# Asymptotics of linear differential systems and application to quasinormal modes of nonrotating black holes

David Langlois,[1] Karim Noui,[2,1] and Hugo Roussille[1,2]

[1]*Université de Paris, CNRS, Astroparticule et Cosmologie, F-75006 Paris, France*
[2]*Institut Denis Poisson (UMR 7013), Université de Tours, Université d'Orléans, Parc de Grandmont, 37200 Tours, France*

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The traditional approach to perturbations of nonrotating black holes in general relativity uses the reformulation of the equations of motion into a radial second-order Schrödinger-like equation, whose asymptotic solutions are elementary. Imposing specific boundary conditions at spatial infinity and near the horizon defines, in particular, the quasinormal modes of black holes. For more complicated equations of motion, as encountered for instance in modified gravity models with different background solutions and/or additional degrees of freedom, we present a new approach that analyses directly the first-order differential system in its original form and extracts the asymptotic behavior of perturbations, without resorting to a second-order reformulation. As a pedagogical illustration, we apply this treatment to the perturbations of Schwarzschild black holes and then show that the standard quasinormal modes can be obtained numerically by solving this first-order system with a spectral method. This new approach paves the way for a generic treatment of the asymptotic behavior of black hole perturbations and the identification of quasinormal modes in theories of modified gravity.

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## I. INTRODUCTION

The oscillations of black holes (BH) have been studied theoretically for several decades. Today, with the first observations of gravitational waves emitted by BH mergers, one can now hope to observe directly these oscillations via their GW signatures, especially in the ringdown phase of the signal when the postmerger black hole relaxes to a Kerr black hole, according to general relativity. One of the major goals of future detections will be to check whether the observed oscillations coincide with the predictions based on general relativity (see e.g., [1,2]). This is also an ideal playground to test alternative theories of gravitation. Indeed, even if the background BH solution may coincide with that of GR, the linear perturbations in general obey different equations of motion.

During the ringdown phase, at least in the linear regime, the GW signal is expected to mainly consist of a superposition of the so-called quasinormal, or resonant, modes (QNMs) which have been excited by the merger and then decay via GW radiation: these modes correspond to the proper oscillation modes of the black hole and are characterized by a complex frequency $\omega$, whose imaginary part quantifies their damping rate.

In the simplest case of nonrotating black holes, i.e., Schwarzschild black holes, the computation of QNMs is based on the classical papers by Regge and Wheeler [3] and later Zerilli [4], who reformulated the linearized Einstein equations in the frequency domain, which are *first-order* with respect to the radial coordinate, as a *second-order* Schrödinger-like equation. This familiar equation, with a specific potential for axial and polar metric perturbations, is the standard starting point for the numerical calculations or semianalytical treatments of QNMs, using for instance well-known methods in quantum mechanics.

Understanding the asymptotic behavior of the perturbations at the horizon and at spatial infinity is crucial for QNMs, which are defined by very specific boundary conditions. Indeed, they correspond to purely outgoing radiation at spatial infinity and ingoing radiation at the horizon. Imposing these specific boundary conditions leads to a discrete set of allowed frequencies.

When the equations of motion of the perturbations are written as a second-order Schrödinger equation, obtaining their asymptotic behavior is immediate, as it simply depends on the asymptotic behavior of the effective potential. In the context of modified gravity however, the problem can become more involved for several reasons. First, the background metric can differ from the standard GR solutions, i.e., be different from Schwarzschild in the nonrotating case. Moreover, modified theories often involve additional fields, such as scalar fields, which increases the number of degrees of freedom and therefore the complexity of the linear equations of motion.

In several interesting cases, the equations of motion can be rewritten as a generalized $N$-dimensional matrix Schrödinger-like system for $N$ fields $\Psi_i$, of the typical form (see e.g., [5])

 

$$f\frac{d}{dr}\left(f\frac{d\Psi_i}{dr}\right) + (\omega^2 - fV_{ij})\Psi_j = 0, \tag{1.1}$$

where $f(r) = 1 - r_s/r$ and the $N \times N$ matrix $V_{ij}$ of radial potentials usually vanishes or becomes a constant diagonal matrix asymptotically. The frequency $\omega$ appears quadratically in the above system, which corresponds to a system of propagation equations if one replaces $\omega$ with $-i\partial/\partial t$. The boundary conditions are still easy to infer from such a differential system.

However, one could also encounter more general situations where such a simple reformulation of the equations of motion is not available or would require an involved and lengthy procedure. Specific examples will be given in a companion paper [6], in the context of degenerate higher-order scalar-tensor (DHOST) theories [7–10] which provide the most general viable set of scalar-tensor theories to date. In those examples, it is not clear whether one can rewrite the polar equations of motion as a second-order Schrödinger-like system of the form (1.1), with its specific dependence on $\omega$. In the specific case of stealth Schwarzschild black holes, a lengthy manipulation of the quadratic Lagrangian for perturbations enabled the authors of [11] to identify master variables, leading to a second-order differential system for the physical degrees of freedom, although of a more complex form than (1.1). To tackle more general situations, it would be very useful to be able to analyze directly the first-order system of equations in its original form and to extract directly from it the asymptotic behavior of perturbations.

The purpose of this paper is to present such a systematic treatment of a general first-order differential system. In order to reach this goal, we use recent developments that appeared in the mathematical literature. These results enable us to determine, via a systematic algorithm, the asymptotic structure of the solutions of a generic first-order differential system. For pedagogical reasons, we use here this algorithm to recover the asymptotic solutions for the axial, or odd-parity, modes and for the polar, or even-parity, modes of the standard Schwarzschild solution. This paper will be completed by a companion paper [12] that applies the same method to a few black hole solutions in DHOST theories.

The outline of the paper is the following. In the next section, we review the standard derivation for the Schwarzschild perturbations, distinguishing as usual the axial and polar modes. In Sec. III, we present our new approach and show explicitly how this new method enables us to recover the usual asymptotic solution, working directly with the first order system. We also show how the quasinormal modes can be computed in this new perspective. We then present, in Sec. IV, the general algorithm, carefully listing the various steps of the algorithm depending on the structure of the system. We give a summary and open some perspectives in the concluding section. A few appendixes contain some additional details.

## II. A SHORT REVIEW ON REGGE-WHEELER AND ZERILLI EQUATIONS

In this section, we review the standard procedure to derive the equations of motion for the perturbations of a Schwarzschild black hole in general relativity, originally obtained by Regge and Wheeler [3] for the axial, or odd-parity, modes and Zerilli [4] for the polar, or even-parity, modes. These equations can be shown to reduce to a Schrödinger-like equation with an effective potential characterising the "dynamics" of the linear perturbations.

### A. Linear perturbations of Einstein equations about the Schwarzschild black hole

We start with the four-dimensional Einstein-Hilbert action in vacuum (with no cosmological constant) for the metric $g_{\mu\nu}$,

$$S[g_{\mu\nu}] = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\,R, \tag{2.1}$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric, $R$ the four-dimensional Ricci scalar and $G_N$ denotes Newton's constant, which actually will not show up in the equations of motion since we are not considering any matter field here.

#### *1. Linearized general relativity*

Given any background metric $\bar{g}_{\mu\nu}$ solution to the Einstein equations, one can introduce the perturbed metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{2.2}$$

where the $h_{\mu\nu}$ denote the linear perturbations of the metric. In order to derive the linear equations of motion that govern the evolution of $h_{\mu\nu}$, one expands the Einstein-Hilbert action (2.1) up to the second order in $h_{\mu\nu}$. The Euler-Lagrange equations associated with the *quadratic* part of this expansion then provide the linearized equations of motion for $h_{\mu\nu}$.

By expanding (2.1), one obtains the following quadratic action for $h_{\mu\nu}$,

$$\begin{aligned} S_{\mathrm{quad}}[h_{\mu\nu}] = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-\bar{g}}\Big\{&-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\bar{R} + \frac{1}{4}h^2\bar{R} \\ &+ hh_{\mu\nu}\bar{R}^{\mu\nu} + 4h_\mu{}^\rho h^{\mu\nu}\bar{R}_{\nu\rho} - 2h^{\mu\nu}h^{\rho\sigma}\bar{R}_{\mu\rho\nu\sigma} \\ &+\frac{1}{2}(\bar{\nabla}_\mu h)(\bar{\nabla}^\mu h) - 2(\bar{\nabla}_\mu h^\mu{}_\nu)(\bar{\nabla}_\rho h_\nu{}^\rho) \\ &- (\bar{\nabla}_\mu h)(\bar{\nabla}_\nu h^{\mu\nu}) + 3(\bar{\nabla}_\nu h_{\mu\rho})(\bar{\nabla}^\rho h^{\mu\nu}) \\ &-\frac{1}{2}(\bar{\nabla}_\rho h_{\mu\nu})(\bar{\nabla}^\rho h^{\mu\nu})\Big\}, \end{aligned} \tag{2.3}$$

where $\bar{R}_{\mu\nu\rho\sigma}$, $\bar{R}_{\mu\nu}$, $\bar{R}$ and $\bar{\nabla}_\mu$ are respectively the Riemann tensor, the Ricci tensor, the Ricci scalar and the covariant derivative associated with the *background* metric $\bar{g}_{\mu\nu}$. The indices are lowered or raised with $\bar{g}_{\mu\nu}$ and $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$ denotes the trace of the metric perturbation. The linearized Einstein equations are then given by the Euler-Lagrange equations of (2.3) and can be written in the form

$$\begin{aligned}\mathcal{E}_{\mu\nu} \equiv\, & \bar{\nabla}_\sigma \bar{\nabla}^\sigma h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h + (\bar{\nabla}_\alpha \bar{\nabla}_\beta h^{\alpha\beta} - \bar{\nabla}_\sigma \bar{\nabla}^\sigma h)\bar{g}_{\mu\nu} \\ & + 2\bar{\nabla}_{(\mu}\bar{\nabla}_\alpha h^\alpha_{\nu)} - 6\bar{\nabla}_\alpha \bar{\nabla}_{(\mu} h^\alpha_{\nu)} + \bar{R}_{\mu\nu} h - \bar{R} h_{\mu\nu} \\ & + \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} h + \bar{R}^{\alpha\beta}\bar{g}_{\mu\nu} h_{\alpha\beta} + 8\bar{R}_{\alpha(\mu} h^\alpha_{\nu)} = 0, \end{aligned} \tag{2.4}$$

where use the standard notation $A_{(\mu\nu)} \equiv (A_{\mu\nu} + A_{\nu\mu})/2$ for the symmetrization of any rank-2 tensor $A_{\mu\nu}$.

Let us now specialize these equations to the case where the background metric is the Schwarzschild metric, expressed as

$$\begin{aligned}\bar{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = & -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 \\ & + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2), \end{aligned} \tag{2.5}$$

where $r_s = 2M_s$ is the Schwarzschild radius, $M_s$ being the mass of the black hole.

Given the spherical symmetry of the background solution, it is convenient to decompose the metric perturbations $h_{\mu\nu}$ into (scalar, vectorial, and tensorial) spherical harmonics that are defined from the standard $Y_{\ell m}(\theta, \varphi)$ functions and their derivatives with respect to $\theta$ and $\varphi$. They are labeled by the two multipole integers $\ell$ and $m$ (with $\ell \geq 0$ and $-\ell \leq m \leq \ell$).

Furthermore, one can distinguish axial and polar modes, which behave differently under the parity transformation $\vec{r} \to -\vec{r}$: the polar, or even-parity, modes transform as $(-1)^\ell$, similarly to the scalar spherical harmonics $Y_{\ell m}(\theta, \varphi)$, whereas the axial, or odd-parity, modes transform as $(-1)^{\ell+1}$. These modes can be treated separately as they are decoupled at linear order. Moreover, we consider here only the modes $\ell \geq 2$. The particular cases of the $\ell = 0$ and $\ell = 1$ modes are briefly discussed in Appendix A 3.

Since the background metric is static, it is also convenient to decompose the time dependence of the perturbations into Fourier modes,

$$F(t, r) = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \tilde{F}(\omega, r) e^{-i\omega t}. \tag{2.6}$$

In the rest of this paper, we will use the same notation for the time-dependent function $F$ and its Fourier transform, as there will be no ambiguity. From a practical point of view, we simply replace every time derivative by a multiplication by $-i\omega$ in the linearized equations, which leads to a system of ordinary differential equations with respect to the radial variable $r$.

In both axial and polar sectors, the equations of motion can be reduced to a system of two first order ordinary differential equations, as we will show below.

### 2. Axial perturbations

We choose the usual Regge-Wheeler gauge [3] to describe the axial modes. As recalled in Appendix A 1, in this gauge the perturbations for $\ell \geq 2$ are parametrized by three families of functions $h_0^{\ell m}$, $h_1^{\ell m}$ and $h_2^{\ell m}$ according to

$$\begin{aligned} h_{t\theta} &= \frac{1}{\sin\theta}\sum_{\ell,m} h_0^{\ell m} \partial_\varphi Y_{\ell m}(\theta, \varphi), \\ h_{t\varphi} &= -\sin\theta \sum_{\ell,m} h_0^{\ell m} \partial_\theta Y_{\ell m}(\theta, \varphi), \\ h_{r\theta} &= \frac{1}{\sin\theta}\sum_{\ell,m} h_1^{\ell m} \partial_\varphi Y_{\ell m}(\theta, \varphi), \\ h_{r\varphi} &= -\sin\theta \sum_{\ell,m} h_1^{\ell m} \partial_\theta Y_{\ell m}(\theta, \varphi), \end{aligned} \tag{2.7}$$

while the other components vanish.

For these perturbations, the equations of motion (2.4) reduce to the following three equations

$$\begin{aligned} \mathcal{E}_{t\theta} &= 2\left(\frac{r_s}{r} - 1 - \lambda\right) h_0(t,r) + r(r - r_s)\frac{\partial^2 h_0}{\partial r^2} \\ &\quad - 2(r - r_s)\frac{\partial h_1}{\partial t} - r(r - r_s)\frac{\partial^2 h_1}{\partial t \partial r} = 0, \\ \mathcal{E}_{r\theta} &= -2\lambda h_1(t,r) - \frac{2r^2}{r - r_s}\frac{\partial h_0}{\partial t} + \frac{r^3}{r - r_s}\frac{\partial^2 h_0}{\partial t \partial r} - \frac{r^3}{r - r_s}\frac{\partial^2 h_1}{\partial t^2} = 0, \\ \mathcal{E}_{\theta\theta} &= 2r_s h_1(t,r) + 2r(r - r_s)\frac{\partial h_1}{\partial r} - \frac{2r^3}{r - r_s}\frac{\partial h_0}{\partial t} = 0, \end{aligned} \tag{2.8}$$

where we have introduced the notation

$$2\lambda \equiv \ell(\ell + 1) - 2, \tag{2.9}$$

as the equations $\mathcal{E}_{t\varphi} = 0$, $\mathcal{E}_{r\varphi} = 0$, $\mathcal{E}_{\varphi\varphi} = 0$ and $\mathcal{E}_{\theta\varphi} = 0$ are identical to the above ones.

Since there are only two independent functions, $h_0$ and $h_1$, one expects one of the above equations to be redundant. This is indeed verified by noting the following relation between the equations (2.8) and their derivatives, written now in the frequency domain,

$$\begin{aligned} & \frac{\mathrm{d}\mathcal{E}_{r\theta}}{\mathrm{d}r} + \frac{i r^2 \omega}{(r - r_s)^2}\mathcal{E}_{t\theta} + \frac{r_s}{r(r - r_s)}\mathcal{E}_{r\theta} \\ & \quad + \frac{\lambda}{r(r - r_s)}\mathcal{E}_{\theta\theta} = 0. \end{aligned} \tag{2.10}$$

This shows that the two equations $\mathcal{E}_{r\theta} = 0$ and $\mathcal{E}_{\theta\theta} = 0$ are sufficient to fully describe the dynamics of axial perturbations. As a consequence, the initial system (2.8) reduces to

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = M(r)Y,$$
$$M(r) = \begin{pmatrix} 2/r & 2i\lambda(r-r_s)/r^3 - i\omega^2 \\ -ir^2/(r-r_s)^2 & -r_s/r(r-r_s) \end{pmatrix}, \qquad (2.11)$$

where the two components of the column vector $Y \equiv {}^T(Y_1, Y_2)$ are $Y_1(r) \equiv h_0(r)$ and $Y_2(r) \equiv h_1(r)/\omega$. Notice that we divided the variable $h_1(r)$ by $\omega$ in the definition of $Y_2$ in order to get a system which does not involve powers of $\omega$ higher than 2, or equivalently which is at most second order in time if one inverts the Fourier transform (2.6).

#### *3. Polar perturbations*

After fixing the gauge, polar perturbations are parametrized by four families of functions $H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$ and $K^{\ell m}$ as shown in Appendix A 2. The nonvanishing metric perturbations then read

$$h_{tt} = A(r)\sum_{\ell,m} H_0^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi),$$
$$h_{tr} = \sum_{\ell,m} H_1^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \qquad (2.12)$$

$$h_{rr} = \frac{1}{A(r)}\sum_{\ell,m} H_2^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi),$$
$$h_{ab} = \sum_{\ell,m} K^{\ell m}(t,r)g_{ab}Y_{\ell m}(\theta,\varphi), \qquad (2.13)$$

where $A(r) \equiv 1 - r_s/r$ is included in the definitions for later convenience, and the indices $a$ or $b$ in the last equation are the angles $\theta$ or $\varphi$.

The linearized Einstein's equations yield seven distinct equations, which can be found in (B1) of Appendix B. After a few manipulation, also discussed in Appendix B, one finds that these equations of motion can be reduced to two first-order equations only. In the frequency domain, they read

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = M(r)Y, \qquad M(r) = \frac{1}{3r_s + 2\lambda r}\begin{pmatrix} \frac{r_s(3r_s+(\lambda-2)r)-2r^4\omega^2}{r(r-r_s)} & \frac{2i(\lambda+1)(r_s+\lambda r)+2ir^3\omega^2}{r^2} \\ \frac{ir(9r_s^2-8\lambda r^2+8(\lambda-1)r_s r)+4ir^5\omega^2}{2(r-r_s)^2} & \frac{2r^4\omega^2-r_s(3r_s+3\lambda r+r)}{r(r-r_s)} \end{pmatrix} \qquad (2.14)$$

where now the two components of $Y$ are defined by $Y_1(r) \equiv K(r)$ and $Y_2(r) \equiv H_1(r)/\omega$. Similarly to the axial sector, the definition of $Y_2$ is motivated by the fact that the resulting system involves at most $\omega^2$ terms.

### B. Schrödinger-like equation and effective potential

In both axial and polar sectors, the equations of motion have been recast in the form of a system consisting simply of two first-order differential equations (with respect to the radial variable), namely (2.11) for axial perturbations and (2.14) for polar perturbations. In both cases, we now recall how this system can be rewritten as a Schrödinger-like equation.

#### *1. From the first order system to the Schrödinger-like equation*

As shown in [3,4], one can rewrite these systems as a single second order (in radial derivatives) Schrödinger-like equation for a unique dynamical variable. Reformulating a first order system of this kind as a Schrödinger equation is, in general, not an easy task because one has to ensure that the Schrödinger equation is second order in time and in space. It requires, in particular, a decoupling of the dynamical variables involved in the original first order system and a "clever" choice for the dynamical variable that should satisfy the second order Schrödinger equation.

In this section, we will describe how this works for the two systems (2.11) and (2.14) which take the general form

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = M(r)Y, \qquad (2.15)$$

where the coefficients of the matrix $M$ are polynomials (of degree at most 2) in $\omega$ and rational functions in $r$.

First, we consider the general (linear) change of vector

$$Y(r) = P(r)\hat{Y}(r), \qquad (2.16)$$

where $\hat{Y}$ is a new column vector and the two dimensional invertible matrix $P$ has not been fixed at this stage. We also define a new radial coordinate $r_*$ and introduce the "Jacobian" of the transformation $n(r) \equiv \mathrm{d}r/\mathrm{d}r_*$. Now, the idea is to show that it is possible to find a matrix $P$ such that the new system satisfied by $\hat{Y}$ takes the canonical form

$$\frac{\mathrm{d}\hat{Y}}{\mathrm{d}r_*} = \begin{pmatrix} 0 & 1 \\ V(r) - \omega^2 & 0 \end{pmatrix}\hat{Y}, \qquad (2.17)$$

where the potential $V(r)$ depends on $r$, but not on $\omega$. Somehow, the first component $\hat{Y}_1$ plays the role of the "momentum" conjugate to the second component $\hat{Y}_2$ which would immediately implies that $\hat{Y}_1$ is the "canonical" variable satisfying the required Schrödinger-like equation

$$\frac{\mathrm{d}^2\hat{Y}_1}{\mathrm{d}r_*^2} + (\omega^2 - V(r))\hat{Y}_1 = 0. \tag{2.18}$$

### 2. Axial modes

Applying this procedure to the system (2.11) for the axial perturbations is rather simple.[1] Indeed, the appropriate transition matrix is given by

$$P(r) = \begin{pmatrix} 1 - r_s/r & r \\ -ir^2/(r - r_s) & 0 \end{pmatrix}, \tag{2.20}$$

while $n(r) = 1 - r_s/r$, which means that $r_*$ coincides with the "tortoise" coordinate,

$$r_* \equiv \int \frac{dr}{1 - r_s/r} = r + r_s \ln(r/r_s - 1). \tag{2.21}$$

Finally the effective potential $V_{\mathrm{odd}}(r)$ for the axial perturbations takes the form

$$V_{\mathrm{odd}}(r) = \left(1 - \frac{r_s}{r}\right)\frac{2(\lambda+1)r - 3r_s}{r^3}. \tag{2.22}$$

Note that this potential vanishes both at spatial infinity ($r \to +\infty$) and at the horizon ($r \to r_s$).

### 3. Polar modes

The case of polar perturbations is slightly more involved. Starting from the system (2.14), we find that the transition matrix leading to a canonical form (2.17) is given by[2]

[1]When one changes variables according to (2.16), the new variable $\hat{Y}$ satisfies the differential equation

$$\frac{\mathrm{d}\hat{Y}}{\mathrm{d}r_*} = \hat{M}\,\hat{Y}, \qquad \hat{M} \equiv n(r)(P^{-1}MP - P^{-1}P'), \tag{2.19}$$

where $P'$ is the derivative of $P$ with respect to $r$, $M$ is the matrix introduced in (2.11) while $\hat{M}$ is the matrix entering in the system (2.17). They take a similar form $M = M_{[0]} + \omega^2 M_{[2]}$ and $\hat{M} = \hat{M}_{[0]} + \omega^2\hat{M}_{[2]}$ where the expressions of $M_{[0]}$, $M_{[2]}$, $\hat{M}_{[0]}$ and $\hat{M}_{[2]}$ are trivially obtained. As $P$ does not depend on $\omega$, the relation between $M$ and $\hat{M}$ translates into the two matricial relations $\hat{M}_{[2]} = n(r)P^{-1}M_{[2]}P$ and $\hat{M}_{[0]} = n(r)(P^{-1}M_{[0]}P - P^{-1}P')$ which can be viewed as 8 equations for the 6 unknowns $n(r)$, $V(r)$ together with the four components of $P$. Interestingly, the system is not overdetermined and admits a solution for $P$ (2.20), for the potential $V(r)$ (2.22) and for the function $n(r)$ which can be shown to be associated with the tortoise coordinate (2.21). Details can be found in the Appendix D of the companion paper.

[2]We follow the same method as the one described in the previous footnote for the axial mode.

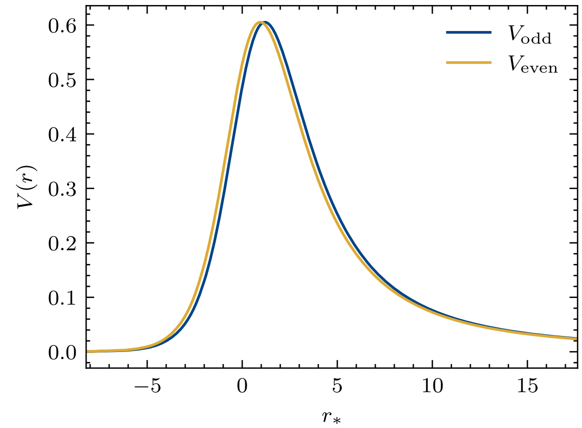


FIG. 1. Illustration of the effective potentials (for axial and polar modes) for a Schwarzschild black hole. The parameters are such that $r_s = 2$ (i.e., the mass of the black is 1 in natural units) and $\ell = 2$ here.

$$P = \begin{pmatrix} \frac{3r_s^2 + 3\lambda r_s r + 2r^2\lambda(\lambda+1)}{2r^2(3r_s + 2\lambda r)} & 1 \\ -i + \frac{ir_s}{2(r-r_s)} + \frac{3ir_s}{3r_s + 2\lambda r} & -\frac{ir^2}{r - r_s} \end{pmatrix}, \tag{2.23}$$

with, in addition, $n(r) = 1 - r_s/r$, which means that $r_*$ is still the tortoise coordinate (2.21). Finally, the corresponding potential $V_{\mathrm{even}}(r)$ reads

$$V_{\mathrm{even}}(r) = \left(1 - \frac{r_s}{r}\right)\frac{9r_s^3 + 18r_s^2 r\lambda + 12r_s r^2\lambda^2 + 8r^3\lambda^2(1+\lambda)}{r^3(3r_s + 2r\lambda)^2}. \tag{2.24}$$

Despite their different analytic forms, we notice in Fig. 1 that the potentials $V_{\mathrm{odd}}(r)$ and $V_{\mathrm{even}}(r)$ are quite similar, although distinct. In fact, there exists an underlying symmetry between these two potentials, further explained in [13], leading to the isospectrality theorem which states that the spectra of axial and polar perturbations are exactly the same.

### 4. Quasinormal modes and boundary conditions

Finding quasinormal modes requires to impose the appropriate boundary conditions: the modes must be outgoing at infinity and ingoing at the horizon.

Since both $V_{\mathrm{odd}}$ and $V_{\mathrm{even}}$ go to zero at infinity and at the horizon, Eq. (2.18) becomes asymptotically

$$\frac{\mathrm{d}^2\hat{X}_1}{\mathrm{d}r_*^2} + \omega^2\hat{X}_1 \approx 0 \quad (r_* \to \pm\infty), \tag{2.25}$$

where $\approx$ is an equality up to subleading corrections.[3] Therefore, at both boundaries, the function $\hat{X}_1$ behaves like

[3]Near the horizon, $V = \mathcal{O}(r - r_s)$ for both potentials, hence we assume $r_s^2\omega^2 \gg r/r_s - 1$. At infinity, $V = \mathcal{O}(1/r^2)$ for both potentials as well, hence we assume $\omega^2 r^2 \gg 1$ in this limit.

$$\hat{X}_1(r) \approx \mathcal{A}e^{i\omega r_*} + \mathcal{B}e^{-i\omega r_*}, \tag{2.26}$$

where $\mathcal{A}$ and $\mathcal{B}$ are integration constants which take different values at the horizon and at infinity.

The physical interpretation of these modes is more transparent if we include their time dependence explicitly, which gives

$$\begin{cases} \hat{X}_1(t,r) \approx \mathcal{A}_{\rm hor} e^{-i\omega(t-r_*)} + \mathcal{B}_{\rm hor} e^{-i\omega(t+r_*)} & \text{when } r \to r_s, \\ \hat{X}_1(t,r) \approx \mathcal{A}_{\infty} e^{-i\omega(t-r_*)} + \mathcal{B}_{\infty} e^{-i\omega(t+r_*)} & \text{when } r \to \infty. \end{cases} \tag{2.27}$$

We can interpret each term as a radially propagating wave: the terms proportional to $\mathcal{A}_{\rm hor}$ and $\mathcal{A}_{\infty}$ are outgoing while the terms proportional to $\mathcal{B}_{\rm hor}$ and $\mathcal{B}_{\infty}$ are ingoing. Imposing a purely outgoing behavior at infinity and a purely ingoing behavior at the horizon, i.e., such that $\mathcal{A}_{\rm hor} = 0$ and $\mathcal{B}_{\infty} = 0$ severely restricts the possible values of $\omega$. These values can be found numerically by integrating the Schrödinger-like equation (see [14] and the reviews [15–18]).

Finally, one can easily deduce the asymptotic expansion of the original gravitational perturbations using the transformations (2.16). For the axial modes, the leading order terms at infinity are thus given by

$$\begin{aligned} h_0(r) &\approx i\omega r(\mathcal{A}_{\infty} e^{i\omega r_*} - \mathcal{B}_{\infty} e^{-i\omega r_*}), \\ h_1(r) &\approx -i\omega r(\mathcal{A}_{\infty} e^{i\omega r_*} + \mathcal{B}_{\infty} e^{-i\omega r_*}), \end{aligned} \tag{2.28}$$

while the leading order terms at the horizon read

$$\begin{aligned} h_0(r) &\approx i\omega r_s(\mathcal{A}_{\rm hor} e^{i\omega r_*} - \mathcal{B}_{\rm hor} e^{-i\omega r_*}), \\ h_1(r) &\approx -\frac{i\omega r_s^2}{\varepsilon}(\mathcal{A}_{\rm hor} e^{i\omega r_*} + \mathcal{B}_{\rm hor} e^{-i\omega r_*}), \end{aligned} \tag{2.29}$$

where we have introduced the variable $\varepsilon \equiv r - r_s$ which satisfies $\varepsilon \ll r_s$ near the horizon.

For the polar modes, the leading order terms at infinity are

$$\begin{aligned} K(r) &\approx i\omega(\mathcal{A}_{\infty} e^{i\omega r_*} - \mathcal{B}_{\infty} e^{-i\omega r_*}), \\ H_1(r) &\approx r\omega^2(\mathcal{A}_{\infty} e^{i\omega r_*} - \mathcal{B}_{\infty} e^{-i\omega r_*}), \end{aligned} \tag{2.30}$$

while the leading terms at the horizon are a bit more involved and read

$$\begin{aligned} K(r) \approx \frac{\lambda + 1 + 2i\omega r_s}{r_s} \mathcal{A}_{\rm hor} e^{i\omega r_*} \\ + \frac{\lambda + 1 - 2i\omega r_s}{r_s} \mathcal{B}_{\rm hor} e^{-i\omega r_*}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} H_1(r) \approx \frac{i r_s \omega(1 - 2i\omega r_s)}{2\varepsilon} \mathcal{A}_{\rm hor} e^{i\omega r_*} \\ + \frac{i r_s \omega(1 + 2i\omega r_s)}{2\varepsilon} \mathcal{B}_{\rm hor} e^{-i\omega r_*}. \end{aligned} \tag{2.32}$$

In the next section, we will recover these asymptotic behaviors in a completely different way.

## III. FIRST ORDER APPROACH TO SCHWARZSCHILD PERTURBATIONS

As we have seen in Sec. II B, finding a (second-order) Schrödinger-like equation for the metric perturbations starting from the Einstein equations requires some manipulations of the equations of motion and an appropriate choice of the function that verifies the Schrödinger-like equation.

The rest of this paper will be devoted to obtaining the asymptotic behaviors of the perturbations by using a different method. Although this is of course not necessary for the perturbations of Schwarzschild in general relativity, our method may prove to be very useful in situations where a Schrödinger-like system is not obvious to find or even impossible to reach. In such a case, one would need an alternative method to determine the asymptotic limits of the solutions of the system, and from them, to compute the quasinormal modes.

The general method will be described in a systematic way in the next section. As the general procedure is somewhat tedious, we have preferred to present it first, in a pedestrian way, for the perturbations of Schwarzschild. A more mathematically minded reader might prefer to jump directly to the next section and later come back to this section to find a particular application of the general method.

### A. Method

Ignoring the traditional Schrödinger reformulation, we now go back to the original first-order system given in (2.11) or (2.14). Schematically, we thus have a first-order system of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = M(r)Y, \tag{3.1}$$

where $Y(r)$ is a column vector and $M(r)$ a square matrix. In order to study the system at spatial infinity, say, i.e., when $r \to \infty$, one can expand the matrix $M(r)$ in powers of $r$,

$$M(r) = M_p r^p + \cdots + M_0 + M_{-1}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{3.2}$$

where all the matrix coefficients $M_i$ are $r$-independent. We stop here the expansion at order $1/r$, which is sufficient for the simplest cases, but higher orders might be needed in general.

If *all* matrices $M_i$ are *diagonal*, it is immediate to integrate the truncated system, which then consists of $n$ ordinary differential equations of the form

$$y'(r) = \left(\lambda_p r^p + \cdots + \lambda_0 + \frac{\mu}{r}\right) y(r), \qquad (3.3)$$

whose solution is

$$y(r) = y_0 e^{q(r)} r^{\mu}, \quad q(r) = \frac{\lambda_p}{p+1} r^{p+1} + \cdots + \lambda_0 r. \qquad (3.4)$$

Putting together these $n$ solutions, we thus get the solution to the system (3.1), assuming all matrices $M_i$ in (3.2) are diagonal, in the form

$$Y(r) = e^{\Upsilon(r)} r^{\mathbf{\Delta}} \mathbf{F}(r) Y_0 \qquad (3.5)$$

where $Y_0$ is a constant vector, corresponding to the $n$ integration constants, $\Upsilon$ is a diagonal matrix whose coefficients are polynomials of degree at most $p+1$, $\mathbf{\Delta}$ is a constant diagonal matrix and $\mathbf{F}(r)$ is a matrix which is regular at infinity (i.e., whose limit is finite).

Of course, in general, the matrices $M_i$ are not diagonal but, remarkably, it is always possible to transform the truncated system into a fully diagonal system, in a *finite* number of steps following an algorithm introduced in [19–22], which we will present in full details in the next section.

At each step in the algorithm, one introduces a new vector $\tilde{Y}$, related to the vector $Y$ of the previous step by

$$Y = P\tilde{Y},$$

where $P$ is an invertible matrix so that the previous system (3.1) is transformed into a new, but equivalent, system of the form

$$\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}r} = \tilde{M}(r)\tilde{Y}, \qquad \tilde{M}(r) \equiv P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}. \qquad (3.6)$$

The idea is then to choose an appropriate transition matrix $P$ at each step in order to diagonalize, order by order, the matrices that appear in the expansion of $M$. Once all the matrices are diagonalized, one can integrate directly the diagonal system, as we have seen earlier, and obtain the general asymptotic solution of the system.

For the asymptotic behavior near the horizon, one proceeds in the same way by noting that the variable $z = 1/(r - r_s)$ goes to infinity when $r \to r_s$. In the rest of this section, we will illustrate the algorithm by considering in turn the asymptotic behaviors of the axial and polar modes.

## B. Axial modes

The analysis of the asymptotic behavior of the first order system (2.11) is relatively simple and instructive. We recall that the system is of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = M(r)Y, \qquad (3.7)$$

with

$$Y(r) \equiv \begin{pmatrix} h_0(r) \\ h_1(r)/\omega \end{pmatrix},$$

$$M(r) \equiv \begin{pmatrix} 2/r & 2i\lambda(r-r_s)/r^3 - i\omega^2 \\ -ir^2/(r-r_s)^2 & -r_s/r(r-r_s) \end{pmatrix}. \qquad (3.8)$$

### *1. Asymptotic analysis at spatial infinity*

We first study the asymptotic behavior at spatial infinity, i.e., when $r \to \infty$. The asymptotic expansion of the matrix $M(r)$ at large $r$ reads

$$M(r) = M_0 + \frac{1}{r}M_{-1} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad M_0 \equiv -i\begin{pmatrix} 0 & \omega^2 \\ 1 & 0 \end{pmatrix},$$
$$M_{-1} \equiv 2\begin{pmatrix} 1 & 0 \\ -ir_s & 0 \end{pmatrix}. \qquad (3.9)$$

The leading term $M_0$ is diagonalizable and one can go to a basis where it is diagonal, by introducing the new vector $Y^{(1)}$ defined by

$$Y \equiv P_{(1)} Y^{(1)}, \qquad P_{(1)} = \begin{pmatrix} \omega & -\omega \\ 1 & 1 \end{pmatrix}. \qquad (3.10)$$

According to (3.6), this gives the new system

$$\frac{\mathrm{d}Y^{(1)}}{\mathrm{d}r} = M^{(1)} Y^{(1)},$$
$$M^{(1)}(r) = M_0^{(1)} + \frac{1}{r}M_{-1}^{(1)} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (3.11)$$

with

$$M_0^{(1)} \equiv \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix},$$
$$M_{-1}^{(1)} \equiv \begin{pmatrix} -i\omega r_s + 1 & i\omega r_s - 1 \\ -i\omega r_s - 1 & i\omega r_s + 1 \end{pmatrix}. \qquad (3.12)$$

We need some extra work to diagonalize the next-to-leading order matrix $M_{-1}^{(1)}$ while keeping the leading order matrix diagonal.

This can be achieved by introducing a new vector $Y^{(2)}$ defined by

$$Y^{(1)} \equiv P_{(2)} Y^{(2)}, \qquad P_{(2)} = I + \frac{1}{r}\Xi, \qquad (3.13)$$

where $I$ is the identity matrix and $\Xi$ a constant matrix. Indeed, it is immediate to see that such a change of variable leads to the equivalent differential system,

$$\frac{\mathrm{d}Y^{(2)}}{\mathrm{d}r} = M^{(2)}Y^{(2)},$$
$$M^{(2)}(r) = M_0^{(2)} + \frac{1}{r}M_{-1}^{(2)} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (3.14)$$

with

$$M_0^{(2)} = M_0^{(1)}, \qquad M_{-1}^{(2)} = M_{-1}^{(1)} + [M_0^{(1)}, \Xi]. \qquad (3.15)$$

The leading matrix remains unchanged while one can easily find a matrix $\Xi$ so that $M_{-1}^{(2)}$ is diagonal. Notice that $\Xi$ appears in (3.15) only in a commutator with the diagonal matrix $M_0^{(1)}$, hence the diagonal part of $\Xi$ is irrelevant and we can already fix the diagonal terms of $\Xi$ to 0. In this case, the solution to (3.15) with $M_{-1}^{(2)}$ diagonal is unique and given by

$$\Xi = \frac{1}{2i\omega}\begin{pmatrix} 0 & i\omega r_s - 1 \\ i\omega r_s + 1 & 0 \end{pmatrix}. \qquad (3.16)$$

We have thus managed to obtain a fully diagonalized system, up to order $1/r$, with the matrix

$$M^{(2)}(r) = \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 1 - i\omega r_s & 0 \\ 0 & 1 + i\omega r_s \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right). \qquad (3.17)$$

This system can be immediately integrated in the form (3.5), and the asymptotic solution reads

$$Y^{(2)}(r) = (1 + \mathcal{O}(1/r))\begin{pmatrix} c_- e^{-i\omega r} r^{1-i\omega r_s} \\ c_+ e^{+i\omega r} r^{1+i\omega r_s} \end{pmatrix}, \qquad (3.18)$$

where $c_\pm$ are integration constants. Taking into account the time dependency $e^{-i\omega t}$ of the modes, the two components $Y_\mp^{(2)}$ of $Y^{(2)}$ are of the form

$$e^{-i\omega t}Y_\mp^{(2)}(r) = (1 + \mathcal{O}(1/r))c_\mp r e^{-i\omega(t \pm (r + r_s \ln r))}$$
$$= c_\mp (r + \mathcal{O}(1)) e^{-i\omega(t \pm r_*)}, \qquad (3.19)$$

where it is convenient to use the "tortoise" coordinate $r_*$, introduced in (2.21), noting that

$$r_* = r + r_s \ln(r/r_s - 1) = r + r_s \ln r + \mathcal{O}(1). \qquad (3.20)$$

As a consequence, one can identify $Y_-^{(2)}$ as an ingoing mode and $Y_+^{(2)}$ as an outgoing mode at spatial infinity.

Finally, we can return to the original vector $Y$ thanks to the transformation

$$Y = P_{(1)}P_{(2)}Y^{(2)} = \begin{pmatrix} \omega & -\omega \\ 1 & 1 \end{pmatrix}\left(1 + \frac{\Xi}{r}\right)Y^{(2)}, \qquad (3.21)$$

in order to obtain the asymptotic expansion of the two original gravitational perturbations $h_0$ and $h_1$ at spatial infinity,

$$h_0(r) = \omega(c_- e^{-i\omega r_*} - c_+ e^{+i\omega r_*})(r + \mathcal{O}(1)), \qquad (3.22)$$

$$h_1(r) = \omega(c_- e^{-i\omega r_*} + c_+ e^{+i\omega r_*})(r + \mathcal{O}(1)). \qquad (3.23)$$

One can immediately check that these expressions agree with the asymptotic expansion (2.28) obtained from the Schrödinger-like equation (with $c_- = -i\mathcal{B}_\infty$ and $c_+ = -i\mathcal{A}_\infty$).

### 2. Asymptotic analysis near the black hole horizon

Let us now study the behavior of the axial modes near the horizon. In this case, it is convenient to introduce the new radial variable $\varepsilon \equiv r - r_s$ and expand the matrix $M$ for the system (3.8) in powers of $\varepsilon$. One finds[4]

$$M(\varepsilon) = \frac{1}{\varepsilon^2}M_2 + \frac{1}{\varepsilon}M_1 + M_0 + \mathcal{O}(\varepsilon), \qquad (3.25)$$

with the matrix coefficients

$$M_2 \equiv \begin{pmatrix} 0 & 0 \\ -ir_s^2 & 0 \end{pmatrix}, \qquad M_1 \equiv \begin{pmatrix} 0 & 0 \\ -2ir_s & -1 \end{pmatrix},$$
$$M_0 \equiv \begin{pmatrix} 2/r_s & -i\omega^2 \\ -i & 1/r_s \end{pmatrix}. \qquad (3.26)$$

An important difference with the previous situation is that the leading term $M_2$ is no longer diagonalizable but nilpotent instead. We thus need to first perform a transformation that yields a diagonalizable leading matrix, taking advantage of the derivative term in (3.6). This can be done with the transformation

$$Y \equiv P_{(1)}Y^{(1)}, \qquad P_{(1)}(\varepsilon) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix}, \qquad (3.27)$$

leading to the new system

[4]Note that $\varepsilon$ goes to zero here, in contrast to the previous case where the variable $r$ was going to infinity. One could work in a fully analogous system by using the variable $z = 1/\varepsilon$, with the system

$$\frac{dY}{dz} = \tilde{M}(z)Y, \quad \tilde{M} = -\frac{1}{z^2}M(z^{-1}) = -M_2 - M_1\frac{1}{z} - M_0\frac{1}{z^2}. \quad (3.24)$$

In the present case, one must push the expansion up to order $1/z^2$ because the leading matrix $M_2$ is nilpotent.

$$\frac{\mathrm{d}Y^{(1)}}{\mathrm{d}\varepsilon} = M^{(1)}Y^{(1)},$$
$$M^{(1)}(\varepsilon) = -\frac{1}{\varepsilon}\begin{pmatrix} 0 & i\omega^2 \\ ir_s^2 & 0 \end{pmatrix} + \mathcal{O}(1). \qquad (3.28)$$

The transformation (3.27) has eliminated the term in $1/\varepsilon^2$ in the expansion and the leading term $M_1^{(1)}$ is now diagonalizable, so that only the expansion of $M^{(1)}$ up to order $1/\varepsilon$ is required (see discussion in the footnote). It is worth noticing that $M_1^{(1)}$ receives contributions from $M_2$, $M_1$ and $M_0$. In particular, some of its coefficients involve the frequency $\omega$ which is originally present only in $M_0$.

The final step of the analysis consists in diagonalizing the system (3.28), via the transformation

$$Y^{(1)} = P_{(2)}Y^{(2)}, \qquad P_{(2)} \equiv \begin{pmatrix} \omega & -\omega \\ r_s & r_s \end{pmatrix}, \qquad (3.29)$$

leading to

$$\frac{\mathrm{d}Y^{(2)}}{\mathrm{d}\varepsilon} = M^{(2)}Y^{(2)},$$
$$M^{(2)}(\varepsilon) \equiv \frac{1}{\varepsilon}\begin{pmatrix} -i\omega r_s & 0 \\ 0 & i\omega r_s \end{pmatrix} + \mathcal{O}(1). \qquad (3.30)$$

Integrating this equation yields

$$Y^{(2)}(\varepsilon) = (1+\mathcal{O}(\varepsilon))\begin{pmatrix} c_-\varepsilon^{-i\omega r_s} \\ c_+\varepsilon^{+i\omega r_s} \end{pmatrix}$$
$$= (1+\mathcal{O}(\varepsilon))\begin{pmatrix} c_-e^{-i\omega r_*} \\ c_+e^{+i\omega r_*} \end{pmatrix}, \qquad (3.31)$$

where we have again expressed the result in terms of the tortoise coordinate $r_*$, which behaves as $r_* = r_s \ln\varepsilon + \mathcal{O}(1)$ near the horizon. One can immediately recognize the ingoing and outgoing modes at the horizon.

Finally, one can return to the original functions, via $Y = P_{(1)}P_{(2)}Y^{(2)}$, and derive the expressions

$$h_0(r) = \omega(c_-e^{-i\omega r_*} - c_+e^{+i\omega r_*})(1+\mathcal{O}(\varepsilon)), \qquad (3.32)$$
$$h_1(r) = \frac{\omega r_s}{\varepsilon}(c_-e^{-i\omega r_*} + c_+e^{+i\omega r_*})(1+\mathcal{O}(\varepsilon)), \qquad (3.33)$$

which coincide with the asymptotic expansions (2.29) obtained from the Schrödinger-like equation (with $c_- = -ir_s\mathcal{B}_{\rm hor}$, $c_+ = -ir_s\mathcal{A}_{\rm hor}$).

## C. Polar modes

The dynamics of the polar perturbations is described by the first-order system (2.14), of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = M(r)Y, \quad \text{with} \quad Y(r) \equiv \begin{pmatrix} K(r) \\ H_1(r)/\omega \end{pmatrix}, \qquad (3.34)$$

and the matrix

$$M(r) = \frac{1}{3r_s + 2\lambda r}\begin{pmatrix} \frac{r_s(3r_s+(\lambda-2)r)-2r^4\omega^2}{r(r-r_s)} & \frac{2i(\lambda+1)(r_s+\lambda r)+2ir^3\omega^2}{r^2} \\ \frac{ir(9r_s^2-8\lambda r^2+8(\lambda-1)r_s r)+4ir^5\omega^2}{2(r-r_s)^2} & \frac{2r^4\omega^2-r_s(3r_s+3\lambda r+r)}{r(r-r_s)} \end{pmatrix}. \qquad (3.35)$$

### *1. Asymptotic analysis at spatial infinity*

Expanding (3.35) in powers of $r$, one gets

$$M(r) = \begin{pmatrix} 0 & 0 \\ \frac{i\omega^2}{\lambda} & 0 \end{pmatrix} r^2 + \begin{pmatrix} -\frac{\omega^2}{\lambda} & 0 \\ \frac{ir_s\omega^2(4\lambda-3)}{2\lambda^2} & \frac{\omega^2}{\lambda} \end{pmatrix} r + \begin{pmatrix} -\frac{(2\lambda-3)r_s\omega^2}{2\lambda^2} & \frac{i\omega^2}{\lambda} \\ -2i + \frac{3i(4\lambda^2-4\lambda+3)r_s^2\omega^2}{4\lambda^3} & \frac{(2\lambda-3)r_s\omega^2}{2\lambda^2} \end{pmatrix}$$
$$+ \frac{1}{r}\begin{pmatrix} -\frac{(4\lambda^2-6\lambda+9)r_s^2\omega^2}{4\lambda^3} & -\frac{3ir_s\omega^2}{2\lambda^2} \\ \frac{i(8(1-2\lambda)\lambda^3 r_s-(27-4\lambda(\lambda(8\lambda-9)+9))r_s^3\omega^2)}{8\lambda^4} & \frac{(4\lambda^2-6\lambda+9)r_s^2\omega^2}{4\lambda^3} \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right). \qquad (3.36)$$

In contrast with the axial modes at spatial infinity, the leading matrix is of order $r^2$ and is nilpotent. So, in principle, one needs to apply a procedure similar to the near-horizon analysis of axial modes, which will be presented in full generality in the next section, and then diagonalize in turn all subsequent orders. All this involves many steps which are straightforward but rather tedious to describe.

To shorten our discussion, we provide directly the transformation that combines all these intermediate steps, given by

$$Y = P\tilde{Y}, \qquad P = \begin{pmatrix} \mathcal{S}+\mathcal{T} & \mathcal{S}-\mathcal{T} \\ \mathcal{U}-\mathcal{V} & \mathcal{U}+\mathcal{V} \end{pmatrix}, \tag{3.37}$$

with the functions

$$\begin{aligned} \mathcal{S}(r) &\equiv \frac{i(r-r_s)((2\lambda-3)r_s+4\lambda r)}{4\lambda r} + \frac{i\lambda}{2r\omega^2}, \\ \mathcal{T}(r) &\equiv \frac{(1-2\lambda)r_s + 2(1+2\lambda)r}{4r\omega}, \\ \mathcal{U}(r) &\equiv r^2 + \frac{2\lambda-3}{4\lambda}r_s r, \qquad \mathcal{V}(r) \equiv \frac{ir}{2\omega}. \end{aligned} \tag{3.38}$$

This leads to the new system

$$\begin{aligned} \frac{\mathrm{d}\tilde{Y}}{\mathrm{d}r} &= \tilde{M}(r)\tilde{Y}, \\ \tilde{M}(r) &= \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} + \begin{pmatrix} -1+i\omega r_s & 0 \\ 0 & -1-i\omega r_s \end{pmatrix}\frac{1}{r} \\ &\quad + \mathcal{O}\left(\frac{1}{r^2}\right), \end{aligned} \tag{3.39}$$

which is diagonal and whose solution is

$$\begin{aligned} \tilde{Y}(r) &= \begin{pmatrix} c_- e^{-i\omega r} r^{-1-i\omega r_s} \\ c_+ e^{+i\omega r} r^{-1+i\omega r_s} \end{pmatrix}(1+\mathcal{O}(1/r)) \\ &= \frac{1}{r}\begin{pmatrix} c_- e^{-i\omega r_*} \\ c_+ e^{+i\omega r_*} \end{pmatrix}(1+\mathcal{O}(1/r)). \end{aligned} \tag{3.40}$$

This result is very similar to that obtained for axial perturbations (3.18), even though the asymptotic expansion of the matrix $M$ is rather different. In terms of the original functions, we find

$$\begin{aligned} K(r) &= \frac{i}{\omega}H_1(r) \\ &= i(c_- e^{-i\omega r_*} + c_+ e^{+i\omega r_*})(1+\mathcal{O}(1/r)), \end{aligned} \tag{3.41}$$

which agree with (2.30) (with $c_- = -\omega\mathcal{B}_\infty$ and $c_+ = \omega\mathcal{A}_\infty$).

### 2. Asymptotic analysis at the black hole horizon

We finally turn to the near-horizon behavior of polar modes. The expansion of the matrix (3.35) in terms of the small parameter $\varepsilon \equiv r - r_s$ yields

$$\begin{aligned} M(\varepsilon) &= \frac{1}{\varepsilon^2}M_2 + \frac{1}{\varepsilon}M_1 + M_0 + \mathcal{O}(\varepsilon), \qquad M_2 = \begin{pmatrix} 0 & 0 \\ \gamma_2 & 0 \end{pmatrix}, \\ M_1 &= \begin{pmatrix} \alpha_1 & 0 \\ \gamma_1 & \delta_1 \end{pmatrix}, \qquad M_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}, \end{aligned} \tag{3.42}$$

where only a few of the coefficients $\alpha_I$, $\beta_I$ and $\gamma_I$ will be needed explicitly.

Once more, the dominant $M_2$ is a nilpotent matrix and, as in the axial case, we use the transformation

$$Y = P_{(1)}Y^{(1)} \quad \text{with} \quad P_{(1)}(\varepsilon) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix}, \tag{3.43}$$

which gives the new system

$$\begin{aligned} \frac{\mathrm{d}Y^{(1)}}{\mathrm{d}\varepsilon} &= M^{(1)}Y^{(1)}, \\ M^{(1)}(\varepsilon) &= \frac{1}{\varepsilon}\begin{pmatrix} \alpha_1 & \beta_0 \\ \gamma_2 & 1+\delta_1 \end{pmatrix} + \mathcal{O}(1), \end{aligned} \tag{3.44}$$

with the coefficients

$$\begin{aligned} \alpha_1 &= -(1+\delta_1) = \frac{1+\lambda-2r_s^2\omega^2}{3+2\lambda}, \\ \beta_0 &= \frac{2i}{r_s^2}\frac{(\lambda+1)^2+r_s^2\omega^2}{3+2\lambda}, \qquad \gamma_2 = \frac{ir_s^2}{2}\frac{1+4r_s^2\omega^2}{3+2\lambda}. \end{aligned} \tag{3.45}$$

The leading matrix can now be diagonalized via the transformation

$$\begin{aligned} Y^{(1)} &= P_{(2)}Y^{(2)}, \quad \text{with} \quad P_{(2)} = \begin{pmatrix} \alpha-\beta & \alpha+\beta \\ 1 & 1 \end{pmatrix} \quad \text{and} \\ \alpha &= \frac{\alpha_1}{\gamma_2}, \qquad \beta = \frac{i\omega r_s}{\gamma_2}. \end{aligned} \tag{3.46}$$

leading to the system

$$\begin{aligned} \frac{\mathrm{d}Y^{(2)}}{\mathrm{d}\varepsilon} &= M^{(2)}Y^{(2)}, \\ M^{(2)} &= \frac{1}{\varepsilon}\begin{pmatrix} -i\omega r_s & 0 \\ 0 & i\omega r_s \end{pmatrix} + \mathcal{O}(1). \end{aligned} \tag{3.47}$$

Note that this expression is extremely simple and does not involve $\lambda$, as expected, even though it appears explicitly in $M^{(1)}$. We obtain immediately the asymptotic behavior of $X^{(2)}$ near the horizon

$$Y^{(2)}(\varepsilon) = (1+\mathcal{O}(\varepsilon))\begin{pmatrix} c_- e^{-i\omega r_*} \\ c_+ e^{+i\omega r_*} \end{pmatrix}, \tag{3.48}$$

which reproduces the same result as for the axial mode (3.31). In terms of the original gravitational functions $H_1(r)$ and $K(r)$, using the transformation $Y = P_{(1)}P_{(2)}Y^{(2)}$, we recover the result (2.32), with

$$\begin{aligned} c_+ &= \frac{i}{2}r_s(1-2i\omega r_s)\mathcal{A}_{\mathrm{hor}} \\ c_- &= -\frac{i}{2}r_s(1+2i\omega r_s)\mathcal{B}_{\mathrm{hor}}. \end{aligned} \tag{3.49}$$

This completes our study of all asymptotic behaviors of Schwarzschild perturbations, demonstrating that one can recover the standard results directly from the linearized Einstein's equations, without resorting to the Schrödinger-like reformulation of the system.

### D. Quasinormal modes

Several powerful numerical methods have been developed for the computation of quasinormal modes when the system is of the form (1.1), but these methods cannot be directly applied to the more general first-order system we are dealing with. In this section, we use a simple numerical method to show how the Schwarzschild quasinormal modes can be recovered numerically, using directly the first-order system instead of the Schrödinger equation (2.18). We restrict ourselves to the polar modes and consider the system (3.34)–(3.35). The computation of the axial quasinormal modes would be completely similar.

By definition of the quasinormal modes, we impose that the solutions are outgoing at spatial infinity and ingoing at the horizon, which means, using the results of Sec. III B, that the two components of the vector $Y$ satisfy

$$Y_1(r) \equiv K(r) = K_\infty(r) e^{i\omega r_*} = \tilde{K}_\infty(r) e^{i\omega r} r^{i\omega r_s} \qquad (3.50)$$

$$= K_{\rm h}(r) e^{-i\omega r_*} = \tilde{K}_{\rm h}(r)(r - r_s)^{-i\omega r_s}, \qquad (3.51)$$

where $K_\infty$ (and $\tilde{K}_\infty$) is finite at infinity while $K_{\rm h}$ (and $\tilde{K}_{\rm h}$) is finite at the horizon, and also

$$Y_2(r) \equiv H_1(r)/\omega = H_\infty(r) r e^{i\omega r_*}$$
$$= \tilde{H}_\infty(r) e^{i\omega r} r^{1+i\omega r_s} \qquad (3.52)$$

$$= H_{\rm h}(r)\varepsilon^{-1} e^{-i\omega r_*} = \tilde{H}_{\rm h}(r)(r - r_s)^{-1-i\omega r_s}, \qquad (3.53)$$

where again $H_\infty$ (and $\tilde{H}_\infty$) is finite at infinity while $H_{\rm h}$ (and $\tilde{H}_{\rm h}$) is finite at the horizon.

Therefore, we look for solutions of (3.34)–(3.35) using the ansatz

$$K(r) = e^{i\omega r} r^{i\omega r_s} \left(\frac{r - r_s}{r}\right)^{-i\omega r_s} f_K(r),$$
$$H_1(r) = e^{i\omega r} r^{1+i\omega r_s} \left(\frac{r - r_s}{r}\right)^{-1-i\omega r_s} f_H(r), \qquad (3.54)$$

where the functions $f_K$ and $f_H$ are supposed to be finite (hence bounded) both at the horizon and at spatial infinity, in agreement with the required boundary conditions. Furthermore, we introduce the new variable

$$u = \frac{2r_s}{r} - 1, \qquad (3.55)$$

so that the black hole horizon is located at $u = 1$ and spatial infinity at $u = -1$. Each function entering in the Eqs. (3.54) is now treated as a function of $u$ and the system of equations (3.34)–(3.35) can be expressed in the form

$$\mathcal{P}_{11}(u) f_K(u) + \mathcal{P}_{12}(u) f_H(u) + \mathcal{Q}_1(u) f'_K(u) = 0,$$
$$\mathcal{P}_{21}(u) f_K(u) + \mathcal{P}_{22}(u) f_H(u) + \mathcal{Q}_2(u) f'_H(u) = 0, \qquad (3.56)$$

where a prime denotes here a derivative with respect to $u$, and the functions $\mathcal{P}_{ij}$ and $\mathcal{Q}_i$ are polynomials in $u$. This is possible because the matrix $M$ given in (3.35) contains only rational fractions of $r$.

In order to solve the system (3.56) numerically, we adapt the spectral method presented in [23] and we decompose $f_K(u)$ and $f_H(u)$ onto a basis of Chebyshev polynomials. The facts that the functions $\mathcal{P}_{ij}$ and the $\mathcal{Q}_i$ are polynomials (hence $C^\infty$-functions) and the Chebyshev polynomials are bounded at the boundaries ensure the boundedness of $f_K(u)$ and $f_H(u)$ which is sufficient to enforce the required boundary conditions. This is called a "behavioral" boundary condition [24].

Then, any smooth and continuous complex-valued function $g(u)$ defined on the interval $[-1, 1]$ can be written as an infinite sum of Chebyshev polynomials $T_n(u)$,

$$g(u) = \sum_{n=0}^{\infty} g_n T_n(u), \qquad (3.57)$$

where $g_n$ are complex coefficients. We can approximate the function $g$ by truncating this series at a given order $N$, the approximation getting better as $N$ is increased. Hence, we decompose the two functions $f_K$ and $f_H$ as follows,

$$f_K(u) \approx \sum_{n=0}^{N} \alpha_n T_n(u), \qquad f_H(u) \approx \sum_{n=0}^{N} \beta_n T_n(u), \qquad (3.58)$$

where $\alpha_n$ and $\beta_n$ are complex coefficients. Notice that the symbol $\approx$ means that we truncated the series at an order $N$, then the equality is not exact.

The next step is to express the differential system (3.56) as a linear system for the coefficients $\alpha_n$ and $\beta_n$, which is always possible due to fundamental relations satisfied by Chebyshev polynomials.[5] As a consequence, the differential system (3.56) can be recast as the following system of algebraic equations

$$M_N(\omega) V_N(\alpha_n, \beta_n) = 0, \qquad (3.60)$$

[5]The Chebyshev polynomials satisfy the properties

$$T'_n(u) = \sum_{m-n=2k+1} \frac{2m}{1+\delta_{n0}} T_m(u),$$
$$(uT_n)(u) = \sum_m \frac{1}{2}((1+\delta_{n,1})\delta_{n-1,m} + \delta_{n+1,m}) T_m(u), \qquad (3.59)$$

where $\delta_{m,n}$ is the Kronecker symbol and $k \in \mathbb{N}$ in the first sum.

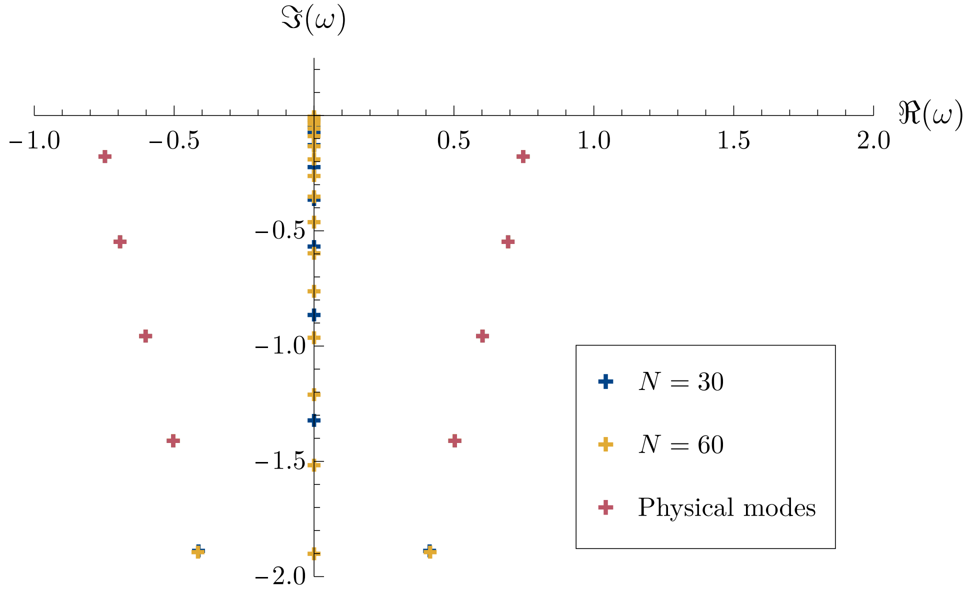


FIG. 2. Quasinormal modes numerically found by *Mathematica* for $r_s = 1$ and $\ell = 2$ ($\lambda = 2$). The blue dots are generalized eigenvalues for $N = 30$, the orange dots generalized eigenvalues for $N = 60$, and the red dots are the modes detected (eigenvalues that change by a factor $10^{-3}$ or less). All the dots present on the imaginary axis correspond to spurious modes. We observe a symmetry with respect to the imaginary axis. The positions of the first modes are $\omega_0 = \pm 0.747 - 0.178i$, $\omega_1 = \pm 0.693 - 0.548i$ and $\omega_2 = \pm 0.602 - 0.957i$.

where $M_N$ is a $2(N+1) \times 2(N+1)$ matrix whose expansion in powers of $\omega$ reads

$$M_N(\omega) = M_{N[0]} + M_{N[1]}\omega + M_{N[2]}\omega^2, \qquad (3.61)$$

while the $2(N+1)$-dimensional vector $V_N(\alpha_n, \beta_n)$ is such that

$${}^T V_N(\alpha_n, \beta_n) \equiv (\alpha_0, \ \cdots, \ \alpha_N, \ \beta_0, \ \cdots, \ \beta_N). \qquad (3.62)$$

Following [23], we can reformulate this system as

$$\tilde{M}_N(\omega)\tilde{V}_N(\alpha_n, \beta_n) = 0, \qquad (3.63)$$

where the matrix $\tilde{M}_N$ is now of dimension $4(N+1)$ and defined by

$$\tilde{M}_N = \tilde{M}_{N[0]} + \tilde{M}_{N[1]}\omega \quad \text{and} \quad \tilde{M}_{N[0]} = \begin{pmatrix} M_{N[0]} & M_{N[1]} \\ 0 & I \end{pmatrix},$$
$$\tilde{M}_{N[1]} = \begin{pmatrix} 0 & M_{N[2]} \\ -I & 0 \end{pmatrix}. \qquad (3.64)$$

Finding the values of $\omega$ such that the system (3.63) is nontrivial is called a *generalized eigenvalues problem* and can be done by a numerical engine such as *Mathematica* or SciPy. In practice, we have computed the eigenvalues for different values of $N$ and identified the ones that (almost) coincide when $N$ varies. There are also nonphysical spurious modes (due to the finite size approximation), which strongly depend on $N$ and must be discarded. The quasinormal modes thus identified, plotted in Fig. 2, coincide with the well-known first quasinormal modes of Schwarzschild.

This result demonstrates that it is feasible to compute quasinormal modes directly from the first-order system, even if our numerical approach is rather crude and gives a very low precision with respect to the sophisticated methods used in the traditional approach.

## IV. GENERAL ANALYSIS

As we have seen in the previous section, it is possible to compute the quasinormal modes of black holes in general relativity without reformulating the linearized Einstein equations in terms of a Schrödinger-like equation. The advantage of this method is that it can be straightforwardly generalized to the study of black holes in theories of modified gravity where it might be difficult or impossible to reduce the linearized equations to a Schrödinger-like form.

In this section, we present a systematic algorithm for a generic first-order system of the form (3.7), which has been developed in the mathematics literature, first in [19] and more recently in [20–22,25,26]. The various steps of the algorithm presented in this section are summarized in the flowchart diagram depicted in Appendix C.

### A. Asymptotic solution: overview

We consider a general system of first-order ordinary differential equations of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = M(z)Y, \tag{4.1}$$

where $Y$ is a $n$-dimensional column vector, $M$ an $n\times n$-dimensional matrix and $z$ a real variable defined in some interval. In the following, we will consider only the asymptotic behavior when $z\to+\infty$, but it is straightforward to extend the algorithm near a finite value $z_0$ where the system is singular, by a suitable change of the variable $z$.

We then assume that one can expand $M$ in powers of $z$, up to some order (depending on the required precision of the asymptotic expansion) as follows,

$$\begin{aligned} M(z) &= M_r z^r + \cdots + M_0 + \ldots M_{r-f} z^{r-f} + \mathcal{O}(z^{r-f-1}) \\ &= z^r \sum_{k=0}^{f} M_{r-k} z^{-k} + \mathcal{O}(z^{r-f-1}), \end{aligned} \tag{4.2}$$

where the integer $r$ is called the Poincaré rank of the system, and the $M_i$ are $z$-independent matrices. In most cases,[6] the general solution to the system (4.1) admits an asymptotic expansion of the form [19]

$$Y(z) = e^{\Upsilon(z)} r^{\mathbf{\Delta}} \mathbf{F}(z) Y_0, \tag{4.3}$$

where $Y_0$ is a constant vector, corresponding to $n$ integration constants, $\Upsilon$ is a diagonal matrix whose coefficients are polynomials of degree at most $r+1$, $\mathbf{\Delta}$ is a constant diagonal matrix and $\mathbf{F}(z)$ is a matrix which is regular at infinity.

The goal of the algorithm presented below is to determine explicitly the expression (4.3) up to some irrelevant sub-leading terms. As we have already seen in the previous section, the guiding principle in order to obtain this expression is to fully diagonalize the differential system, up to the appropriate order, by using iteratively transformations of the vector $Y$ into a new vector $\tilde{Y}$, of the form

$$Y(z) = P(z)\tilde{Y}(z),$$

where $P$ is an invertible matrix. The system (4.1) is then transformed into a new but equivalent differential system, given by

$$\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}z} = \tilde{M}(z)\tilde{Y}, \qquad \tilde{M}(z) \equiv P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}. \tag{4.4}$$

The endpoint of this procedure is a system where the matrix coefficients in the expansion of the form (4.2) are diagonal at each order. It is then immediate to integrate the system and to find the solution in the form (4.3), as discussed in Sec. III A.

In the following subsections, we describe the algorithm step by step. We have also inserted two subsections that contain examples chosen to illustrate some of the finer points of the algorithm. The algorithm contains several branches, depending on whether the leading term $M_r$ in the expansion of $M(z)$ is diagonalizable or not.

### B. Case 1: The leading term is diagonalizable

The simplest situation is when the leading matrix $M_r$ is diagonalizable, with each eigenvalue of multiplicity 1. In this case, one first uses the transformation $Y = P_{(1)}Y^{(1)}$ where $P_{(1)}$ is a constant matrix that diagonalizes $M_r$, which gives the new system

$$\begin{aligned} &\frac{\mathrm{d}Y^{(1)}}{\mathrm{d}z} = M^{(1)}Y^{(1)}, \\ &M^{(1)}(z) = D_r z^r + M^{(1)}_{r-1} z^{r-1} + \cdots + M^{(1)}_0 \\ &\qquad + M^{(1)}_{-1}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \end{aligned} \tag{4.5}$$

where the matrix $D_r$ is diagonal.

One then seeks to transform the next-to-leading matrix $M^{(1)}_{r-1}$ into a diagonal matrix (if it is not already) without affecting the diagonal form of the leading order. This can be accomplished with a new transformation

$$Y^{(1)} = P_{(2)}Y^{(2)}, \qquad P_{(2)}(z) = I + \frac{1}{z}\Xi_{(2)}, \tag{4.6}$$

where $\Xi_{(2)}$ is a constant matrix. Indeed, this yields the new system

$$\begin{aligned} &\frac{\mathrm{d}Y^{(2)}}{\mathrm{d}z} = M^{(2)}Y^{(1)}, \\ &M^{(2)}(z) = D_r z^r + D_{r-1} z^{r-1} + M^{(2)}_{r-2} z^{r-2} + \cdots \\ &\qquad + M^{(2)}_{-1}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \end{aligned} \tag{4.7}$$

with

$$D_{r-1} = M^{(1)}_{r-1} + [D_r, \Xi^{(2)}], \tag{4.8}$$

[6]Note that, in some cases, the variable $z$ in the expression (4.3) differs from the variable $z$ in the original system (4.1), because a change of variable is necessary, as will be discussed around Eq. (4.21). Moreover, the special case where $M(z) = M_{-1}/z + \mathcal{O}(z^{-2})$ with $M_{-1}$ nilpotent leads to a $\ln z$ behavior at large $z$, as discussed at the end of Sec. IV C.

which is imposed to be diagonal via an appropriate choice[7] for $\Xi_{(2)}$. Furthermore, $D_{r-1}$ is the diagonal part of $M^{(1)}_{r-1}$.

One can proceed similarly to "diagonalize" all the other terms, order by order, until one gets a system of the form[8]

$$\frac{\mathrm{d}Y^{(r+2)}}{\mathrm{d}z} = M^{(r+2)}Y^{(r+2)},$$
$$M^{(r+2)}(z) = D_r z^r + \cdots + D_0 + D_{-1}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad (4.9)$$

where all matrices are diagonal up to order $1/z$. The system can then be immediately integrated, to yield

$$Y^{(r+2)}(z) = e^{\Upsilon(z)} z^{\mathbf{\Delta}} \mathbf{F}(z) Y_0, \qquad \mathbf{\Delta} \equiv D_{-1},$$
$$\Upsilon(z) \equiv D_r \frac{z^{r+1}}{r+1} + \cdots + D_0 z, \qquad (4.10)$$

where $Y_0$ is a constant vector.

The asymptotic expansion of the original vector $Y$ can be simply deduced from the combined transformations, i.e.,

$$Y = P_{(1)}P_{(2)}\cdots P_{(r+2)}Y^{(r+2)}. \qquad (4.11)$$

Since the $P_{(j)}$ are polynomials of $1/z$, $Y$ has exactly the same exponential behavior (in its asymptotic expansion) as $Y^{(r+2)}$.

The above procedure is not directly applicable if the leading matrix $M_r$ has eigenvalues of multiplicity higher than one. In such a case, writing $M_r$ in a block diagonal form, with eigenvalues $\lambda_i$ of multiplicity $m_i$, one applies a transformation

$$Y^{(1)} = P_{(2)}Y^{(2)}, \qquad (4.12)$$

where $P_{(2)}$ has the same block structure as $M_r$, with the blocks $B_i$ of size $m_i \times m_i$ defined as $B_i = \exp(\frac{\lambda_i}{r+1} z^{r+1})$ if $m_i \geq 2$ and $B_i = 1$ if $m_i = 1$. For example, if the leading matrix is $M_r = \mathrm{Diag}(\lambda_1, \lambda_1, \lambda_2)$, with $r = 1$, then the transformation is $P_{(2)} = \mathrm{Diag}(\exp(\lambda_1 \frac{z^2}{2}), \exp(\lambda_1 \frac{z^2}{2}), 1)$.

Such a transformation puts the multidimensional blocks to zero, allowing one to pursue the algorithm with the subleading terms. One must however be careful when coming back to the original variable $Y^{(1)}$, since the transformation $P_{(2)}$ will greatly affect the computed asymptotic behavior.

### C. Case 2: The leading term is nondiagonalizable, similar to a single-block Jordan matrix

Solving asymptotically a system where the dominant term $M_r$ is not diagonalizable is more challenging. The basic idea consists in finding a transformation where the leading term of the new matrix becomes diagonalizable. This can be done by reducing progressively the Poincaré rank of the system until the leading term is diagonalizable, in which case the procedure of the previous subsection becomes applicable. If the leading term never gets diagonalizable down to the rank $r = -1$, then the general formula (4.3) for the asymptotic expansion is not valid but the system can nevertheless be integrated explicitly.

The reduction of the Poincaré rank together with the diagonalization of the leading term is done in different steps, which we now describe, first when the leading term is similar to a Jordan matrix with a single block. The case of a Jordan matrix with several blocks will be discussed later, in section IV E.

#### 1. Step 1. Transformation to a Jordan block

Starting from the asymptotic expansion (4.2) of the matrix $M$, we use the transformation $X = P_{(1)}X^{(1)}$ to write $M^{(1)}_r = P^{-1}_{(1)} M_r P_{(1)}$ in a Jordan canonical form (although with a lower triangular matrix). We assume here that $M^{(1)}_r$ contains a single (lower triangular) Jordan block with eigenvalue $\lambda$, i.e., of the form

$$M^{(1)}_r = \begin{pmatrix} \lambda & 0 & \cdots & & \\ 1 & \lambda & 0 & \cdots & \\ 0 & 1 & \lambda & 0 & \cdots \\ \vdots & & & & \end{pmatrix} \equiv \lambda I + J(n), \qquad (4.13)$$

where $J(n)$ has the property to be nilpotent (we recall that $n$ is the dimension of the matrix).

#### 2. Step 2. Transformation to a nilpotent matrix

We then apply the transformation

$$Y^{(1)} = P_{(2)}Y^{(2)}, \qquad P_{(2)}(z) \equiv \exp\left(\frac{\lambda}{r+1} z^{r+1}\right) I, \qquad (4.14)$$

[7]To find $\Xi$ such that the matrix $\tilde{D} = M + [D, \Xi]$ is diagonal, $M$ being arbitrary and $D$ diagonal, one notices that $[D, \Xi]_{ij} = (d_i - d_j)\Xi_{ij}$ where $d_i$ are the eigenvalues of $D$. Consequently, $\tilde{D}$ is given by the diagonal component of $M$ and the coefficients of $\Xi$ satisfy $(d_i - d_j)\Xi_{ij} + M_{ij} = 0$, which always admit at least one solution for each $\Xi_{ij}$ as long as all $d_i$ are different.

[8]Note that we could have proceeded in a single step by introducing the new variable $\tilde{Y}$ defined by $Y = P(z)\tilde{Y}$ with $P(z) = P_0 + \frac{1}{z}P_1 + \cdots + \frac{1}{z^{r+1}}P_{r+1}$ and determining the constant matrices $P_j$ so that $\tilde{M}(z)$ is equal to (4.9). The calculation we have just done proves this is possible with $\tilde{Y} = Y^{(r+2)}$.

which renders the leading term nilpotent[9]

$$M^{(2)}(z) = J(n)z^r + M^{(2)}_{r-1}z^{r-1} + \cdots + M^{(2)}_0 + M^{(2)}_{-1}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right). \quad (4.16)$$

### 3. Step 3. Normalization and reduction of the Poincare´ rank

The next step consists in reducing the Poincaré rank of the system by using the transition matrix

$$P(z) = D(n,z) \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & z & 0 & \cdots & \cdots & 0 \\ 0 & 0 & z^2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & & \cdots & z^{n-1} \end{pmatrix}, \quad (4.17)$$

which satisfies the useful property

$$P^{-1}J(n)P = \frac{1}{z}J(n). \quad (4.18)$$

A transformation with the above $P$ will thus reduce the order of the leading term $J(n)z^r$, but will also affect the sub-dominant terms in the expansion (4.16) of $M^{(2)}$, in particular $M^{(2)}_{r-1}$ which could generate terms whose order is higher than $r-1$ in the new matrix.

To avoid this situation, we need first to "normalize" the system, with the transformation

$$P_{(3)}(z) = I + \frac{1}{z}\Lambda_{(3)}, \quad (4.19)$$

where $\Lambda_{(3)}$ is a constant matrix, chosen such that such that the next-to-leading order matrix $M^{(3)}_{r-1}$ in the new matrix expansion contains only zeros except possibly in the first row. Let us stress that this transformation leaves the leading term of the expansion unchanged. The new system associated with $M^{(3)}$ is said to be *normalized*.

One can then perform the transformation generated by the transition matrix

$$P_{(4)}(z) = D(n,z), \quad (4.20)$$

which, in *most* cases, gives a reduced Poincaré rank. There are however a few exceptions where the reduction does not work. These special cases require a more general transformation, with a transition matrix of the form

$$P_{(4)}(z) = D(n, z^{p/q}), \qquad (1 \le p \le q \le n) \quad (4.21)$$

where $p$ and $q$ are co-prime integers. For example, when $n=4$, the possible choices are $\{1/4, 1/3, 1/2, 2/3, 3/4, 1\}$, where the last value corresponds to the generic case (4.20). To identify the appropriate value of $p/q$, one must test successively the possible values, in decreasing order, until the transformation (4.21) effectively leads to a system with a lower Poincaré rank. The corresponding value of $p/q$ is said to be "admissible". In practice, this can be understood as a change of variable,[10] $z$ being replaced by $u = z^{p/q}$.

### 4. Step 4. Diagonalizable or not diagonalizable?

The next step depends on the nature of the system $(Y^{(4)}, M^{(4)})$, which possesses a lower Poincaré rank than the initial system. If the leading term of $M^{(4)}$ is diagonalizable, one proceeds as in Sec. IV B.

If $M^{(4)}$ is not diagonalizable, one needs to reduce again the Poincaré rank of the system, unless one has already reached $r=-1$, in which case one can jump directly to the next paragraph. Otherwise, one must distinguish the following different cases.

(i) If the leading term is similar to a single-block Jordan matrix and we took $p/q = 1$ in the previous step, we repeat the procedure of this subsection.

(ii) If the leading term is similar to a single-block Jordan matrix but we took $p/q < 1$ in the previous step, we discard the last step, and start again with the normalized system $M^{(3)}$. However, this time, we normalize the system up to second order: after having normalized $M_{-1}$, we repeat the procedure with $z^2$ instead of $z$ in $P_{(3)}$ (4.19) and require that $M_{-2}$ has a specific form. Details can be found in [20]. If necessary, one can pursue the normalization to higher orders.

(iii) If the Jordan canonical form of the leading term contains several blocks, we go to Sec. IV E.

Eventually we obtain either a system with a diagonalizable leading term, which can be solved following Sec. IV B, or a system of Poincaré rank $r=-1$ with a nilpotent leading term. In the latter case, the solution is equivalent to a polynomial of $\ln z$ at large $z$. Indeed, a system of the form

[9] This follows from the relation

$$P^{-1}_{(2)}(z^r(\lambda I + J(n))P_{(2)} - P^{-1}_{(2)}\frac{\mathrm{d}P_{(2)}}{\mathrm{d}z} = z^r(M^{(1)}_r - \lambda I) = z^r J(n). \quad (4.15)$$

[10] In this case, the asymptotic expansion of the solution may have an exponential behavior where the argument $Q(z)$ is not a polynomial of $z$ but rather a polynomial of $z^{1/q}$.

$$\frac{\mathrm{d}Y}{\mathrm{d}z}=\frac{\mu_0}{z}\begin{pmatrix}0 & 0 & \cdots & & \\ 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & & & & \end{pmatrix}Y, \tag{4.22}$$

where $\mu_0$ is an arbitrary constant, is easily integrated. The components $Y_i$ (for $1 \leq i \leq n$) are obtained iteratively and are given by $Y_1(z) = \xi_1$, $Y_2(z) = \xi_1 \ln z + \xi_2$ and more generally,

$$Y_i(z) = \sum_{j=1}^{i} \frac{\xi_j}{(i-j)!}(\mu_0 \ln z)^{i-j}, \tag{4.23}$$

where the $\xi_i$ are $n$ constants of integration. All the components of $Y$ are thus polynomials of $\ln z$ at large $z$.

### D. An example with a nilpotent leading term

Let us give a concrete example of the procedure used for systems with a nilpotent leading term. We consider the two-dimensional system defined by

$$\frac{\mathrm{d}Y}{\mathrm{d}z}=M(z)Y, \qquad M(z)=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}z^2+\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{4.24}$$

and let us determine its asymptotic solution at large $z$, following the algorithm described above.

We first put the leading term in its lower triangular Jordan form:

$$P_{(1)} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$\Rightarrow M^{(1)}(z) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}z^2 + \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.25}$$

Since the leading term is already nilpotent, step 2 is irrelevant. Moreover, the system is already normalized since the next-to-leading order term vanishes.

We can thus move directly to the reduction of the order of the system and consider the transformation of the form (4.17):

$$P_{(2)}(z) = \begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}$$
$$\Rightarrow M^{(2)}(z) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}z + \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.26}$$

The order has been reduced but the leading term is still nilpotent. Since the reduction was obtained via a transformation with $p/q = 1$, we continue the process by doing a new iteration of the algorithm. We first normalize the system with a transformation of the form (4.19),

$$P_{(3)}(z) = I + \frac{1}{z}\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}$$
$$\Rightarrow M^{(3)}(z) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}z + \begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\frac{1}{z} + \begin{pmatrix}0 & -2\\ 0 & 0\end{pmatrix}\frac{1}{z^2}, \tag{4.27}$$

and again reduce the order of the system with the transformation

$$P_{(4)}(z) = \begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}$$
$$\Rightarrow M^{(4)}(z) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -2\\ 0 & -2\end{pmatrix}\frac{1}{z}. \tag{4.28}$$

The leading term is now diagonalizable. We diagonalize it explicitly, via

$$P_{(5)} = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$
$$\Rightarrow M^{(5)}(z) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ -2 & -2\end{pmatrix}\frac{1}{z}, \tag{4.29}$$

then we diagonalize the next-to-leading term, with a transformation of the form (4.6),

$$P_{(6)}(z) = \begin{pmatrix}1 & 0\\ 1/z & 1\end{pmatrix}$$
$$\Rightarrow M^{(6)}(z) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{4.30}$$

We have thus managed to fully diagonalize the system, which immediately gives us the asymptotic solution

$$Y^{(6)}(z) = (1+\mathcal{O}(1/z))\begin{pmatrix}\exp(-z) & 0\\ 0 & \frac{1}{z^2}\exp(z)\end{pmatrix}Y_0,$$
$$Y_0 \equiv \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \tag{4.31}$$

where $Y_0$ is a constant column vector. As a consequence, to obtain the behavior of $Y$ in the original system, we use the combined transformations

$$Y = \left(\prod_{j=1}^{6} P_{(j)}\right)Y^{(6)}, \tag{4.32}$$

which implies

$$Y(z) = (1+\mathcal{O}(1/z))\begin{pmatrix}\xi_1\exp(-z)z^2 + \xi_2\exp(z)\\ -2\xi_1\exp(-z)\end{pmatrix}. \tag{4.33}$$

For this particular example, it turns out that the original system (4.24) can be solved exactly, with the solution

$$Y(z) = \begin{pmatrix} \frac{1}{2}\xi_1 \exp(-z)(1+2z+2z^2) + \xi_2 \exp(z) \\ -2\xi_1 \exp(-z) \end{pmatrix}. \quad (4.34)$$

One can thus check that the asymptotic solution (4.33) agrees with the asymptotic behavior of the exact solution.

### E. Case 3: $M_r$ is similar to a Jordan matrix with several blocks

We now briefly discuss (without entering into too many details, which can be found in [20]) the more general case where $M_r$ is block diagonalizable and its canonical Jordan form admits several Jordan blocks. The first two steps of Sec. IV C still apply to this case and one can find a transformation (with a constant matrix $P$) such that the new system associated with $M^{(2)}$ (we use the same notation as in Sec. IV C) has a block diagonal leading term $M_r^{(2)}$ with Jordan lower triangular blocks, each block being either nilpotent or 1-dimensional:

$$M_r^{(2)} = \begin{pmatrix} J(n_1) & 0 & \cdots & & & & \\ 0 & J(n_2) & 0 & \cdots & & & \\ \vdots & 0 & \ddots & 0 & \cdots & & \\ & \vdots & 0 & \lambda_1 & 0 & \cdots & \\ & & \vdots & 0 & \lambda_2 & 0 & \\ & & & \vdots & 0 & \ddots & \end{pmatrix},$$

$$J(n) \equiv \begin{pmatrix} 0 & 0 & \cdots & & & \\ 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & 0 & \cdots & \\ \vdots & & & & & \end{pmatrix}. \quad (4.35)$$

The Jordan form is chosen so that the blocks $J(n)$ are ordered by decreasing size ($n_1 \geq n_2 \geq \cdots$). We will use this block structure as a layout for the block structure of the other matrices that appear in the expansion of $M^{(2)}$. And each block will be denoted by two indices, $(KL)$, corresponding to a submatrix of dimensions $n_K \times n_L$.

The principle of the diagonalization procedure is similar to what was done in Secs. IV B and IV C. However, it is now possible to have both diagonalizable blocks and nilpotent blocks. Those must be dealt with separately to get the full asymptotic behavior of the system. In order to do this, one can generalize the order-by-order procedure of Sec. IV B: this is called the "splitting lemma" in [20]. It is not detailed here, but can be understood by considering blocks instead of scalars in the computations of Sec. IV B.[11]

One can use this lemma to block diagonalize $M^{(2)}$, order by order: the two global blocks considered will be the nilpotent part of $M_r^{(2)}$ and its diagonalizable part. The latter can be dealt with using the procedure given in Sec. IV B, while the former must be addressed using a generalized version of the procedure given in Sec. IV C. We give here more details about the last part and, in the rest of this section, assume without loss of generality that $M_r^{(2)}$ contains only nilpotent blocks, such that

$$M_r^{(2)} = \begin{pmatrix} J(n_1) & 0 & \cdots & & \\ 0 & J(n_2) & 0 & \cdots & \\ \vdots & 0 & J(n_3) & 0 & \cdots \\ & \vdots & & & \ddots \end{pmatrix}$$

$$(\text{with} \quad n_1 \geq n_2 \geq \cdots \geq n_{\text{last}}). \quad (4.37)$$

The procedure in such a case requires to put the system in a specific normalized form. For a matrix $M$, obtained at a generic step in the algorithm, one says that the matrix is "normalized up to order $s$" if all its leading terms $M_r, \cdots M_{r-s}$ have their $(KL)$ blocks verifying the following properties:

(i) either all rows are equal to zero except possibly the first one if $K \leq L$,

(ii) or all columns are equal to zero except possibly the last one if $K > L$.

In order to reach this normalized form, one must use a succession of transformations[12] $P_{\text{norm}}(k)$ of the form

$$P_{\text{norm}}(k) = I + \frac{1}{z^k}\Lambda, \quad (4.38)$$

where $k$ varies from 1 to $s$. The matrix $\Lambda$ is a constant matrix, whose coefficients must be chosen, similarly to $\Xi$ in (4.6), such that the new matrix $M$ is normalized, in the sense defined above ($\Lambda$ is uniquely defined if one requires that all its blocks $\Lambda^{KL}$ have zero last row if $K \leq L$ and zero

[11] In the case where $M_r^{(2)}$ consists of a 2-block Jordan matrix, one would use a transformation of the form

$$P = \begin{pmatrix} I & \sum_{j=1}^{p} \Xi_j z^{-j} \\ \sum_{j=1}^{p} \Lambda_j z^{-j} & I \end{pmatrix}, \quad (4.36)$$

where the $\Xi_j$ and $\Lambda_j$ are constant matrices. Such a transformation, which generalizes (4.6), enables us to transform each $M_{r-j}^{(2)}$ in the same block diagonal form as $M_r^{(2)}$ with a convenient choice of $\Xi_i$ and $\Lambda_i$. Therefore, the initial system gives two decoupled subsystems and, for each one, we proceed along the same lines as in the previous section.

[12] Let us emphasize on the fact that the hierarchy $n_1 \geq n_2 \geq \cdots$ is crucial for this step to succeed.

first column if $K > L$). The procedure is iterative: if the system is normalized up to order $k$, it is possible to normalize it up to order $k+1$ by applying a transformation $P_{\rm norm}(k+1)$. Indeed, this transformation will not modify any term of order higher than $r-k-1$.

The complete procedure to reduce the Poincaré rank of the matrix is then the following:

1. one starts with $s = 1$;
2. one normalizes the system up to order $s$ using $P_{\rm norm}(k)$ transformations;
3. if $M_{r-s}$ is not block-diagonal, one uses a transformation $P_u(n) = {\rm diag}(I_{n_1}, I_{n_2}, \ldots, z^s I_{n_{\rm last}})$ and one goes back to step 1[13];
4. if it is block-diagonal, one uses a $P_{p/q}$ transformation, which is a block form of (4.17) or (4.21):

$$P_{p/q} = \begin{pmatrix} D(n_1, z^{p/q}) & 0 & \cdots & & \\ 0 & D(n_2, z^{p/q}) & 0 & \cdots & \\ \vdots & 0 & D(n_3, z^{p/q}) & 0 & \cdots \\ & \vdots & & & \ddots \end{pmatrix}, \tag{4.39}$$

where the matrices $D(n, z)$ have been defined in (4.17) and $p$ and $q$ are either co-prime integers (with $1 \leq p \leq q \leq n_1$) or equal in the case $p/q = 1$;
5. if no $P_{p/q}$ transformation is admissible (see the definition after 4.21), one goes back to step 1 with $s$ increased by one. Otherwise, one stops here.

Thanks to the above procedure, one obtains either a system depending on $z$ with a reduced Poincaré rank, or a new system depending on $z^{p/q}$ with a non-nilpotent leading term. In the former case, one can simply pursue with the algorithm. In the latter case, one can change variables by writing $w = z^{p/q}$ and start the algorithm again.

### F. A higher dimensional example with $p/q \neq 1$

We now present a higher dimensional ($n = 5$) example, adapted from [22], where the dominant term in the asymptotic expansion of the matrix $M$ has a non trivial canonical Jordan form with two Jordan blocks. The matrix $M(z)$ is given by

$$M(z) = \begin{pmatrix} 0 & z^3 & -z & 1 & 2z \\ -z^2 & z & 0 & -z & 0 \\ z & 1 & 0 & z^3 & 1 \\ 1 & -z & 1 & z & z^3 \\ z & 0 & -3z & 0 & -1 \end{pmatrix} \equiv M_3 z^3 + M_2 z^2 + M_1 z + M_0, \tag{4.40}$$

where the leading term $M_3$ is nilpotent and has a 2-block Jordan structure.

We perform a first transformation $Y = P_{(1)} Y^{(1)}$ so that the leading term has now the following Jordan (lower triangular) canonical form (the matrix $P_{(1)}$ can easily been deduced):

$$M^{(1)}(z) = \begin{pmatrix} -1 & 0 & -3z & 0 & z \\ z^3 & z & 1 & -z & 1 \\ 1 & z^3 & 0 & 1 & z \\ 0 & -z & 0 & z & -z^2 \\ 2z & 1 & -z & z^3 & 0 \end{pmatrix} \Rightarrow M_3^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.41}$$

The block structure of $M_3^{(1)}$ defines the layout that we will be using to compute the asymptotic expansion of the solution.

We notice that the next-to-leading term $M_2^{(1)}$ in the expansion of $M^{(1)}$ is already normalized. Therefore, we can immediately try to reduce the order of the system thanks to a new transformation $Y^{(1)} = P_{(2)} Y^{(2)}$,

$$P_{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} \Rightarrow M^{(2)} = \begin{pmatrix} -1 & 0 & -3z^3 & 0 & z^2 \\ z^2 & z - \frac{1}{z} & z & -1 & 1 \\ \frac{1}{z^2} & z^2 & -\frac{2}{z} & \frac{1}{z^2} & 1 \\ 0 & -z^2 & 0 & z & -z^3 \\ 2 & 1 & -z^2 & z^2 & -\frac{1}{z} \end{pmatrix}. \tag{4.42}$$

[13]It is proved in [20] that after a finite number of steps, one always gets a block-diagonal subleading term, which means that this procedure stops at some point and that one can go on with step 4.

However, we immediately see that the order of the system has not diminished. This example falls in the cases where we need to change the variable $z$ or, equivalently, i.e., to make a transformation of the form (4.21) for each Jordan block, We must therefore cancel the previous transformation (4.42) and instead consider $Y^{(1)} = \tilde{P}_{(2)}\tilde{Y}^{(2)}$, with

$$\tilde{P}_{(2)}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & z^{p/q} & 0 & 0 & 0 \\ 0 & 0 & z^{2p/q} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z^{p/q} \end{pmatrix}. \quad (4.43)$$

Following the method described below Eq. (4.21), we note that the largest Jordan block is of dimension 3, therefore we should take 2 coprime integers between 1 and 3 for $p$ and $q$ with $p \leq q$. The possible choices for the ratio $p/q$ belong to the set $\{1/3, 1/2, 2/3\}$, since $p/q = 1$ does not work. The largest value is $p/q = 2/3$, which gives for the matrix $M^{(3)}$ the expression

$$\begin{pmatrix} -1 & 0 & -3z^{7/3} & 0 & z^{5/3} \\ z^{7/3} & z - \frac{2}{3z} & z^{2/3} & -z^{1/3} & 1 \\ \frac{1}{z^{4/3}} & z^{7/3} & -\frac{4}{3z} & \frac{1}{z^{4/3}} & z^{1/3} \\ 0 & -z^{5/3} & 0 & z & -z^{8/3} \\ 2z^{1/3} & 1 & -z^{5/3} & z^{7/3} & -\frac{2}{3z} \end{pmatrix}. \quad (4.44)$$

We observe that the subdiagonal terms have order 7/3. To keep this value of $p/q$, we must make sure that no other term behaves like $z^{\alpha}$ with $\alpha > 7/3$. However in this case there is a $z^{8/3}$ term. Therefore, the value 2/3 is not admissible and we have to consider the next possible choice which is $p/q = 1/2$. Such a change of variable leads to the matrix

$$\tilde{M}^{(2)} = \begin{pmatrix} -1 & 0 & -3z^2 & 0 & z^{3/2} \\ z^{5/2} & z - \frac{1}{2z} & \sqrt{z} & -\sqrt{z} & 1 \\ \frac{1}{z} & z^{5/2} & -\frac{1}{z} & \frac{1}{z} & \sqrt{z} \\ 0 & -z^{3/2} & 0 & z & -z^{5/2} \\ 2\sqrt{z} & 1 & -z^{3/2} & z^{5/2} & -\frac{1}{2z} \end{pmatrix}. \quad (4.45)$$

Now, it verifies the requirements and we thus keep the value $p/q = 1/2$ and continue the process.

The previous change of variable leads to a differential system where the coefficients of $M^{(3)}$ are noninteger powers functions of $z$. To apply the algorithm, we have to make a change of coordinate so that the system involves only integer powers of $z$. This can easily be done by introducing the new coordinate $u$ defined by $z = u^2$. As a consequence, the new differential system is now given by

$$\frac{dY^{(3)}}{du} = M^{(3)}(u)Y^{(3)},$$

$$M^{(3)}(u) = \begin{pmatrix} -2u & 0 & -6u^5 & 0 & 2u^4 \\ 2u^6 & \frac{2u^4-1}{u} & 2u^2 & -2u^2 & 2u \\ \frac{2}{u} & 2u^6 & -\frac{2}{u} & \frac{2}{u} & 2u^2 \\ 0 & -2u^4 & 0 & 2u^3 & -2u^6 \\ 4u^2 & 2u & -2u^4 & 2u^6 & -\frac{1}{u} \end{pmatrix}, \quad (4.46)$$

where $Y^{(3)}(u) \equiv \tilde{Y}^{(2)}(z)$ and $M^{(3)}(u) \equiv 2u\tilde{M}^{(2)}(z)$ with $z = u^2$. As the leading term is not nilpotent, we keep the value of $p/q$. If it had been nilpotent, we would have had to go back one step and normalize up to the next order.

We can continue the algorithm with this new system: we will to do a new change of variables, reduce the order, and decouple the system... We will not present more steps as the rest of the computations is similar to what was done here and in previous sections. Nonetheless, for the sake of completeness, we give the final result. We show that, after enough steps of the algorithm, the initial system can be equivalently reformulated as

$$\frac{dY^{(4)}}{dw} = M^{(4)}(w)Y^{(4)}, \quad (4.47)$$

where $w = z^{1/6}$ and $M^{(4)}(w)$ is the following diagonal matrix

$$M^{(4)}(w) = \mathrm{Diag}[3^{4/3}(1 - i\sqrt{3})w^{19} + 2w^{11}, -2 \times 3^{4/3}w^{19} + 2w^{11}, 3^{4/3}(1 + i\sqrt{3})w^{19} + 2w^{11}, 6iw^{20} + 3w^{11}, -6iw^{20} + 3w^{11}] + \mathcal{O}(w^9), \quad (4.48)$$

up to order $\mathcal{O}(w^9)$. Integrating such a system is immediate and yields the leading orders of the asymptotic expansion of $Y^{(4)}$ from which we can extract the asymptotic expansion of the original variable $Y$.

## V. CONCLUSION

In this work, we have studied the asymptotic behaviors, both at spatial infinity and near the horizon, of the linear perturbations about Schwarzschild black holes. Instead of following the traditional approach that consists in rewriting the equations of motion in the form of a stationary Schrödinger-like equation, which is second-order with respect to the radial coordinate, we have worked directly with the first-order equations of motion (in the frequency domain). For this direct approach to the asymptotic behavior, we have used an algorithm that has been developed in several recent articles published in mathematical journals.

The principle of this algorithm is to transform the differential system, via successive changes of functions, until it can be written in an explicitly diagonal form, up to the required order (in the small parameter characterising the asymptotic regime). This procedure automatically provides the combination of the metric perturbations that encapsulates the physical degree of freedom in this asymptotic region and enables one to separate the ingoing and outgoing physical modes. Although we have worked in the standard Regge-Wheeler gauge, the same approach would work similarly for any other gauge choice.

Beyond its application to the perturbations of black holes, this systematic approach to the asymptotic behavior could be very useful for similar problems in other domains of physics. This is why we have devoted the last part of this paper to a pedagogical presentation of the algorithm, with a few illustrative examples.

For black holes, the knowledge of the asymptotic behavior of the perturbations is an indispensable first step in the determination of the quasinormal modes. Indeed, these modes are characterized by the following boundary conditions: a purely outgoing behavior at spatial infinity and purely ingoing behavior at the horizon. Imposing these boundary conditions, we have shown that the known quasinormal modes can be recovered numerically, without resorting to the Schrödinger-like formulation, thus providing an alternative approach to the standard method. We stress that our rudimentary numerical calculation was simply to illustrate the feasibility of this new approach, without trying to reach the precision and efficiency of the powerful numerical methods that have been developed in the traditional approach.

This novel approach could be especially useful in the context of generalized black hole solutions, for instance in modified gravity theories, where the equations of motion for the perturbations are different and extra fields can be present. In a companion paper, we have applied the same algorithm to a few black holes solutions within scalar-tensor theories that belong to the most general known family: DHOST (degenerate higher-order scalar-tensor) theories. The same method could be applied to the study of other types of black holes, or even completely different physical systems.

As a final remark, let us stress that this approach could be used to get some analytical insight on the asymptotic behavior of the modes by looking directly at the structure of the matrix coefficients that are relevant. In this sense, it might provide a prediagnosis tool to explore the healthiness of some black hole solutions without resorting to a full numerical investigation.

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## APPENDIX A: GAUGE TRANSFORMATIONS

For completeness, we summarise in this Appendix the gauge fixing procedure for polar and axial perturbations about a Schwarzschild black hole in general relativity, as originally discussed in [3,4].

Due to the invariance of the theory under space-time diffeomorphisms, the metric perturbations are not completely determined $h_{\mu\nu}$. Indeed, any infinitesimal change of coordinates $x^\mu \to x^\mu + \xi^\mu$ induces the transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{A1}$$

at the linear level. These transformations can be “projected” in the axial or polar sectors, which we examine in turn.

### 1. Axial perturbations: Regge-Wheeler gauge

Before gauge fixing, axial perturbations are parametrized by three families of functions $h_0^{\ell m}$, $h_1^{\ell m}$ and $h_2^{\ell m}$ of the variables $(r, t)$, according to

$$\begin{aligned} h_{t\theta} &= \frac{1}{\sin\theta}\sum_{\ell,m} h_0^{\ell m}(t,r)\partial_\varphi Y_{\ell m}(\theta,\varphi), \\ h_{t\varphi} &= -\sin\theta \sum_{\ell,m} h_0^{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\varphi), \\ h_{r\theta} &= \frac{1}{\sin\theta}\sum_{\ell,m} h_1^{\ell m}(t,r)\partial_\varphi Y_{\ell m}(\theta,\varphi), \\ h_{r\varphi} &= -\sin\theta \sum_{\ell,m} h_1^{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\varphi), \\ h_{ab} &= \sin\theta \sum_{\ell,m} h_2^{\ell m}(t,r)\epsilon_{c(a}D^c\partial_{b)} Y_{\ell m}(\theta,\varphi), \end{aligned} \tag{A2}$$

where, in the last equation, the indices $a$ and $b$ belong to the set $\{\theta, \varphi\}$, $\epsilon_{ab}$ is the totally antisymmetric symbol such that $\epsilon_{\theta\varphi} = +1$ and $D_a$ is the 2-dimensional covariant derivative associated with the metric of the 2-sphere $d\theta^2 + \sin^2\theta d\varphi^2$. More explicitly, the angular components of the metric can be written

$$h_{\theta\theta} = \sum_{\ell,m} \frac{1}{\sin\theta} h_2^{\ell m}(t,r)(\partial_\theta\partial_\varphi - \cot\text{an}\theta\,\partial_\varphi) Y_{\ell m}(\theta,\varphi), \tag{A3}$$

$$h_{\theta\varphi} = h_{\varphi\theta} = -\sum_{\ell,m} \sin\theta h_2^{\ell m}(t,r)\left(\frac{\ell(\ell+1)}{2} + \partial_\theta^2\right) Y_{\ell m}(\theta,\varphi), \quad \text{(A4)}$$

$$h_{\varphi\varphi} = -\sum_{\ell,m} h_2^{\ell m}(t,r)\sin\theta\,(\partial_\theta\partial_\varphi - \text{cotan}\theta\,\partial_\varphi) Y_{\ell m}(\theta,\varphi). \quad \text{(A5)}$$

All the other components of the axial perturbations vanish.

In the axial sector, the nonzero components of the generator $\xi^\mu$ that preserves the odd parity of the perturbations can be decomposed into spherical harmonics as follows,

$$\xi_\theta = \sum_{\ell,m} \xi^{\ell m}(t,r)\partial_\theta Y_{\ell,m}(\theta,\varphi),$$
$$\xi_\varphi = \sum_{\ell,m} \xi^{\ell m}(t,r)\partial_\varphi Y_{\ell,m}(\theta,\varphi), \quad \text{(A6)}$$

and the induced gauge transformations on the functions $h_0$, $h_1$ and $h_2$ are given, according to (A1), by

$$h_0 \to h_0 - \dot{\xi}, \quad h_1 \to h_1 - \xi' + \frac{2}{r}\xi, \quad h_2 \to h_2 - 2\xi, \quad \text{(A7)}$$

where we have dropped the indices $(\ell m)$ for simplicity. A dot and a prime denote a derivative with respect to $t$ and $r$, respectively.

As a consequence, one can always choose a gauge in which $h_2^{\ell m} = 0$ which is the well-known Regge-Wheeler gauge for the axial perturbations [3]. Notice that this gauge choice is possible for $\ell \geq 2$ only (the cases $\ell = 0$ and $\ell = 1$ will be discussed later below).

### 2. Even-parity or polar perturbations: Zerilli gauge

Before gauge fixing, polar perturbations of the metric are parametrized by seven families of functions $H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$, $\alpha^{\ell m}$, $\beta^{\ell m}$, $K^{\ell m}$ and $G^{\ell m}$ of the variables $(r,t)$ which appear in the components of the metric perturbations as follows,

$$h_{tt} = A(r)\sum_{\ell,m} H_0^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi),$$
$$h_{tr} = \sum_{\ell,m} H_1^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \quad \text{(A8)}$$
$$h_{rr} = \frac{1}{A(r)}\sum_{\ell,m} H_2^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \quad \text{(A9)}$$
$$h_{ta} = \sum_{\ell,m} \beta^{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi),$$
$$h_{ra} = \sum_{\ell,m} \alpha^{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi), \quad \text{(A10)}$$
$$h_{ab} = \sum_{\ell,m} K^{\ell m}(t,r)g_{ab}Y_{\ell m}(\theta,\varphi) + \sum_{\ell,m} G^{\ell m}(t,r)D_a D_b Y_{\ell m}(\theta,\varphi). \quad \text{(A11)}$$

More precisely, the angular part of the metric can be written as

$$h_{\theta\theta} = \sum_{\ell,m} K^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi) + \sum_{\ell,m} G^{\ell m}(t,r)\partial_\theta^2 Y_{\ell m}(\theta,\varphi), \quad \text{(A12)}$$

$$h_{\theta\varphi} = h_{\varphi\theta} = -\sum_{\ell,m} G^{\ell m}(t,r)\text{cotan}\theta\,\partial_\varphi Y_{\ell m}(\theta,\varphi), \quad \text{(A13)}$$

$$h_{\varphi\varphi} = \sum_{\ell,m} \sin^2\theta K^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi) + \sum_{\ell,m} G^{\ell m}(t,r)(\partial_\varphi^2 + \sin\theta\cos\theta\partial_\theta)Y_{\ell m}(\theta,\varphi). \quad \text{(A14)}$$

Similarly to the axial sector, this parametrization is redundant and can be simplified by gauge fixing. Now, linear diffeomorphisms which preserve even-parity of the metric components are generated by vector fields $\xi$ whose components decompose into spherical harmonics as follows,

$$\xi_t = \sum_{\ell,m} T^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi),$$
$$\xi_r = \sum_{\ell,m} R^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi),$$
$$\xi_\theta = \sum_{\ell,m} \Theta^{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\varphi),$$
$$\xi_\varphi = \sum_{\ell,m} \Theta^{\ell m}(t,r)\partial_\varphi Y_{\ell m}(\theta,\varphi). \quad \text{(A15)}$$

Here $T^{\ell m}$, $R^{\ell m}$ and $\Theta^{\ell m}$ are arbitrary functions of $(t,r)$. These linear diffeomorphisms induce gauge transformations on the functions that parametrize metric perturbations according to

$$H_0^{\ell m}(t,r) \to H_0^{\ell m}(t,r) + \frac{2}{A(r)}\dot{T}^{\ell m}(t,r) + A'(r)R^{\ell m}(t,r),$$
$$H_1^{\ell m}(t,r) \to H_1^{\ell m}(t,r) + \dot{R}^{\ell m}(t,r) + T'^{\ell m}(t,r) + \frac{A'(r)}{A(r)}T^{\ell m}(t,r),$$
$$H_2^{\ell m}(t,r) \to H_2^{\ell m}(t,r) + 2A(r)R'^{\ell m}(t,r) - A'(r)R_{\ell m}(t,r),$$
$$\beta^{\ell m}(t,r) \to \beta^{\ell m}(t,r) + T^{\ell m}(t,r) + \dot{\Theta}^{\ell m}(t,r),$$
$$\alpha^{\ell m}(t,r) \to \alpha^{\ell m}(t,r) + R^{\ell m}(t,r) + \Theta'^{\ell m}(t,r) - \frac{2}{r}\Theta^{\ell m}(t,r),$$
$$K^{\ell m}(t,r) \to K^{\ell m}(t,r) + \frac{2A(r)}{r}R^{\ell m}(t,r),$$
$$G^{\ell m}(t,r) \to G^{\ell m}(t,r) + 2\Theta^{\ell m}(t,r). \quad \text{(A16)}$$

An immediate consequence of the gauge transformations is that one can choose the gauge parameter $\xi$ such that $G^{\ell m} = 0$ by fixing $\Theta^{\ell m}$, then $\alpha^{\ell m} = 0$ and $\beta^{\ell m} = 0$ by fixing $R^{\ell m}$ and $T^{\ell m}$ respectively, in the case where $\ell \geq 2$. This gauge is known as the Zerilli gauge [4] (see [27] for a recent presentation in the context of modified gravity).

#### 3. Monopole and dipole perturbations

We consider here the special cases $\ell = 0$ and $\ell = 1$.

##### a. Axial modes

For the axial modes, the components $h_{ab}$ vanish identically for $\ell = 1$ (axial perturbations do not have $\ell = 0$ components) which means that $h_2$ does not show up in the components of the metric. Hence, when $\ell = 1$, it is necessary to make a different gauge choice. In general, one chooses $h_1 = 0$ which fixes the gauge parameter $\xi$ up to a function of the form $C(t)r^2$. Therefore, $h_0$ inherits a residual gauge invariance given by $h_0 \to h_0 + F(t)r^2$ where $F(t)$ is an arbitrary function. Then $h_0$ can be shown to satisfy the equation of motion,

$$2h_0(r) - rh_0'(r) = 0. \tag{A17}$$

Therefore, the mode $h_0$ is not propagating.

##### b. Polar modes

Let us now turn to polar perturbations. In the case $\ell = 0$, $H_0$, $H_1$, $H_2$ and $K$ are the only nonvanishing components of the metric perturbations whereas $T$ and $R$ are the only nonvanishing components of the gauge parameter (so that the gauge transformation preserves the monopole). As in the general case, one can choose $R$ to fix $K = 0$. Then, one can in principle make use of $T$ to get rid of $H_1$ (we could have also set $H_0 = 0$). Finally, we are left with only two nonvanishing functions which are either $H_2$ or $H_0$ and we will compute the corresponding equations of motion in the next section.

The main difference, concerning the gauge fixing, between the general case and the case $\ell = 1$ lies in the fact that, in the latter, $h_{ab}$ can be shown to depend on the difference $G - K$ only, so that one can fix $K = 0$ without loss of generality. Furthermore, one can make the gauge fixing $G = 0$ by an appropriate choice of $\Theta$. Then, one makes use of $T$ to fix $\beta = 0$. Finally, one uses the remaining free gauge function $R$ to fix $\alpha = 0$. At the end, we are left with the three nonvanishing functions $H_0$, $H_1$ and $H_2$. The dynamics of these three free parameters will be studied in the next section as well.

Concerning the monopole ($\ell = 0$), we showed in Sec. II A 3 that its dynamics is fully described in terms of the functions $H_0$ and $H_2$ only, as all the others can be sent to 0 by gauge fixing. Thus the equations of motion simplifies drastically and, after some calculations, give

$$H_0(r) - H_2(r) = 0, \quad H_2(r) + (r - r_s)H_2'(r) = 0. \tag{A18}$$

The solution reads $H_2(r) = C/(r - r_s)$ and the mode is not propagating.

Finally, the dynamics of the polar dipole ($\ell = 1$) is described by the three nonvanishing functions $H_0$, $H_1$ and $H_2$ which satisfy the three independent equations,

$$\begin{aligned} &2H_2(r) + (r - r_s)H_2'(r) = 0, \qquad H_1(r) + i\omega H_2(r) = 0, \\ &H_0(r) + (r_s - r)H_0'(r) - 2ir\omega H_1(r) + H_2(r) = 0. \end{aligned} \tag{A19}$$

Indeed, the full set of the original Einstein equations is equivalent to this one which can easily be solved explicitly but its solution is not relevant for our purpose. Nonetheless, we see immediately from the equations that, like the monopole, the polar dipole does not propagate. This is why we do not consider it in the rest of the paper.

## APPENDIX B: EQUATIONS OF MOTION FOR THE POLAR PERTURBATIONS

In this appendix, we present the equations of motion satisfied by the polar perturbations and show how the system (2.14) is obtained. The Euler-Lagrange equations equations of motion (2.4) yield, in the polar sector,

$$\begin{aligned}
\mathcal{E}_{tt} &= -2(\lambda+2)\left(1-\frac{r_s}{r}\right)H_2(t,r) - 2\lambda\left(1-\frac{r_s}{r}\right)K(t,r) \\
&\quad - \frac{2}{r}(r-r_s)^2\frac{\partial H_2}{\partial r} + \left(6r - 11r_s + \frac{5r_s^2}{r}\right)\frac{\partial K}{\partial r} \\
&\quad + 2(r-r_s)^2\frac{\partial^2 K}{\partial r^2} = 0, \\
\mathcal{E}_{tr} &= -2(\lambda+1)H_1(t,r) - 2r\frac{\partial H_2}{\partial t} + r\frac{2r-3r_s}{r-r_s}\frac{\partial K}{\partial t} \\
&\quad + 2r^2\frac{\partial^2 K}{\partial t\partial r} = 0, \\
\mathcal{E}_{rr} &= -2\frac{\lambda+1}{1-r_s/r}H_0(t,r) + \frac{2}{1-r_s/r}H_2(t,r) \\
&\quad + \frac{2\lambda}{1-r_s/r}K(t,r) + 2r\frac{\partial H_0}{\partial r} - r\frac{2r-r_s}{2(r-r_s)}\frac{\partial K}{\partial r} \\
&\quad - \frac{4r^2}{r-r_s}\frac{\partial H_1}{\partial t} + \frac{2r^4}{(r-r_s)^2}\frac{\partial^2 K}{\partial t^2} = 0, \\
\mathcal{E}_{t\theta} &= -\frac{r_s}{r}H_1(t,r) - (r-r_s)\frac{\partial H_1}{\partial r} + r\frac{\partial H_2}{\partial t} + r\frac{\partial K}{\partial t} = 0, \\
\mathcal{E}_{r\theta} &= \frac{2r-3r_s}{2(r-r_s)}H_0(t,r) - \frac{2r-r_s}{2(r-r_s)}H_2(t,r) - r\frac{\partial H_0}{\partial r} \\
&\quad + r\frac{\partial K}{\partial r} + \frac{r^2}{r-r_s}\frac{\partial H_1}{\partial t} = 0,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{E}_{\theta\theta} &= \frac{2r+r_s}{2}\frac{\partial H_0}{\partial r} + \frac{2r-r_s}{2}\frac{\partial H_2}{\partial r} - (2r-r_s)\frac{\partial K}{\partial r} \\
&\quad + r(r-r_s)\frac{\partial^2 H_0}{\partial r^2} - r(r-r_s)\frac{\partial^2 K}{\partial r^2} - r\frac{2r-r_s}{r-r_s}\frac{\partial H_1}{\partial t} \\
&\quad - 2r^2\frac{\partial^2 H_1}{\partial t\partial r} + \frac{r^3}{r-r_s}\frac{\partial^2 H_2}{\partial t^2} + \frac{r^3}{r-r_s}\frac{\partial^2 K}{\partial t^2} = 0, \\
\mathcal{E}_{\theta\varphi} &= H_0(t,r) - H_2(t,r) = 0. \qquad \text{(B1)}
\end{aligned}
$$

The equations of motion $\mathcal{E}_{t\varphi}=0$, $\mathcal{E}_{r\varphi}=0$ and $\mathcal{E}_{\varphi\varphi}=0$ are identical to $\mathcal{E}_{t\theta}=0$, $\mathcal{E}_{r\theta}=0$ and $\mathcal{E}_{\theta\theta}=0$, respectively.

We can immediately solve the last equation of the system (B1) and replace $H_2$ by $H_0$ in all the other equations. We thus get six equations for only three independent functions $K$, $H_0$ and $H_1$, and we want to extract three "simple" independent equations out of them. One can then note that the combination

$$
\mathcal{E} \equiv \frac{ir_s}{4\omega r(r-r_s)}\mathcal{E}_{tr} + \frac{1}{2}\mathcal{E}_{rr} + \mathcal{E}_{r\theta} \qquad \text{(B2)}
$$

is purely algebraic, i.e., it does not involve any derivatives of the functions. Moreover, we find that the system $\mathcal{E}_{tr}$, $\mathcal{E}_{t\theta}$, $\mathcal{E}_{r\theta}$, $\mathcal{E}$ enables us to recover $\mathcal{E}_{tt}$ and $\mathcal{E}_{\theta\theta}$ so that we can restrict immediately to the system formed by these four equations which, after some simple calculations, are given by the system of differential equations

$$
\begin{aligned}
K'(r) - \frac{1}{r}H_0(r) - \frac{i(\lambda+1)}{\omega r^2}H_1(r) + \frac{1}{r}\frac{2r-3r_s}{2(r-r_s)}K(r) &= 0, \\
H_1'(r) + \frac{i\omega r}{r-r_s}H_0(r) + \frac{r_s}{r(r-r_s)}H_1(r) + \frac{i\omega r}{r-r_s}K(r) &= 0, \\
H_0'(r) - K'(r) + \frac{r_s}{r(r-r_s)}H_0(r) + \frac{i\omega r}{r-r_s}H_1(r) &= 0,
\end{aligned}
\qquad \text{(B3)}
$$

together with the algebraic equation

$$
\left(\frac{3r_s}{r}+2\lambda\right)H_0(r) + \left(\frac{ir_s(\lambda+1)}{\omega r^2} - 2i\omega r\right)H_1(r) + \frac{3r_s^2 + 2r_s(2\lambda-1)r - 4\lambda r^2 + 4\omega^2 r^4}{2r(r-r_s)}K(r) = 0.
$$

One equation is still redundant. However, we can solve the algebraic equation for $H_0$ and substitute its expression into the first three equations. This shows that the third is not independent from the first two. Finally, we obtain

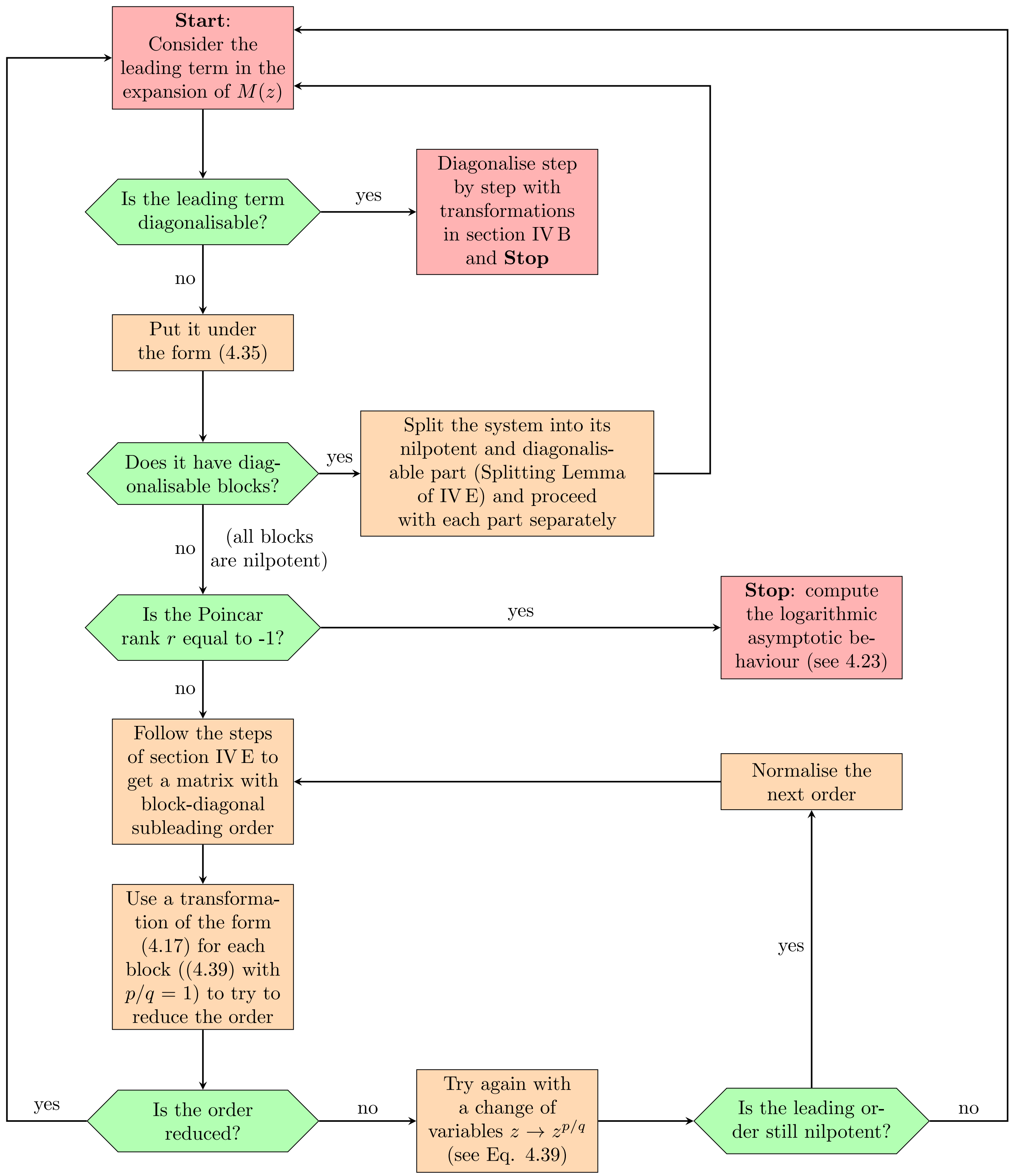
$$
\begin{aligned}
K'(r) &= \frac{3r_s^2 + r_s(\lambda-2)r - 2\omega^2 r^4}{r(r-r_s)(3r_s+2\lambda r)}K(r) \\
&\quad + \frac{i}{\omega r^2}\left(\lambda + 1 + \frac{-r_s(\lambda+1) + 2\omega^2 r^3}{3r_s + 2\lambda r}\right)H_1(r), \\
H_1'(r) &= \frac{ir(9r_s^2 + 8r_s(\lambda-1)r - 8\lambda r^2 + 4\omega^2 r^4)}{2(r-r_s)^2(3r_s+2\lambda r)}\omega K(r) \\
&\quad - \frac{3r_s^2 + r_s(1+3\lambda)r - 2\omega^2 r^4}{r(r-r_s)(3r_s+2\lambda r)}H_1(r), \qquad \text{(B4)}
\end{aligned}
$$

and we obtain the required form (2.14) with the definitions $X_1(r) \equiv K(r)$ and $X_2(r) \equiv H_1(r)/\omega$.

## APPENDIX C: FLOWCHART FOR THE ALGORITHM

In this appendix, we draw a flowchart to illustrate the algorithm that we are using to compute the asymptotic behavior of a solution of a first order system.

It should be noted that, in principle, one can skip the first question *"Is the leading term diagonalizable?"* and put directly the leading order term in its Jordan form. Indeed, when the leading term is diagonalizable, putting it into its Jordan form is equivalent to diagonalizing it and the resulting Jordan matrix is made of $d$ one-dimensional blocks where $d$ is the dimension of the system, thus of the matrix. Therefore, the procedure for splitting the system into several subsystems described in Sec. IV E is in this case equivalent to the procedure described in Sec. IV B where we are treating several blocks.


**Start**:
Consider the
leading term in the
expansion of $M(z)$
Is the leading term
diagonalisable?
yes
Diagonalise step
by step with
transformations
in section IV B
and **Stop**
no
Put it under
the form (4.35)
Does it have diag-
onalisable blocks?
yes
Split the system into its
nilpotent and diagonalis-
able part (Splitting Lemma
of IV E) and proceed
with each part separately
no
(all blocks
are nilpotent)
Is the Poincar
rank $r$ equal to -1?
yes
**Stop**: compute
the logarithmic
asymptotic be-
haviour (see 4.23)
no
Follow the steps
of section IV E to
get a matrix with
block-diagonal
subleading order
Normalise the
next order
Use a transforma-
tion of the form
(4.17) for each
block ((4.39) with
$p/q = 1$) to try to
reduce the order
yes
yes
Is the order
reduced?
no
Try again with
a change of
variables $z \to z^{p/q}$
(see Eq. 4.39)
Is the leading or-
der still nilpotent?
no

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