# Darboux covariance: A hidden symmetry of perturbed Schwarzschild black holes

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We determine and characterize the full space of master equations describing the dynamics of perturbed Schwarzschild black holes and show that all of them are connected via Darboux transformations. This reveals the presence of a hidden symmetry, Darboux covariance, which preserves the spectrum of quasinormal modes and the continuous spectrum associated with black hole scattering processes. This picture is shown to share a deep connection with the Korteweg-de Vries equation and inverse scattering methods which leads to an infinite hierarchy of conserved quantities.

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## I. INTRODUCTION

General relativistic perturbation theory of spherically-symmetric spacetimes is of paramount importance since it applies to a wide variety of physical phenomena: From structure formation in the homogeneous and isotropic standard cosmological models [1–4] to the dynamics of perturbed Schwarzschild black holes (BHs) and spherical relativistic stars [5–8] (see [9] for the impact on gravitational wave physics). In the case of BHs [10–13], perturbation theory describes scattering processes [14–16] and quasinormal modes [5,6,17] (QNMs), which are crucial for the last stage of the emission of gravitational radiation from BH binary coalescence.

## II. MASTER FUNCTIONS AND EQUATIONS

Spherically-symmetric spacetimes have a warped geometry and as such the metric has the form: $g^{(4)} = h \times_r \Omega$, where $h_{ab}$ $[x^a = (t, r)]$ is a Lorentzian metric, $r$ is the area radial coordinate, and $\Omega_{AB}$ $[x^A = (\theta, \varphi)]$ is the metric of the unit 2-sphere. For a Schwarzschild BH in Schwarzschild coordinates: $h_{ab} = \mathrm{diag}(-f, 1/f)$ with $f = 1\text{–}2M/r$ and $M$ is the BH mass. The warped geometry allows us to decompose the metric perturbations in spherical harmonics in such a way that modes with different harmonic numbers $(\ell, m)$ and different parity (odd/even parity) decouple from each other. We can find linear combinations of the metric perturbations and their first-order derivatives, the master functions $\Psi_{\ell m}^{\mathrm{even/odd}}$, so that the perturbative Einstein equations become wave equations of the form: $(\Box_2 - V_\ell/f)\Psi_{\ell m}^{\mathrm{even/odd}} = 0$, where $\Box_2\Psi = h^{ab}\Psi_{:ab}$ and $V_\ell(r) \equiv V(r)$ is the $\ell$-dependent potential. Considering the BH exterior, where there is a timelike Killing vector, $t^a = \partial/\partial t$, and using the radial tortoise coordinate, $dx/dr = 1/f(r)$, the master equations become

$$(-\partial_t^2 + L_V)\Psi = (-\partial_t^2 + \partial_x^2 - V)\Psi = 0, \qquad (1)$$

where $L_V = \partial_x^2 - V$ is the well-known Schrödinger time-independent operator. Physical quantities like the gravitational-wave fluxes of energy and momenta can be estimated exclusively from the master functions, which are gauge-invariant. Other nongauge invariant quantities like the self-force [18,19] require the reconstruction of all metric perturbations, which depends on the choice of perturbative gauge.

We recently determined [20] the space of possible master equations assuming the master functions are linear combinations (with coefficients depending only on $r$) of the metric perturbations and their first-order derivatives. There are two branches of possible pairs of potentials/master functions, $\{(V, \Psi)\}$: (i) *The standard branch*. We call it the standard branch because it contains a single potential for each parity: The Regge-Wheeler potential $V_{\mathrm{RW}}$ for odd-parity perturbations [10] and the Zerilli potential $V_{\mathrm{Z}}$ for even-parity perturbations [11]. The most general master function is a linear combination (with constant coefficients) of two master functions:

$$_S\Psi_{\mathrm{odd}}^{\mathrm{even}} = \mathcal{C}_1\Psi_{\mathrm{CPM}}^{\mathrm{ZM}} + \mathcal{C}_2\Psi_{\mathrm{RW}}^{\mathrm{NE}}, \qquad (2)$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two arbitrary constants. In the odd-parity case the two master functions turn out to be the well-known Regge-Wheeler [10] and Cunningham-Price-Moncrief master functions [21–23]: $(\Psi_{\rm RW}, \Psi_{\rm CPM})$. In the even-parity case, the two master functions are the Zerilli-Moncrief master function $\Psi_{\rm ZM}$ [11,12,24] and another function that to our knowledge was unknown, which we called $\Psi_{\rm NE}$ (new even-parity master function). All of these master functions are gauge-invariant. The two master functions in each parity are related by a time derivative: $t^a\Psi^{\rm ZM}_{{\rm CPM},a} = 2\Psi^{\rm NE}_{\rm RW}$. (ii) *The Darboux branch*. In this branch, for each parity, there is an infinite set of possible potentials and master functions, $\{(V, \Psi)\}$. The set of possible potentials is determined by a nonlinear second-order ordinary differential equation. Then, for each potential, the master function can be written as [20]

$$ {}_D\Psi^{\rm even}_{\rm odd} = \mathcal{C}_1\Psi^{\rm ZM}_{\rm CPM} + \mathcal{C}_2(\Sigma^{\rm even}_{\rm odd}\Psi^{\rm ZM}_{\rm CPM} + \Phi^{\rm even}_{\rm odd}), \qquad (3) $$

where $\Sigma^{\rm even}_{\rm odd} = \Sigma^{\rm even}_{\rm odd}(x)$ is a function that contains the integral of the potential; $\Phi^{\rm even}_{\rm odd} = \Phi^{\rm even}_{\rm odd}(t, r)$ are linear combinations of the metric perturbations and their first-order derivatives, but only the combination with $\Psi^{\rm ZM}_{\rm CPM}$ in Eq. (3) is a true master function.

## III. DARBOUX COVARIANCE

To understand the landscape of master equations, or pairs $\{(V, \Psi)\}$, let us first consider the standard branch, where we have only the Regge-Wheeler and Zerilli potentials. It was first noted by Chandrasekhar [25,26] that these potentials, for both Schwarzschild and Reissner-Nordström BHs, lead to the same transmission and reflection coefficients (see also [27,28]) as well as the same spectra of QNMs frequencies. However, it has only been recently realized [29] (see also [30]) that this is a consequence of the master equations being related by a Darboux transformation (DT) [31,32] (see also [33,34]).

Two any master equations, characterized by pairs $(v, \varphi)$ and $(V, \Psi)$, are related by a DT if the two pairs are related by a transformation of the form

$$ \Psi = \varphi_{,x} + g\varphi, \qquad V = v + 2g_{,x} \qquad (4) $$

where the DT generating function, $g(x)$, must satisfy the following Riccati equation

$$ g_{,x} - g^2 + v = \mathcal{C}, \qquad (5) $$

where $\mathcal{C}$ is an arbitrary constant. We can write the DT generating function as $g = (V + v)_{,x}/(2(V - v))$. Then, the consistency between the expressions for $g(x)$ and $g_{,x}(x)$ is a second-order nonlinear equation

$$ \left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0, \qquad (6) $$

where $\delta V = V - v$. This is precisely the equation that any potential in the Darboux branch should satisfy [20], where $v = V^{\rm Z}_{\rm RW}$. Hence, all master equations in the Darboux branch are connected via a DT to the standard branch, with DT generating functions given by:

$$ g^{\rm even}_{\rm odd} = \frac{1}{2}\int dx\,(V - V^{\rm Z}_{\rm RW}), \qquad (7) $$

while the two parities in the standard branch are connected by a DT with generating function:

$$ g^{\rm odd\to even} = \frac{1}{2}\int dx\,(V_{\rm Z} - V_{\rm RW}) = -g^{\rm even\to odd}. \qquad (8) $$

In conclusion, we have an infinite set of master equations linked by DTs, showing the existence of a hidden symmetry in the perturbations of spherically symmetric BHs: Darboux covariance [33].

In this work we have adopted a view of the DT that is more general than the original one, as introduced by Crum [35], based on Sturm-Liouville problems and where the generating function of the DT is constructed from an eigenfunction. Instead, we apply the DT to wave-type equations (1) and consider generating functions that only have to satisfy Eq. (5). We can make contact with the Crum approach by working in the frequency domain and study single-frequency solutions: $\Psi(t, r) = e^{i\omega t}\psi(x;\omega)$, which obey a time-independent Schrödinger equation

$$ L_V\psi(x;\omega) = -\omega^2\psi(x;\omega). \qquad (9) $$

Given a solution $\psi_o(x;\omega_o)$ with eigenvalue $-\omega_o^2$, the function $g(x) = -(\ln\psi_o)_{,x}$ generates a DT that transform Eq. (9) into another equation of the same form with the same eigenvalue $-\omega^2$, therefore showing the isospectral character of the DT. The Riccati equation (5) is automatically satisfied with $\mathcal{C} = -\omega_o^2$, and so is Eq. (6). The new master function from (4), say $\phi$, can be written as $\phi = W[\psi, \psi_o]/\psi_o$ where $W[\psi, \psi_o]$ denotes the Wronskian of $\psi$ and $\psi_o$. It turns out that the DT generating function between the Regge-Wheeler and Zerilli-Moncrief master equations, Eq. (8), can be constructed from one of the algebraically special solutions of the Regge-Wheeler equation [36,37] (see [29]), namely

$$ \psi_o = \frac{\lambda(r)}{2}{\rm e}^{-i\omega_* x}, \qquad \omega_* = -i\frac{n_\ell(n_\ell + 1)}{3M}, \qquad (10) $$

where $n_\ell = (\ell + 2)(\ell - 1)/2$ and $\lambda(r) = 2n_\ell + 6M/r$. The generating function itself is

$$ G(x) = \frac{6Mf(r)}{\lambda(r)r^2}. \qquad (11) $$

Following [27,28], the Regge-Wheeler and Zerilli potentials can be written in terms of $G(x)$ as

$$V_{\mathrm{RW}}^{\mathrm{Z}} = \pm G_{,x} + \alpha G + G^2, \qquad \alpha = \frac{1}{6M}\frac{(\ell+2)!}{(\ell-2)!}, \tag{12}$$

which can be seen as a Riccati equation for $G(x)$. This form of the potentials is reminiscent of supersymmetric quantum mechanics (SUSY QM) [38–40] where the quantum description of systems with double degeneracy of energy levels is realized. This is related to the fact that the Schrödinger equation (9), for two DT-related potentials [Eq. (4)], can be written in the form

$$(\partial_x^2 - V_\pm)\psi = -\hat{\omega}^2\psi, \qquad V_\pm = \pm g_{,x} + g^2, \tag{13}$$

where $g(x)$ plays the role of the SUSY QM *superpotential*, $V_\pm$ are *partner* potentials, $\hat{\omega}^2 = \omega^2 - \mathcal{C}$ is the energy eigenvalue, and we can introduce ladder operators $A = \partial_x - g$ and $A^\dagger = -\partial_x - g$ that factorize the Hamiltonians $H_- = A \cdot A^\dagger$ and $H_+ = A^\dagger \cdot A$ ($H_\pm = -\partial_x^2 + V_\pm$). In the standard branch: $g = G + \alpha/2$ and $\mathcal{C} = -\alpha^2/4$. This factorization is the key ingredient of the intertwining operator method used in [41] (see also [42,43]) to look for simpler potentials yet equivalent to the Regge-Wheeler and Zerilli ones.

## IV. DTs AND THE KORTEWEG-DE VRIES EQUATION

In the frequency-domain we can establish the connection with inverse scattering theory following the work by Gardner, Green, Miura and Kruskal [44] (see also [45,46]), where they discovered a way to solve the initial-value problem for the Korteweg-de Vries (KdV) equation [47]

$$V_{,\tau} = 6VV_{,x} - V_{,xxx}. \tag{14}$$

by identifying $V$ with the potential of the time-independent Schrödinger equation (9). We now show how this connection reveals interesting properties of our Darboux-covariant master equations in the frequency domain. The spectrum of $L_V$ is twofold [48,49]: It has a continuous part, the *scattering states*, and a discrete part made out of a finite number of discrete negative eigenvalues. In the case of the Schwarzschild BH, the potentials $V_{\mathrm{RW}}^{\mathrm{ZM}}(x)$ are positive everywhere and decay to zero at both ends ($x \to \pm\infty$). Therefore, there are no discrete normalizable states. Let us now deform the Schrödinger equation by introducing the KdV time $\tau$ in the following way: $V(x) \to V(\tau, x)$, $\psi(x) \to \psi(\tau, x)$, and $\omega \to \omega(\tau)$. If $V(\tau, x)$ follows the KdV flow we can show that

$$[\partial_x^2 - (V - \omega^2)]\Xi = -(\omega^2)_{,\tau}\psi, \tag{15}$$

where $\Xi(\tau, x) = \psi_{,\tau} + V_{,x}\psi - 2(V + 2\omega^2)\psi_{,x}$. In the hypothetical case of bound states (not our case), and assuming that $\psi$ and $V$ decay sufficiently fast at $x \to \pm\infty$, one can show, by multiplying by $\psi$ and integrating over $x \in (-\infty, \infty)$, that $(\omega^2)_{,\tau} = 0$. For non-normalizable states we can adopt an approach due to Lax [50] consisting in the introduction of a pair of operators, $P_V$ and $L_V$ (*Lax pair*), defined by

$$\psi_{,\tau} = P_V\psi = -4\psi_{,xxx} + 6V\psi_{,x} + 3V_{,x}\psi, \tag{16}$$

and Eq. (9) respectively. A remarkable fact about this Lax pair is that the relation between differential operators, $dL_V/d\tau = [P_V, L_V]$, yields the KdV equation [Eq. (14)]. Following [33], one can show that the pair of equations $(L_V + \omega^2, -\partial_\tau + P_V)\psi = 0$ is invariant under a DT provided the DT generating function satisfies Eq. (5) and is KdV-deformed according to

$$g_{,\tau} = -g_{,xxx} + 6(V + g_{,x})g_{,x}. \tag{17}$$

On the other hand, using the KdV-deformation of $\psi$, Eq. (16), we rewrite Eq. (15) in the form

$$(V_{,\tau} - 6VV_{,x} + V_{,xxx} - (\omega^2)_{,\tau})\psi = 0. \tag{18}$$

Therefore, if $(V, \psi)$ are KdV-deformed according to Eqs. (14) and (16), we must have $\psi(\omega^2)_{,\tau} = 0$, which means that $\omega$ is preserved by the KdV flow. This argument can be applied to the discrete and continuous spectra as well as to the QNM frequencies.

## V. DTs AND THE KdV HIERARCHY

It was shown by Lax [50] that equations that are equivalent to a relation between a Lax pair of operators, like the KdV equation, have an infinite set of first integrals. Gardner showed [51] that these first integrals are associated with symmetries of the KdV equation that yield higher-order KdV equations, and all of them can be formulated as a Hamiltonian system with infinite degrees of freedom. Zakharov and Fadeev [52] showed that the hierarchy of KdV equations leads to a completely integrable Hamiltonian system that admits canonical action-angle variables constructed from the scattering data of the Schrödinger equation. Here, we use this point of view to study the KdV hierarchy of first integrals for the infinite set of master equations for BH perturbations.

The scattering states of the continuum spectrum coming either from $x \to -\infty$ or from $x \to +\infty$ toward the potential barrier described by $V$ are part reflected and part transmitted. For plane waves coming from $x \to \infty$, the solution of the Schrödinger equation has the *Jost* asymptotic behavior [53]:

$$\psi \to \begin{cases} e^{i\omega x} & \text{for } x \to -\infty, \\ a(\tau,\omega)e^{i\omega x} + b(\tau,\omega)e^{-i\omega x} & \text{for } x \to +\infty, \end{cases} \quad (19)$$

where the complex coefficients $a(\tau,\omega)$ and $b(\tau,\omega)$, which fully determine the S-matrix, satisfy: $|a|^2 - |b|^2 = 1$. The reflection coefficient is $R(\tau,\omega) = b(\tau,\omega)/a(\tau,\omega)$, and in our case, it completely characterizes the scattering data so that the mapping $V(\tau,x) \to s(\tau)$ is uniquely invertible [54]. The transmission coefficient is $T(\tau,\omega) = 1/a(\tau,\omega)$ so that $|R|^2 + |T|^2 = 1$. Under the KdV flow they evolve as [44]: $T_{,\tau} = 0$ and $R_{,\tau} = 8i\omega^3 R$, which implies:

$$a_{,\tau} = 0, \quad \text{and} \quad b_{,\tau} = 8i\omega^3 b. \quad (20)$$

In the inverse scattering method, given the initial value of the potential $V(\tau=0,x)$ we construct the associated scattering data, $s(0)$, evolve it according the KdV flow, thus obtaining $s(\tau)$, and we recover $V(\tau,x)$ from $s(\tau)$ using the Gelfand-Levitan-Marchenko method [55–57].

Let us look at the consequences of the conservation law for $a(\tau,\omega)$ [Eq. (20)] under the KdV flow. Following [52], let us write

$$\psi(\tau,x,\omega) = \exp\left\{i\omega x + \int_{-\infty}^{x} dx'\Phi(\tau,x',\omega)\right\}, \quad (21)$$

so that $a(\tau,\omega)$ becomes [see Eq. (19)]

$$a(\tau,\omega) = \lim_{x\to\infty} e^{-i\omega x}\psi = \exp\left\{\int_{-\infty}^{+\infty} dx'\Phi(\tau,x',\omega)\right\}. \quad (22)$$

It turns out [52] that $\ln a(\tau,\omega)$ admits an expansion in inverse powers of $\omega$ for $|\omega| \to \infty$. We can then write

$$\Phi(\tau,x,\omega) = \sum_{n=1}^{\infty} \frac{f_n(\tau,x)}{(2i\omega)^n}. \quad (23)$$

Therefore, $a_{,\tau} = 0$ implies that each coefficient $f_n(\tau,x)$ yields a conserved quantity, the KdV integrals [46]: $I_n(\tau) = \int_{-\infty}^{+\infty} dx f_n(\tau,x)$ with $dI_n/d\tau = 0$. After inserting Eq. (21) into the Schrödinger equation we get

$$\Phi_{,x} + 2i\omega\Phi + \Phi^2 = V. \quad (24)$$

This is a complex Riccati equation. Introducing the expansion for $\Phi$ here we find that $f_1(\tau,x) = V(\tau,x)$ and a recursion for the rest of coefficients $f_n(\tau,x)$ $(n > 1)$, which turn out to be differential polynomials in $V(\tau,x)$:

$$\frac{df_n}{dx} + f_{n+1} + \sum_{m=1}^{n-1} f_m f_{n-m} = 0. \quad (25)$$

It is convenient to split $\Phi$ into its real and imaginary parts

$$\Phi = \sum_{N=1}^{\infty} \frac{f_{2N}}{(2i\omega)^{2N}} + \sum_{M=0}^{\infty} \frac{f_{2M+1}}{(2i\omega)^{2M+1}} = \chi_{\rm R} + i\chi_{\rm I}. \quad (26)$$

From Eq. (24), $\chi_{\rm R}(\tau,x)$ and $\chi_{\rm I}(\tau,x)$ satisfy

$$\chi_{{\rm R},x} - 2\omega\chi_{\rm I} + \chi_{\rm R}^2 - \chi_{\rm I}^2 = V, \quad (27)$$

$$\chi_{{\rm I},x} + 2\omega\chi_{\rm R} + 2\chi_{\rm R}\chi_{\rm I} = 0, \quad (28)$$

and from here we get an expression for $\chi_{\rm R}$

$$\chi_{\rm R} = -\frac{1}{2}\frac{d}{dx}\ln(\chi_{\rm I} + \omega), \quad (29)$$

that is, $\chi_{\rm R}$ is a gradient involving only $\chi_{\rm I}$. This, together with the decaying behavior of the potential $V$, which follows from the decaying properties of $V_{\rm RW}$ and $V_{\rm Z}$ and the Riccati equation (5), implies the known result [52,54] that all the even KdV integrals, $I_{2N}$, vanish. To study the odd KdV integrals, let us integrate Eq. (24) over the real line $x \in (-\infty, +\infty)$ and use the decaying properties of our potentials and derivatives to obtain:

$$2i\omega\int_{-\infty}^{+\infty} dx\Phi + \int_{-\infty}^{+\infty} dx\Phi^2 = \int_{-\infty}^{+\infty} dxV. \quad (30)$$

For the standard branch, the potential $V = V_{\rm RW}^{\rm Z}$ admits the form in Eq. (12). Therefore, using the behavior of $G(x)$ at $x \to \pm\infty$, the right-hand side of Eq. (30) becomes

$$\int_{-\infty}^{+\infty} dxV = \int_{-\infty}^{+\infty} dx(\alpha G + G^2), \quad (31)$$

and hence Eq. (30) is the same for the whole standard branch. Any potential of the Darboux branch can be written as $V = V_{\rm odd}^{\rm even} = V_{\rm RW}^{\rm Z} + 2g_{,x}$. Then, using again the decaying properties of the DT generating functions, we deduce that any potential of the Darboux branch also satisfies Eq. (31), thus Eq. (30) is universal. We can write it in terms of $(\chi_{\rm R},\chi_{\rm I})$ and then use Eq. (29) and

$$\chi_{\rm R}\chi_{\rm I} = -\frac{1}{2}\partial_x(\chi_{\rm I} - \omega\ln(\chi_{\rm I} + \omega)), \quad (32)$$

which is a total derivative. Then, we arrive at

$$-2\omega\int_{-\infty}^{+\infty} dx\chi_{\rm I} + \int_{-\infty}^{+\infty} dx(\chi_{\rm R}^2 - \chi_{\rm I}^2) = \int_{-\infty}^{+\infty} dx(\alpha G + G^2). \quad (33)$$

When we introduce the expansions for $\chi_{\rm R}$ and $\chi_{\rm I}$ [Eq. (26)] this becomes a universal recurrent relation for the odd KdV integrals. Then, we conclude that all the KdV integrals associated with the potentials of the infinite set of master equations are the same. A first hint of this result appears in

Chandrasekhar's work [14,28], where some evidence is given that the KdV integrals should be the same for a pair of potentials of the form in Eq. (12) (but not necessarily related to BH perturbations), although a full proof was not provided.

It is interesting to mention that this infinite set of KdV integrals, which makes the KdV equation completely integrable [52], has been connected to a recurrence between the infinite KdV hierarchy of equations, initially suggested by Lenard [58], and is rooted in the fact that the KdV equation admits a bi-Hamiltonian structure [51,59]. On the other hand, Gelfand and Dickii [60] showed that these conserved quantities are connected with trace formulas for half-integer powers of the operator $L_V$.

## VI. DTS AND QNMS

We have seen that all the possible potentials associated with the infinite set of master equations found in [20] have the same set of KdV integrals when studying the continuous spectrum associated with the Sturm-Liouville problem that emerges when one considers scattering problems. QNMs are not associated with a Sturm-Liouville problem, they rather appear as *scattering resonances* [61], poles in the meromorphic continuation of the resolvent/Green function (related to $L_V$ in our case). They can also be seen as the poles of the S-matrix and the associated residues [62]. We can use the argument given by Chandrashekar [14] to show that the frequencies of QNMs are the same for all possible potentials, provided they have similar decaying behavior at $x \to \pm\infty$. This is the case for our set of potentials by virtue the Riccati equation (5) for the DT generating function. Finally, thanks to Eq. (18) we can state that QNM frequencies and damping times are preserved by the KdV flow provided the potential is KdV-deformed and the radial master function $\psi$ is KdV-deformed according to Eq. (16). Apart from these results, it would be interesting to explore the structure of the resolvent associated with our time-independent master equations and their behavior and properties under the KdV flow.

## VII. CONCLUSIONS

The study of the general structure of the full space of master functions and equations has revealed a hidden symmetry in the theory of perturbations of (spherically symmetric) BHs, Darboux covariance. The implications are diverse and here we have shown that, given the decaying properties of the potentials at both infinities (horizon and spatial infinity), DTs preserve the spectrum of QNMs and the continuous spectrum associated with scattering processes around the BH. The DT also preserves the infinite set of KdV conserved quantities that appear as a consequence of the invariance of the scattering transmission coefficient under KdV deformations of the master equations. A large part of the developments shown in this work can be applied to other spherically symmetric backgrounds and to other theories of gravity. The main changes may come from different boundary conditions and their implications for the asymptotic behavior of the potentials.

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