Gravitational waves in Hořava-Lifshitz anisotropic gravity

J. Mestra-Páez,* J. M. Peña,† and A. Restuccia‡
Departamento de Física, Universidad de Antofagasta, Aptdo 02800, Chile

(Received 30 August 2021; accepted 12 November 2021; published 27 December 2021)

We show that in anisotropic Hořava-Lifshitz gravity there is a well-defined wave zone where the physical degrees of freedom propagate according to a nonrelativistic linear evolution equation of high order in spatial derivatives, which reduces to the wave equation at low energy. This is so, provided the coupling parameters satisfy some restrictions which we study in detail. They are imposed to obtain a finite Arnowitt-Deser-Misner gravitational energy, which depends manifestly on the terms which break the Lorentz symmetry of the formulation. The analysis we perform is beyond the linearized approach and includes all high-order terms of the Hamiltonian potential.

DOI: 10.1103/PhysRevD.104.124073

I. INTRODUCTION

The Hořava-Lifshitz gravity theory [1,2] has been proposed as a candidate of a renormalizable gravity theory 16–3]]. Following Lifshitz, the time and the spatial coordinates scale differently, in a way that the overall coupling of the theory becomes dimensionless. In this sense the theory is anisotropic and nonrelativistic. It introduces interaction terms with high-order spatial derivatives in the potential that break the relativistic symmetry in a manifest way, but improve the ultraviolet (UV) behavior in comparison to general relativity (GR). The theory is renormalizable by power counting.

There are different versions of Hořava-Lifshitz gravity: the projectable version has interesting applications to cosmology (see Refs. [17,18] and references therein), and in the nonprojectable version the full number of gravitational physical degrees of freedom—the transverse-traceless tensorial modes at the linearized level—become dynamical. Among the nonprojectable Hořava-Lifshitz versions, the propagating degrees of freedom differ according to the value of the dimensionless coupling constant λ on the kinetic term of the Hořava-Lifshitz action (see Ref. [1]). For $\lambda \neq 1/3$ the theory propagates a scalar degree of freedom, in addition to the transverse-traceless tensorial ones. Several works have discussed the problem of strong coupling of this scalar mode [3,5,19].

Some restrictions on the couplings of the theory have to be imposed in order to justify the existence of this scalar mode without violating the well-established gravitational data [20]. For $\lambda = 1/3$, the kinetic term of the Hořava-Lifshitz action (see Ref. [1]) has an additional conformal symmetry. In this case, the propagating degrees of freedom exactly coincide with transverse-traceless tensorial modes of GR. No additional scalar field is present in the theory. At low energies the theory depends on two coupling constants, β and α; when $\beta = 1$ and $\alpha = 0$ the field equations are exactly the GR equations in a particular gauge. The restrictions on the coupling constants are in this case less stringent. In both cases there is a range of values for the coupling constants for which the theory fits the known gravitational experimental data satisfied by GR [21].

The linearized theory at low energies coincides with the corresponding linearized GR formulation [22]. Additionally, it satisfies the well-known Einstein quadrupole formula [23]. The coupling to the Maxwell theory in four dimensions was recently studied using a Kaluza-Klein approach in Hořava-Lifshitz theory in five dimensions [24,25]. The speed of propagation of the gravitational and electromagnetic physical modes, at low energies, is in agreement with the recent experimental data arising from the detection of gravitational and electromagnetic waves generated by the same source [26]. An analysis of the theory including all higher-order spatial derivative terms of the potential was performed in Ref. [13].

Although at the linearized level Hořava-Lifshitz theory and GR coincide, the full theories behave in different ways, as can be seen from the static spherically symmetric solutions of the field equations. In this case, in the Hořava-Lifshitz theory there are solutions with a throat, connecting an asymptotically flat manifold with a nonasymptotically flat one [27]; see also Ref. [28] where a charged throat was considered. The solutions depend on one coupling constant α in a way that in the limit $\alpha \to 0$ the geometry outside the throat, on the

*jarvin.mestra@ua.cl
†joselen@yahoo.com
‡alvaro.restuccia@uantof.cl

asymptotically flat side, tends to the geometry of the Schwarzchild solutions in GR outside the black hole. The asymptotic behavior of these solutions defines the asymptotically flat behavior in Hořava-Lifshitz gravity.

Nonlinear interaction terms in gravity theories are essential in the propagation of the physical degrees of freedom. However, in GR in the wave zone, the propagation of the physical degrees of freedom, at leading order, is free from such nonlinear terms. We may wonder if this is the case for Hořava-Lifshitz gravity, since both theories propagate exactly the same physical degrees of freedom. In particular, we are interested in the effect of the terms that break the relativistic symmetry on the propagation in the wave zone.

In Ref. [29] we analyzed the wave zone in Hořava-Lifshitz gravity at low energies, that is, when terms in the potential with high-order spatial derivatives are dismissed. We found that the propagation of the physical degrees of freedom is described by the wave equation as in GR, with the speed of propagation given in terms of the coupling parameter β.

In this paper we show that there exists a wave zone in the complete Hořava-Lifshitz theory. We include in our analysis all high-order derivative terms in the potential. We consider the theory at the kinetic conformal point, where $\lambda = 1/3$. We notice that the value $\lambda = 1/3$ is protected from quantum corrections by the presence of a second-class constraint which arises directly from the formulation of the theory [22].

In this zone the constraints of the theory can be solved in powers of $1/r$. In distinction to GR where the Hamiltonian constraint is of first class, in this theory there are two second-class constraints which allow to obtain the components g^T and N in terms of the transverse-traceless (TT) tensorial modes. The lapse N is not a Lagrange multiplier in this theory. We analyze the general coordinate transformations in the wave zone and obtain the field equations for the physical degrees of freedom described by the TT modes of the metric. At order $1/r$, the field equations become a linear partial differential equation, which reduces to the wave equation at low energies. Although this field equation at order $1/r$ is the same one that arises from the linearized theory, the nontrivial solution for the g^T and N fields provides the existence of a Newtonian background that is absent in the linearized theory and relevant in the asymptotic behavior of the theory, in particular in the determination of the gravitational energy. In the formulation of the theory in the wave zone, we follow the approach rigorously established in Ref. [30].

In distinction to the Arnowitt-Deser-Misner (ADM) analysis of GR, the solution of the constraints and the dynamical equations involve an elliptic operator of sixth order, yielding a higher-order dispersion relation. The terms that break the Lorentz symmetry in the action of the theory contribute to the Newtonian background and appear explicitly in the expression of the gravitational energy. In order to have a finite gravitational energy, some restrictions have to be imposed on the coupling parameters of the potential. In fact, the solution of the constraints involve an evolution operator with high-order spatial derivatives, which may have zero modes. They produce inconsistent contributions to the gravitational energy. In order to avoid them, we impose restrictions on the coupling parameters.

The relevance of the existence of a wave zone in a theory describing gravity is directly related to the detection of gravitational waves (GWs) in the last five years. In 2010, they were measured indirectly through the relative reduction of the distance between the members of the binary pulsar system PSR B1913 + 16 by continuous emission of GWs [31]. In 2015, the LIGO/Virgo Collaboration directly detected the first signal of a GW produced from the black hole–black hole (BHBH) coalescence GW150914 [32–34]. After that, there were multiple detections of GWs due to the coalescence of compact binary systems, such as BHBH, neutron star–neutron star, and neutron star–black hole systems [35–39].

The electromagnetic signal of GW170817 was also detected—the γ-ray burst GBR170817A [40]—which has been very important for comparing the speeds of both electromagnetic and gravitational signals, as it constrains the difference between the speed of gravity and the speed of light to be between -3×10^{-15} and 7×10^{-16} times the speed of light [41,42]. In Hořava-Lifshitz theory at the kinetic conformal point, both interactions have the same speed of propagation [24,25].

A main difference between the wave zone in Hořava-Lifshitz theory and that in GR is the modified dispersion relation involving a polynomial in the square of the wave number k, which at low energy and for $\beta = 1$ reduces to the relativistic relation. Modifications of dispersion relations in theories with Lorentz invariance violations (LIVs) have been discussed in the literature; for a review of experimental tests of LIV in astroparticle physics scenarios, see Ref. [43] and references therein. The Large High Altitude Air Shower Observatory recently detected ultra-high-energy photons (γ rays) at the PeV energy scale [44]. These observations suggest the existence of subluminal LIV beyond special relativity in the photon sector at the scale of 3.6×10^{17} GeV [45]. In accordance with conclusions of Refs. [46–48] in which it was stated that, from data on γ-ray bursts observed by the Fermi Gamma-ray Space Telescope, to this same energy scale the speed of light is energy dependent.

II. FOLIATIONS IN HOŘAVA GRAVITY

We consider a three-dimensional foliation of a four-dimensional manifold M. It is a decomposition of M into a union of disjoint three-dimensional submanifolds—the leaves of the foliation—such that the covering of M by charts U_i together with homeomorphisms $\varphi_i : U_i \Leftrightarrow \mathcal{N}_i \subset \mathbb{R}^4$ for overlapping pairs U_i, U_j satisfy

$$\varphi_j \circ \varphi_i^{-1}\colon\ \mathbb{R}^4 \to \mathbb{R}^4, \tag{1}$$

which is a C^∞ bijection from $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$, and

$$\varphi_j \circ \varphi_i^{-1}\colon\ (t,x) \to (\tilde{t}(t), \tilde{x}(t,x)), \tag{2}$$

where x are coordinates on the three-dimensional leaves Σ_t, and t is the coordinate on the codimension-1 manifold which we take to be $\mathbb{R}$. We assume that the leaves are Riemannian manifolds with metric g_{ij}. In the Hořava-Lifshitz formulation, t and x^i, $i = 1, 2, 3$ have different dimensions and there is no *a priori* universal constant of dimension $[T]^{-1}[L]$. Although there are couplings with this dimension, it's numerical value depend of the energy scale, and hence only in the infrared limit (or UV one) we can have a fixed numerical value for such parameter. Although there is not a metric on the manifold M, there are intrinsic geometrical objects, which we define as

$$N(t,x)dt \tag{3}$$

with dimension $[T]$, and

$$dx^i + N^i(t,x)dt \tag{4}$$

with dimension $[L]$. They transform under Eq. (2) as

$$\tilde{N}(\tilde{t},\tilde{x})d\tilde{t} = N(x,t)dt, \tag{5}$$

$$d\tilde{x}^i + \tilde{N}^i(\tilde{t},\tilde{x})d\tilde{t} = \frac{\partial \tilde{x}^i}{\partial x^j}[dx^j + N^j(t,x)dt] \tag{6}$$

and allow to have an intrinsic volume element,

$$N dt \wedge (dx^1 + N^1 dt) \wedge (dx^2 + N^2 dt) \wedge (dx^3 + N^3 dt)\sqrt{g}$$
$$= N dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \sqrt{g}, \tag{7}$$

where g is the determinant of the three-dimensional metric g_{ij} on the leaves. The metric is taken to be a dimensionless tensor under the diffeomorphisms on the leaves.

From Eqs. (5)–(6) we obtain the transformation law of N, a dimensionless density, and N^i. In the Hořava-Lifshitz formulation of gravity, t scales as b^z, while x^i scale as b^1; consequently, the dimension of t is $[t] = [L]^z$ and that of N^i is $[N^i] = [L]^{1-z}$. We notice that the above construction does not depend on the value of z. In the four-dimensional Hořava-Lifshitz gravitational theories, $z = 3$ in order to have a dimensionless overall coupling constant. Hence, $[N^i] = [L]^{-2}$ and, as we have determined, $[N] = [L]^0$.

The transformation laws for N and N^i under

$$\tilde{t} = \tilde{t}(t), \qquad \tilde{x}^i = \tilde{x}^i(t,x) \tag{8}$$

are then

$$\tilde{N}(\tilde{t},\tilde{x})\dot{\tilde{t}}(t) = N(t,x), \tag{9}$$

$$\tilde{N}^i(\tilde{t},\tilde{x})\dot{\tilde{t}}(t) = -\dot{\tilde{x}}^i(t,x) + \frac{\partial \tilde{x}^i}{\partial x^j}N^j(t,x), \tag{10}$$

respectively, where $\dot{()} \equiv \frac{\partial}{\partial t}()$. We notice that N and N^i transform as densities under time reparametrization; N^i does not transform as a vector field under diffeomorphisms on the leaves. The first term on the right-hand side of Eq. (10) is characteristic of the transformation law of a Lagrange multiplier. If we rewrite the general finite transformation of coordinates as

$$\tilde{t} = t + f(t), \qquad \tilde{x}^i = x^i + \xi^i(t,x), \tag{11}$$

where f and ξ^i are C^1 arbitrary functions, we obtain

$$\tilde{N}(\tilde{t},\tilde{x})(1 + \dot{f}(t)) = N(t,x), \tag{12}$$

$$\tilde{N}^i(\tilde{t},\tilde{x})(1 + \dot{f}(t)) = -\dot{\xi}^i(t,x) + N^i(t,x) + \frac{\partial \xi^i}{\partial x^j}N^j(t,x), \tag{13}$$

and also

$$g_{ij}(t,x) = \frac{\partial \tilde{x}^l}{\partial x^i}\frac{\partial \tilde{x}^m}{\partial x^j}\tilde{g}_{lm}(\tilde{t},\tilde{x}). \tag{14}$$

A. T + L decomposition

In the following analysis we will use the T + L decomposition of symmetric tensors fields (f_{ij}) vanishing at infinity in terms of linear orthogonal symmetric parts [49],

$$f_{ij} = f_{ij}^{TT} + f_{ij}^{T} + 2f_{(i,j)}. \tag{15}$$

The transverse part $f_{ij}^T \equiv \frac{1}{2}[\delta_{ij}f^T - \frac{1}{\Delta}f^T_{,ij}]$ satisfies $f^T_{ij,j} = 0$. The transverse-traceless f_{ij}^{TT} satisfies $f^{TT}_{ij,j} = 0$ and $f^{TT}_{ii} = 0$. The remaining term, $2f_{(i,j)}$, is its longitudinal part. $f^T = f^T_{ii}$ is the trace of the transverse part of f_{ij}. $\frac{1}{\Delta}$ is the inverse of the flat-space Laplacian, defined on the space of functions that vanish at infinity.

The tensor components are

$$f_{i,j} = \frac{1}{\Delta}\left\{f_{ik,kj} - \frac{1}{2}\left[\frac{1}{\Delta}f_{lm,lm}\right]_{,ij}\right\}, \tag{16}$$

$$f_{ij}^T = \frac{1}{2}\left[f^T\delta_{ij} - \frac{1}{\Delta}f^T_{,ij}\right], \tag{17}$$

$$f^T = \frac{1}{\Delta}(f_{ll,mm} - f_{lm,lm}), \tag{18}$$

$$f_{ij}^{TT} = f_{ij} - f_{ij}^{T} - 2f_{(i,j)}. \tag{19}$$

III. 3 + 1 HOŘAVA-LIFSHITZ GRAVITY AT THE CONFORMAL KINETIC POINT

The $3 + 1$ anisotropic Hořava-Lifshitz Hamiltonian at the conformal kinetic point is given by

$$H = \int_{\Sigma_t} d^3x \left\{ N\sqrt{g}\left[\frac{\pi^{ij}\pi_{ij}}{g} - \mathcal{V}(g_{ij}, N)\right] - N_j H^j - \sigma P_N - \mu\pi \right\} + \beta E_{\text{ADM}}, \tag{20}$$

where (N_i, σ, μ) are Lagrange multipliers, (H^j, P_N, π) are primary constraints, and the potential $\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)} + \mathcal{V}^{(3)}$ is the most general scalar constructed from the three-dimensional spacelike metric and the lapse N function. It is independent of the conjugate momenta and of the $N_i \equiv g_{ij}N^j$. The potential, where only the interacting terms that will contribute to the wave zone are explicitly given, is expressed as follows:

$$\mathcal{V}^{(1)} = \beta R + \alpha a_i a^i, \tag{21}$$

$$\mathcal{V}^{(2)} = \alpha_1 R\nabla_i a^i + \alpha_2 \nabla_i a_j \nabla^i a^j + \beta_1 R_{ij}R^{ij} + \beta_2 R^2 + O^{(2)}(a_i a^i), \tag{22}$$

$$\mathcal{V}^{(3)} = \alpha_3 \nabla^2 R \nabla_i a^i + \alpha_4 \nabla^2 a_i \nabla^2 a^i + \beta_3 \nabla_i R_{jk} \nabla^i R^{jk} + \beta_4 \nabla_i R \nabla^i R + O^{(3)}(R_{ij}, a_k), \tag{23}$$

where $a_i \equiv \frac{1}{N}\partial_i N$ is a 3-vector under the transformations (8) with dimension $[L]^{-1}$ [2], and the α's and β's are coupling constants. We notice that the couplings involved in Eq. (23) are dimensionless. The terms $O^{(2)}$ and $O^{(3)}$ contain products of more than two fields. They will not contribute to the leading-order or next-to-leading-order terms in the wave zone.

The surface integral,

$$E_{\text{ADM}} \equiv \oint_{\partial\Sigma_t} (\partial_j g_{ij} - \partial_i g_{jj}) dS_i, \tag{24}$$

is added in order to ensure the differentiability of the Hamiltonian with respect to the metric; see Ref. [50] where this idea was introduced for GR. E_{ADM} is the well-known ADM energy in general relativity.

The primary constraints,

$$\pi \equiv g_{ij}\pi^{ij} = 0, \tag{25}$$

$$P_N = 0, \tag{26}$$

are second class, while the momentum constraints,

$$H^j \equiv 2\nabla_i \pi^{ij} = 0, \tag{27}$$

are first class.

The conservation of the primary constrains (25)–(26) yields the constraints

$$H_P \equiv \frac{3}{2}\frac{N}{\sqrt{g}}\pi^{ij}\pi_{ij} + \frac{1}{2}N\sqrt{g}\beta R + N\sqrt{g}\left(\frac{\alpha}{2} - 2\beta\right)a_i a^i - 2\beta N\sqrt{g}\nabla^i a_i + \{\pi, \mathcal{U}\}_{PB} = 0, \tag{28}$$

$$H_N \equiv -\frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \beta g R) - \alpha\sqrt{g}a_i a^i - 2\alpha\sqrt{g}\nabla_i a^i + \{P_N, \mathcal{U}\}_{PB} = 0, \tag{29}$$

where

$$\mathcal{U} \equiv -\int_{\Sigma_t} d^3x N\sqrt{g}(\mathcal{V}^{(2)} + \mathcal{V}^{(3)}). \tag{30}$$

These are second-class constraints. The conservation of Eqs. (28)–(29) determines the Lagrange multipliers. Then, Eqs. (25)–(29) is the complete set of constraints.

The dynamical field equations are

$$\dot{g}_{ij} = \frac{2N}{\sqrt{g}}\pi_{ij} + 2\nabla_{(i}N_{j)} - \mu g_{ij}, \tag{31}$$

$$\dot{\pi}^{ij} = \frac{N}{2}\frac{g^{ij}}{\sqrt{g}}\pi^{kl}\pi_{kl} - \frac{2N}{\sqrt{g}}\pi^{ik}\pi^{j}{}_{k} + N\sqrt{g}\beta\left(\frac{R}{2}g^{ij} - R^{ij}\right) - \alpha N\sqrt{g}\left(a^i a^j - \frac{1}{2}g^{ij}a_k a^k\right) - \nabla_k[2\pi^{k(i}N^{j)} - \pi^{ij}N^k] + \beta\sqrt{g}[\nabla^{(i}\nabla^{j)}N - g^{ij}\nabla^2 N] + \mu\pi^{ij} + \{\pi^{ij}, \mathcal{U}\}_{PB}, \tag{32}$$

where the last term in Eq. (32) will contribute terms with spatial derivatives higher than second order.

The Hamiltonian (20) can be rewritten in terms of the constraints of the theory, up to a total divergence. In fact, Eq. (29) can be written as

$$H_N \equiv \sqrt{g}\left(\frac{\pi^{ij}\pi_{ij}}{g} - \mathcal{V}\right) - \frac{\sqrt{g}}{N}\nabla_i\left[-N\frac{\partial\mathcal{V}}{\partial a_i} + \nabla_j\left(N\frac{\partial\mathcal{V}}{\partial\nabla_j a_i}\right) - \nabla_k\nabla_j\left(N\frac{\partial\mathcal{V}}{\partial\nabla_j\nabla_k a_i}\right) + \cdots\right] = 0, \tag{33}$$

where the first term arises from the variation of the action with respect to the global factor N, while the second term comes from the variation of the potential with respect to N. It follows then that

$$N\sqrt{g}\left(\frac{\pi^{ij}\pi_{ij}}{g}-\mathcal{V}\right)=NH_N+\sqrt{g}\nabla_i\left[-N\frac{\partial\mathcal{V}}{\partial a_i}+\nabla_j\left(N\frac{\partial\mathcal{V}}{\partial\nabla_j a_i}\right)-\nabla_k\nabla_j\left(N\frac{\partial\mathcal{V}}{\partial\nabla_j\nabla_k a_i}\right)+\cdots\right]. \quad (34)$$

Under asymptotically flat conditions, $g_{ij}=\delta_{ij}+\mathcal{O}(\frac{1}{r})$, $N=1+\mathcal{O}(\frac{1}{r})$, $N_i=\mathcal{O}(\frac{1}{r})$, $\pi^{ij}=\mathcal{O}(\frac{1}{r})$, and all first spatial derivatives of fields of order $\mathcal{O}(\frac{1}{r^2})$, the total divergence that contributes at infinity arises from the first term in the parentheses. It is

$$-2\alpha\sqrt{g}\nabla_i(g^{ij}\nabla_j N). \quad (35)$$

We may now evaluate the physical Hamiltonian density in a particular coordinate system. We proceed as in Ref. [29]. The Lagrangian evaluated on the constraint submanifold is given by

$$L=\int dtd^3x(\pi^{ij}\dot{g}_{ij}-\mathcal{H})-E_{\mathrm{ADM}}, \quad (36)$$

where $\mathcal{H}\equiv-2\alpha\sqrt{g}\nabla_i\nabla^i N+$ total divergence. The generic terms "total divergence" do not contribute to the Lagrangian under the asymptotically flat conditions. The Lagrangian L can be expressed in terms of the $\mathrm{T}+\mathrm{L}$ components as in Ref. [29]. We may consider the coordinate condition

$$g_i=x^i+\left(\frac{1}{4\Delta}\right)g^T_{,i}, \qquad \Delta\equiv\delta^{ij}\partial_i\partial_j, \quad (37)$$

where g_i is the longitudinal part of the metric, and (following Ref. [29]) we end up with

$$\mathcal{H}=-\beta\Delta g^T-2\alpha\sqrt{g}\nabla_i(g^{ij}\partial_j N) \quad (38)$$

as the Hamiltonian density. Δg^T and ΔN are obtained from the constraints (28)–(29) in terms of the g^{TT}_{ij} and π^{ijTT} physical components.

The gravitational energy is then given by

$$E=\int dtd^3x\mathcal{H}=\oint_{\partial\Sigma_t}(-\beta g^T_{,i}-2\alpha N_{,i})dS_i, \quad (39)$$

which is, of course, independent of the coordinate condition (37).

In order to obtain g^T and N from the constraints (28)–(29), a condition on the invertibility of the higher-order derivative operator must be imposed. We discuss this point in Sec. VI C. The invertibility condition, expressed as the absence of the zero modes, is assured once several conditions on the coupling parameters are imposed.

IV. WAVE ZONE DEFINITION

The "wave zone" is an asymptotic region in space within which the degrees of freedom of the theory represent free radiation. That is, the leading order satisfies a wave equation and represents traveling spherical waves that escape to infinity without being affected by the source.

In the wave zone the self-interactions do not affect the dominant order. In the analysis the nonlinearities are not eliminated *a priori*; indeed, there can exist nontrivial static terms of the same order as the leading canonical one, but nevertheless they do not affect its propagation as free radiation. This approach is different from the first-order perturbation analysis where nonlinear effects are eliminated from the beginning. The existence of a wave zone is a very important property of nonlinear theories describing long-range interactions.

In linear theories like classical electrodynamics, nonlinearities may be due to sources and they do not occur if the wave zone is far enough from sources. Then, in this case, if radiation has a wave number k, the wave zone coincides with the "far zone" $kr\gg1$. That is, the distance from the sources to the wave zone is a very large number of wavelength and it guarantees that gradients and temporal derivatives of canonical modes are $\mathcal{O}(1/r)$. In general, we can apply the time Fourier transformation to the wave equation; it becomes a Helmholtz equation $\Delta u+k^2u=0$ and one can write its solution in the form $u=\frac{e^{ikr}}{r}\sum_{n=0}\frac{f_n(\theta,\phi)}{r^n}$. Then, it is straightforward to prove that if the f_0 coefficient of the $1/r$ term is zero, then $u=0$, that is, in the wave zone the only nonzero solution of the wave equation always contains the $\mathcal{O}(1/r)$ contribution. This is different from the solution of the Laplace equation. In fact, even if the monopole, dipole, etc., contributions are zero, the multipolar contributions can be nonzero.

In nonlinear theories the propagating fields may act as effective sources of itself. This self-interaction phenomena breaks up the desired free-radiation behavior. Therefore, the far zone condition is not sufficient to define the wave zone. The additional requirements have to guarantee that the field amplitudes are small such that the nonlinear terms are negligible. Here "negligible" means that they do not affect the free-wave propagation.

For GR there exists a well-defined wave zone, in which the background curvature does not affect the radiation of canonical modes. The gauge-invariant fields have the following behavior: the oscillatory part of $g^{TT}_{ij}\sim\pi^{ijTT}=\mathcal{O}(1/r)$, the oscillatory part of $g^T_{ij}\sim\mathcal{O}(1/r^2)$, the static part of $g^T_{ij}=\mathcal{O}(1/r)$ and of $\pi^{ijL}=\mathcal{O}(1/r^2)$. The remaining parts of g_{ij} and π^{ij} and ADM variables (N, N^i) are gauge dependent and do not affect the propagation of canonical modes [30].

The wave zone splits the space into three regions: the interior, the wave zone, and the exterior. The "interior" region bounded by the sphere of radius R_0 centered at the origin, $B_{R_0}(0)$, contains the sources and possibly space-time singularities; here, the phenomena of self-interaction are not negligible even at points where sources are not located. Then there is the wave zone, $B_{R_1}(0)/B_{R_0}(0)$ with $R_0 < r < R_1$, where the curvature does not affect radiation. Finally, there is the "exterior" region, $\mathbb{R}^3/B_{R_1}(0)$, where the propagating fields decay rapidly. This represents an asymptotically flat space-time.

In this work we prove that in the $3+1$ nonprojectable Hořava-Lifshitz theory at the kinetic conformal point there exists a wave zone as in GR. That is, the $(g_{ij}^{TT}, \pi^{ijTT}) \sim f(\theta,\phi)\exp i(kx-\omega(k)t)/r$, with $f(\theta,\phi) = \mathcal{O}(1)$, propagate without being affected by the static parts in the same way as in the linearized theory. Due to the anisotropic character of this theory the dispersion relation contains, besides the terms ω^2 and k^2, k^4 and k^6 terms associate to higher-order derivative terms, this represents a propagating dispersive signal. At low energies, the dispersion reduces to the usual nondispersive wave and differs from GR in the speed of propagation. The speed of propagation at low energies is equal to $\sqrt{\beta}$ and the recent gravitational-wave observations fix it to 1 the GR value, up to an error of the order of one part in 10^{15} [20,21,41,42].

We define the wave zone in Hořava-Lifshitz theory, as in GR, to be the spacelike domain far away from the sources satisfying the following conditions:

(1) $kr \gg 1$, where k is the wave number and $r = (x_i x^i)^{1/2}$.

(2)

$$|g_{ij} - \delta_{ij}| \lesssim A/r, \tag{40}$$

$$|N - 1| \lesssim A/r, \tag{41}$$

$$|N^i| \lesssim A/r, \tag{42}$$

where $A(t,\theta,\phi)$ represents a function of time and angles such that A and all its derivatives are bounded.

(3) The conditions

$$|\partial g_{ij}/\partial(kr)|^2 \ll |g_{ij} - \delta_{ij}|, \tag{43}$$

$$|\partial N_i/\partial(kr)|^2 \ll |N_i|, \tag{44}$$

$$|\partial N/\partial(kr)|^2 \ll |N - 1| \tag{45}$$

are fulfilled. Here we use the shorthand notation $\lesssim A/r \equiv \mathcal{O}(A/r)$, and thus "$\lesssim A/r$" means the left part decreases at least as A/r, and (g,δ) are symbolic representations of any component of the metric and the Euclidean three-dimensional metric tensor.

The three conditions can be satisfied by choosing a region with a large enough distance from the source. The first condition is the same as in linear theories and indicates that the wave zone is quite far from the sources (with distance measured in units of wavelength) and guarantees that the gradients of the canonical modes are $\mathcal{O}(1/r)$ and exclude the wave number $k \to 0$ in the wave zone. The second and third conditions are necessary to ensure that, in the wave zone, there are no self-interactions and the canonical variables propagate freely. The second condition is imposed in order to ensure that both the perturbations of the dynamic modes and the Newtonian parts are $\mathcal{O}(1/r)$, and guarantees that the terms of higher order than the leading one in $|g_{ij} - \delta_{ij}|$ remain negligible compared to the leading order. The third condition is imposed in order to ensure that the nonlinear terms containing spatial derivatives are small compared to the leading order. This implies that for a wave number $k > k_{\min}$ the radiation can be treated as free radiation if $k_{\min} \gg k_{\max}(A/r)^{1/2}$, where $|g-\delta| \sim A/r$ and $k_{\max}$ is the maximum frequency. This ensures that the interference of two or more subleading $\mathcal{O}(1/r^2)$ modes do not generate an effect comparable to the leading order $\mathcal{O}(1/r)$ modes.

The conditions (40)–(42) are preserved under the change of coordinates (8) if and only if

$$|\dot{\xi}^i| \lesssim \frac{A^i}{r} \ll 1, \quad |\xi^i{}_{,j}| \lesssim \frac{A^i{}_j}{r} \ll 1, \quad |\dot{f}(t)| \lesssim \frac{A}{r} \ll 1. \tag{46}$$

In addition to Eqs. (12)–(14), we have

$$\tilde{N}(1+\dot{f}) = N, \tag{47}$$

$$\tilde{N}^i(\tilde{t},\tilde{x})(1+\dot{f}(t)) = N^i(t,x) - \dot{\xi}^i(t,x) + \mathcal{O}(1/r^2), \tag{48}$$

$$\tilde{g}_{ij}(\tilde{t},\tilde{x}) - g_{ij}(t,x) = -(\xi_{i,j}(x,t) + \xi_{j,i}(x,t)) + \mathcal{O}(1/r^2). \tag{49}$$

We remark that $\frac{\mathcal{O}(1/r^2)}{1/r} \ll 1$.

We then conclude that the longitudinal part of g_{ij} is a gauge-dependent field. It can be fixed by choosing particular coordinates on the leaves. We notice that the admissible coordinate transformations are not necessarily infinitesimal ones. In fact, one may have a parameter $\xi \sim \log r$. Also, one may have parameters with dependence $1/r$ or e^{ikr}/r or $\mathcal{O}(1/r^2)$. Consequently, the continuous transformations in the wave zone are an extension of the infinitesimal ones. These have the usual form

$$\tilde{N}(t,x) - N(t,x) = -\xi^k\partial_k N(t,x) - \dot{f}(t)N(t,x) - f(t)\dot{N}(t,x), \tag{50}$$

$$\tilde{N}^i(t,x) - N^i(t,x) = -\xi^k \partial_k N^i(t,x) + \partial_j \xi^i N^j(t,x) - \dot{\xi}^i - \dot{f}(t) N^i(t,x) - f(t)\dot{N}^i(t,x), \quad (51)$$

$$\tilde{g}_{ij}(t,x) - g_{ij}(t,x) = -\xi^k \partial_k g_{ij}(t,x) - g_{ki}\partial_j \xi^k - g_{kj}\partial_i \xi^k - f\dot{g}_{ij}(t,x), \quad (52)$$

where in this case $f(t)$ and $\xi^i(t,x)$ are infinitesimal parameters. We notice that these transformations contain terms that are of order $\mathcal{O}(1/r^2)$, and hence in the wave zone at order $\mathcal{O}(1/r)$ they can be eliminated.

V. BEHAVIOR IN THE WAVE ZONE

In the wave zone the metric, lapse, and shift fields have the behavior

$$g_{ij} - \delta_{ij} \sim N_i \sim N - 1 \lesssim \frac{B_{ij}}{r} + \frac{A_{ij}e^{ikr}}{r}, \quad (53)$$

$$N_{i,j} \sim g_{ij,l} \sim \Gamma^i_{jl} \lesssim \frac{B}{r^2} + k\frac{Ae^{ikr}}{r}, \quad (54)$$

where A and B are a generic functions of time and angles with bounded derivatives.

A. Laplacian solutions in the wave zone

Let $\varphi \in C^2(\Omega)$ be a solution of the Poisson equation $\Delta\varphi = -4\pi\rho$ in $\Omega \equiv \mathbb{R}^3/B_{R_0}(0)$ such that $\varphi \to 0$ when $r \to \infty$. Expanding $\varphi(r,\theta,\phi)$ into spherical harmonics $Y_{lm}(\theta,\phi)$, a standard result gives

$$\varphi(r,\theta,\phi) = \sum_{l=0}^{l=\infty}\sum_{m=-l}^{m=l} \chi_{lm}(r) r^l Y_{lm}(\theta,\phi), \quad (55)$$

$$\chi_{lm}(r) \equiv \int_r^\infty \frac{M_{lm}(r')}{r'^{2l+2}} dr', \quad (56)$$

$$M_{lm}(r) \equiv \int_{B_r(0)} \rho(r',\theta',\phi') r'^l Y'_{lm}(r',\theta',\phi') d^3r'. \quad (57)$$

Following Arnowitt, Deser, and Misner [30], in the wave zone we have the following.

(1) If in the region $r > R_0$ the source ρ has an oscillatory asymptotic behavior $\rho \sim Y_{lm}e^{ikr}/r^n$, then

$$\varphi \sim (1/k^2) Y_{lm}\frac{e^{ikr}}{r^n}[1 + \mathcal{O}(1/kr)] + \sum_{l=0}^{l=\infty}\sum_{m=-l}^{m=l} c_{lm} Y_{lm} \frac{1}{r^{l+1}}. \quad (58)$$

Here the c_{lm} do not depend on the spatial coordinates and they are determined from the behavior of ρ in the region $r \leq R_0$. Hence, an oscillatory source produced in $r > R_0$ a source-independent oscillatory solution of the same order of ρ plus a static source-dependent part $\lesssim 1/r$. By "static" we mean non-oscillatory on the spatial dependence.

(2) If in the region $r > R_0$,

$$\psi \sim B_m(t,\theta,\phi)/r^m + A_n(t,\theta,\phi)e^{ikr}/r^n, \quad (59)$$

with $|B_m(t,\theta,\phi)|$ and $|A_n(t,\theta,\phi)|$ bounded, then

$$\frac{1}{\Delta}\psi_{,j} \sim \frac{\langle B\rangle}{r^{m-1}} + \frac{1}{ik}\frac{\langle A\rangle e^{ikr}}{r^n} + \sum_{l=1}^{\infty}\sum_{p=-l}^{p=l} c_{lp} Y_{lp}\frac{1}{r^{l+1}}, \quad (m \geq 2). \quad (60)$$

$$\frac{1}{\Delta}\psi_{,ij} \sim \frac{\langle B\rangle}{r^m} + \frac{\langle A\rangle e^{ikr}}{r^n} + \sum_{l=2}^{\infty}\sum_{p=-l}^{p=l} c_{lp} Y_{lp}\frac{1}{r^{l+1}}, \quad (m \geq 1). \quad (61)$$

$\langle A\rangle$, etc., are generic angular integrals of A and other angular functions. For $m \geq 3$, Eqs. (60) and (61) may also have contributions of the form $r^{-m+1}\log r$ and $r^{-m}\log r$, respectively, which are source independent.

VI. WAVE ZONE IN HOŘAVA-LIFSHITZ GRAVITY AT THE KINETIC CONFORMAL POINT

We now analyze the solution of the field equations in the wave zone and show that the equations of the canonical transverse-traceless modes describe freely propagating fields.

A. Solution of the primary constraints

Taking a derivative with respect to the j coordinate of the $\mathrm{T}+\mathrm{L}$ decomposition of the momentum constraint in Eq. (27), we obtain

$$\Delta\pi^j{}_{,j} = -\frac{1}{2}(\pi^{lm}\Gamma^j{}_{lm})_{,j}. \quad (62)$$

If we use the estimation of the previous section, the result (60), and the same argument of finite momenta used in the wave zone for RG (see Appendix C of Ref. [30]), then

$$\pi^j{}_{,j} \lesssim \frac{B}{r^2} + k\frac{\tilde{A}e^{ikr}}{r^2}, \quad (63)$$

where B is source dependent while $\tilde{A}$ is source independent. Taking now a derivative with respect to the i coordinate of the $\mathrm{T}+\mathrm{L}$ decomposition of the momentum constraint (27), we get

$$\Delta\pi^j{}_{,i} = (\pi^l{}_{,l})_{,ij} - (\pi^{lm}\Gamma^j{}_{lm})_{,i}. \tag{64}$$

Thus, we can use Eqs. (60)–(61) to finally obtain

$$\pi^j{}_{,i} \lesssim \frac{B_i^j}{r^2} + k\frac{\hat{A}_i^j e^{ikr}}{r^2} \tag{65}$$

for new functions B_i^j and $\tilde{A}_i^j$.

The constraint (25) expressed in the T + L decomposition becomes $\pi^T + 2\pi^i{}_{,i} = 0$, and then

$$\pi^T \lesssim \frac{B}{r^2} + k\frac{\hat{A}e^{ikr}}{r^2}, \qquad \pi^{ijT} \lesssim \frac{B^{ij}}{r^2} + k\frac{\hat{A}^{ij}e^{ikr}}{r^2}, \tag{66}$$

where we have used the estimate given by Eq. (61).

We note that the TT term of the T + L decomposition is the dominant one among the terms with oscillatory behavior,

$$\pi^{ijTT} \lesssim \frac{B^{ij}}{r} + k\frac{\hat{A}^{ij}e^{ikr}}{r}. \tag{67}$$

In the following section we will prove that the static parts of $\pi^{ij} \lesssim B^{ij}/r^2$, where B^{ij} does not depend on time.

B. Solution of the secondary constraints

Now we calculate the constraints $H_N = 0$ and $H_P = 0$. We include the $\mathcal{V}$-potential terms up to $\mathcal{O}(1/r)$, that is the relevant terms in the wave zone. We get the estimate $\pi^{ij}\pi_{ij} \lesssim B/r^2 + Ae^{ikr}/r^2$, where $B = B^{ij}B_{ij}$. Up to order $\mathcal{O}(1/r)$ the second-class constraints $H_N = 0$ and $H_P = 0$ become

$$\begin{aligned} &-\beta R + 2\alpha\Delta N - \alpha_1\Delta R - 2\alpha_2\Delta^2 N - \alpha_3\Delta^2 R \\ &\quad + 2\alpha_4\Delta^3 N \lesssim \frac{\tilde{B}}{r^2} + k^2 P_2(k^2)\frac{\hat{A}e^{ikr}}{r^2}, \end{aligned} \tag{68}$$

$$\begin{aligned} &\frac{\beta}{2}R - 2\beta\Delta N - 2\alpha_1\Delta^2 N - 2\alpha_3\Delta^3 N \\ &\quad - \beta_1 R^{ij}_{,ij} + \beta_3\Delta R^{ij}_{,ij} - (\beta_1 + 4\beta_2)\Delta R \\ &\quad + (\beta_3 + 4\beta_4)\Delta^2 R \lesssim \frac{\hat{B}}{r^2} + k^2 P_2(k^2)\frac{\hat{A}e^{ikr}}{r^2}, \end{aligned} \tag{69}$$

respectively, where $\tilde{B}$ and $\hat{B}$ are proportional to B. We have used $P_2(k^2)$ to denote a generic second-order polynomial in k^2 with real coefficients.

If we use the gauge condition $g_{ij,j} = 0$, we get

$$R_{ik} + \frac{1}{2}\Delta g_{ik}^{TT} + \frac{1}{2}\Delta g_{ik}^T + \frac{1}{2}g^T_{,ik} \lesssim \frac{\mathcal{B}}{r^4} + k^2\frac{Ae^{ikr}}{r^2}, \tag{70}$$

$$R + \Delta g^T \lesssim \frac{\mathcal{B}}{r^4} + k^2\frac{Ae^{ikr}}{r^2}, \tag{71}$$

$$R^{ij}_{,ij} + \frac{1}{2}\Delta^2 g^T \lesssim \frac{\mathcal{B}}{r^6} + k^4\frac{Ae^{ikr}}{r^2}, \tag{72}$$

$$\Delta R^{ij}_{,ij} + \frac{1}{2}\Delta^3 g^T \lesssim \frac{\mathcal{B}}{r^8} + k^6\frac{Ae^{ikr}}{r^2}. \tag{73}$$

Then, from Eqs. (68) and (69) we obtain the coupled system of two sixth-order partial differential equations for N and g^T:

$$\begin{aligned} &(\beta\Delta + \alpha_1\Delta^2 + \alpha_3\Delta^3)g^T + 2(\alpha\Delta - \alpha_2\Delta^2 + \alpha_4\Delta^3)N \\ &\lesssim \frac{\tilde{B}}{r^2} + k^2 P_2(k^2)\frac{\hat{A}e^{ikr}}{r^2}, \end{aligned} \tag{74}$$

$$\begin{aligned} &\left[-\frac{\beta}{2}\Delta + \frac{1}{2}(3\beta_1 + 8\beta_2)\Delta^2 - \frac{1}{2}(3\beta_3 + 8\beta_4)\Delta^3\right]g^T \\ &\quad - 2[\beta\Delta + \alpha_1\Delta^2 + \alpha_3\Delta^3]N \lesssim \frac{\hat{B}}{r^2} + k^2 P_2(k^2)\frac{\tilde{A}e^{ikr}}{r^2}. \end{aligned} \tag{75}$$

From Eqs. (74) and (75), it follows that the B factor must vanish, and the nonoscillatory contribution is of order $O(1/r^3)$. This means that the nonoscillatory part of the momentum is of order $O(1/r^2)$. The above conclusion is very important for the discussions in the following sections.

By assumption, the fields decay as in Eq. (53) in the wave zone. From Eqs. (74)–(75), it then follows that the nonoscillatory part of the solution for g^T and N can contribute to this order, since on the right-hand side there may be terms of order $1/r^3$. On the other side, the oscillatory terms with dependence e^{ikr}/r satisfies the equation

$$(\Delta + k^2)\frac{e^{ikr}}{r} = 0. \tag{76}$$

Considering $g^T \sim g_o^T\frac{e^{ikr}}{r}$ and $N \sim N_o\frac{e^{ikr}}{r}$ in the equations (74) and (75) we obtain

$$D_1(k^2)g_o^T + D_2(k^2)N_o = 0, \tag{77}$$

$$D_2(k^2)g_o^T + D_3(k^2)N_o = 0, \tag{78}$$

where we define polynomials

$$D_1(k^2) \equiv \frac{1}{8}[-(3\beta_3 + 4\beta_4)k^6 - (3\beta_1 + 8\beta_2)k^4 - \beta k^2], \tag{79}$$

$$D_2(k^2) \equiv \frac{1}{2}[-\alpha_3 k^6 + \alpha_1 k^4 - \beta k^2], \tag{80}$$

$$D_3(k^2) \equiv -\alpha_4 k^6 - \alpha_2 k^4 - \alpha k^2, \tag{81}$$

equations (77), (78) yield $g_o^T = N_o = 0$ for these zero modes, except when

$$D(k^2) \equiv D_1(k^2)D_3(k^2) - D_2^2(k^2) = 0. \tag{82}$$

In the case where the zero modes vanish, the behavior of g^T and N in the wave zone, taking into account the asymptotic flatness condition $N-1 \to 0$, is

$$N - 1 \sim g^T \lesssim \frac{B}{r} + \frac{\hat{A}e^{ikr}}{r^2}, \tag{83}$$

and hence

$$a_i \lesssim \frac{B_i}{r^2} + k\frac{\hat{A}_i e^{ikr}}{r^2}, \tag{84}$$

$$a_i a^i \lesssim \frac{B}{r^4} + k^2\frac{\hat{A}e^{ikr}}{r^4}, \tag{85}$$

$$\nabla^i a_i \lesssim \frac{B}{r^3} + k^2\frac{\hat{A}e^{ikr}}{r^2}. \tag{86}$$

On the other side, there are a finite number of solutions of Eq. (82) for which there are nontrivial solutions for g_0^T and N_0 and hence contributions of the form $\frac{e^{ikr}}{r}$ for g^T and N. In the low-energy case, Eq. (82) reduces to a term proportional to k^2 with the factor $-\frac{\beta^2}{4}(\beta - \frac{\alpha}{2})$, which is different from zero for the values of β and α determined from experimental data. At low energies [29], we always have $g_0^T = N_0 = 0$. In the next section we analyze the general solution of Eq. (82).

C. Constraint resolution

The polynomial of Eq. (82) has $D(k^2) = k^4 Q(k^2)$, where we defined the quartic polynomial in k^2 as

$$Q(k^2) \equiv ak^8 + bk^6 + ck^4 + dk^2 + e, \tag{87}$$

$$a \equiv \alpha_4(3\beta_3 + 4\beta_4) - 2\alpha_3^2, \tag{88}$$

$$b \equiv \alpha_2(3\beta_3 + 4\beta_4) + \alpha_4(3\beta_1 + 8\beta_2) + 4\alpha_1\alpha_3, \tag{89}$$

$$c \equiv \alpha(3\beta_3 + 4\beta_4) + \alpha_2(3\beta_1 + 8\beta_2) + \beta(\alpha_4 - 4\alpha_3) - 2\alpha_1^2, \tag{90}$$

$$d \equiv \alpha(3\beta_1 + 8\beta_2) + \beta(4\alpha_1 + \alpha_2), \tag{91}$$

$$e \equiv \beta(\alpha - 2\beta). \tag{92}$$

For the theory to be power-counting renormalizable, we demand $a \neq 0$ [13]. Experimental tests at low energies require that $e \leq 0$. Note that $k^2 = 0$ is a root of Eq. (82), and this root is not considered in the analysis in the wave zone due to the condition $kr \gg 1$.

We are interested in the roots of $Q(k^2) = 0$, and so we use the Descartes's method. The change of variable $k^2 = s - \frac{b}{4a}$ yields the depressed polynomial associated to $Q(k^2) = 0$,

$$P(s) \equiv a(s^4 + ps^2 + qs + r) = 0, \tag{93}$$

$$p \equiv \frac{c}{a} - \frac{3}{8}\frac{b^2}{a^2}, \tag{94}$$

$$q \equiv \frac{1}{8}\frac{b^3}{c^3} + \frac{d}{a} - \frac{bc}{2a^2}, \tag{95}$$

$$r \equiv -\frac{3}{256}\frac{b^4}{a^4} + \frac{1}{16}\frac{cb^2}{a^3} - \frac{bd}{a^3} + \frac{e}{a}. \tag{96}$$

We have factorized

$$P(s) = a(s^2 + fs + g)(s^2 - fs + h), \tag{97}$$

$$g + h = f^2 + p, \tag{98}$$

$$f(h - g) = q, \tag{99}$$

$$gh = r, \tag{100}$$

where f^2 satisfies the equation

$$R(f^2) \equiv f^6 + 2pf^4 + (p^2 - 4r)f^2 - q^2 = 0. \tag{101}$$

There always exists a real positive solution for Eq. (101). In fact, if $q^2 > 0$, then Eq. (101) always has a strictly positive solution. If $q = 0$ we can chose $f^2 = 0$ as a root, and $P(s)$ is a biquadratic polynomial whose roots can be determined using the general quadratic formula for s^2. If $f \neq 0$, from Eq. (97) we can calculate g and h via

$$h = \frac{1}{2}\left(f^2 + p + \frac{q}{f}\right), \tag{102}$$

$$g = \frac{1}{2}\left(f^2 + p - \frac{q}{f}\right). \tag{103}$$

Finally, we can express Eq. (87) in a factorized form,

$$Q(k^2) = a(k^4 + Ak^2 + B)(k^4 + Ck^2 + D), \tag{104}$$

where the constants A, B, C, D are related to the coupling parameters by

$$A = f + \frac{1}{2}\frac{b}{a}, \qquad B = \frac{1}{4}\frac{bf}{a} + \frac{1}{16}\frac{b^2}{a^2} + g, \tag{105}$$

$$C = -f + \frac{1}{2}\frac{b}{a}, \qquad D = -\frac{1}{4}\frac{bf}{a} + \frac{1}{16}\frac{b^2}{a^2} + h. \quad (106)$$

If B and D have nonzero imaginary parts, then there are no zero modes. If they are real, then there are no zero modes if and only if the coupling parameters satisfy the following conditions:

$$a<0, \;\; B>0, \;\; D>0, \;\; -2\sqrt{B}<A, \;\; -2\sqrt{D}<C. \quad (107)$$

Generically, the zero modes have a divergent contribution to the total gravitational energy. In fact, the energy density obtained from the boundary terms in the Hamiltonian, evaluated at the zero modes, decays as $1/r$.

We have thus shown that the conditions (107) must be satisfied in order to have a consistent formulation of Hořava-Lifshitz gravity.

D. Dynamical equations in the wave zone

The T + L components of Eq. (31) in the gauge $g_{ij,j} = 0$ become

$$\dot{g}^{TT}_{ij} - 2\pi^{TT}_{ij} + (\mu g_{ij})^{TT} \lesssim \frac{B}{r^2} + k^2\frac{\hat{A}e^{ikr}}{r^2}, \quad (108)$$

$$-2N_{(i,j)} + (\mu g_{ij})^{L} \lesssim \frac{B}{r^2} + k^2\frac{\hat{A}e^{ikr}}{r^2}, \quad (109)$$

$$\dot{g}^{T}_{ij} + (\mu g_{ij})^{T} \lesssim \frac{B}{r^2} + k^2\frac{\hat{A}e^{ikr}}{r^2}, \quad (110)$$

where $\mu g_{ij} = (\mu g_{ij})^{TT} + (\mu g_{ij})^{T} + (\mu g_{ij})^{L}$, $(\mu g_{ij})^{T} = \mu\delta_{ij} - \frac{1}{\Delta}\mu_{,ij} + \mathcal{O}(1/r^2)$, $(\mu g_{ij})^{L} = \frac{1}{\Delta}\mu_{,ij} + \mathcal{O}(1/r^2)$, and $(\mu g_{ij})^{TT} = \mathcal{O}(1/r^2)$. Then, from Eq. (110) we get $\mu \lesssim \frac{B}{r^2} + k^2\frac{\hat{A}e^{ikr}}{r^2}$ and consequently

$$\dot{g}^{TT}_{ij} - 2\pi^{TT}_{ij} \lesssim \frac{B}{r^2} + k^2\frac{\hat{A}e^{ikr}}{r^2}, \quad (111)$$

$$N_{(i,j)} \lesssim \frac{B}{r^2} + k^2\frac{\hat{A}e^{ikr}}{r^2}. \quad (112)$$

We then have $N_i \lesssim \frac{B}{r} + k\frac{\hat{A}e^{ikr}}{r^2}$.

Besides, we get the following contributions in Eq. (32):

$$\beta N\frac{\delta R}{\delta g_{ij}} = \frac{\beta}{2}\Delta g^{T} + \mathcal{O}(1/r^2), \quad (113)$$

$$\beta_1 N\frac{\delta}{\delta g_{ij}}(R_{ij}R^{ij}) = \frac{\beta_1}{2}\Delta^2 g^{TT}_{ij} + \mathcal{O}(1/r^2), \quad (114)$$

$$\beta_3 N\frac{\delta}{\delta g_{ij}}(\nabla_i R_{jk}\nabla^i R^{jk}) = -\frac{\beta_3}{2}\Delta^3 g^{TT}_{ij} + \mathcal{O}(1/r^2). \quad (115)$$

Then, we obtain

$$\dot{\pi}^{TT}_{ij} = \frac{1}{2}(\beta\Delta + \beta_1\Delta^2 - \beta_3\Delta^3)g^{TT}_{ij} + \mathcal{O}(1/r^2). \quad (116)$$

Equations (111) and (116) show that in the wave zone the TT part of the fields up to order $\mathcal{O}(1/r)$ satisfies the sixth-order partial differential equation

$$\ddot{g}^{TT}_{ij} - (\beta\Delta + \beta_1\Delta^2 - \beta_3\Delta^3)g^{TT}_{ij} = 0, \quad (117)$$

$$\ddot{\pi}^{TT}_{ij} - (\beta\Delta + \beta_1\Delta^2 - \beta_3\Delta^3)\pi^{TT}_{ij} = 0. \quad (118)$$

The terms with spatial derivative of order greater than two, are due to the contributions of the potential terms $\mathcal{V}^{(2)}$ and $\mathcal{V}^{(3)}$.

We can obtain the low-energy limit of Eqs. (117) and (118) if we assume that $k_{\max} \ll K_{\mathrm{UV}}$, where K_{UV} is the ultraviolet cutoff. Then, the Laplacian acting on the TT modes is the leading term among the spatial-derivative ones. In this case, Eqs. (117) and (118) reduce to the wave equation for the TT modes,

$$\ddot{g}^{ijTT} - \beta\Delta g^{ijTT} = 0, \quad (119)$$

$$\ddot{\pi}^{ijTT} - \beta\Delta\pi^{ijTT} = 0. \quad (120)$$

The canonical modes travel with a speed $\sqrt{\beta}$, and it only differs from that in the wave zone of general relativity by the value of the velocity of propagation. Gravitational-wave experiments restrict this value to be between -3×10^{-15} and 7×10^{-16} times the speed of light [41,42].

VII. SOLUTION OF THE HOŘAVA-LIFSHITZ WAVE EQUATION AT THE KINETIC CONFORMAL POINT

The solution to Eq. (117) can be written in terms of the spherical Hankel functions $h^{(1)}_l(kr)$ and $h^{(2)}_l(kr)$. In fact,

$$u(r,k) \equiv \sum_{l,m}[A^{(1)}_{lm}h^{(1)}_l(kr) + A^{(2)}_{lm}h^{(2)}_l(kr)]Y_{lm}(\theta,\phi) \quad (121)$$

satisfies

$$-\Delta u(r,k) = k^2 u(r,k), \quad (122)$$

and hence

$$[\beta(-\Delta) - \beta_1(-\Delta)^2 - \beta_3(-\Delta)^3]u(r,k) = (\beta k^2 - \beta_1 k^4 - \beta_3 k^6)u(r,k). \quad (123)$$

Finally, $\Psi(r,t) = e^{i\omega t}u(r,k)$ satisfies Eq. (117) provided

$$\beta k^2 - \beta_1 k^4 - \beta_3 k^6 = \omega^2. \quad (124)$$

For a given l,

$$h_l^{(1)}(kr) = \left[(-i)^{l+1}\sum_{s=0}^{l}\frac{i^s}{s!(2kr)^s}\frac{(l+s)!}{(l-s)!}\right]\frac{e^{ikr}}{r}, \quad (125)$$

and $h_l^{(2)}(kr) = \overline{h_l^{(1)}}(kr)$, where $\bar{h}$ denotes complex conjugation.

In the wave zone

$$h_l^{(1)}(kr) \approx (-i)^{l+1}\frac{e^{ikr}}{r}, \qquad h_l^{(2)}(kr) \approx (i)^{l+1}\frac{e^{-ikr}}{r}, \quad (126)$$

and hence

$$u(r,k) = \sum_{l,m}[A_{lm}^{(1)}(-i)^{l+1}Y_{lm}(\theta,\phi)]\frac{e^{ikr}}{r} + \sum_{l,m}[A_{lm}^{(2)}(i)^{l+1}Y_{lm}(\theta,\phi)]\frac{e^{-ikr}}{r}. \quad (127)$$

Finally, the outgoing solution to Eq. (117) in the wave zone is

$$\Psi = \frac{e^{-i\omega t+ikr}}{r}\sum_{l,m}A_{lm}^{(1)}(-i)^{l+1}Y_{lm}(\theta,\phi), \quad (128)$$

where k is the solution of the dispersion relation (124).

In order to have a positive left-hand side in Eq. (124), the couplings must satisfy [13]

$$\alpha \neq 2\beta, \quad \beta > 0, \quad \beta_3 < 0, \quad \beta_1 < 2\sqrt{\beta|\beta_3|}. \quad (129)$$

In this case, for a given ω^2 there always exists a unique k^2 satisfying the dispersion relation. Consequently, for a given ω there exists an outgoing solution that depends on the radial coordinate $e^{-i\omega t+ikr}/r$. The relation between ω and k can be rewritten as

$$\frac{\omega^2}{k^2} = \beta - \beta_1 k^2 - \beta_3 k^4, \quad (130)$$

which tells us that at low energies the speed of propagation of the gravitational wave is $\sqrt{\beta}$, while at high energies the speed depends on the energy of the wave. Under the above assumptions about the coupling constants, the right-hand side of Eq. (130) is always positive.

VIII. DISCUSSION

We showed that in Hořava-Lifshitz theory at the kinetic conformal point there exists a wave zone in which the physical degrees of freedom—the transverse-traceless tensorial modes—propagate as free waves. In that region of space, provided the set of restrictions on the coupling parameters (107) is assumed, the other components of the metric on the spacelike leaves of the foliation as well as the lapse function are of the same order in powers of $1/r$, with r being the distance to a bounded source. They constitute a static Newtonian background which does not interact with the propagation of the wave modes. At larger distances from the source than the wave zone, the TT components of the metric decay with higher powers of $1/r$, while the T modes vary in such a way that the energy of the system is finite. The set of restrictions (107) is a necessary and sufficient condition for the nonexistence of zero modes to the solutions of the constraints. The zero modes would have a divergent contribution to the gravitational energy and must be avoided. The wave solution describing the propagation of the physical degrees of freedom satisfies a linear partial differential equation quadratic in time derivatives but cubic in the Laplace operator. At low energies, that is in the infrared limit of the renormalization flow of the Hořava-Lifshitz theory, the equation reduces to the wave equation, with a speed of propagation characterized by the square root of a coupling constant, which should take at this limit a value very near to the speed of light c. At higher energies, the high spacelike derivative terms in the potential of the Hamiltonian become relevant and the wave solutions are characterized by a dispersion relation, which means that the speed of propagation of the physical modes becomes energy dependent. The existence of real and positive solutions of the dispersion relation Eq. (124) requires additional restrictions on the coupling parameters a, β, β_1 and β_3. They must satisfy Eq. (129).

The low-energy limit of the Hořava-Lifshitz theory at the kinetic conformal point has only two coupling constants, β and α, which may have a bounded range of values in order that the predictions of the theory become consistent with the known experimental data, for $\beta = 1$ and $\alpha = 0$ at low energies the theory is the same as GR in a particular gauge. For $\alpha \neq 0$, the static solution with spherical symmetry does not describe a black hole, but rather a throat connecting an asymptotically flat space and a space with an essential singularity [13]. That is, the physics of the theory is different from GR, as predicted in Ref. [1], but in the limit $\beta \to 1$ and $\alpha \to 0$ the geometry external to the throat is very similar to the geometry external to the black hole of the Schwarzschild solution in GR; however, only in the limit $\beta = 1$ and $\alpha = 0$ is there a black hole. An analogous relation to GR occurs with the gravitation radiation. We have also shown that the gravitational radiation predicted by the theory at low energies has similar properties as the radiation in GR, but at higher energies the wave solutions satisfy a dispersion relation characteristic of Hořava-Lifshitz theory. The higher-order spacelike derivative terms in the evolution equation are related to the higher-order derivative terms in the potential that determine, in the UV regime, the power-counting renormalizability of the theory, which is the main virtue of the Hořava proposal.

An interesting study would be to analyze the radiative properties of the gravitational-electromagnetic interaction in the wave zone. It is known that in a perturbative approach the gravitational and electromagnetic degrees of freedom propagate with the same speed, but their interaction behavior in the wave zone is unknown [24,25].

ACKNOWLEDGMENTS

J. Mestra-Páez is partially supported by CONICYT/PFCHA/DOCTORADO BECAS CHILE/Number2019-21191442. J. Mestra-Páez acknowledge the PhD program Doctorado en Física mención Física Matemática de la Universidad de Antofagasta for continous support and encouragement. J.M. P is supported by the projects ANT1956 and SEM18-02 of the University of Antofagasta, Chile.

[1] P. Hořava, Quantum gravity at a Lifshitz point, Phys. Rev. D **79**, 084008 (2009).
[2] D. Blas, O. Pujolàs, and S. Sibiryakov, Consistent Extension of Hořava Gravity, Phys. Rev. Lett. **104**, 181302 (2010).
[3] C. Charmousis, G. Niz, A. Padilla, and P. M. Saffin, Strong coupling in Hořava gravity, J. High Energy Phys. 08 (2009) 070.
[4] M. Visser, Power-counting renormalizability of generalized Hořava gravity, arXiv:0912.4757.
[5] A. Papazoglou and T. P. Sotiriou, Strong coupling in extended Hořava-Lifshitz gravity, Phys. Lett. B **685**, 197 (2010).
[6] D. Orlando and S. Reffert, The renormalizability of Hořava-Lifshitz-type gravities, Classical Quantum Gravity **26**, 155021 (2009).
[7] F.-W. Shu and Y.-S. Wu, Stochastic quantization of the Hořava gravity, arXiv:0906.1645.
[8] D. Benedetti and F. Guarnieri, One-loop renormalization in a toy model of Hořava-Lifshitz gravity, J. High Energy Phys. 03 (2014) 078.
[9] A. Contillo, S. Rechenberger, and F. Saueressig, Renormalization group flow of Hořava-Lifshitz gravity at low energies, J. High Energy Phys. 12 (2013) 017.
[10] G. D'Odorico, F. Saueressig, and M. Schutten, Asymptotic Freedom in Hořava-Lifshitz Gravity, Phys. Rev. Lett. **113**, 171101 (2014).
[11] G. D'Odorico, J.-W. Goossens, and F. Saueressig, Covariant computation of effective actions in Hořava-Lifshitz gravity, J. High Energy Phys. 10 (2015) 126.
[12] A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov, and C. F. Steinwachs, Renormalization of Hořava gravity, Phys. Rev. D **93**, 064022 (2016).
[13] J. Bellorín and A. Restuccia, Quantization of the Hořava theory at the kinetic-conformal point, Phys. Rev. D **94**, 064041 (2016).
[14] A. Wang, Hořava gravity at a Lifshitz point: A progress report, Int. J. Mod. Phys. D **26**, 1730014 (2017).
[15] S. Shin and M.-I. Park, On gauge invariant cosmological perturbations in UV-modified Hořava gravity, J. Cosmol. Astropart. Phys. 12 (2017) 033.
[16] M. Pospelov and Y. Shang, Lorentz violation in Hořava-Lifshitz-type theories, Phys. Rev. D **85**, 105001 (2012).
[17] S. Mukohyama, Hořava-Lifshitz cosmology: A review, Classical Quantum Gravity **27**, 223101 (2010).
[18] M. Jamil and E. N. Saridakis, New agegraphic dark energy in Hořava-Lifshitz cosmology, J. Cosmol. Astropart. Phys. 07 (2010) 028.
[19] D. Blas, O. Pujolas, and S. Sibiryakov, Comment on Strong coupling in extended Horava–Lifshitz gravity [Phys. Lett. B **685**, 197 (2010)]; Phys. Lett. B **688**, 350 (2010).
[20] O. Ramos and E. Barausse, Constraints on Hořava gravity from binary black hole observations, Phys. Rev. D **99**, 024034 (2019).
[21] A. Emir Gümrükçüoğlu, M. Saravani, and T. P. Sotiriou, Hořava gravity after GW170817, Phys. Rev. D **97**, 024032 (2018).
[22] J. Bellorín, A. Restuccia, and A. Sotomayor, Consistent Hořava gravity without extra modes and equivalent to general relativity at the linearized level, Phys. Rev. D **87**, 084020 (2013).
[23] J. Bellorín and A. Restuccia, Einstein's quadrupole formula from the kinetic-conformal Hořava theory, Int. J. Mod. Phys. D **27**, 1750174 (2018).
[24] J. Bellorín, A. Restuccia, and F. Tello-Ortiz, Anisotropic coupling of gravity and electromagnetism in Hořava-Lifshitz theory, Phys. Rev. D **98**, 104018 (2018).
[25] A. Restuccia and F. Tello-Ortiz, Pure electromagnetic-gravitational interaction in Hořava-Lifshitz theory at the kinetic conformal point, Eur. Phys. J. C **80**, 86 (2020).
[26] B. P. Abbott *et al.*, GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).
[27] J. Bellorín, A. Restuccia, and A. Sotomayor, Wormholes and naked singularities in the complete Hořava theory, Phys. Rev. D **90**, 044009 (2014).
[28] A. Restuccia and F. Tello-Ortiz, Charged throats in the Hořava-Lifshitz theory, Eur. Phys. J. C **81**, 447 (2021).
[29] J. Mestra-Páez, J. Peña, and A. Restuccia, Wave zone in the Hořava-Lifshitz theory at the kinetic-conformal point in the low energy regime, Eur. Phys. J. C **81**, 923 (2021).
[30] R. Arnowitt, S. Deser, and C. W. Misner, Wave zone in general relativity, Phys. Rev. **121**, 1556 (1961).
[31] J. M. Weisberg, D. J. Nice, and J. H. Taylor, Timing measurements of the relativistic binary pulsar PSR B1913 + 16, Astrophys. J. **722**, 1030 (2010).

[32] B. P. Abbott *et al.*, Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[33] B. P. Abbott *et al.*, Tests of General Relativity with GW150914, Phys. Rev. Lett. **116**, 221101 (2016).

[34] B. P. Abbott *et al.*, Erratum: Tests of General Relativity with GW150914 [Phys. Rev. Lett. **116**, 221101 (2016)]; Phys. Rev. Lett. **121**, 129902 (2018).

[35] M. C. Miller and N. Yunes, The new frontier of gravitational waves, Nature (London) **568**, 469 (2019).

[36] B. P. Abbott *et al.*, GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs, Phys. Rev. X **9**, 031040 (2019).

[37] B. P. Abbott *et al.*, Observation of gravitational waves from two neutron star–black hole coalescences, Astrophys. J. Lett. **915**, L5 (2021).

[38] B. P. Abbott *et al.*, GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo during the First Half of the Third Observing Run, Phys. Rev. X **11**, 021053 (2021).

[39] A. H. Nitz, C. D. Capano, S. Kumar, Y.-F. Wang, S. Kastha, M. Schäfer, R. Dhurkunde, and M. Cabero, 3OGC: Catalog of gravitational waves from compact-binary mergers, arXiv:2105.09151.

[40] B. P. Abbott *et al.*, Multi-messenger observations of a binary neutron star merger, Astrophys. J. **848**, L12 (2017).

[41] B. P. Abbott *et al.*, Gravitational waves and gamma-rays from a binary neutron star merger: GW170817 and GRB 170817a, Astrophys. J. **848**, L13 (2017).

[42] B. P. Abbott *et al.*, Tests of General Relativity with GW170817, Phys. Rev. Lett. **123**, 011102 (2019).

[43] H. Martínez-Huerta, R. G. Lang, and V. de Souza, Lorentz invariance violation tests in astroparticle physics, Symmetry **12**, 1232 (2020).

[44] Z. Cao, F. Aharonian, Q. An, L. Bai, Y. Bai, Y. Bao, D. Bastieri, X. Bi, Y. Bi, H. Cai *et al.*, Ultrahigh-energy photons up to 1.4 petaelectronvolts from 12γ-ray galactic sources, Nature (London) **594**, 33 (2021).

[45] C. Li and B.-Q. Ma, Ultrahigh-energy photons from LHAASO as probes of Lorentz symmetry violations, Phys. Rev. D **104**, 063012 (2021).

[46] H. Xu and B.-Q. Ma, Light speed variation from gamma-ray bursts, Astropart. Phys. **82**, 72 (2016).

[47] H. Xu and B.-Q. Ma, Light speed variation from gamma ray burst GRB 160509A, Phys. Lett. B **760**, 602 (2016).

[48] H. Xu and B.-Q. Ma, Regularity of high energy photon events from gamma ray bursts, J. Cosmol. Astropart. Phys. 01 (2018) 050.

[49] R. L. Arnowitt, S. Deser, and C. W. Misner, Republication of: The Dynamics of General Relativity, Gen. Relativ. Gravit. **40**, 1997 (2008).

[50] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88**, 286 (1974).