Stationary generalizations for the Bronnikov-Ellis wormhole and for the vacuum ring wormhole

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(Received 7 October 2021; accepted 2 December 2021; published 21 December 2021)

We analyze possibilities to obtain a globally regular stationary generalization for the ultrastatic wormhole with a repulsive scalar field found by Bronnikov and by Ellis in 1973. The extreme simplicity of this static solution suggests that its spinning version could be obtainable analytically and should be globally regular, but no such generalization has been found. We analyze the problem and find that the difficulty originates in the vacuum theory, since the scalar field can be eliminated within the Eris-Gurses procedure. The problem then reduces to constructing the spinning generalization for the vacuum wormhole sourced by a thin ring of negative tension. Solving the vacuum Ernst equations determines the g_{00}, $g_{0\varphi}$ metric components and hence the AMD mass M and angular momentum J, all of these being specified by the ring source. The scalar field can be included into consideration afterwards, but this only affects g_{rr} and $g_{\vartheta\vartheta}$ without changing the rest. Within this approach, we analyze a number of exact stationary generalizations for the wormhole, but none of them are satisfactory. However, the perturbative expansion around the static vacuum background contains only bounded functions and presumably converges to an exact solution. Including the scalar field screens the singularity at the ring source and renders the geometry regular. This solution describes a globally regular spinning wormhole with two asymptotically flat regions. Even though the source itself is screened and not visible, the memory of it remains in g_{00} and $g_{0\varphi}$ and accounts for the $\mathrm{M} \propto \mathrm{J}^2$ relation typical for a rotating extended source. Describing stationary spacetimes with an extended source is a complicated problem, which presumably explains the difficulty in finding the solution.

DOI: 10.1103/PhysRevD.104.124064

I. INTRODUCTION

Wormholes are bridges or tunnels between different universes or different parts of the same universe. They were first introduced by Einstein and Rosen (ER) [1], who noticed that the Schwarzschild black hole actually has two exterior regions connected by a spacelike bridge. One can also discuss traversable wormholes accessible for ordinary classical particles or light (see [2] for a review), but their existence requires [3,4] that the null energy condition (NEC) must be violated. Therefore, traversable wormholes are possible if only the energy density becomes negative, for example due to vacuum polarization [5] or due to exotic matter [6,7].

Since the energy is normally supposed to be positive, the traversable wormholes were for a longtime considered as something odd. The situation changed after the discovery of the cosmic acceleration [8,9], which invoked a large number of alternative gravity models in which the energy is not necessarily positive definite. Wormholes have been found in many such theories, as for example in the Gauss-Bonnet theory [10,11], in the braneworld models [12], in theories with nonminimally coupled fields [13], in massive (bi)gravity [14], etc. As a result, wormholes have become quite popular nowadays.

We do not intend to argue that wormholes actually exist, neither shall we advocate the opposite viewpoint. We are merely interested in the problem of constructing solutions describing *spinning* wormholes in the theory with a gravity-coupled phantom scalar field Φ. This theory presents extremely simple wormhole solutions found in 1973 independently by Bronnikov and by Ellis (BE) [6,7]. Their simplest version is

$$ds^2 = -dt^2 + d\mathrm{r}^2 + (\mathrm{r}^2 + \mu^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \qquad (1.1)$$

with the scalar field $\Phi = \arctan(\mathrm{r}/\mu)$. The parameter μ determines the size of the wormhole throat, the radial coordinate $\mathrm{r} \in (-\infty, \infty)$, and the limits $\mathrm{r} \to \pm\infty$ correspond to two asymptotically flat regions connected through the wormhole throat.

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The theory also admits exact axially symmetric solutions describing superpositions of several wormholes [15–18], as well as solutions describing axially symmetric deformations of a single wormhole [18,19]. The latter are all singular, and one can prove that the BE solutions do not admit globally regular generalizations in the *static* sector [20].

At the same time, nothing forbids the existence of globally regular *stationary* generalizations for the BE solutions which would describe spinning wormholes. The extreme simplicity of the solution (1.1) suggests that its stationary version could be easily obtainable analytically, and it is natural to expect this spinning solution to be globally regular. However, even now, almost 50 years later, this solution is still unknown.

Spinning wormholes are in fact often discussed in the literature (see, e.g., [21]), but what is usually meant are not exact solutions but some model geometries [22]. Exact stationary solutions are also known, but they show singularities, for example of the NUT type [23] or present other problems [24]. At the same time, there exist perturbative (up to the second order terms) [25,26] and numerical [27,28] indications in favor of existence of *globally regular* spinning generalizations for the BE wormholes. However, their analytical form is unknown.

Exact stationary solutions may sometimes be obtained by applying the generating methods like dualities [29–31] or by using some other tricks [32], but none of these methods help to construct regular spinning wormholes. Therefore, in what follows we are trying to analyze the situation to understand why the problem is so difficult and what can be done. It seems that the problem originates already in the vacuum theory and can be summarized as follows.

Already before the BE discovery, it was known that the vacuum general relativity admits the *ring wormholes* described by the *oblate* metrics of Zipoy and Vorhees [33,34], whose simplest version is

$$ds^2 = -dt^2 + \frac{\mathrm{r}^2 + \mu^2\cos^2\vartheta}{\mathrm{r}^2 + \mu^2}[d\mathrm{r}^2 + (\mathrm{r}^2 + \mu^2)d\vartheta^2] + (\mathrm{r}^2 + \mu^2)\sin^2\vartheta d\varphi^2 \tag{1.2}$$

with $\mathrm{r} \in (-\infty, \infty)$. Although looks complicated, this would be just the Minkowski metric expressed in spheroidal coordinates, if the radial coordinate was restricted to $\mathrm{r} \in [0, \infty)$. For $\mathrm{r} \in (-\infty, \infty)$ this metric is only locally flat and describes a wormhole made of two copies of Minkowski space glued to each other through the disk bounded by the circle $\mathrm{r} = 0$, $\vartheta = \pi/2$ in the equatorial plane. The circle carries a distributional singularity of the Ricci tensor that can be viewed as a singular matters source: a ring (loop) made of a cosmic string of *negative* tension with the negative angle deficit of -2π [18,19]. As was noticed in [35], the metric (1.2) is a special limit of the Kerr geometry. All of this will be explained below.

Now, the BE solution (1.1) can be obtained from the ring metric (1.2). Assuming that $\Phi = \Phi(\mathrm{r})$, the scalar field equation admits the same solution $\Phi = \arctan(\mathrm{r}/\mu)$ for both metrics. Adding this as the source to the Einstein equations only modifies (removes) the conformal factor in front of the r, ϑ part of the metric (1.2), after which the metric reduces to (1.1). We shall call this procedure "dressing," hence the BE wormhole is the ring wormhole "with scalar dressing". A similar procedure works also in the stationary case [36], hence finding a stationary generalization for the BE solution (1.1) reduces to solving the same problem for the vacuum ring metric (1.2). The vacuum Ernst equations determine the g_{00}, $g_{0\varphi}$, $g_{\varphi\varphi}$ metric components yielding the ADM mass and angular momentum, all of these being insensitive to the scalar field. The latter only modifies g_{rr}, $g_{\vartheta\vartheta}$ not affecting the rest. As a result, all the essential features of the system are encoded already in the vacuum theory.

Therefore, all we need to do is to solve the vacuum Ernst equations. This is not always easy but still simpler than to directly attack the full system of coupled Einstein and scalar field equations as was done in [25,26] and in [27,28]. Following this logic, we apply in what follows the procedure based on vacuum Ernst equations to construct stationary solutions with the scalar field. Our ultimate goal is to try and possibly obtain the globally regular rotating wormholes exactly. We do obtain indeed new exact solutions, but these are not globally regular. At the same time, we construct the perturbative expansion for the globally regular solution of the Ernst equations which determine the asymptotically flat stationary generalizations for both the ring wormhole (1.2) and for the BE wormhole (1.1). However, promoting this perturbative solution to an exact one does not seem to be obvious.

Intuitively, the explanation of the difficulty is that the ring metric (1.2) has en extended source—the cosmic string loop. This source is hidden also in the spinning version of the regular BE solution (1.1), although not directly visible there, being screened by the scalar. However, since its g_{00} and $g_{0\varphi}$ metric components are the same as for the ring wormhole, the BE wormhole shows the same relation between the mass and angular momentum which is typical for a rotating extended source—the ring. The spinning BE wormhole "knows" about this source. However, finding a stationary solution with an extended source is a more difficult problem than finding it for a pointlike source, say (the Kerr metric). Therefore, although the spinning version of the BE wormhole can be constructed perturbatively or numerically, it may be not expressible in a compact analytical form. This presumably explains why this solution has never been obtained, despite the apparent simplicity of its static limit described by (1.1).

In what follows we describe the approach based on solving the vacuum Ernst equations first and including the scalar field afterwords. We consider a number of stationary generalizations for the wormhole. The most obvious one is the Kerr metric, however, adding to it the scalar dressing yields a singular result. We then analyze a special ansatz reducing the Ernst equations to a harmonic equation. This yields exact solutions which are "almost" perfect but unfortunately are not globally regular. We then consider the perturbative expansion around the static ring metric (1.2) and find that it contains unbounded functions and hence is ill-defined. However, reformulating the Ernst equations in terms of the axial Killing vector instead of the timelike one yields a better result, and we explicitly construct the globally regular perturbative expansion up to the fourth order terms. This expansion presumably converges to an exact solution describing the correct stationary generalization for the ring wormhole and for the BE wormhole. We finish by discussing chances to get this solution exactly.

II. THE THEORY

We consider the theory with a minimally coupled to gravity scalar field with a "wrong" sign in front of the kinetic term. It is convenient to introduce from the very beginning the length scale μ and represent the line element as

$$d\mathrm{s}^2 = \mu^2 g_{\mu\nu} dx^\mu dx^\nu \equiv \mu^2 ds^2, \tag{2.1}$$

where the metric $g_{\mu\nu}$ and coordinates x^μ are dimensionless; their dimensionful analogues will be denoted by roman symbols. The action of the theory is

$$\begin{aligned} S &= \frac{1}{2}\mu^2 \mathrm{M}_{\mathrm{Pl}}^2 \int (R + 2\partial_\mu \Phi \partial^\mu \Phi)\sqrt{-g}d^4x \\ &\equiv \frac{1}{2}\mu^2 \mathrm{M}_{\mathrm{Pl}}^2 \int \mathcal{L}_4 \sqrt{-g} d^4x, \end{aligned} \tag{2.2}$$

which yields upon varying the equations

$$R_{\mu\nu} = -2\partial_\mu \Phi \partial_\nu \Phi, \qquad \nabla_\mu \nabla^\mu \Phi = 0. \tag{2.3}$$

Assuming the system to be stationary, the metric is chosen in the Papapetrou form,

$$ds^2 = -e^{2U}(dt - w_k dx^k)^2 + e^{-2U} h_{ik} dx^i dx^k, \tag{2.4}$$

where the Newtonian potential U, the rotation field w_k, and the 3-metric h_{ik} depend on the spatial coordinates x^k. Inserting this to (2.2) yields

$$\mathcal{L}_4 \sqrt{-g} = \mathcal{L}_3 \sqrt{h} + \text{total derivative} \tag{2.5}$$

with

$$\mathcal{L}_3 = \overset{(3)}{R}(h) - 2(\partial U)^2 + \frac{1}{4} e^{4U} F_{ik} F^{ik} + 2(\partial \Phi)^2, \tag{2.6}$$

were $F_{ik} = \partial_i w_k - \partial_k w_i$ and $(\partial U)^2 = \partial_k U \partial^k U$ with the indices moved by h_{ik}. Varying this Lagrangian yields the equations

$$\nabla^k \nabla_k U + \frac{1}{4} e^{4U} F_{ik} F^{ik} = 0, \tag{2.7a}$$

$$\nabla_i (e^{4U} F^{ik}) = 0, \tag{2.7b}$$

$$\nabla^k \nabla_k \Phi = 0, \tag{2.7c}$$

$$\overset{(3)}{R}_{ik} - 2\partial_i U \partial_k U + \frac{1}{2} e^{4U} F_{is} F_k{}^s + 2\partial_i \Phi \partial_k \Phi = 0, \tag{2.7d}$$

where ∇_k is the covariant derivative with respect to h_{ik}.

A. Static case

In the static case, when $F_{ik} = 0$, the Lagrangian (2.6) and equations (2.7) are invariant under global rotations,

$$\begin{aligned} U &\to U \cosh\alpha + \Phi \sinh\alpha, \\ \Phi &\to \Phi \cosh\alpha + U \sinh\alpha, \\ h_{ik} &\to h_{ik}. \end{aligned} \tag{2.8}$$

This allows one to generate nontrivial solutions from a vacuum seed metric. Let us see how this works. The Schwarzschild metric of unit mass can be described by

$$\begin{aligned} e^{2U} &= \frac{x-1}{x+1}, \\ h_{ik} dx^i dx^k &= dx^2 + (x^2 - 1)(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \\ \Phi &= 0. \end{aligned} \tag{2.9}$$

Applying to this the transformation (2.8) with $\cosh\alpha = \delta$ yields the solution with a nontrivial scalar,

$$\begin{aligned} ds^2 = &-\left(\frac{x-1}{x+1}\right)^\delta dt^2 \\ &+ \left(\frac{x+1}{x-1}\right)^\delta [dx^2 + (x^2-1)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)], \\ \Phi = &\ \frac{1}{2}\sqrt{1-\delta^2} \ln\left(\frac{x-1}{x+1}\right), \end{aligned} \tag{2.10}$$

which reduces back to (2.9) if $\delta = 1$. Performing the analytic continuation,

$$t \to it, \qquad x \to ix, \qquad \delta \to i\delta, \tag{2.11}$$

and also replacing $\mu \to i\mu$ in (2.1), the line element $ds^2 = d\mathrm{s}^2/\mu^2$ and the scalar become

$$ds^2 = -e^{2\delta\Psi}dt^2 + e^{-2\delta\Psi}[dx^2 + (x^2+1)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)],$$
$$\Phi = \sqrt{1+\delta^2}\Psi, \tag{2.12}$$

with $\Psi = \arctan(x)$. Setting $t = \mathrm{t}/\mu$ and $x = \mathrm{r}/\mu$, this describes the static BE wormholes [6,7], whose ultrastatic version (1.1) is obtained when $\delta \to 0$.

We are looking for the stationary generalization of these solutions. Unfortunately, the global symmetry (2.8) is lost in the stationary case and one cannot play the same game again and generate solutions with a nontrivial scalar field starting from the Kerr metric, say.

B. Stationary case

In this case there exist other global symmetries. Defining the twist potential χ via

$$\partial^i\chi = \frac{e^{4U}}{\sqrt{h}}\epsilon^{ijk}\partial_j w_k, \tag{2.13}$$

whose integrability is insured by (2.7b), the Lagrangian assumes the form

$$\mathcal{L}_3 = \overset{(3)}{R}(h) - 2(\partial U)^2 - \frac{1}{2}e^{-4U}(\partial\chi)^2 + 2(\partial\Phi)^2$$
$$\equiv \overset{(3)}{R}(h) - 2\mathcal{G}_{AB}\partial_i Y^A\partial^i Y^B, \tag{2.14}$$

where the target space coordinates are $Y^A = (U, \chi, \Phi)$ and the target space metric is

$$\mathcal{G}_{AB}dY^A dY^B = (\partial U)^2 + \frac{1}{4}e^{-4U}(\partial\chi)^2 - (\partial\Phi)^2. \tag{2.15}$$

Introducing the complex Ernst potential

$$\mathcal{E} = e^{2U} + i\chi \equiv \pm\frac{1-\xi}{1+\xi} \tag{2.16}$$

(we shall be choosing either plus or minus sign in this formula, depending on the context, because $\mathcal{E} \to -\mathcal{E}$ is a symmetry), one has

$$dU^2 + \frac{1}{4}e^{-4U}d\chi^2 = \frac{d\mathcal{E}d\bar{\mathcal{E}}}{\mathcal{E}+\bar{\mathcal{E}}} = \frac{d\xi d\bar{\xi}}{(\xi\bar{\xi}-1)^2}, \tag{2.17}$$

which is the metric on the hyperbolic space H^2 (Lobachevsky plane). Therefore, the target space (2.15) is the pseudo-Euclidean direct product $H^2 \otimes R^1$ with the same geometry as the one induced on the hyperboloid

$$X_0^2 - X_1^2 - X_2^2 = \frac{1}{4} \tag{2.18}$$

in the 4-dimensional space with the metric

$$dS^2 = -d\Phi^2 - dX_0^2 + dX_1^2 + dX_2^2. \tag{2.19}$$

Isometries of this space are the shifts

$$\Phi \to \Phi + \Phi_0 \tag{2.20}$$

and the H^2 isometries, which can be represented in the form

$$\mathcal{E} \to \mathcal{E} + i\alpha, \qquad \frac{1}{\mathcal{E}} \to \frac{1}{\mathcal{E}} + i\beta, \qquad \xi \to e^{i\gamma}\xi. \tag{2.21}$$

These isometries can be used to produce solutions with a NUT charge, which however does not help to construct globally regular spinning wormholes.

C. Stationary and axially symmetric case

Let us choose the spatial coordinates as $x^k = (\rho, z, \varphi)$ and assume that nothing depends on φ. The 3-metric can be represented in the form

$$dl^2 = h_{ik}dx^i dx^k = e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2, \tag{2.22}$$

while $w_k dx^k = w d\varphi$ where w, k, as well as U, Φ depend only on ρ, z. This form of the metric is possible only in the vacuum theory, otherwise one should replace ρ^2 in front of $d\varphi^2$ by a function of ρ, z (an introduction into the theory of stationary gravitational fields can be found, e.g., in [37]). However, since the Ernst equations considered below correspond to the vacuum theory, the choice (2.22) of the 3-metric is legitimate.

The function k drops out from the first three equations in (2.7), since one has, for example,

$$\nabla^k\nabla_k U = \frac{1}{\sqrt{h}}\partial_i(\sqrt{h}h^{ik}\partial_k U)$$
$$= e^{-2k}\left(\partial_{\rho\rho}U + \frac{1}{\rho}\partial_\rho U + \partial_{zz}U\right)$$
$$\equiv e^{-2k}\Delta U, \tag{2.23}$$

where Δ is the standard flat space Laplace operator expressed in cylindrical coordinates. As a result, the first two equations in (2.7) decouple from the rest and comprise a closed system

$$\Delta U + \frac{e^{4U}}{2\rho^2}[(\partial_\rho w)^2 + (\partial_z w)^2] = 0,$$
$$\rho\partial_\rho\left(\frac{e^{4U}}{\rho}\partial_\rho w\right) + \partial_z(e^{4U}\partial_z w) = 0, \tag{2.24}$$

which can be written compactly as

$$\Delta U + \frac{e^{4U}}{2\rho^2}(\vec{\nabla} w)^2 = 0, \qquad \vec{\nabla}\left(\frac{e^{4U}}{\rho^2}\vec{\nabla} w\right) = 0. \quad (2.25)$$

Using the definition of the twist potential (2.13),

$$\partial_\rho \chi = \frac{1}{\rho} e^{4U}\partial_z w, \qquad \partial_z \chi = -\frac{1}{\rho} e^{4U}\partial_\rho w, \quad (2.26)$$

these two equations can be represented in the form

$$\Delta U + \frac{1}{2} e^{-4U}[(\partial_\rho \chi)^2 + (\partial_z \chi)^2] = 0,$$
$$\frac{1}{\rho}\partial_\rho(\rho e^{-4U}\partial_\rho \chi) + \partial_z(e^{-4U}\partial_z \chi) = 0, \quad (2.27)$$

or, using the compact notation, as

$$\Delta U + \frac{1}{2} e^{-4U}(\vec{\nabla}\chi)^2 = 0, \qquad \vec{\nabla}(e^{-4U}\vec{\nabla}\chi) = 0. \quad (2.28)$$

Using the Ernst potential (2.16), these two equations can be combined to one complex-valued equation

$$(\xi\bar{\xi} - 1)\Delta\xi = 2\bar{\xi}(\vec{\nabla}\xi)^2 \quad (2.29)$$

usually called in the literature Ernst equation [38]. However, in what follows we shall for simplicity call "Ernst" also equations in the form (2.28) or (2.25). When these equations are solved and the equation $\Delta\Phi = 0$ is solved as well, the metric function k is obtained from (2.7d). The latter contains two first order equations for k and one second order equation. The first order equations read

$$\frac{1}{\rho}\partial_\rho k = (\partial_\rho U)^2 - (\partial_z U)^2 + \frac{1}{4\rho^2} e^{4U}[(\partial_z w)^2 - (\partial_\rho w)^2]$$
$$- (\partial_\rho \Phi)^2 + (\partial_z \Phi)^2,$$
$$\frac{1}{2\rho}\partial_z k = \partial_\rho U \partial_z U - \frac{1}{4\rho^2} e^{4U}\partial_\rho w \partial_z w - \partial_\rho \Phi \partial_z \Phi, \quad (2.30)$$

which can equivalently be rewritten in terms of the twist χ instead of rotation w. The integrability conditions for these equations are insured by the Ernst equations and by the equation for the scalar field. The second order equation can be represented in the form

$$\Delta k + 2(\partial_z U)^2 - 2(\partial_z \Phi)^2 + \frac{e^{4U}}{2\rho^2}(\partial_\rho w)^2 = 0. \quad (2.31)$$

One can check that this is a differential consequence of the other equations.

III. THE DRESSING PROCEDURE OF ERIS AND GURSES—SUPERPOSING THE SOLUTIONS

The above equations split into two independent groups, since the Ernst equation and the scalar field equation are independent from each other. As a result, the solution can be constructed in two steps. The first step is to consider the purely vacuum problem described by the Ernst equation, whose solution determines U, χ and w,

$$\text{Step I: Ernst} \Rightarrow U, \chi, w. \quad (3.1)$$

This solution is used to compute the amplitude k_{I} defined by (2.30), where one sets $\Phi = 0$. This yields explicitly

$$\frac{1}{\rho}\partial_\rho k_{\mathrm{I}} = (\partial_\rho U)^2 - (\partial_z U)^2 + \frac{1}{4\rho^2} e^{4U}[(\partial_z w)^2 - (\partial_\rho w)^2],$$
$$\frac{1}{2\rho}\partial_z k_{\mathrm{I}} = \partial_\rho U \partial_z U - \frac{1}{4\rho^2} e^{4U}\partial_\rho w \partial_z w. \quad (3.2)$$

The second step is to solve the scalar field equation,

$$\text{Step II: } \Delta\Phi = 0, \quad (3.3)$$

and compute the amplitude k_{II} from equations obtained from (2.30) by keeping there only terms with Φ,

$$\frac{1}{\rho}\partial_\rho k_{\mathrm{II}} = -(\partial_\rho \Phi)^2 + (\partial_z \Phi)^2,$$
$$\frac{1}{2\rho}\partial_z k_{\mathrm{II}} = -\partial_\rho \Phi \partial_z \Phi. \quad (3.4)$$

Taking the sum,

$$k = k_{\mathrm{I}} + k_{\mathrm{II}}, \quad (3.5)$$

finally yields the solution U, χ, k, Φ of the equations. One can say that the solution is obtained by superposing ("dressing") a vacuum metric with the scalar field. This was first noticed by Eris and Gurses [36]. Notice that the dressing only affects the g_{rr} and $g_{\vartheta\vartheta}$ metric components while g_{00}, $g_{0\varphi}$, $g_{\varphi\varphi}$ are determined by the vacuum equations. If one wants the solution to be asymptotically flat, then the scalar field should be bounded, but there is only one bounded harmonic function, as we shall see. Therefore, the scalar field is already known up to a constant factor and the problem reduces to finding a suitable Ernst potential in the vacuum sector.

IV. SPHEROIDAL COORDINATES

Let us pass from ρ, z to the spheroidal coordinates x, $y = \cos\vartheta$ via

$$\rho=\sqrt{x^2+\nu}\sin\vartheta=\sqrt{(x^2+\nu)(1-y^2)},$$
$$z=x\cos\vartheta=xy, \tag{4.1}$$

where $\nu=0,\pm1$. One has

$$\frac{\rho^2}{x^2+\nu}+\frac{z^2}{x^2}=1, \tag{4.2}$$

hence the spheroidal coordinates are oblate if $\nu=1$, prolate if $\nu=-1$, and spherical if $\nu=0$. The 3-metric (2.22) becomes

$$dl^2=e^{2k}(d\rho^2+dz^2)+\rho^2d\varphi^2$$
$$=e^{2K}\left[dx^2+\frac{x^2+\nu}{1-y^2}dy^2\right]+(x^2+\nu)(1-y^2)d\varphi^2 \tag{4.3}$$

with

$$e^{2K}=\frac{x^2+\nu y^2}{x^2+\nu}e^{2k}, \tag{4.4}$$

and the 4-metric is

$$ds^2=-e^{2U}(dt-wd\varphi)^2+e^{-2U}dl^2. \tag{4.5}$$

Equations (2.25) assume the form

$$[(x^2+\nu)U_{,x}]_{,x}+[(1-y^2)U_{,y}]_{,y}$$
$$+\frac{e^{4U}}{2\rho^2}[(x^2+\nu)w_{,x}^2+(1-y^2)w_{,y}^2]=0,$$
$$(x^2+\nu)w_{,xx}+(1-y^2)w_{,yy}$$
$$+4[(x^2+\nu)w_{,x}U_{,x}+(1-y^2)w_{,y}U_{,y}]=0. \tag{4.6}$$

The relations (2.26) between the rotation field and the twist now read

$$w_{,x}=(y^2-1)e^{-4U}\chi_{,y},\qquad w_{,y}=(x^2+\nu)e^{-4U}\chi_{,x}, \tag{4.7}$$

the integrability condition $\partial_y\chi_{,x}=\partial_x\chi_{,y}$ being insured by the second equation in (4.6). Using the twist potential χ instead of w, Eqs. (4.6) assume the form (2.28),

$$[(x^2+\nu)U_{,x}]_{,x}+[(1-y^2)U_{,y}]_{,y}$$
$$+\frac{1}{2}e^{-4U}[(x^2+\nu)\chi_{,x}^2+(1-y^2)\chi_{,y}^2]=0,$$
$$[(x^2+\nu)\chi_{,x}]_{,x}+[(1-y^2)\chi_{,y}]_{,y}$$
$$-4[(x^2+\nu)\chi_{,x}U_{,x}+(1-y^2)\chi_{,y}U_{,y}]=0, \tag{4.8}$$

while the complex Ernst equation (2.29) becomes

$$(\xi\bar\xi-1)\{[(x^2+\nu)\xi_{,x}]_{,x}+[(1-y^2)\xi_{,y}]_{,y}\}$$
$$=2\bar\xi[(x^2+\nu)\xi_{,x}^2+(1-y^2)\xi_{,y}^2]. \tag{4.9}$$

The scalar field equation reads

$$[(x^2+\nu)\Phi_{,x}]_{,x}+[(1-y^2)\Phi_{,y}]_{,y}=0. \tag{4.10}$$

Finally, the metric function K defined by (4.4) can be represented as

$$K=K_{\rm I}+K_{\rm II} \tag{4.11}$$

where, using (3.2), $K_{\rm I}$ is defined by

$$\partial_xK_{\rm I}=\frac{1-y^2}{x^2+\nu y^2}\left(\Gamma(U)+\frac{1}{4}e^{-4U}\Gamma(\chi)+\frac{\nu x}{x^2+\nu}\right),$$
$$\partial_yK_{\rm I}=\frac{x^2+\nu}{x^2+\nu y^2}\left(\Lambda(U)+\frac{1}{4}e^{-4U}\Lambda(\chi)+\frac{\nu y}{x^2+\nu}\right), \tag{4.12}$$

or equivalently

$$\partial_xK_{\rm I}=\frac{1-y^2}{x^2+\nu y^2}\left(\Gamma(U)-\frac{e^{4U}}{4\rho^2}\Gamma(w)+\frac{\nu x}{x^2+\nu}\right),$$
$$\partial_yK_{\rm I}=\frac{x^2+\nu}{x^2+\nu y^2}\left(\Lambda(U)-\frac{e^{4U}}{4\rho^2}\Lambda(w)+\frac{\nu y}{x^2+\nu}\right), \tag{4.13}$$

with the following definitions

$$\Gamma(f)\equiv x(x^2+\nu)f_{,x}^2-2y(x^2+\nu)f_{,x}f_{,y}+x(y^2-1)f_{,y}^2,$$
$$\Lambda(f)\equiv y(x^2+\nu)f_{,x}^2+2x(1-y^2)f_{,x}f_{,y}+y(y^2-1)f_{,y}^2. \tag{4.14}$$

The second part of the amplitude, $K_{\rm II}$, is defined by Eq. (3.4),

$$\partial_xK_{\rm II}=-\frac{1-y^2}{x^2+\nu y^2}\Gamma(\Phi),$$
$$\partial_yK_{\rm II}=-\frac{x^2+\nu}{x^2+\nu y^2}\Lambda(\Phi). \tag{4.15}$$

A straightforward verification confirms that the condition $\partial_x\partial_yK_{\rm I}=\partial_y\partial_xK_{\rm I}$ is guaranteed by the Ernst equations (4.6), (4.8), while the similar condition for $K_{\rm II}$ follows from the scalar field equation (4.10). The second order equation (2.31) can be represented in the form [after combining it with (2.30)]

$$
\begin{aligned}
&((x^2+\nu)\partial^2_{xx}+x\partial_x+(1-y^2)\partial^2_{yy}-y\partial_y)K+\frac{\nu}{x^2+\nu}\\
&\quad+(x^2+\nu)\left[(\partial_x U)^2-(\partial_x\Phi)^2+\frac{e^{4U}}{4\rho^2}(\partial_x w)^2\right]\\
&\quad+(1-y^2)\left[(\partial_y U)^2-(\partial_y\Phi)^2+\frac{e^{4U}}{4\rho^2}(\partial_y w)^2\right]=0. \qquad (4.16)
\end{aligned}
$$

As a result, to solve the problem, the first step is to integrate (4.6) or (4.8) to find U and w, χ and then compute $K_{\rm I}$ from (4.12) or from (4.13). This determines the vacuum metric (4.5) with $K = K_{\rm I}$. The second step is to solve the scalar field equation (4.10) and compute the dressing amplitude $K_{\rm II}$ from (4.15). Finally one promotes the vacuum metric to the "dressed" one via replacing $K = K_{\rm I} \to K_{\rm I} + K_{\rm II}$ while U, w do not change. The second step of this procedure is essentially trivial, as we shall now see.

V. HARMONIC FUNCTIONS

Solutions of the scalar field equation (4.10) are harmonic functions

$$
\Phi(x,y)=\sum_{l=0}^{\infty}X_l(x)P_l(y), \qquad (5.1)
$$

where

$$
\begin{aligned}
&[(x^2+\nu)X_l(x)']'=l(l+1)X_l(x),\\
&[(1-y^2)P_l(y)']'=-l(l+1)P_l(y). \qquad (5.2)
\end{aligned}
$$

Solutions of the latter equation are the Legendre polynomials, $P_0(y)=1$, $P_1(y)=y$, $P_2(y)=3y^2-1$, etc. Harmonic functions are generically unbounded, but there is one exceptional solution obtained in oblate coordinates, where $\nu = 1$, in which case one has

$$
\begin{aligned}
&X_0(x)=A+C\arctan(x),\\
&X_1(x)=Ax+C[x\arctan(x)+1],\\
&X_2(x)=A(3x^2+1)+C[(3x^2+1)\arctan(x)+3x],\ldots \qquad (5.3)
\end{aligned}
$$

The mode $X_0(x)$ is bounded while all the others are unbounded. For example, one can choose the integration constants A, C such that

$$
\begin{aligned}
&X_1(x)=A^+X_1^+(x)+A^-X_1^-(x) \quad \text{with}\\
&X_1^{\pm}(x)=\frac{x}{2}\pm\frac{1}{\pi}(x\ \arctan(x)+1), \qquad (5.4)
\end{aligned}
$$

and when $-\infty \leftarrow x \to \infty$ one has, respectively,

$$
\begin{aligned}
&\frac{1}{3\pi x^2}+\ldots\leftarrow X_1^+(x)\to x+\frac{1}{3\pi x^2}+\ldots\\
&x-\frac{1}{3\pi x^2}+\ldots\leftarrow X_1^-(x)\to -\frac{1}{3\pi x^2}+\ldots \qquad (5.5)
\end{aligned}
$$

so that each $X_1^{\pm}$ stays finite either for $x\to\infty$ or for $x\to\infty$ but not in both limits.

If one is interested in globally regular solutions, then the scalar field should be bounded. Therefore, the only acceptable solution for the scalar field and the corresponding dressing amplitude defined by (4.15) are

$$
\nu=1:\quad \Phi=C\arctan(x)\Rightarrow K_{\rm II}=-\frac{C^2}{2}\ln\frac{x^2+y^2}{x^2+1}. \qquad (5.6)
$$

VI. STATIC WORMHOLES

Let us see how the dressing procedure works for static wormholes. As a first step, we choose the simplest solution of the Ernst equations (4.6),

$$
U=\chi=0, \qquad (6.1)
$$

in which case Eqs. (4.12) yield

$$
K_{\rm I}=\frac{1}{2}\ln\frac{x^2+\nu y^2}{x^2+\nu}. \qquad (6.2)
$$

Setting $\nu = 1$ and $K = K_{\rm I}$ yields the vacuum metric

$$
\begin{aligned}
ds^2=&-dt^2+\frac{x^2+y^2}{x^2+1}\left[dx^2+\frac{x^2+1}{1-y^2}dy^2\right]\\
&+(x^2+1)(1-y^2)d\varphi^2. \qquad (6.3)
\end{aligned}
$$

Assuming that $x\in(-\infty,\infty)$ and $t = \mathrm{t}/\mu$, $x = \mathrm{r}/\mu$, this is precisely the ring wormhole (1.2). This metric is locally flat and the curvature is zero everywhere apart from the conical singularity at the ring $x = y = 0$. The singularity is detected by noting that the x, y part of the metric reduces in the vicinity of $x = y = 0$ to

$$
\begin{aligned}
(x^2+y^2)(dx^2+dy^2)&=r^2(dr^2+r^2d\phi^2)\\
&=dR^2+R^2d\psi^2, \qquad (6.4)
\end{aligned}
$$

where $x = r\cos\phi$, $y = r\sin\phi$, $R = r^2/2$, $\psi = 2\phi$. This is the flat 2D metric in polar coordinates R, ψ, however, since $\phi\in[0,2\pi]$, the angular variable $\psi\in[0,4\pi]$. Therefore, one revolution around $x = y = 0$ in the x, y space corresponds to two revolutions in the R, ψ space, hence (6.4) describes the geometry of a cone with a negative angle deficit of -2π. This conical singularity can be interpreted as a result of the presence of a distributional matter source: a cosmic string of *negative* tension extending along the azimuthal φ-direction [18,19]. In other words, this is a loop or ring made of an infinitely thin cosmic string.

Notice that if the range of x was $x \geq 0$, then $x = y = 0$ would be at the boundary of the x, y space and then one would have $\phi \in [0, \pi]$ and $\psi \in [0, 2\pi]$ in (6.4) so that the conical singularity would be absent. Then (6.3) would be just the Minkowski metric in spheroidal coordinates.

The presence of a distributional source can also be detected by the equations. The second order equation for K in (4.16) reduces for $U = w = \Phi = 0$ to

$$((x^2+1)\partial^2_{xx} + x\partial_x + (1-y^2)\partial^2_{yy} - y\partial_y)K + \frac{1}{x^2+1} = 0. \tag{6.5}$$

If K is given by (6.2) then this equation is apparently fulfilled. However, there is a subtlety due to the fact that

$$(\partial^2_{xx} + \partial^2_{yy})\frac{1}{2}\ln(x^2+y^2) = 2\pi\delta(x)\delta(y). \tag{6.6}$$

This implies that injecting K given (6.2) to (6.5) does not actually give zero on the right but the delta function instead. This corresponds to a distributional source that should be added to the Einstein equations in order that (6.3) be the solution. As a result, the metric (6.3) indeed has a singular source. A more detailed analysis reveals that the distributional singularity is contained only in the G_{00} and $G_{\varphi\varphi}$ components of the Einstein tensor, hence one needs to introduce a source $T_{\mu\nu}$ with the only nonvanishing T_{00} and $T_{\varphi\varphi}$ components. This corresponds to a cosmic string along the azimuthal direction.

Let us now add the scalar field. Choosing the solution (5.6) for the scalar, one has

$$K = K_{\rm I} + K_{\rm II} = \frac{1-C^2}{2}\ln\frac{x^2+y^2}{x^2+1}. \tag{6.7}$$

Therefore, the $\ln(x^2+y^2)$ term rendering the metric singular can be removed by setting $C^2 = 1$, in which case

$$K = 0, \qquad \Phi = \pm\arctan(x), \tag{6.8}$$

and the metric becomes

$$ds^2 = -dt^2 + dx^2 + \frac{x^2+1}{1-y^2}dy^2 + (x^2+1)(1-y^2)d\varphi^2. \tag{6.9}$$

This is precisely the ultrastatic BE wormhole (1.1). We obtain it via adding the scalar field to the vacuum ring wormhole and the scalar screens the singular ring source. The scalar itself is regular and the screening simply means that the resulting geometry with the scalar is globally regular and the curvature is everywhere bounded, so that no extra sources in the equations are needed.

Two remarks are in order. First, the phantom scalar does not create the wormhole as one might think but only makes it regular, while the wormhole itself exists already in the vacuum theory. Second, although the ring source in the static solution seems to be completely screened by the scalar, the situation is different in the stationary case, as we shall see below. For stationary solutions the scalar field also removes the singularity and makes the geometry regular, but the memory of the ring source remains visible in the metric. Therefore, the regular solutions “remember” their descendance from the singular vacuum ring.

The other static BE solutions can be obtained similarly. Choosing the complex Ernst potential ξ to be real and setting

$$\xi = \tanh(\psi) \Rightarrow \mathcal{E} = \frac{1-\xi}{1+\xi} = e^{-2\psi} = e^{2U}, \tag{6.10}$$

the Ernst equation (2.29) reduces to [38]

$$\Delta\psi = 0 = \Delta U, \tag{6.11}$$

hence the solution is the bounded harmonic function. In the oblate coordinates, with $\nu = 1$, one has, with δ being an integration constant,

$$U = \delta\arctan(x), \qquad \Rightarrow K_{\rm I} = \frac{\delta^2+1}{2}\ln\frac{x^2+y^2}{x^2+1}, \tag{6.12}$$

which determines the vacuum metric with $K = K_{\rm I}$,

$$ds^2 = -e^{2\delta\Psi}dt^2 + e^{-2\delta\Psi}dl^2, \quad dl^2 = \left(\frac{x^2+y^2}{x^2+1}\right)^{1+\delta^2}\left[dx^2 + \frac{x^2+1}{1-y^2}dy^2\right] + (x^2+1)(1-y^2)d\varphi^2. \tag{6.13}$$

This is the *oblate* ZV metric describing a singular vacuum ring [33,34]. It reduces to the locally flat metric (6.3) if $\delta = 0$. The metric singularity at the ring can be removed by adding the scalar dressing (5.6) with $C^2 = 1 + \delta^2$, which yields $K = K_{\rm I} + K_{\rm II} = 0$. Therefore, setting

$$U = \delta\Psi, \qquad \Phi = \pm\sqrt{1+\delta^2}\Psi, \qquad K = 0, \tag{6.14}$$

transforms (6.13) to

$$ds^2 = -e^{2\delta\Psi}dt^2 + e^{-2\delta\Psi}\left(\left[dx^2 + \frac{x^2+1}{1-y^2}dy^2\right] + (x^2+1)(1-y^2)d\varphi^2\right), \tag{6.15}$$

which coincides with the regular BE metric (2.12). Finding its stationary version requires to find a spinning generalization for the vacuum ZV metric (6.13) with a subsequent dressing.

VII. SPINNING WORMHOLES—THE RELATION TO THE KERR METRIC

Let us now start considering stationary generalizations for the wormholes. As discussed above, to obtain a spinning version of the ultrastatic BE solution (1.1) one has to solve the same problem for the vacuum ring wormhole (1.2) and then add the scalar field.

What is the stationary version for the ring wormhole? The answer seems to be obvious because, as has already been said and will be shown below, the static ring wormhole (1.2) is the special limit of the Kerr metric [35]. Hence its stationary generalization is the Kerr metric itself. Therefore, there remains just to add the scalar field to the Kerr metric to obtain a spinning version of the ultrastatic BE wormhole. Let us see, however, what this gives.

The Kerr metric is obtained from the following solution of the Ernst equation (4.9) [38],

$$\xi = px + iqy \quad \text{where } q^2 - \nu p^2 = 1 \quad \text{with} \quad \nu = \pm 1. \tag{7.1}$$

Reading off U, χ from

$$\mathcal{E} = e^{2U} + i\chi = \frac{\xi - 1}{\xi + 1} \tag{7.2}$$

and computing w and K_I from (4.7), (4.12), one obtains

$$\begin{aligned} e^{2U} &= 1 - \frac{2(px+1)}{(px+1)^2 + q^2y^2}, \qquad \chi = \frac{2qy}{(px+1)^2 + q^2y^2}, \\ w &= \frac{2q}{p} \times \frac{(px+1)(1-y^2)}{p^2x^2 + q^2y^2 - 1}, \\ K_{\rm I} &= \frac{1}{2}\ln\frac{p^2x^2 + q^2y^2 - 1}{x^2 + \nu} + K_0. \end{aligned} \tag{7.3}$$

Injecting this to (4.5) and then to (2.1), setting $K = K_{\rm I}$ and choosing

$$\begin{aligned} \mu &= \sqrt{\nu(\mathrm{a}^2 - \mathrm{M}^2)}, \qquad p = \frac{\mu}{\mathrm{M}}, \qquad q = \frac{\mathrm{a}}{\mathrm{M}}, \qquad K_0 = -\ln p, \\ x &= \frac{\mathrm{r} - \mathrm{M}}{\mu}, \qquad t = \frac{\mathrm{t}}{\mu}, \qquad y = \cos\vartheta, \end{aligned} \tag{7.4}$$

yields the Kerr metric in the standard dimensionful form,

$$\begin{aligned} ds^2 = &-dt^2 + \frac{\Sigma}{\Delta}d\mathrm{r}^2 + \Sigma d\vartheta^2 + (\mathrm{r}^2 + \mathrm{a}^2)\sin^2\vartheta d\varphi^2 \\ &+ \frac{2\mathrm{Mr}}{\Sigma}(d\mathrm{t} - \mathrm{a}\sin^2\vartheta d\varphi)^2, \end{aligned} \tag{7.5}$$

where $\Sigma = \mathrm{r}^2 + \mathrm{a}^2\cos^2\vartheta$ and $\Delta = \mathrm{r}^2 - 2\mathrm{Mr} + \mathrm{a}^2$.

It is clear from (7.4) that $\nu = -1$ corresponds to the $\mathrm{a} < \mathrm{M}$ case when the black hole angular momentum is not very high, while $\nu = 1$ corresponds to the supercritical case when $\mathrm{a} > \mathrm{M}$ and the even horizon is absent so that the singularity is naked.

As known [39], the Kerr metric describes a wormhole geometry with two asymptotic regions corresponding to the limits $\mathrm{r} \to \infty$ and $\mathrm{r} \to -\infty$. Geodesics can interpolate between these two regions, unless they hit the curvature singularity located at the ring in the equatorial plane, $\mathrm{r} = 0$, $\theta = \pi/2$, where one has $\Sigma = 0$.

Taking the $\mathrm{M} \to 0$ limit with a fixed a, which corresponds to the oblate $\nu = 1$ regime and to $\mu = a$, the last term in (7.5) disappears while the remaining three terms reduce exactly to the ring metric (1.2). The range of the radial coordinate remains the same as for the original Kerr metric, hence the geometry still describes a wormhole but becomes *locally* flat (and not flat as often stated in the literature). As a result, the $\mathrm{M} \to 0$ limit of the Kerr metric is the static ring wormhole (1.2) [35]. Therefore, the natural stationary generalization for the latter is the Kerr metric itself.

It follows that a spinning generalization for the BE wormhole will be obtained if we add the scalar dressing to the supercritical Kerr metric. Assuming that $\mathrm{M} < \mathrm{a}$ in (7.5), the original Papapetrou form of the Kerr metric corresponds to the oblate coordinates, $\nu = 1$, and then the dressing procedure is prescribed by (5.6),

$$\begin{aligned} &\Phi = C\arctan(x), \\ &K = K_{\rm I} \to K = K_{\rm I} - \frac{C^2}{2}\ln\frac{x^2 + y^2}{x^2 + 1}. \end{aligned} \tag{7.6}$$

The agreement with the $\mathrm{M} \to 0$ limit requires that $C^2 = 1$ and the resulting metric is obtained by giving to the r, ϑ part of the geometry (7.5) the conformal factor $e^{2K_{\rm II}} = (x^2 + 1)/(x^2 + y^2)$. This amounts to the replacing in (7.5)

$$\begin{aligned} &\frac{\Sigma}{\Delta}d\mathrm{r}^2 + \Sigma d\vartheta^2 \\ &\to \frac{\Delta}{(\mathrm{r} - \mathrm{M})^2 + (\mathrm{a}^2 - \mathrm{M}^2)\cos^2\vartheta}\left(\frac{\Sigma}{\Delta}d\mathrm{r}^2 + \Sigma d\vartheta^2\right). \end{aligned} \tag{7.7}$$

For $\mathrm{M} = 0$ the denominator of the conformal factor cancels against Σ and there remains $dr^2 + (\mathrm{r}^2 + \mathrm{a}^2)d\vartheta^2$ hence the 4-metric reduces to that for the ultrastatic BE wormhole in (1.1). However, for $\mathrm{M} \neq 0$ the denominator introduces a curvature singularity at $\mathrm{r} = \mathrm{M}$, $\vartheta = \pi/2$, in addition to the original singularity at $r = 0$, $\vartheta = \pi/2$. As a result, we do get an exact stationary generalization of the BE solution, but it is doubly singular. Therefore, one has to study other stationary generalizations.

VIII. THE RELATION TO THE TOMIMATSU-SATO METRICS

This relation is suggested by the following observation. The oblate vacuum ZV metric (6.13) can be obtained by the analytic continuation

$$x \to ix, \qquad t \to it, \qquad \delta \to i\delta, \tag{8.1}$$

assuming also $\mu \to i\mu$ in (2.1), from the *prolate* ZP metric,

$$\begin{aligned} ds^2 &= -\left(\frac{x-1}{x+1}\right)^{\delta} dt^2 + \left(\frac{x-1}{x+1}\right)^{-\delta} dl^2, \\ dl^2 &= \left(\frac{x^2-y^2}{x^2-1}\right)^{1-\delta^2}\left[dx^2 + \frac{x^2-1}{1-y^2}dy^2\right] \\ &\quad + (x^2-1)(1-y^2)d\varphi^2. \end{aligned} \tag{8.2}$$

This reduces to the Schwarzschild metric for $\delta = 1$. Its stationary generalizations are explicitly known for $\delta = 1$ (Kerr metric) and for $\delta = 2, 3\ldots$. These are the Tomimatsu-Sato (TS) metrics [40,41] obtained from solutions of the vacuum Ernst equation (4.9) in the *prolate* ($\nu = -1$) case with the complex Ernst potential of the form

$$\xi_{\mathrm{TS}}(p, q, \delta, x, y) = \frac{P(x, y)}{Q(x, y)}. \tag{8.3}$$

Here P, Q are polynomials in x, y with coefficients depending on two real parameters p, q subject to $p^2 + q^2 = 1$. The powers of the polynomials depend on δ, originally assumed to be integer, but the analysis can be extended to arbitrary real δ [42]. In the static limit, $q \to 0$, one has

$$\mathcal{E}_{\mathrm{TS}} = \frac{\xi_{\mathrm{TS}} - 1}{\xi_{\mathrm{TS}} + 1} \to \left(\frac{x-1}{x+1}\right)^{\delta} \quad \text{as } q \to 0, \tag{8.4}$$

which corresponds to the prolate ZV solution (8.2).

This suggests the following procedure: take the stationary TS solution for an arbitrary real δ, then perform the analytic continuation (8.1), and finally add the scalar dressing. This will give a spinning version of the BE wormhole. The problem, however, is that the TS solution for an arbitrary real δ is known only in a very implicit form [42–50] which does not allow us to perform the analytic continuation.

Any TS solution for $\nu = -1$ can also be analytically continued via [51]

$$p \to -ip, \qquad x \to ix, \tag{8.5}$$

which yields a solution of the Ernst equation (4.9) for $\nu = +1$. However, this continuation is different from (8.1) and does not give what we need. For the TS solution with $\delta = 0$ the rule (8.1) would reduce just to $x \to ix$, and this would give a stationary extension for the ultrastatic vacuum ring (1.2). However, the $\delta = 0$ TS solution is not known explicitly either.

Last but not least, the "correct" stationary solution that will be obtained below perturbatively does not have the TS form of the Ernst potential. Therefore, the TS metrics are not useful for us, although they do provide some stationary solutions for our problem.

IX. SOLUTIONS OBTAINED WITH THE HARMONIC ANSATZ

Exact stationary solutions can be obtained within the special ansatz which reduces the nonlinear Ernst equations to a single harmonic equation. Choosing the Ernst potential in the form

$$\xi = e^{i\alpha}\tanh(\psi), \tag{9.1}$$

one has

$$\mathcal{E} = \frac{1-\xi}{1+\xi} = e^{2U} + i\chi, \tag{9.2}$$

where

$$\begin{aligned} e^{-2U} &= \cosh(2\psi) + \cos(\alpha)\sinh(2\psi), \\ \chi &= -\frac{\sin(\alpha)}{\coth(2\psi) + \cos(\alpha)}. \end{aligned} \tag{9.3}$$

The Ernst equations (2.29) assume the form

$$\begin{aligned} \Delta\psi - \frac{1}{4}(\vec{\nabla}\alpha)^2\sinh(4\psi) &= 0, \\ \Delta\alpha + 4(\vec{\nabla}\alpha\vec{\nabla}\psi)\coth(2\psi) &= 0. \end{aligned} \tag{9.4}$$

If $\alpha = \mathrm{const}$, these reduce simply to

$$\Delta\psi = 0. \tag{9.5}$$

A. NUT wormholes

The simplest stationary wormhole, first found in [23], can be obtained by assuming that $\alpha = \mathrm{const}$, setting $\nu = 1$, and choosing

$$\psi = -\delta\arctan(x), \qquad \Phi = \sqrt{\delta^2 + 1}\arctan(x). \tag{9.6}$$

Injecting this to (9.3) and computing the rotation amplitude via (4.7) and the K-amplitude from (4.12), (4.15) yields

$$w = 2\delta\sin(\alpha)(y - y_0), \qquad K = K_{\rm I} + K_{\rm II} = 0. \tag{9.7}$$

Choosing the integration constant $y_0 = 1$, the metric is

$$ds^2 = -e^{2U}\left(dt + 4\delta\sin(\alpha)\sin^2\frac{\vartheta}{2}d\varphi\right)^2 + e^{-2U}[dx^2 + (x^2+1)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)], \tag{9.8}$$

with U defined by (9.3), (9.6). This stationary solution reduces to the static BE solution (2.12) when $\alpha \to 0$, but for $\alpha \neq 0$ it contains the Misner string—the conical singularity along the $\theta = \pi$ axis where $\sin^2(\vartheta/2)$ does not vanish. The singularity appears because U and χ do not depend on y, in which case the rotation field w obtained from (4.7) is linear in $y = \cos\vartheta$ and so cannot vanish both for $y = 1$ and for $y = -1$. Although this singularity is actually quite harmless [52,53], still its appearance is unpleasant.

B. Removing the NUT singularity

Still keeping $\alpha = \text{const}$, one can avoid the Misner string by letting ψ depend both on x and y. In this case the rotation field w is no longer a liner function of y and one can adjust it to vanish both for $y = 1$ and for $y = -1$. However, since y-depending harmonic functions are unbounded, the solution will no longer be asymptotically flat in both limits.

As the simplest choice, we consider the dipole mode

$$\psi = \mathcal{A}X_1^-(x)y \tag{9.9}$$

with $X_1^-(x)$ defined by (5.4). Using (5.5), we see that

$$\mathcal{A}\left(x - \frac{1}{3\pi x^2} + \ldots\right)y \leftarrow \psi \to -\frac{\mathcal{A}y}{3\pi x^2} + \ldots, \tag{9.10}$$

and injecting this to (9.3) it follows that U ranges in the limits

$$|\mathcal{A}y|x + \ldots \leftarrow U \to \frac{\mathcal{A}\cos\alpha}{3\pi x^2}y + \ldots \quad \text{as } -\infty \leftarrow x \to \infty. \tag{9.11}$$

As U tends to minus infinity for $x \to -\infty$, the geometry is not asymptotically flat in this limit. The rotation field is determined from (4.7),

$$w = \mathcal{A}\sin(\alpha)f(x)(1 - y^2) \tag{9.12}$$

with

$$f(x) = (x^2+1)\left[\frac{1}{2} - \frac{1}{\pi}\arctan(x)\right] - \frac{x}{\pi}, \tag{9.13}$$

which ranges within the limits

$$x^2 + 1 + \frac{2}{3\pi x} \leftarrow f(x) \to \frac{2}{3\pi x} + \ldots \quad \text{as } -\infty \leftarrow x \to \infty. \tag{9.14}$$

We see that the rotation field approaches zero for $x \to \infty$ but diverges in the opposite limit. At the same time, the K-amplitude will be everywhere regular if the scalar field is chosen to be the superposition of the dipole and monopole modes,

$$\Phi = \arctan(x) + \mathcal{A}X_1^-(x)y. \tag{9.15}$$

Equations (4.12), (4.15) then yield

$$K = K_{\rm I} + K_{\rm II} = \frac{\mathcal{A}}{\pi}\left[\pi(1-y) + 2\arctan(x)y - 2\arctan\left(\frac{x}{y}\right)\right]. \tag{9.16}$$

This function is bounded and ranges in the following limits

$$2\mathcal{A}(1-y) + \ldots \leftarrow K \to \frac{2\mathcal{A}y(1-y^2)}{3\pi x^3} + \ldots \quad \text{as } -\infty \leftarrow x \to \infty. \tag{9.17}$$

The resulting stationary geometry

$$ds^2 = -e^{2U}(dt - \mathcal{A}\sin(\alpha)f(x)\sin^2\vartheta d\varphi)^2 + e^{-2U}\{e^{2K}[dx^2 + (x^2+1)d\vartheta^2] + (x^2+1)\sin^2\vartheta d\varphi^2\} \tag{9.18}$$

is free from the Misner string. It seems this solution has not been described before. The geometry is asymptotically flat for $x \to \infty$, with the angular momentum

$$\mathrm{J} = \frac{\mathcal{A}\mu}{3\pi}\sin(\alpha). \tag{9.19}$$

Curiously, the ADM mass vanishes because $U = \mathcal{O}(1/x^2)$ for $x \to \infty$. This solution reduces to the ultrastatic BE wormhole when $\mathcal{A} \to 0$ *pointwise* for $x > -\infty$. Therefore, at least when restricted to the $x > -\infty$ region, it can be viewed as a rotating generalization for the BE wormhole. However, U and w diverge as $x \to -\infty$ hence the second flat asymptotic is lost.

C. $\alpha \neq$ const

Let us assume that $\alpha = \alpha(\psi)$. Equations (9.4) then reduce to

$$\Delta\psi = \frac{1}{4}\alpha'^2\sinh(4\psi)(\vec{\nabla}\psi)^2,$$
$$\tanh(2\psi)\alpha'' + \frac{1}{2}\sinh^2(2\psi)\alpha'^3 + 4\alpha' = 0, \tag{9.20}$$

which can be solved in the parametric form,

$$\cosh(2\psi) = \sqrt{1+\eta^2}\cosh(Y),$$
$$\tan(\alpha-\alpha_0) = \eta\coth(Y), \tag{9.21}$$

where η, α_0 are integration constants and Y is a harmonic function, $\Delta Y = 0$. This yields a family of new exact stationary solutions. The case of constant $\alpha = \alpha_0$ considered above is recovered when $\eta \to 0$. Injecting to (4.7) and defining $\mathcal{S} = \eta - \sqrt{1+\eta^2}\sin(\alpha_0)$ gives

$$w_{,x} = \mathcal{S}(y^2-1)Y_{,y}, \qquad w_{,y} = \mathcal{S}(x^2+\nu)Y_{,x}. \tag{9.22}$$

Therefore, in order to avoid the Misner string, the harmonic function Y should be y-dependent and hence unbounded. The ψ-amplitude is then also unbounded and U in (9.3) is unbounded too. Hence the solution cannot be asymptotically flat in both limits.

Other exactly solvable cases which similarly reduce to the Laplace equation are $U = U(\chi)$ and $S = S(w)$ with $S = \rho^2 e^{-2U}$. They always show the same problem—solutions are not asymptotically flat.

X. SLOWLY ROTATING WORMHOLES—EXPANSION AROUND THE TRIVIAL VACUUM

One can try and approach the problem differently by assuming the deviation from the static limit to be small, without restricting the form of the fields. Let us start form the static ring (1.2) described by the Ernst potential

$$\xi = 0 \tag{10.1}$$

and try constructing its slowly rotating version. A slowly rotating solution is expected to be a small deformation of the static one, hence the Ernst potential ξ should be small. Therefore, in the first order of the perturbation theory, it should fulfill the linearized Ernst equation (2.29), hence

$$\Delta\xi = 0. \tag{10.2}$$

As discussed above, the solution must depend both on x and y to avoid the NUT singularity. Therefore, it should be unbounded, which means that the perturbative approach breaks down. This may look like the no-go proof forbidding the existence of slowly rotating wormholes. Nevertheless, slowly rotating wormholes can be constructed since, in fact, $\xi = 0$ is a "wrong vacuum" to expand around. As we shall see below, the same static ring wormhole can also be described by a different solution of the Ernst equation,

$$\xi = \tanh(\ln(\rho)), \tag{10.3}$$

and the perturbation theory around this vacuum is well defined. We shall see this in the next sections, while at the time being let us see what happens if we expand around the trivial vacuum $\xi = 0$.

Let us choose the U, w variables and consider the ultrastatic background (1.2) for which $\nu = 1$ and

$$U = w = 0. \tag{10.4}$$

Small deformations of this solutions are described by

$$U = \overset{(1)}{U} + \overset{(2)}{U} + \ldots, \qquad w = \overset{(1)}{w} + \overset{(2)}{w} + \ldots \tag{10.5}$$

Inserting this to (2.25) yields in the first order of perturbation theory

$$\Delta\overset{(1)}{U} = 0, \qquad \vec{\nabla}\left(\frac{1}{\rho^2}\vec{\nabla}\overset{(1)}{w}\right) = 0, \tag{10.6}$$

where one can set $\overset{(1)}{U} = 0$, while the w-equation explicitly reads

$$(x^2+1)\overset{(1)}{w}_{,xx} + (1-y^2)\overset{(1)}{w}_{,yy} = 0. \tag{10.7}$$

Its solution that vanishes at $y^2 = 1$ and is free from the Misner string is

$$\overset{(1)}{w} = \mathcal{A}f(x)(1-y^2), \tag{10.8}$$

where $f(x)$ is the same is in (9.13). This solution coincides with that in (9.12) up to redefining the integration constant, hence it shows the same asymptotics which can be written as

$$\mathcal{A}\rho^2 + \ldots \leftarrow \overset{(1)}{w} \to \frac{2\mathcal{A}\rho^2}{3\pi x^3} + \ldots \quad \text{as } -\infty \leftarrow x \to \infty, \tag{10.9}$$

where $\rho^2 = (x^2+1)\sin^2\vartheta$. The solution diverges for $x \to -\infty$.

In the second order of perturbation theory one has

$$\Delta\overset{(2)}{U} + \frac{1}{2\rho^2}(\vec{\nabla}\overset{(1)}{w})^2 = 0, \qquad \vec{\nabla}\left(\frac{1}{\rho^2}\vec{\nabla}\overset{(2)}{w}\right) = 0. \tag{10.10}$$

This is solved by setting $\overset{(2)}{w} = 0$ and choosing

$$\overset{(2)}{U} = F_0(x) + F_2(x)y^2, \tag{10.11}$$

which yields two ordinary differential equations (ODE) for $F_0(x)$ and $F_2(x)$. Integrating these equations, the integration constants can be adjusted such that $F_0(x) \to 0$ and $F_2(x) \to 0$ for $x \to +\infty$, but in the opposite limit these function inevitably diverge, which yields

$$-\frac{\mathcal{A}^2}{4}x^2(1+y^2)+\ldots \leftarrow \overset{(2)}{U} \rightarrow \frac{\mathcal{A}^2}{45\pi x^3}(y^2-1)$$
$$\text{as} \ -\infty \leftarrow x \rightarrow \infty. \tag{10.12}$$

Therefore, the rotating excitations cannot be small and diverge for $x \to -\infty$. However, there is a different way to carry out the perturbation theory that allows one to keep everything finite.

XI. DUALIZATION

Let (U, w) fulfill the Ernst equations

$$\Delta U + \frac{e^{4U}}{2\rho^2}(\vec{\nabla} w)^2 = 0, \qquad \vec{\nabla}\left(\frac{e^{4U}}{\rho^2}\vec{\nabla} w\right) = 0. \tag{11.1}$$

The corresponding spacetime metric can be expressed in two different forms, which we shall call the dual t and φ forms:

$$\begin{aligned} \boldsymbol{t}\colon\ ds^2 &= -e^{2U}(dt - wd\varphi)^2 \\ &\quad + e^{-2U}\{e^{2k}(d\rho^2+dz^2)+\rho^2 d\varphi^2\} \\ \boldsymbol{\varphi}\colon\ &= -\rho^2 e^{-2\mathbf{U}}dt^2 + e^{-2\mathbf{U}}e^{2\mathbf{k}}(d\rho^2+dz^2) \\ &\quad + e^{2\mathbf{U}}(d\varphi - \mathbf{w}dt)^2, \end{aligned} \tag{11.2}$$

where

$$e^{2\mathbf{U}} = \rho^2 e^{-2U} - w^2 e^{2U}, \quad \mathbf{w}e^{2\mathbf{U}} = -we^{2U}, \quad \mathbf{k}-\mathbf{U} = k - U. \tag{11.3}$$

Notice that e^{2U} is the norm of the timelike Killing vector $\partial/\partial t$ while $e^{2\mathbf{U}}$ is the norm of the azimuthal Killing vector $\partial/\partial\varphi$. In addition, $(\mathbf{U}, \mathbf{w})$ fulfill exactly the same Ernst equations as (U, w) in (12.10),

$$\Delta \mathbf{U} + \frac{e^{4\mathbf{U}}}{2\rho^2}(\vec{\nabla}\mathbf{w})^2 = 0, \qquad \vec{\nabla}\left(\frac{e^{4\mathbf{U}}}{\rho^2}\vec{\nabla}\mathbf{w}\right) = 0, \tag{11.4}$$

while $\mathbf{k}$ fulfills the same equation as in (2.30), up to replacing $U \to \mathbf{U}$, $w \to \mathbf{w}$, $k \to \mathbf{k}$. As a result, the amplitudes $(\mathbf{U}, \mathbf{w}, \mathbf{k})$ determine not only the φ-form of the same solution (11.2), but also a new solution with the metric

$$\begin{aligned} \boldsymbol{t}\colon\ ds^2 &= -e^{2\mathbf{U}}(dt - \mathbf{w}d\varphi)^2 \\ &\quad + e^{-2\mathbf{U}}\{e^{2\mathbf{k}}(d\rho^2+dz^2)+\rho^2 d\varphi^2\} \\ \boldsymbol{\varphi}\colon\ &= -\rho^2 e^{-2U}dt^2 + e^{-2U}e^{2k}(d\rho^2+dz^2) \\ &\quad + e^{2U}(d\varphi - wdt)^2. \end{aligned} \tag{11.5}$$

This solution can formally be obtained from (11.2) by the complex change of coordinates

$$t \to i\varphi, \qquad \varphi \to it. \tag{11.6}$$

The inverse transformation $(\mathbf{U}, \mathbf{w}) \to (U, w)$ has exactly the same structure as (11.3),

$$e^{2U} = \rho^2 e^{-2\mathbf{U}} - \mathbf{w}^2 e^{2\mathbf{U}}, \quad we^{2U} = -\mathbf{w}e^{2\mathbf{U}}, \quad k - U = \mathbf{k} - \mathbf{U}. \tag{11.7}$$

Summarizing, solutions of the Ernst equations come in pairs (U, w) and $(\mathbf{U}, \mathbf{w})$ related to each other via (11.3), (11.7). Each pair determines two different geometries (11.2) and (11.5). Equivalently, each solution (U, w) of the Ernst equations determines two different geometries: either the t-geometry defined in (11.2) or the φ-geometry defined in (11.5).

Asymptotically flat geometries correspond to solutions (U, w) of the Ernst equations for which $e^{2U} \to 1$ at infinity, but also to solutions $(\mathbf{U}, \mathbf{w})$ of the Ernst equations for which $\rho^2 e^{-2\mathbf{U}} \to 1$ at infinity. We have considered above the first option by choosing $U = w = 0$ as the background vacuum configuration corresponding to the ultrastatic wormhole. However, the same background can be described in the dual way by $e^{2\mathbf{U}} = \rho^2$, $\mathbf{w} = 0$.

A. Exact solutions

Let us first see if the dual description allows one to obtain new exact solutions. Introducing the twist potential $\boldsymbol{\chi}$ related to $\mathbf{U}, \mathbf{w}$ in the same way as in (2.26), one can use for $\mathbf{U}$, $\boldsymbol{\chi}$ the same harmonic ansatz as in (9.3). This ansatz expresses the solution in terms of ψ, α and finally in terms of a harmonic function Y via (9.21). To preserve the asymptotic condition $e^{2\mathbf{U}} \to \rho^2$, one may choose, for example,

$$Y = B\ln(\rho) + A\ \arctan(x), \tag{11.8}$$

with a suitably adjusted coefficient B. This yields a family of new exact stationary solutions. However, injecting into (9.22) determines the rotation field

$$\mathbf{w} = \left(\eta - \sqrt{1+\eta^2}\sin(\alpha_0)\right)(A + 2Bx)y + \mathbf{w}_0, \tag{11.9}$$

and this is an unbounded function of x that spoils the asymptotic flatness. This function becomes bounded if $B = 0$ but then the condition $e^{2\mathbf{U}} \to \rho^2$ is not fulfilled. Therefore, the φ-version of the solution is not asymptotically flat, while its t-version is similar to (9.8) and contains the Misner string. If one modifies (11.8) by adding extra terms similar to (9.9), this destroys the asymptotic flatness in both settings. Therefore, one might conclude that the dual formulation does not give anything interesting, at least within the harmonic ansatz. However, the situation changes if one abandons the ansatz.

XII. PERTURBATIVE ANALYSIS IN THE DUAL SETTING

Let us start from the metric (11.2) in the φ-form. Defining

$$e^{2V}=\rho^2 e^{-2\mathbf{U}},\qquad e^{2\gamma}=\rho^2 e^{2\mathbf{k}-4\mathbf{U}},\qquad (12.1)$$

the metric becomes

$$ds^2=-e^{2V}dt^2+e^{-2V}(e^{2\gamma}(d\rho^2+dz^2)+\rho^2(d\varphi-\mathbf{w}dt)^2). \qquad (12.2)$$

The Ernst equations (11.4) assume the form

$$\Delta V=\frac{\rho^2}{2}e^{-4V}(\vec{\nabla}\mathbf{w})^2,\qquad \vec{\nabla}(\rho^2 e^{-4V}\vec{\nabla}\mathbf{w})=0,\qquad (12.3)$$

while Eq. (2.30) reduces to

$$\frac{1}{\rho}\partial_\rho\boldsymbol{\gamma}=(\partial_\rho V)^2-(\partial_z V)^2+\frac{\rho^2}{4}e^{-4V}[(\partial_z w)^2-(\partial_\rho w)^2]-(\partial_\rho\Phi)^2+(\partial_z\Phi)^2,$$
$$\frac{1}{2\rho}\partial_z\boldsymbol{\gamma}=\partial_\rho V\partial_z V-\frac{\rho^2}{4}e^{-4V}\partial_\rho w\partial_z w-\partial_\rho\Phi\partial_z\Phi. \qquad (12.4)$$

Passing to the spheroidal coordinates (4.1) yields

$$ds^2=-e^{2V}dt^2+e^{-2V}\left(e^{2\mathbf{K}}\left[dx^2+\frac{x^2+\nu}{1-y^2}dy^2\right]+(x^2+\nu)(1-y^2)(d\varphi-\mathbf{w}dt)^2\right) \qquad (12.5)$$

with

$$e^{2\mathbf{K}}=\frac{x^2+\nu y^2}{x^2+\nu}e^{2\gamma}. \qquad (12.6)$$

The Ernst equations (12.3) read

$$[(x^2+\nu)V_{,x}]_{,x}+[(1-y^2)V_{,y}]_{,y}=\frac{1}{2}e^{-4V}(x^2+\nu)(1-y^2)[(x^2+\nu)\mathbf{w}_{,x}^2+(1-y^2)\mathbf{w}_{,y}^2],$$
$$\frac{[(x^2+\nu)^2\mathbf{w}_{,x}]_{,x}}{x^2+\nu}+\frac{[(1-y^2)^2\mathbf{w}_{,y}]_{,y}}{1-y^2}=4[(x^2+\nu)\mathbf{w}_{,x}V_{,x}+(1-y^2)\mathbf{w}_{,y}V_{,y}]. \qquad (12.7)$$

Setting $\mathbf{K}=\mathbf{K}_{\mathrm{I}}+\mathbf{K}_{\mathrm{II}}$ one obtains from Eq. (12.4)

$$\partial_x\mathbf{K}_{\mathrm{I}}=\frac{1-y^2}{x^2+\nu y^2}\left(\Gamma(V)-\frac{1}{4}(x^2+\nu)(1-y^2)e^{-4V}\Gamma(\mathbf{w})+\frac{\nu x}{x^2+\nu}\right),$$
$$\partial_y\mathbf{K}_{\mathrm{I}}=\frac{x^2+\nu}{x^2+\nu y^2}\left(\Lambda(V)-\frac{1}{4}(x^2+\nu)(1-y^2)e^{-4V}\Lambda(\mathbf{w})+\frac{\nu y}{x^2+\nu}\right), \qquad (12.8)$$

with the same notation as in (4.14), whereas $\mathbf{K}_{\mathrm{II}}$ fulfills the same equation as K_{II} in (4.15),

$$\partial_x\mathbf{K}_{\mathrm{II}}=-\frac{1-y^2}{x^2+\nu y^2}\Gamma(\Phi),$$
$$\partial_y\mathbf{K}_{\mathrm{II}}=-\frac{x^2+\nu}{x^2+\nu y^2}\Lambda(\Phi). \qquad (12.9)$$

A. Expansion around the dual vacuum

Let us start from the static solution of (12.7),

$$\nu=1,\qquad V=\mathbf{w}=0. \qquad (12.10)$$

This corresponds not the trivial Ernst potential $e^{2U}+i\chi=1$ but rather to $e^{2\mathbf{U}}+i\chi=\rho^2$, which still describes the ultrastatic BE wormhole or the vacuum ring wormhole, depending on whether the scalar field is added or not. Consider small deformations of (12.10),

$$V=\overset{(1)}{V}+\overset{(2)}{V}+\ldots,\qquad \mathbf{w}=\overset{(1)}{\mathbf{w}}+\overset{(2)}{\mathbf{w}}+\ldots \qquad (12.11)$$

Inserting this to (12.3) yields $\Delta\overset{(1)}{V}=0$, whose solution can be chosen to be $\overset{(1)}{V}=0$, while $\overset{(1)}{\mathbf{w}}$ satisfies the equation

$$\vec{\nabla}(\rho^2\vec{\nabla}\overset{(1)}{\mathbf{w}})=0, \qquad (12.12)$$

which reads explicitly

$$\frac{[(x^2+1)^2\overset{(1)}{\mathbf{w}}_{,x}]_{,x}}{x^2+1}+\frac{[(1-y^2)^2\overset{(1)}{\mathbf{w}}_{,y}]_{,y}}{1-y^2}=0. \qquad (12.13)$$

We remember that the rotation field w in the t-form of the metric should be proportional to $\sin^2\vartheta$ to avoid the NUT singularity. However, the same condition is not needed for the rotation field $\mathbf{w}$ in the φ-form, since the $(d\varphi-\mathbf{w}dt)^2$

element of the metric (12.5) is multiplied by $1-y^2 = \sin^2\vartheta$. Therefore, we can assume $\overset{(1)}{\mathbf{w}}$ to depend only on x (otherwise solutions of (12.13) are unbounded), which yields

$$\overset{(1)}{\mathbf{w}} = \mathcal{A}\frac{f(x)}{x^2+1} \equiv \mathcal{A}W(x), \tag{12.14}$$

with $f(x)$ defined in (9.13). When compared with the previously studied cases (9.12), (10.8), the rotation field now contains an additional factor of $1/(x^2+1)$, hence one has

$$\mathcal{A}\left(1+\frac{2}{3\pi x^3}+\ldots\right) \leftarrow \overset{(1)}{\mathbf{w}} \rightarrow \mathcal{A}\frac{2}{3\pi x^3}+\ldots \quad \text{as } -\infty \leftarrow x \rightarrow \infty, \tag{12.15}$$

so that $\overset{(1)}{\mathbf{w}}$ is everywhere bounded and approaches a constant value as $x\to-\infty$. This may seem surprising, since the previously obtained perturbative solution (10.8) for the rotation field $\overset{(1)}{w}$ was unbounded for $x\to-\infty$, where $\overset{(1)}{w}\sim\rho^2$. However, the two results actually agree, because the duality transformation (11.3) reduces in the first order of perturbation theory to

$$e^{2\mathbf{U}} = \rho^2 e^{-2U} = e^{2V} = 1, \qquad \overset{(1)}{\mathbf{w}} = -\frac{\overset{(1)}{w}}{\rho^2}. \tag{12.16}$$

Dividing $\overset{(1)}{w}$ in (10.8) by ρ^2 yields precisely $\overset{(1)}{\mathbf{w}}$ in (12.14), hence calculations in the t-setting and in the φ-setting agree. Since the rotation field $\overset{(1)}{w}$ is unbounded whereas $\overset{(1)}{\mathbf{w}}$ is bounded, the perturbation theory applies in the φ-setting but not in the t-setting.

Now, in the φ-setting there is an important "twisting" symmetry of the line element (12.2),

$$\mathbf{w} \rightarrow \underline{\mathbf{w}} = \mathbf{w}-\omega, \qquad \varphi \rightarrow \underline{\varphi} = \varphi - \omega t, \tag{12.17}$$

with a constant ω, which amounts to passing to a rotating frame. Applying this with $\omega = \overset{(1)}{\mathbf{w}}(-\infty) = \mathcal{A}$ yields the rotation field in the new frame,

$$\mathcal{A}\frac{2}{3\pi x^3}+\ldots \leftarrow \overset{(1)}{\underline{\mathbf{w}}} \rightarrow \mathcal{A}\left(-1+\frac{2}{3\pi x^3}+\ldots\right). \tag{12.18}$$

Comparing with (12.15), we see that

$$\overset{(1)}{\mathbf{w}}(-x) = -\overset{(1)}{\underline{\mathbf{w}}}(x) \tag{12.19}$$

so that the rotation field is antisymmetric under the combined action of the reflection in the wormhole throat, $x\to -x$, and the twisting (12.17). This suggests using two frames: the frame where the rotation fulfills (12.15) should be used in the $x>0$ region, while the frame where the rotation fulfills (12.18) should be used in the $x<0$ region. Using these two frames, the rotation field approaches zero in both limits, for $x\to\infty$ and for $x\to-\infty$.

Let us now continue to the second order of perturbation theory, where one can set $\overset{(2)}{\mathbf{w}}=0$, while the amplitude $\overset{(2)}{V}$ fulfills the equation

$$\Delta\overset{(2)}{V} = \frac{\rho^2}{2}(\vec{\nabla}\overset{(1)}{\mathbf{w}})^2, \tag{12.20}$$

or explicitly

$$[(x^2+1)\overset{(2)}{V}_{,x}]_{,x} + [(1-y^2)\overset{(2)}{V}_{,y}]_{,y} = \frac{2\mathcal{A}^2}{\pi^2}\frac{1-y^2}{(x^2+1)^2}. \tag{12.21}$$

The variables here can be separated by setting

$$\overset{(2)}{V}(x,y) = \mathcal{A}^2[\overset{(2)}{V}_0(x) + \overset{(2)}{V}_2(x)y^2]. \tag{12.22}$$

This yields ordinary differential equations for $\overset{(2)}{V}_0(x)$ and $\overset{(2)}{V}_2(x)$ which admit an everywhere bounded solution. The procedure can than be continued to higher orders of the perturbation theory. In every order the variables can be separated similarly, and the integration constants of the ordinary differential equations which appear are *uniquely* fixed by the requirement that the solution should be bounded. As a result, $\mathcal{A}$ is the only integration constant which remains.

B. Fourth order perturbative solution

Skipping the details, here is the solution up to the fourth order terms

$$\begin{aligned}
\mathbf{w} &= \mathcal{A}\overset{(1)}{w}_0(x) + \mathcal{A}^3[\overset{(3)}{w}_0(x) + \overset{(3)}{w}_2(x)y^2] + \mathcal{O}(\mathcal{A}^5),\\
V &= \mathcal{A}^2[\overset{(2)}{V}_0(x) + \overset{(2)}{V}_2(x)y^2]\\
&\quad + \mathcal{A}^4[\overset{(4)}{V}_0(x) + \overset{(4)}{V}_2(x)y^2 + \overset{(4)}{V}_4(x)y^4]\\
&\quad + \mathcal{O}(\mathcal{A}^6),
\end{aligned} \tag{12.23}$$

where the coefficient functions can be represented as follows,

$$\overset{(1)}{w}_0(x) = \frac{1}{2}(1 - W),$$
$$\overset{(3)}{w}_0(x) = -\frac{1}{8}(x^2+1)(W^2-1)W - \frac{x}{3X},$$
$$\overset{(3)}{w}_2(x) = \frac{1}{8}(5x^2+1)(W^2-1)W + \frac{x(6W^2-1)}{3X} + \frac{2W}{X^2},$$
$$\overset{(2)}{V}_0(x) = \frac{1}{8}(x^2+1)(W^2-1),$$
$$\overset{(2)}{V}_2(x) = -\frac{1}{8}(3x^2+1)(W^2-1) - \frac{xW}{X} - \frac{1}{X^2},$$
$$\overset{(4)}{V}_0(x) = \frac{1}{64}(x^2+1)^2(W^2+3)(W^2-1) + \frac{xW}{6\pi},$$
$$\overset{(4)}{V}_2(x) = -\frac{1}{32}(x^2+1)(5x^2+1)(W^2+3)(W^2-1) - \frac{xW(3W^2+7)}{6\pi} - \frac{3W^2+5}{6\pi X},$$
$$\overset{(4)}{V}_4(x) = \frac{1}{192}(35x^4+30x^2+3)(W^2+3)(W^2-1) + \frac{xW}{18X}[3(5x^2+3)W^2+20x^2+14] + \frac{1}{18X^2}[9(3x^2+1)W^2+13x^2+7] + \frac{4xW}{3X^3} + \frac{2}{3X^4}, \tag{12.24}$$

with the abbreviations $X = \pi(x^2+1)$ and

$$W = \frac{2}{\pi}\left(\arctan(x) + \frac{x}{x^2+1}\right) = \frac{2}{\pi}[x\ \arctan(x)]'. \tag{12.25}$$

This function is antisymmetric, $W(x) = -W(-x)$, and one has for $-\infty \leftarrow x \rightarrow \infty$

$$-1 - \frac{4}{3\pi x^3} + \ldots \leftarrow W(x) \rightarrow +1 - \frac{4}{3\pi x^3} + \ldots \tag{12.26}$$

As a result, $\mathbf{w}(-x,y) = \mathcal{A} - \mathbf{w}(x,y)$ and $V(-x,y) = V(x,y)$. This yields the lowest terms of the perturbative expansion of the solution of the Ernst equations, and higher orders can be included similarly. All terms in (12.23) are bounded. One has for $x \rightarrow \pm\infty$

$$V = -\mathcal{A}^2\left(1 + \frac{2}{5}\mathcal{A}^2 + \ldots\right) \times \frac{1}{3\pi|x|} + \mathcal{O}\left(\frac{1}{x^3}\right),$$
$$\mathbf{w} = \mathbf{w}(\pm\infty) + 2\mathcal{A}\left(1 + \frac{2}{5}\mathcal{A}^2 + \ldots\right) \times \frac{1}{3\pi x^3} + \mathcal{O}\left(\frac{1}{x^4}\right), \tag{12.27}$$

where $\mathbf{w}(\infty) = 0$ and $\mathbf{w}(-\infty) = \mathcal{A}$. Computing $e^{2\mathbf{U}} = \rho^2 e^{-2V}$ and injecting together with $\mathbf{w}$ to (4.7) yields the twist potential

$$\chi = \frac{2\mathcal{A}}{3\pi}y(3-y^2) + \mathcal{A}^3 y\left(\frac{4}{15\pi}(2y^4-5y^2+5) + (1-y^2)^2\left[\frac{X}{4\pi^2}(W^2-1)(xXW+2) + \frac{2x^2}{3\pi}\right]\right) + \mathcal{O}(\mathcal{A}^5), \tag{12.28}$$

from where one can see that the complex Ernst potential $e^{2\mathbf{U}} + i\chi = \rho^2 e^{-2V} + i\chi$ is not of the Tomimatsu-Sato type since it contains powers of x, y and also of $\arctan(x)$.

To complete the line element, there remains to determine the $\mathbf{K}$-amplitude. Integrating (12.8) yields

$$\mathbf{K}_{\mathrm{I}} = \mathbf{K}_{\mathrm{reg}} + \mathbf{K}_{\mathrm{sing}}, \tag{12.29}$$

where the regular part is

$$\mathbf{K}_{\mathrm{reg}} = \left(\mathcal{A}^2 + \mathcal{A}^4\left[\frac{1}{3}(2-y^2) + y^2W^2 + y^2xXW(W^2-1) + \frac{X^2}{32}[y^2-1+x^2(9y^2-1)](W^2-1)^2\right] + \mathcal{O}(\mathcal{A}^6)\right) \times \frac{1}{2\pi^2}\frac{y^2-1}{x^2+1}, \tag{12.30}$$

and the singular part

$$\mathbf{K}_{\mathrm{sing}} = \left(1 - \frac{\mathcal{A}^2}{\pi^2} + \frac{\mathcal{A}^4}{3\pi^2} + \mathcal{O}(\mathcal{A}^6)\right) \times \frac{1}{2}\ln\frac{x^2+y^2}{x^2+1}. \tag{12.31}$$

If we assume that $\Phi = 0$ then $\mathbf{K} = \mathbf{K}_{\mathrm{I}}$ and the above solution describes the spinning generalization of the vacuum ring wormhole. If $\mathcal{A} = 0$, then the 4-metric (12.5) becomes flat everywhere away from the circle $x = y = 0$ where the singularity of the Ricci tensor is located. As explained above, this singularity can be interpreted as a result of the distributional ring source. For $\mathcal{A} \neq 0$ the ring rotates in the equatorial plane, the geometry is then no longer flat and in addition to the distributional part the curvature develops a volume part containing components of the Riemann tensor which diverge as one approaches the ring. However, the solution remains a wormhole with two asymptotically flat regions and the geodesics can interpolate between these regions, unless they hit the ring singularity.

Let us finally add the scalar field according to (5.6),

$$\Phi = C\ \arctan(x) \Rightarrow \mathbf{K}_{\mathrm{II}} = -\frac{C^2}{2}\ln\frac{x^2+y^2}{x^2+1}. \tag{12.32}$$

Comparing with (12.31), we see that $\mathbf{K}_{\mathrm{II}} + \mathbf{K}_{\mathrm{sing}} = 0$ and hence the metric singularity contained in $\mathbf{K}_{\mathrm{sing}}$ is cancelled if the integration constant is chosen as

$$C^2 = 1 - \frac{\mathcal{A}^2}{\pi^2} + \frac{\mathcal{A}^4}{3\pi^2} + \mathcal{O}(\mathcal{A}^6). \tag{12.33}$$

One has then $\mathbf{K} = \mathbf{K}_{\rm reg}$ and the spacetime geometry becomes everywhere regular because all three metric functions V, $\mathbf{w}$, $\mathbf{K}$ in the lime element (12.5) are bounded. One has for $x \to \pm\infty$

$$\mathbf{K} = \mathbf{K}_{\rm reg} = \left(\mathcal{A}^2 + \frac{4}{9}\mathcal{A}^4 + \mathcal{O}(\mathcal{A}^6)\right) \times \frac{y^2 - 1}{2\pi x^2} + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{12.34}$$

One can say that the singular ring source is screened by the scalar, which yields the globally regular spinning generalization for the ultrastatic BE wormhole. Its curvature is everywhere bounded and approaches zero in the two asymptotic regions.

C. The ADM mass and angular momentum

If one uses just one coordinate frame, than the metric components

$$g_{00} = -e^{2U} = -e^{2V} + \rho^2 \mathbf{w}^2 e^{-2V},$$
$$g_{0\varphi} = -\rho^2 \mathbf{w} e^{-2V} \tag{12.35}$$

are unbounded since for $-\infty \leftarrow x \to \infty$ one has

$$\mathbf{w}^2(-\infty) \times \rho^2 \leftarrow g_{00} \to -1,$$
$$-\mathbf{w}(-\infty) \times \rho^2 \leftarrow g_{0\varphi} \to 0. \tag{12.36}$$

The spacetime contains an ergoregion where $g_{00} = -e^{2U}$ becomes positive and the timelike Killing vector becomes spacelike. Therefore, the Newtonian potential U is not globally defined. However, there exists a linear combination of the two Killing vectors which remains timelike in the ergoregion, while the metric components can be made finite for $x < 0$ by the twisting transformation (12.17), (12.18).

This suggests using two rotation fields $\mathbf{w}_+ = \mathbf{w}$ and $\mathbf{w}_- = \mathbf{w} - \mathbf{w}(-\infty)$ related to each other via (12.17), (12.18) with $\omega = \mathbf{w}(-\infty)$ such that for $-\infty \leftarrow x \to \infty$ one has

$$\mathcal{A} \leftarrow \mathbf{w}_+ \to 0, \qquad 0 \leftarrow \mathbf{w}_- \to -\mathcal{A}. \tag{12.37}$$

One uses $\mathbf{w}_+$ to compute the metric components g_{00} and $g_{0\varphi}$ in the $x > 0$ region and one uses $\mathbf{w}_-$ to compute them in the $x < 0$ region. The metric components are then finite everywhere and one can compute the ADM mass M and angular momentum J. These are the same for the ring wormhole and for the BE wormhole, since g_{00} and $g_{0\varphi}$ are the same. One has for $x \to \pm\infty$

$$-g_{00} = e^{2V} - \rho^2 \mathbf{w}_\pm^2 e^{-2V} = 1 - \frac{2M_\pm}{|x|} + \ldots,$$
$$-g_{0\varphi} = \rho^2 \mathbf{w}_\pm e^{-2V} = \frac{2J_\pm \sin^2\vartheta}{|x|} + \ldots \tag{12.38}$$

This determines the mass $M_\pm$ and angular momentum $J_\pm$ measured, respectively, at $x \to \pm\infty$. Notice that the denominators in (12.38) should contain $|x|$ and not x since the mass and angular momentum should be invariant under the coordinate transformation $x \to -x$.

It is important to emphasize that g_{00} and $g_{0\varphi}$ and hence $M_\pm$ and $J_\pm$ are determined by the vacuum equations and are insensitive to the scalar field. Including the latter only modifies the $\mathbf{K}$-amplitude without affecting g_{00} and $g_{0\varphi}$.

Using (12.27) in (12.38) and restoring the length scale gives the dimensionful values,

$$\mathrm{M} \equiv \mathrm{M}_\pm = \frac{\mu}{3\pi}\mathcal{A}^2\left(1 + \frac{2}{5}\mathcal{A}^2 + \ldots\right),$$
$$\mathrm{J}_\pm = \pm\frac{\mu}{3\pi}\mathcal{A}\left(1 + \frac{2}{5}\mathcal{A}^2 + \ldots\right), \tag{12.39}$$

so that the mass is the same and positive in each asymptotic region, while the angular momentum changes sign when one passes from one region to the other one. The latter property is clear, since if the hole rotates in the clockwise direction, say, when viewed from one asymptotic region, then it rotates in the opposite direction when viewed from the other region.

One can establish an exact Smarr-type relation between M and J [27]. The two Ernst equations (12.3) can be combined to yield

$$\Delta V = \vec{\nabla}\left(\frac{\rho^2}{2} e^{-4V} \mathbf{w} \vec{\nabla} \mathbf{w}\right). \tag{12.40}$$

Integrating this over x from x_1 to x_2 and over y from -1 to 1 one obtains

$$\int_{-1}^{1} dy (x^2+1)\partial_x V \Big|_{x_1}^{x_2} = \frac{1}{2}\int_{-1}^{1} dy (1-y^2)(x^2+1)^2 e^{-4V} \mathbf{w}\partial_x \mathbf{w}\Big|_{x_1}^{x_2}. \tag{12.41}$$

Choosing first $x_1 = 0$, $x_2 = \infty$ and next $x_1 = -\infty$, $x_2 = 0$, using (12.27) and the fact that V is a symmetric function of x, it is not difficult to see that

$$\mathrm{M} = \mathcal{A}\mathrm{J}_+. \tag{12.42}$$

Since $\mathbf{w}(-x, y) = \mathcal{A} - \mathbf{w}(x, y)$, it follows that $\mathbf{w}(0, y) = \mathcal{A}/2 \equiv \mathbf{w}^0$ hence the rotation field assumes a constant value in the wormhole throat—the throat angular velocity.

This depends on the frame. For the two rotation fields $\mathbf{w}_+ = \mathbf{w}$ and $\mathbf{w}_-$ one has $\mathbf{w}^0_\pm = \pm\mathcal{A}/2$, which allows one to represent (12.42) as

$$\mathrm{M} = 2\mathbf{w}^0_\pm \mathrm{J}_\pm. \tag{12.43}$$

This, however, does not mean that M is a linear function of J, since the proportionality coefficient is J-dependent. As seen from (12.39), one has for small J

$$\mathrm{M} = \frac{3\pi}{\mu}\mathrm{J}^2 + \mathcal{O}(\mathrm{J}^4). \tag{12.44}$$

This is worth comparing with the expression for the rotational energy of a nonrelativistic rigid body,

$$\mathrm{E}_{\mathrm{rot}} = \frac{\mathrm{J}^2}{2I}, \tag{12.45}$$

where I is the moment of inertia. Such a relation is expected for the ring wormhole, since it contains the extended matter source—the cosmic string loop. For a slow rotation, this spinning string should exhibit the standard nonrelativistic relation. It is however quite remarkable that the spinning BE wormhole also shows exactly the same relation (12.44), because it has the same M and J as the ring wormhole. Although it is globally regular, it "remembers" its descendance from the ring wormhole.

This observation is important. Although the BE wormhole contains the scalar field, the latter is spherically symmetric and does not carry rotational degrees of freedom. The rotation is supported by the ring source and is encoded in the g_{00} and $g_{0\varphi}$ metric components which are insensitive to the scalar. The scalar only modifies the g_{rr} and $g_{\vartheta\vartheta}$ components to hide the source and make the geometry regular. However, the source is still visible in g_{00}, $g_{0\varphi}$ and the BE wormhole "knows" about it. Therefore, the essential features, such as the wormhole structure itself and the rotation, originate in the vacuum theory and exist due to the ring source and not due to the phantom scalar field as one might have thought. The only role of the scalar is to render the geometry regular.

XIII. CONCLUDING REMARKS—TOWARD THE EXACT SOLUTION?

Summarizing, we have described a number of possible ways to construct the stationary generalization for the static BE wormholes supported by the phantom scalar field. Perhaps not immediately interesting physically, since the BE wormhole is unstable [54], this problem is important conceptually. Indeed, it is important to understand why it is so difficult to construct the stationary version of the solution whose static limit (1.1) looks much simpler than the Schwarzschild geometry.

We find that the difficulty is actually not related to the scalar field, which can be eliminated within the Eris-Gurses procedure. The problem reduces to constructing the stationary generalization for the vacuum ring wormhole via solving the vacuum Ernst equations, and it is the latter step which is difficult. Even though the static wormhole geometry (1.2) is *locally flat*, its stationary generalization is difficult to obtain and it is of a previously unknown type.

Using the special ansatz to solve the Ernst equations, we have constructed exact solutions, but they are not globally regular. The perturbative expansion around the trivial solution of the Ernst equation which describes the static limit, $\mathcal{E} = e^{2U} + i\chi = 1$, contains unbounded functions and is ill defined. However, the static limit is also described by $e^{2\mathbf{U}} + i\boldsymbol{\chi} = \rho^2$ where $e^{2\mathbf{U}}$ and $\boldsymbol{\chi}$ are the norm and twist of the axial Killing vector. The perturbative expansion around this vacuum is described by (12.23) and contains only bounded functions. Although not a proof, this gives a good indication for the existence of a fully nonperturbative solution. An additional indication is provided by the numerical analysis in [27,28], which shows a numerical solution whose properties seem to correspond to our solution. This gives an extra evidence in favor of its existence.

The solution describes the spinning generalization for the locally flat vacuum ring wormhole and that for the ultrastatic BE wormhole, depending on whether the scalar field is included or not. The spinning wormhole interpolates between two asymptotically flat regions and is characterized by a nonzero ADM mass proportional to the square of the angular momentum, which is typical for a rotating extended source. The ring wormhole shows the ring singularity but the spinning BE wormhole is globally regular. Apart from this difference, the singular and regular solutions have identically the same g_{00} and $g_{0\varphi}$ metric components and the same ADM mass and angular momentum determined by the ring source and not by the phantom scalar field as one might have expected. The only role of the scalar is to screen the metric singularity at the ring source and make the geometry globally regular, but the memory of the source remains in g_{00} and $g_{0\varphi}$.

Let us finally remember that our initial intension was to obtain the solution exactly. Therefore, there remains the question of weather the perturbative solution (12.23) could be promoted to an exact one. However, since the expansion contains powers of x and y and also of $\arctan(x)$, there is little hope to guess the exact form of the solution, while the other known methods to get the solution do not seem to work. For example, it is known that the Ernst equations are equivalent to one fourth order PDE for the metric function k [44]. For the Tomimatsu-Sato solutions, as for example for the Kerr metric in (7.3), one always has $k(x,y) = k(\eta)$ with

$$\eta = \frac{x^2 + \nu y^2}{x^2 + \nu}.$$

This implies that the fourth order equation for k actually becomes an ODE, which allows one to obtain exact solutions [42–45]. However, neither the function $\mathbf{K}$ in (12.30), (12.31) nor the amplitude k or its φ-counterpart $\mathbf{k}$ defined by (11.2) expressed by

$$k = \mathbf{K} + \ln(1 - \rho^2 \mathbf{w}^2 e^{-4V}) - \frac{1}{2}\ln(\eta),$$
$$\mathbf{k} = \mathbf{K} - 2V + \frac{1}{2}\ln\frac{\rho^2}{\eta}$$

depend exclusively on η. Therefore, neither k not $\mathbf{k}$ satisfy an ODE, hence this approach does not allow to get the solution exactly.

The static ring wormholes described by the oblate ZV metrics can be promoted to the stationary sector by applying the solution generating methods, but this yields nonasymptotically flat solutions with an electric field [29,30]. It is also not obvious if the inverse scattering method [55] could be helpful, although this possibility deserves a separate study.

The reason for the difficulties in finding the exact solution is clear. The analytically known stationary metrics like Kerr describe spinning states of zero-dimensional objects—massive points. However, the static vacuum geometry (1.2) has an extended one-dimensional source: the ring. Therefore, constructing its stationary version should be a more complex problem that may not have an analytical solution.

ACKNOWLEDGMENTS

It is a pleasure to thank Gary Gibbons for discussions. This work was partly supported by the French CNRS/RFBR PRC Grant No. 289860, by the Russian Foundation for Basic Research on the Project No. 20-52-18012, and also by the Kazan Federal University Strategic Academic Leadership Program.

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