}{F_0(\lambda)} - \sum_{a,b=1}^{N-1} \frac{L_a}{\sqrt{m_a}} \frac{L_b}{\sqrt{m_b}} + \sum_{a=1}^{N-1} \left[\hat{h}^a \frac{R_a''}{R_a} + \frac{\hat{h}^a}{\lambda_a} \frac{R_a'}{R_a} - \sum_{i=1}^{N-1} \frac{\hat{h}^a}{m_i - \lambda_a} \frac{R_a'}{R_a} - \frac{k_{\varphi_a}^2}{x_a^2}\right]\right) \Phi = M^2 \Phi. \quad (61)$$

Using the relations (A2), (A3), and (A4), we see that the Klein-Gordon equation is separable:

$$\sum_{a=1}^{N-1} \frac{\hat{Q}_{\lambda_a}(\lambda_a) - M^2(-\lambda_a)^{N-2}}{\prod_{b=1;a\neq b}^{N-1} (\lambda_b - \lambda_a)} = 0, \quad (62)$$

where

$$\hat{Q}_{\lambda_a}(\lambda_a) = 4\left(\frac{R_a''}{R_a} + \frac{1}{\lambda_a} \frac{R_a'}{R_a} - \sum_{b=1}^{N-1} \frac{R_a'/R_a}{m_b - \lambda_a}\right) \prod_{c=1}^{N-1} (m_c - \lambda_a) - \frac{4\hat{\mathscr{C}}_2}{\lambda_a} \prod_{b=1}^{N-1} m_b + \sum_{b=1}^{N-1} \frac{\hat{q}_{\varphi_b}}{m_b - \lambda_a} + \hat{q}_0 (-\lambda_a)^{N-2}, \quad (63)$$

and

$$\hat{q}_{\varphi_a} \equiv (-1)^{N-1} k_{\varphi_a}^2 \prod_{\substack{b=1 \\ b\neq a}}^{N-1} (m_a - m_b), \qquad \hat{q}_0 \equiv \left(\sum_{a=1}^{N-1} \frac{L_a}{\sqrt{m_a}}\right)^2. \quad (64)$$

Equation (62) is only satisfied when

$$\hat{Q}_{\lambda_a}(\lambda_a) = \sum_{\alpha=1}^{N-1} k_\alpha \lambda_a^{\alpha-1}, \quad (65)$$

where $k_\alpha (\alpha = 1, \ldots, N-2)$ are arbitrary constants and $k_{N-1} = (-1)^{N-2} M^2$.

Therefore, the R_{λ_a} satisfies the following ordinary differential equation:

$$R_a'' + \left(\frac{1}{\lambda_a} - \sum_b^{N-1} \frac{1}{m_b - \lambda_a}\right) R_a' + \frac{R_a}{4\prod_{c=1}^{N-1}(m_c - \lambda_a)} \left[4\hat{\mathscr{C}}_2 \prod_{b=1}^{N-1} m_b + \hat{q}_0(-\lambda_a)^{N-2} + \sum_{i=b}^{N-1} \frac{\hat{q}_{\varphi_b}}{(m_b - \lambda_a)} - \sum_{\alpha=1}^{N-1} k_\alpha \lambda_a^{\alpha-1}\right] = 0, \quad (66)$$

which depends on only λ_a coordinate.

IV. DISCUSSION

We studied the separability of the Klein-Gordon equation on two near horizon geometries in d dimensions: generic extremal and extremal vanishing horizon cases. Since the latter case exists only in odd dimensions, we take $d = 2n+1$ in both cases. We do not expect that the separability of the Klein-Gordon equations changes for the first case if we take even dimensions. This expectation roots from the fact that the separability of geodesic equation has been shown for both even and odd dimensions [16] and the Klein-Gordon and geodesic equation are related in the semiclassical limit: if we write the solution to the

Klein-Gordon equation Φ as $\Phi = \mathcal{N} \exp(\frac{iS}{\alpha})$, we get the Hamilton-Jacobi equation for S in the so-called semi-classical limit, $\alpha \to 0$. (see [38] for an explicit example.)

It is worth mentioning that since the extremal/EVH limit does not commute with near horizon limit, we *could not* get the Klein-Gordon equation on near horizon EVH geometry by taking the EVH limit on NHEMP metric. Therefore, we studied the Klein-Gordon equation on near horizon EVH geometry, independently.

In the parameter of the near horizon extremal/EVH geometry, we kept the rotation parameters generic. However, one can ask what will happen if we set some or all of the rotation parameters equal. This is one of the interesting problems to which we will come back in future.

Regarding the solution to the radial equations, we discussed the asymptotic behavior for the generic values of the parameters. There are some special limits in the parameter space which can lead to different asymptotic behavior (like $\mu = 0$ of $\mathcal{M}_{K,\mu}$ or $\mathcal{W}_{K,\mu}$). These special limits are known for Whittaker and Bessel functions. We are interested in the interpretation of their consequences in near horizon geometries.

To examine these near horizon geometries better, we should study the other probes such as Dirac, Maxwell, and gravitational perturbation fields and investigate the separability of their field equations in the future.

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APPENDIX: USEFUL IDENTITIES

The following identity holds between $N-1$ independent variables λ_a and any parameter κ:

$$\sum_{a=1}^{N-1} \frac{1}{\prod_{\substack{b=1\\ b\neq a}}^{N-1}(\lambda_a - \lambda_b)} \frac{(\lambda_a)^\alpha}{\kappa - \lambda_a} = \frac{(\kappa)^\alpha}{\prod_{a=1}^{N-1}(\kappa - \lambda_a)} - \delta_{\alpha,N-1}, \quad \text{(A1)}$$

where $\alpha = \{0, \ldots, N-1\}$. For $\alpha = 0$, this equation reduces to

$$\frac{1}{\prod_{a=1}^{N-1}(\lambda_a - \kappa)} = \sum_{a=1}^{N-1} \frac{1}{\prod_{b=1;a\neq b}^{N-1}(\lambda_b - \lambda_a)} \frac{1}{\lambda_a - \kappa}, \quad \text{(A2)}$$

and for an additional condition of $\kappa = 0$ we get

$$\frac{1}{\lambda_1 \ldots \lambda_{N-1}} = \sum_{a=1}^{N-1} \frac{1}{\prod_{b=1;b\neq a}^{N-1}(\lambda_b - \lambda_a)} \frac{1}{\lambda_a}. \quad \text{(A3)}$$

On the other hand, setting $\kappa = 0$ in (A1) results in

$$\sum_{i=1}^{N} \frac{\lambda_i^\beta}{\prod_{j=1;i\neq j}^{N}(\lambda_i - \lambda_j)} = \delta_{\beta,N-1} \quad \text{for } 0 \leq \beta \leq N-1. \quad \text{(A4)}$$

Another relation that we use in this paper is the so-called matrix determinant lemma, which states

$$\det(\mathrm{I} + \boldsymbol{x}\boldsymbol{y}^T) = 1 + \boldsymbol{y}^T\boldsymbol{x}, \quad \text{(A5)}$$

where I is a unit matrix and $\boldsymbol{x}\boldsymbol{y}^T$ is the outer product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

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