

Hamiltonian formulation of an effective modified gravity with nondynamical background fields

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The current paper is dedicated to developing a $(3 + 1)$ decomposition for the minimal gravitational Standard-Model Extension. Our setting is explicit diffeomorphism violation and we focus on the background fields known in the literature as u and $s^{\mu\nu}$. The Hamiltonian formalism is developed for these contributions, which amounts to deriving modified Hamiltonian and momentum constraints. We then study the connection between these modified constraints and the modified Einstein equations. Implications are drawn on the form of the background fields to guarantee the internal consistency of the corresponding modified-gravity theories. In the course of our analysis, we obtain a set of consistency requirements for u and certain sectors of $s^{\mu\nu}$. We argue that the constraint structure remains untouched when these conditions are satisfied. Our results shed light on explicit violations of diffeomorphism invariance and local Lorentz invariance in gravity. They may turn out to be valuable for developing a better understanding of effective modified-gravity theories.

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I. INTRODUCTION

Lorentz invariance is one of the foundations of the current scientific paradigm that has shaped our understanding of nature at both small and large length scales. The Standard Model (SM) rests on global Lorentz symmetry and provides a description of elementary particles in terms of quantum fields defined in Minkowski spacetime. This fundamental symmetry implies that measurements performed in two identical experiments moving uniformly with respect to each other provide results based on the same laws of nature connected by a Lorentz transformation. Thus, the form of the laws of nature does not depend on the state of uniform motion. An analogous property exists for measurements made with an apparatus and an identical rotated one. The latter viewpoint is called particle Lorentz invariance in the literature.¹

General Relativity (GR) is a theory based on a dynamical spacetime in which the physical laws are invariant with respect to diffeomorphisms. It also exhibits local Lorentz symmetry in the tangent space T_p at a point p of a spacetime manifold \mathcal{M} [1,2], such that the results of

two measurements of the same quantity are connected with each other by a local Lorentz transformation. This applies, in particular, when the first is performed in a freely falling inertial reference frame and the second in a boosted frame with respect to the first. The same holds true for two freely falling inertial frames whose axes enclose a fixed angle. The vierbein (tetrad) formalism allows us to transform from a general spacetime frame described by the metric $g_{\mu\nu}$ of the (curved) manifold \mathcal{M} to a freely falling frame at a particular spacetime point where the metric corresponds to that of Minkowski spacetime. Such a transformation makes local Lorentz invariance explicit.

GR also exhibits invariance under general coordinate transformations. The latter relate the same objects in the manifold, such as points and curves, expressed in different coordinates with each other. In contrast, diffeomorphisms establish relations between different objects without changing the coordinates. They can be interpreted as spacetime-dependent translations and are the active counterparts of general coordinate transformations (see, e.g., page 133 of [2]). In particular, diffeomorphisms are represented by differentiable maps from a manifold onto itself, $\mathcal{M} \rightarrow \mathcal{M}$, where the corresponding inverse maps are also required to be differentiable.²

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¹The concept of “invariance under active Lorentz transformations” is frequently used in Minkowski spacetime, but such a notion does not correspond to particle Lorentz invariance in the presence of background fields. A background field would transform like a four-tensor under active Lorentz transformations, but remains fixed under particle Lorentz transformations, since it is beyond control through experimentalists.

²Manifolds that are related by a diffeomorphism can be considered as geometrically equivalent. A diffeomorphism may change how a manifold is embedded into an ambient space, but its intrinsic geometry remains untouched.

Because of diffeomorphism invariance, only two of the ten metric components correspond to physical, propagating degrees of freedom. This property is made transparent in the $(3+1)$ decomposition of spacetime developed by Arnowitt, Deser, and Misner (denoted as the ADM decomposition in the remainder of the paper) [3,4]. In the ADM formulation, four-dimensional spacetime is foliated with spacelike hypersurfaces (of constant time) that evolve with respect to time. The foliation itself is governed by a lapse function N and a shift vector with components N^i and permits constructing a Hamiltonian associated with GR. The conjugate momenta for the spatial metric components $g_{ij} \equiv q_{ij}$ are nonzero, whereas those associated with N and N^i are zero showing that there are four primary first-class constraints in GR [5]. The latter exist due to the covariant structure of this theory. Therefore, N and N^i are unphysical and represent gauge degrees of freedom. In addition, the Hamiltonian and momentum constraints arise as a set of four secondary first-class constraints. They generate spacetime diffeomorphisms and spatial diffeomorphisms, respectively.

A violation of (local) Lorentz invariance is the most prominent signal for physics at the Planck scale, which was shown to arise in particular string field theories [6–10] as well as in loop quantum gravity [11,12]. Furthermore, this effect occurs in other settings such as noncommutative spacetime structures [13–15], spacetime foams [16–18], nontrivial spacetime topologies [19–22], and Hořava-Lifshitz gravity [23]. The strong suppression of Lorentz violation effects at low energy scales has led to the necessity of conceiving ultrasensitive tests for their possible detection [24].

The Standard-Model Extension (SME) [25–27] is a comprehensive field theory framework to parametrize deviations from Lorentz invariance, and it allows us to compare the results of different experiments with each other. The incorporation of Lorentz violation in the SME is through background fields arising as vacuum expectation values of tensor-valued fields. The latter imply preferred spacetime directions and involve controlling coefficients that describe the strength of Lorentz violation. Suitable contractions of background fields with SM field operators result in expressions invariant under coordinate transformations (observer Lorentz transformations). The minimal SME includes field operators of mass dimensions 3 and 4, whereas the nonminimal SME contains field operators of mass dimensions ≥ 5 [28–30]. Since the controlling coefficients in a nongravitational context are usually³ assumed to be independent of the spacetime coordinates, the SME exhibits translation invariance. As a consequence, energy and momentum are conserved quantities.

³One of the few studies on effects related to spacetime-dependent coefficients in Minkowski spacetime is provided by [31].

In the presence of gravity, the situation is more subtle. The notions of global Lorentz violation and translation noninvariance in Minkowski spacetime are replaced by two fundamentally distinct concepts: local Lorentz violation and diffeomorphism violation. To incorporate these concepts into the SME language, a generic background field in a curved spacetime manifold \mathcal{M} must have both contributions defined in a spacetime frame and contributions given in a local inertial reference frame. A gravitational version of the SME has been put forward in a series of papers [32–39] to study these aspects. The spacetime manifold itself is described by the metric tensor $g_{\mu\nu}$ and formal transitions between a spacetime frame and a local inertial frame are provided by vierbeins e_μ^a . In the pure-gravity sector, background fields in spacetime frames are suitably contracted with objects built from the Riemann curvature tensor, covariant derivatives, the Levi-Civita tensor, and the spin connection ω_μ^{ab} that endows spacetime with a spin structure. These terms are constructed in a way such that general coordinate invariance is maintained. The presence of a background field in a local inertial frame implies that the form of the laws of nature is different in frames that are boosted or rotated with respect to the original one.

The most recent article [39] rests on a better understanding of diffeomorphism and local Lorentz violation in gravity acquired since the base of the gravitational SME was laid in [32]. It widely extends the findings of [32] and introduces additional concepts such as global local Lorentz transformations and manifold Lorentz transformations that are combinations of diffeomorphisms and local Lorentz transformations. The latter can be considered analogous to global Lorentz transformations in Minkowski spacetime. An almost flat spacetime setting, which is sufficient for various studies in practice such as propagating gravitational waves [35,38,40,41] or modified dispersion relations in linearized gravity [42], is also introduced. Therefore, [39] provides many additional possibilities of how to construct terms leading to observable pure-gravity effects that are not in accordance with GR. By employing a powerful notation, an infinite number of such terms covering the mass dimensions ≤ 8 is stated. Furthermore, contributions involving individual matter fields as well as those endowed with the entire SM gauge symmetry are compiled, too.

It is paramount to emphasize that in the setting of nondynamical background fields in gravity, the physics of a background field depends on whether it is described by a contravariant, covariant or mixed observer tensor, i.e., the position of spacetime indices plays a crucial role. The reason is that the spacetime metric changing the index position is a dynamical object proper, i.e., it must be taken into account in variations of the action. The authors of [39] emphasize this issue again (although not for the first time). Thus, explicitly Lorentz-violating contributions formulated in terms of a covariant background field or a contravariant

one (in a spacetime frame) must be considered as distinct models. To avoid conflicts, we will state background fields with upper indices only as done in [32]. Such a distinction is unnecessary for local Lorentz indices, since the Minkowski metric, which is used to lower and raise these indices, is a nondynamical object by definition.

In a gravitational field, the notion of a constant background field loses its meaning. While a covariantly constant background field cannot even be defined in most curved manifolds that are of interest in gravity [32], even such a field would depend on the spacetime coordinates. Therefore, apart from local Lorentz violation, the gravitational SME may exhibit diffeomorphism violation,⁴ in general [32,39]. For the effective modified-gravity theory⁵ that we will be considering, any variation in the number of degrees of freedom and any symmetry departure, including a violation of diffeomorphism invariance, are expected to show up in the constraint structure as well as the Poisson algebra between the canonical variables and the constraints. This assertion is particularly true when counting the number of degrees of freedom, which depends crucially on the number of first- and second-class constraints.

In the current paper, we intend to apply the ADM decomposition to two sectors of the minimal gravitational SME to understand the implications of explicit diffeomorphism violation on the gravitational degrees of freedom. The ADM decomposition is more than suitable for such an analysis, since it renders the constraint structure transparent. Within the effective framework, we consider the following three points as crucial:

- (i) Obtain the standard results of GR in the limit of vanishing controlling coefficients.
- (ii) Maintain the same number of physical, propagating degrees of freedom (2) in the modified-gravity theory.
- (iii) Implement the diffeomorphism group $\text{Diff}(\mathcal{M})$ in the sense of GR as differentiable mappings from a manifold onto itself, $\mathcal{M} \rightarrow \mathcal{M}$, given by the transformation $f: x^\mu \rightarrow f^\mu(x)$.

An investigation of the minimal gravitational SME by means of the ADM formulation [46,47] has been published recently, i.e., our study has some overlap with the latter article. However, we will focus on other aspects—including the points mentioned above—and present the results in a different manner. Note also that our background fields carry upper spacetime indices compared to those used in the latter papers. Thus, in light of the comments made

previously on the position of spacetime indices in theories with explicit diffeomorphism violation, our model is physically nonequivalent to the one studied in [46,47]. Furthermore, applying the ADM formalism does not require working in a weak-field regime, as it was done in [48] to find the Hamiltonian using modified Poisson brackets and deformed constraints. Thus, scenarios of strong gravitational fields in the presence of diffeomorphism violation could be studied occurring, e.g., during the creation of gravitational waves [49,50].

The paper is organized as follows. In Sec. II we explain the concepts and mathematical relationships in the ADM formalism with an emphasis on those that are of direct relevance for us. Section III provides a summary of the minimal gravitational SME. It is followed by Sec. IV that constitutes the foundation of the article for the subsequent calculations. Here we review how to derive the Hamiltonian of GR and carry out analogous computations for both the $s^{\mu\nu}$ and the u term of the gravitational SME. These studies imply modified Hamiltonian and momentum constraints in the presence of the aforementioned background fields. In Sec. V we intend to understand how the constraints and the modified Einstein equations are related with each other. This analysis will enable us to derive requirements for the internal consistency of a modified-gravity theory resting on explicit diffeomorphism violation. Section VI is dedicated to a brief investigation of the constraint structure as well as the Hamilton equations. By evaluating suitable Poisson brackets between canonical variables and constraints we will demonstrate that the modified constraints still generate both spacetime diffeomorphisms and spatial diffeomorphisms, respectively. Last but not least, our findings are concluded on in Sec. VII.

The main body of the text is dedicated to presenting and interpreting the central results as well as to providing conceptual discussions. Detailed derivations and computations are relegated to the Appendices. The latter can be skipped by readers who are primarily interested in the results and their implications, but they may be valuable to researchers who want to base their investigations on the findings of this article. Appendix A gives an account on the most important geometrical formulas that are indispensable to carry out the ADM decomposition of the SME. Computational details on constructing the Hamiltonians of the gravitational SME are presented in Appendix B. Appendix C states some remarks on a modified ADM decomposition that plays a role for a subset of the $s^{\mu\nu}$ coefficients. In the course of the investigations, it has turned out that suitable boundary terms must be included in the action and the derivation of those is shown in Appendix D. Appendix E presents the most crucial steps in deriving the modified Einstein equations from the action, as they are valuable to understand the necessity of the boundary terms mentioned before. Subsequently, in Appendix F we show in a very detailed manner how to relate the Hamiltonian and

⁴Illustrations of the concepts of general coordinate transformations and diffeomorphisms in the presence of background fields can be found in [43].

⁵Reviews on modified-gravity theories are provided by [44,45]. The latter references also include material on the gravitational SME, but the focus is on models of spontaneous diffeomorphism and Lorentz violation in this context.

momentum constraints to the modified Einstein equations. Useful formulas on the ADM decomposition of covariant derivatives of background fields are derived here that are probably not to be found anywhere in the existing literature. Furthermore, in Appendix G we give calculational details on how to compute functional derivatives of the ADM-decomposed action. These results provide further support for our arguments. Last but not least, in Appendix H we introduce a helpful counting scheme based on the canonical variables of the ADM decomposition. The latter allowed us to check any relation involving ADM variables for (dimensional) consistency and turned out to be a useful tool for finding calculational errors.

II. THE ADM DECOMPOSITION

The ADM formulation of General Relativity [1,3,4] furnishes a decomposition of spacetime into space and time leading to a description of the gravitational phase space by means of a Hamiltonian. The Hamiltonian formulation starts with selecting a special foliation of the generic spacetime manifold \mathcal{M} that is to be covered by a chart of coordinates X^μ . We start by choosing a temporal coordinate that we call t where X^0 does not necessarily correspond to t . The spatial coordinates that are employed for the decomposition will be denoted as x^i . By considering a flow of time

$$t^\mu \equiv \frac{\partial X^\mu}{\partial t}, \quad (1)$$

the four-dimensional manifold \mathcal{M} decomposes into space-like hypersurfaces Σ_t at fixed instants of time t .

We pick two spacelike hypersurfaces Σ_t and Σ_{t+dt} with constant t and $t + dt$, respectively. The lapse of proper time between the lower and upper hypersurface is Ndt , which is why the scalar function $N = N(t, x, y, z)$ is called the lapse in the literature. We define the unit timelike vector n^μ that indicates the path of proper time τ . In general, as long as t does not correspond to proper time, there is a misalignment between t^μ and n^μ :

$$N^\mu \equiv t^\mu - Nn^\mu. \quad (2)$$

The latter vector is called the shift and it depends on both time and the spatial coordinates.

To grasp a better understanding of the foliation, we provide an illustrative interpretation (see Ref. [1]) of the lapse N and the shift N^μ , cf. Fig. 1. Let the two hypersurfaces Σ_t, Σ_{t+dt} be modeled by metal sheets and let them be separated by connectors welded at both sheets. The connectors have a well-defined length Ndt that the lapse is characteristic for. The latter is not necessarily constant, but depends on which hypersurfaces are connected to each other as well as where the connectors are placed.

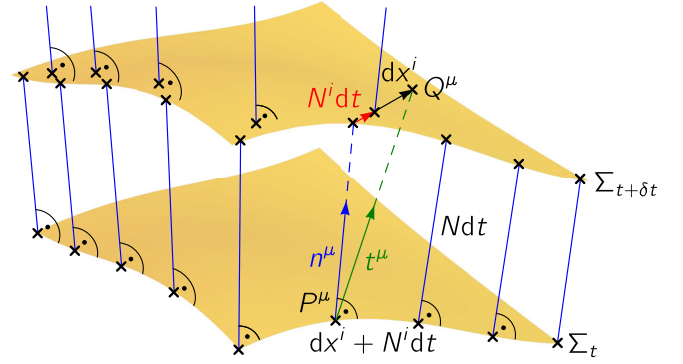


FIG. 1. Two hypersurfaces Σ_t and Σ_{t+dt} that are linked via connectors illustrated by blue lines. A connector links the point $P^\mu = (t, x^i)$ on the lower sheet to the point $Q^\mu = (t + dt, x^i - N^i dt)$ on the upper one. The axis of a connector points along the direction n^μ . The four-vector t^μ indicates the direction between the point (t, x^i) on the lower hypersurface and $(t + dt, x^i + dx^i)$ on the upper one. The points where the connectors are welded are represented by crosses (see also Ref. [1]).

We consider a particular connector welded to the lower sheet at a point with spatial coordinates x^i .

To construct a stable and rigid structure, the connectors must be orthogonal to the lower sheet at the point x^i . As the upper sheet differs from the lower one, this connector is not necessarily orthogonal to the upper sheet, as well. This also means that the connector linking Σ_{t+dt} and Σ_{t+2dt} cannot be placed at the same spatial coordinates x^i on the upper sheet, as it would not be orthogonal to this sheet at x^i . For the connector to be orthogonal, it must be placed at a point $x^i - N^i dt$ on Σ_{t+dt} . This requirement introduces a vector N with components $N^i = N^i(t, x, y, z)$ that is tangent to the sheet and corresponds to the spatial components of N^μ introduced in Eq. (2). We deduce that $N^0 = 0$, i.e., N^μ is purely spacelike. Also, the shift vector is zero when the connector is automatically perpendicular to the second sheet such that the next connector can be placed directly above the previous one. This scenario does not occur in general, though.

With this construction in mind, we consider a point (t, x^i) on Σ_t and move to another point $(t + dt, x^i + dx^i)$ on Σ_{t+dt} by following the flow of time; see Eq. (1). Let the geometry of the hypersurface Σ_t be described by the three-metric $q_{ij} = q_{ij}(t, x^i)$. The latter corresponds to the spatial components of the metric $g_{\mu\nu} = g_{\mu\nu}(t, x^i)$ describing the geometry of the ambient spacetime. To compute the infinitesimal path length interval squared between the points (t, x^i) and $(t + dt, x^i + dx^i)$, we need the following ingredients. The infinitesimal distance perpendicular to the lower hypersurface is Ndt , as we have already argued above. Furthermore, the infinitesimal vector pointing from x^i to $x^i + dx^i$ in the lower hypersurface is $dx^i + N^i dt$. The path length interval squared then reads

$$\begin{aligned} ds^2 &= q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - N^2 dt^2 \\ &= q_{ij}dx^i dx^j + 2N_i dt dx^i + (N_a N^a - N^2) dt^2. \end{aligned} \quad (3)$$

The spatial metric q_{ij} is employed to pull indices up and down of the shift vector, as the latter lives completely in the hypersurface. Identifying the path length interval of Eq. (3) with

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (4)$$

we identify the decomposed spacetime metric as

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_j \\ N_i & q_{ij} \end{pmatrix}. \quad (5)$$

Its inverse can be cast into the form

$$g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N^i/N^2 & q^{ij} - N^i N^j/N^2 \end{pmatrix}. \quad (6)$$

The four-vector n^μ points along the difference between the two points P^μ and Q^μ of Fig. 1 lying on top of each other:

$$Q^\mu - P^\mu \equiv N n^\mu dt = \begin{pmatrix} 1 \\ -N^i \end{pmatrix} dt, \quad (7)$$

which is why

$$n^\mu = \begin{pmatrix} 1/N \\ -N^i/N \end{pmatrix}. \quad (8)$$

Lowering the index with the spacetime metric implies that n_μ is purely timelike:

$$n_\mu = \begin{pmatrix} -N \\ 0^i \end{pmatrix}. \quad (9)$$

We define

$$q^{\mu\nu} \equiv g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0^j \\ 0^i & q^{ij} \end{pmatrix}, \quad (10)$$

which is interpreted as the induced (inverse) metric on a spatial hypersurface Σ_t generalized to \mathcal{M} . The lower (3×3) block can contain nonzero entries only. Pulling down the second index with the metric of Eq. (5) implies

$$q^\mu{}_\nu = \begin{pmatrix} 0 & 0^j \\ N^i & \delta^i_j \end{pmatrix}. \quad (11)$$

The latter tensor satisfies the important property

$$q^\mu{}_\nu q^\nu{}_\rho = q^\mu{}_\rho. \quad (12)$$

i.e., it is a projector. Furthermore, it obeys

$$q^\mu{}_\nu n^\nu = 0, \quad (13)$$

which is why it can be employed to project vectors and tensors defined on \mathcal{M} into the hypersurface Σ_t . Note that the projector as stated in Eq. (11) only involves non-dynamical objects by definition.

In this context we would like to comment on a set of coordinates that can be very valuable when dealing with particular problems: Gaussian normal coordinates (also known as synchronous coordinates). The latter are characterized by the choices $N = 1$ and $N^i = 0$, i.e., the lapse function is a coordinate-independent scalar and the shift vector is discarded. Then, the time coordinate corresponds to proper time for an observer remaining at fixed spatial coordinates. Furthermore, the unit vector pointing along time is perpendicular to the unit vectors pointing along each spatial dimension (see, e.g., page 717 of [1]). In Gaussian normal coordinates it holds that $n^\mu = t^\mu$ [see Eq. (2)]. We will be referring to these coordinates at some points in the paper.

III. THE SME GRAVITY SECTOR

The action of the minimal gravitational SME is a modification of the Einstein-Hilbert (EH) action that is invariant with respect to general coordinate transformations [32,39]. It is written as

$$S_g = \int_{\mathcal{M}} d^4x (\mathcal{L}^{(0)} + \mathcal{L}'), \quad (14)$$

with

$$\mathcal{L}^{(0)} = \frac{\sqrt{-g}}{2\kappa} {}^{(4)}R, \quad (15a)$$

$$\mathcal{L}' = \frac{\sqrt{-g}}{2\kappa} (k_R)^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}. \quad (15b)$$

Here, $\mathcal{L}^{(0)}$ is the EH Lagrangian without cosmological constant and \mathcal{L}' is a minimal SME term containing a background field $(k_R)^{\mu\nu\rho\sigma}$ that transforms as a four-tensor under general coordinate transformations. Besides, $\kappa = 8\pi G_N$ with Newton's constant G_N , $g \equiv \det(g_{\mu\nu})$ and ${}^{(4)}R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor defined in the four-dimensional spacetime manifold \mathcal{M} . Its single contraction ${}^{(4)}R_{\mu\nu} \equiv {}^{(4)}R^{\rho}{}_{\mu\rho\nu}$ is the Ricci tensor and its double contraction ${}^{(4)}R \equiv {}^{(4)}R^\mu{}_\mu$ corresponds to the Ricci scalar.

We work in a scenario of explicit diffeomorphism violation in gravity, i.e., $(k_R)^{\mu\nu\rho\sigma}$ is a nondynamical tensor-valued background field defined in a spacetime frame of the curved manifold \mathcal{M} . For simplicity and as we do not consider spontaneous diffeomorphism violation

in this article, no confusion should arise by omitting the bar proposed to be put on top of such coefficients [39]. However, to clarify the physics in local frames, we can benefit from considering a background vierbein denoted by $\langle e \rangle_\mu^a$ that arises, in principle, from solving the Einstein field equations in the vacuum [39].

In contrast to a scenario in Minkowski spacetime, a globally constant tensor in the sense of a vanishing covariant derivative at each point in \mathcal{M} does not exist for a general manifold. Therefore, we must assume that the background field is coordinate dependent: $(k_R)^{\mu\nu\rho\sigma} = (k_R)^{\mu\nu\rho\sigma}(x)$. According to the third line of Table II in [39], a term of the form of Eq. (15b) does not imply local Lorentz violation at the level of the Lagrange density. However, suitable combinations of the coefficients with background vierbeins may give rise to preferred directions in local frames at each point x^μ of the manifold implying local Lorentz violation. We will come back to this point later.

The no-go result of the SME gravity sector [32,39] implies that explicit diffeomorphism and local Lorentz violation clash with specific properties of (pseudo-)Riemannian geometry such as the second Bianchi identity of the Riemann curvature tensor. This finding requires that certain Noether identities linked to the invariance under general coordinate transformations must be satisfied for a consistent setting [51,52]. A descriptive interpretation of the no-go result is that a nondynamical background field cannot absorb or emit momentum, which is why it cannot account for the momentum transfer of a test particle moving along a geodesic in \mathcal{M} [51]. This issue is neatly avoided by considering spontaneous Lorentz violation, i.e., a dynamical background field that satisfies its own field equations. In such scenarios both massless and massive propagating modes of the background field can be excited. The massless modes correspond to propagating fluctuations of preferred directions, whereas the massive modes are interpreted as fluctuations of the strength of Lorentz violation or, in other words, the size of the controlling coefficients.

Bjorken initially proposed the idea of spontaneous Lorentz violation back in 1963 [53] to explain the photon as a massless Goldstone boson. In the aftermath, further physicists took this idea over to gravity to interpret the graviton as a Goldstone boson linked to a spontaneous breakdown of local Lorentz and diffeomorphism invariance [54,55]. More recent works in the context of electrodynamics are [56–58] and for gravity we refer to [59–62] where the mechanism suggested in [59] is even considered as a solution of the cosmological-constant problem. In the context of gravity, it was demonstrated that an alternative gravity theory called cardinal gravity [61,62] can be constructed by means of a bootstrap method from a linearized theory with a two-tensor field that undergoes spontaneous Lorentz violation. At energies much lower

than the Planck scale, this theory corresponds to GR, but it significantly differs from GR near the Planck energy.

Spontaneous Lorentz violation was investigated in great detail in toy theories known as bumblebee models in Minkowski spacetime [63–67] as well as in the presence of gravity [68,69]. The focus in the latter works lies on a profound understanding of the Goldstone and Higgs-like modes. A particular model of a purely timelike vector field and its implications for matter particles was analyzed in [70]. References [71–73] give an account of a scenario of an antisymmetric two-tensor acquiring a vacuum expectation value. Papers have also been written on black-hole solutions in the presence of bumblebee-type Lorentz violation [74–78]. Note that spontaneous Lorentz violation was demonstrated to occur in open-string field theory, too [6–10]. The latter finding was a motivation for constructing a comprehensive low-energy effective field theory framework for Lorentz violation that we now know as the SME.

An alternative to spontaneous Lorentz and diffeomorphism violation could be explicit symmetry violation in a more general geometry that does not rely on the quadratic restriction of path length functionals in Riemannian geometry [79,80]. A promising framework is Finsler geometry [81–83], as it was shown that classical-particle analogs based on the SME [84] move along geodesics in certain Finsler spaces [85,86]. This discovery stimulated a vast series of articles on classical-particle descriptions in Lorentz-violating background fields as well as their connections to Finsler geometry [87–100].

However, recent findings suggest that the no-go result is not as restrictive as it was believed to be for almost 15 years. There may be scenarios of explicit Lorentz violation and/or diffeomorphism violation in gravity where the Stückelberg trick can be used to create massless propagating modes by introducing a set of additional scalar fields [101]. This construction allows for a consistent description of explicit symmetry violations that does not contradict the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ (where $G^{\mu\nu}$ is the Einstein tensor). Also, it is possible to satisfy the consistency conditions perturbatively in certain cases [102]. Recently, even analyses of experimental data [103,104] were carried out to constrain coefficients giving rise to explicit symmetry violation in gravity. We take these findings as a justification for considering explicit diffeomorphism violation in a Riemannian setting. In this work, we will also provide new insights and precise results in such a context.

The minimal-gravity modification of Eq. (15b) is usually rewritten as follows [32]:

$$\mathcal{L}' = \frac{\sqrt{-g}}{2\kappa} (\mathcal{L}'^{(u)} + \mathcal{L}'^{(s)} + \mathcal{L}'^{(t)}), \quad (16a)$$

with

$$\mathcal{L}'^{(u)} = -u^{(4)}R, \quad (16b)$$

$$\mathcal{L}'^{(s)} = s^{\mu\nu(4)}R_{\mu\nu}, \quad (16c)$$

$$\mathcal{L}'^{(t)} = t^{\mu\nu\rho\sigma(4)}R_{\mu\nu\rho\sigma}. \quad (16d)$$

The decomposition of \mathcal{L}' shown above has turned out to be valuable. First, $u = u(x)$ is a Lorentz scalar dependent on spacetime position, which is why it implies diffeomorphism violation. Derivatives of $u(x)$ for the coordinates can give rise to a preferred direction in a local frame [39,105]. Therefore, this contribution is also able to induce local Lorentz violation without this being obvious at a first place. In the setting of spontaneous diffeomorphism violation with only weak gravitational fields present, u can be removed by a field redefinition (see Sec. III in [33]). We will see that the u term is more subtle in the presence of explicit diffeomorphism violation. Second, $s^{\mu\nu} = s^{\mu\nu}(x)$ is a two-tensor-valued background field that can be taken as symmetric due to the symmetry of the Ricci tensor.

The four-tensor-valued background field $t^{\mu\nu\rho\sigma} = t^{\mu\nu\rho\sigma}(x)$ has the symmetries of the Riemann tensor. It does not seem to play a role in physical observables in the post-Newtonian limit and this interesting observation was coined the “t puzzle” [106]. The reason for that peculiar property seems to be the approximative scheme employed in most phenomenological studies of local Lorentz and diffeomorphism violation in gravity, because it relies on an asymptotically flat spacetime. If this assumption is not made such as in cosmology, $t^{\mu\nu\rho\sigma}$ can give rise to significant effects providing tight constraints on these coefficients [102,107].

Note that the following form of the Lagrange density expressed in terms of irreducible pieces of the Riemann curvature tensor is sometimes employed [33]:

$$\mathcal{L}'' = \frac{\sqrt{-g}}{2\kappa} (-u^{(4)}R + s^{\mu\nu(4)}R_{\mu\nu}^T + t^{\mu\nu\rho\sigma(4)}C_{\mu\nu\rho\sigma}), \quad (17)$$

with the trace-free Ricci tensor ${}^{(4)}R_{\mu\nu}^T$ and the four-dimensional Weyl tensor ${}^{(4)}C_{\mu\nu\rho\sigma}$. The latter inherits all the symmetries from the Riemann tensor, but it does not involve nonvanishing traces, anymore. Hence, by using Eq. (17), the trace of $s^{\mu\nu}$ as well as the single and double traces of $t^{\mu\nu\rho\sigma}$ have been extracted, which leaves nine independent components of $s^{\mu\nu}$ and ten of $t^{\mu\nu\rho\sigma}$ [32]. However, throughout the paper we will be employing the form of Eq. (16). As will become clear later, special care has to be taken in our setting when performing field redefinitions to move such traces from one term to another. In the context of spontaneous diffeomorphism violation, Eq. (17) can be interpreted as following directly from Eq. (15b) by extracting the single and double traces of the Riemann curvature tensor. When diffeomorphism invariance is violated explicitly, though, the background fields u ,

$s^{\mu\nu}$, and $t^{\mu\nu\rho\sigma}$ in Eq. (16) should be taken as definitions independent of $(k_R)^{\mu\nu\rho\sigma}$ occurring in Eq. (15b). Furthermore, u , $s^{\mu\nu}$, and $t^{\mu\nu\rho\sigma}$ of Eq. (16) are assumed to be independent of each other, i.e., we will leave traces where they are and avoid transferring them between different contributions.

Although the background fields u , $s^{\mu\nu}$, and $t^{\mu\nu\rho\sigma}$ do not imply local Lorentz violation at the level of the Lagrange density, quantities like $s^{ab} \equiv s^{\mu\nu} \langle e \rangle_\mu^a \langle e \rangle_\nu^b$ and $t^{abcd} \equiv t^{\mu\nu\rho\sigma} \langle e \rangle_\mu^a \langle e \rangle_\nu^b \langle e \rangle_\rho^c \langle e \rangle_\sigma^d$ (that must be interpreted as mere definitions) can give rise to preferred orientations in a local frame where an experiment is performed [39]. Considering the modification of the dispersion relation of gravitational waves in the regime of weak gravitational fields, $t^{\mu\nu\rho\sigma}$ leads to birefringence in contrast to $s^{\mu\nu}$ that does not do so [35]. Therefore, there is an analogy between $s^{\mu\nu}$ ($t^{\mu\nu\rho\sigma}$) and the nonbirefringent (birefringent) part of the *CPT*-even background field $(k_F)^{\mu\nu\rho\sigma}$ of the SME photon sector. Indeed, appropriate field redefinitions allow for transforming between photon sector and gravity sector coefficients such that matter-gravity experiments are only sensitive to combinations of such coefficients [101].

IV. HAMILTONIAN FORMULATION OF A GRAVITY THEORY

The Hamiltonian formulation of GR is the foundation for many prototypes of quantum gravity [108] as well as field theories coupled to gravity (see [109] for an example involving Schrödinger theory) and modified-gravity theories, in general [110,111]. The Hamiltonian itself is a powerful tool to define the total mass, momentum, and angular momentum of a gravitational system. The ADM formalism provides a set of suitable canonical variables and a means to obtain the GR Hamiltonian. It permits gaining a better understanding of the physics on hypersurfaces characteristic for a particular spacetime, e.g., the event horizon of a black hole, which is a null hypersurface in the corresponding four-dimensional spacetime [112,113].

The Hamiltonian formulation also uncovers that GR is characterized by constraints [5]. Theories subject to constraints play a pivotal role in physics [114]. A constraint is a relation between the canonical variables that reduces the number of variables that can be considered as physical. Hence, in a constrained theory not every canonical variable does necessarily describe a physical degree of freedom, but there are some variables that correspond to mere gauge degrees of freedom. Constraints appear, e.g., in classical (non)relativistic mechanics, electrodynamics, and GR [5]. In the following sections, we will first of all obtain modified constraints where their structure will only be revealed in Sec. VI towards the end of the main text.

Note that the constraints of GR are given in terms of the canonical variables. However, a Hamiltonian in the context of the ADM formulation is first obtained as a function of

the extrinsic curvature describing the embedding of space-like hypersurfaces Σ_t into the spacetime manifold \mathcal{M} . The latter must be eliminated in favor of the canonical momentum density to obtain the relationships that are usually interpreted as constraints in phase space. Nevertheless, we will sometimes also employ the terminology ‘‘constraint’’ for the original relationships depending on the extrinsic curvature.

A. General relativity

We start by briefly reviewing how to derive the Hamiltonian associated with GR in the context of the ADM formalism. Consider the EH action without cosmological constant:

$$S^{(0)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} {}^{(4)}R. \quad (18)$$

With the help of the expression in Eq. (A19), the EH action can be written as [108]

$$S^{(0)} = \int_{\mathcal{M}} d^4x \frac{N\sqrt{q}}{2\kappa} [R - K^2 + K_{ij}K^{ij} + 2\nabla_\mu(n^\mu K - a^\mu)], \quad (19)$$

where $q \equiv \det(q_{ij})$, R is the Ricci scalar associated with Σ_t , and K_{ij} the extrinsic-curvature tensor defined by

$$K_{ij} = \frac{1}{2N} (\dot{q}_{ij} - D_i N_j - D_j N_i). \quad (20)$$

Furthermore, $K \equiv K^i_i$ corresponds to the trace of the latter and a_μ is the acceleration that is linked to the derivative of the lapse function [see Eq. (A1)]. In what follows, we will discard the total covariant derivative (last term) in Eq. (19), which leads to a boundary term when the integral over Σ_t is computed. Although this procedure seems to be of minor importance, it turns out to be an essential point and we will come back to it in Sec. IV C.

There are ten canonical variables in GR: the lapse function N , the three shift vector components N^i and the six spatial-metric components q_{ij} . To obtain the Hamilton density associated with the EH Lagrange density, we need the time derivatives \dot{q}_{ij} that follow from Eq. (20) and are given by

$$\dot{q}_{ij} = 2NK_{ij} + D_i N_j + D_j N_i. \quad (21)$$

The canonical momentum density associated with \dot{q}_{ij} is

$$(\pi_0)^{ij} = \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{q}_{ij}} = \frac{\sqrt{q}}{2\kappa} (K^{ij} - q^{ij}K). \quad (22)$$

Considering the Legendre transformation and using Eq. (21), we obtain a preliminary form of the Hamilton density:

$$\mathcal{H}^{(0)} = -\frac{\sqrt{q}}{2\kappa} [N(R + K^2 - K_{ij}K^{ij}) + 2(KD_i N^i - K^{ij}D_i N_j)]. \quad (23)$$

An integration over the spatial hypersurface Σ_t provides the Hamiltonian. Performing suitable partial integrations and expressing the extrinsic curvature in terms of the canonical momentum leads to

$$H^{(0)} = \int_{\Sigma_t} d^3x \mathcal{H}^{(0)} = - \int_{\Sigma_t} d^3x \frac{\sqrt{q}}{2\kappa} (NC_0 + N^i C_i), \quad (24a)$$

with

$$C_0 = R + K^2 - K_{ij}K^{ij}, \quad (24b)$$

$$C_i = 2(D_j K_i^j - D_i K). \quad (24c)$$

To carry out the canonical analysis of the EH Hamilton density in Eq. (24), we must express the extrinsic curvature in terms of the phase space variables, in particular, the canonical momentum $(\pi_0)^{ij}$. By doing so, we obtain

$$K^{ij} = \frac{2\kappa}{\sqrt{q}} \left[(\pi_0)^{ij} - \frac{\pi_0}{2} q^{ij} \right], \quad K = -\frac{\kappa}{\sqrt{q}} \pi_0, \quad (25)$$

as well as the Hamilton density

$$\mathcal{H}^{(0)} = NC_0 + N^i C_i, \quad (26a)$$

with the Hamiltonian and momentum constraint

$$C_0 = \frac{2\kappa}{\sqrt{q}} \left[(\pi_0)^{ij} (\pi_0)_{ij} - \frac{(\pi_0)^2}{2} \right] - \frac{\sqrt{q}}{2\kappa} R, \quad (26b)$$

$$C_i = -2D_j (\pi_0)^j_i, \quad (26c)$$

where π_0 is understood as the trace of the canonical momentum: $\pi_0 \equiv (\pi_0)^i_i$. Note that the prefactor in Eq. (24) has been absorbed into C_0 and C_i . The contribution in Eq. (24a) proportional to the lapse function involves the spatial part of the Ricci scalar as well as the extrinsic curvature. The part in Eq. (24a) linear in the shift vector does not depend on the internal geometry of the spatial hypersurface, but only on the way it is embedded into the four-dimensional manifold.

B. Minimal gravitational SME

At first, we will focus on the $s^{\mu\nu}$ sector of the minimal gravitational SME defined by Eq. (16c). The background field $s^{\mu\nu}$ can be decomposed into three parts:

$$s^{\alpha\beta} = q^\alpha_\mu q^\beta_\nu s^{\mu\nu} - (q^\alpha_\nu n^\beta + q^\beta_\nu n^\alpha) s^{\nu n} + n^\alpha n^\beta s^{nn}, \quad (27)$$

where we introduce the notation $s^{ij} \equiv q^i_\mu q^j_\nu s^{\mu\nu}$ for the part projected entirely into Σ_t . In addition, we define a mixed (vectorial) part via $s^{in} \equiv q^i_\mu n_\nu s^{\mu\nu}$ as well as a scalar part $s^{nn} \equiv n_\mu n_\nu s^{\mu\nu}$ projected completely along the direction perpendicular to Σ_t . Equation (27) has to be considered as an identity. Note that we will interpret s^{in} and s^{nn} as new degrees of freedom independent of $s^{\mu\nu}$ and the lapse function N or the shift vector N^i . As long as diffeomorphism invariance is violated explicitly by the non-dynamical field $s^{\mu\nu}$, we can think of s^{in} and s^{nn} as being *defined* in the manner above. When diffeomorphism invariance is violated spontaneously, it could be questioned whether such definitions make sense, as they would mix dynamical and nondynamical objects.

In the second contribution to the Lagrange density \mathcal{L}' of Eq. (16) the background field $s^{\mu\nu}$ is contracted with the four-dimensional Ricci tensor. As a first step, Eq. (10) enables us to express the Lagrange density in terms of quantities defined on Σ_t :

$$\begin{aligned} s^{\alpha\beta(4)} R_{\alpha\beta} &= g^{\alpha\gamma} g^{\beta\delta} s_{\gamma\delta}{}^{(4)} R_{\alpha\beta} \\ &= (q^{\alpha\gamma} - n^\alpha n^\gamma) (q^{\beta\delta} - n^\beta n^\delta) s_{\gamma\delta}{}^{(4)} R_{\alpha\beta} \\ &= s^{\gamma\delta} q_\gamma^\alpha q_\delta^\beta R_{\alpha\beta} - 2n_\delta s^{\gamma\delta} q_\gamma^\alpha n^\beta R_{\alpha\beta} \\ &\quad + n^\gamma n^\delta s_{\gamma\delta}{}^{(4)} R_{\alpha\beta} n^\alpha n^\beta. \end{aligned} \quad (28)$$

In the second step, we will benefit from the following relations that give possible contractions of the Ricci tensor with the projector q^μ_ν and the four-vector n^μ :

$$\begin{aligned} q^\nu_\beta q^\sigma_\delta{}^{(4)} R_{\nu\sigma} &= \frac{1}{N} \mathcal{L}_m K_{\beta\delta} - \frac{1}{N} D_\beta D_\delta N + R_{\beta\delta} \\ &\quad + K K_{\beta\delta} - 2K^\mu{}_\beta K_{\mu\delta}, \end{aligned} \quad (29a)$$

$$q^\mu_\beta n^\nu{}^{(4)} R_{\mu\nu} = D_\mu K^\mu{}_\beta - D_\beta K, \quad (29b)$$

$$n^\nu n^\sigma{}^{(4)} R_{\nu\sigma} = -\frac{1}{N} \mathcal{L}_m K + \frac{1}{N} D_\beta D^\beta N - K^{\mu\nu} K_{\mu\nu}, \quad (29c)$$

where \mathcal{L}_m is the Lie derivative [see Eqs. (5.31) and (5.32) of [2]] along the four-vector $m^\mu \equiv N n^\mu$ and D_μ denotes the three-dimensional covariant derivative. We refer to Refs. [108,115,116] for details on their derivation and to Appendix A for a compilation of valuable formulas in this context. Equation (29a) describes how the four-dimensional Ricci tensor is projected completely into a spacelike hypersurface Σ_t . The result involves the three-dimensional Ricci tensor, products of the extrinsic-curvature tensor as well as suitable derivatives of the extrinsic curvature and the lapse function. Equation (29b) is the contracted Codazzi-Mainardi relation. It describes a partial projection of the Ricci tensor into Σ_t and involves three-dimensional covariant derivatives of the extrinsic curvature only. Last but not least, Eq. (29c) states the complete projection of the

Ricci tensor along the direction orthogonal to Σ_t . This particular projection contains contributions similar to those in Eq. (29a), but it is devoid of the three-dimensional Ricci tensor or its trace.

Performing the decomposition of Eqs. (16b) and (16c) implies

$$\mathcal{L}' = \sum_{\alpha=1}^4 \mathcal{L}'^{(\alpha)}, \quad (30a)$$

where

$$\begin{aligned} \mathcal{L}'^{(1)} &= \frac{N\sqrt{q}}{2\kappa} s^{ij} \left(\frac{1}{N} \mathcal{L}_m K_{ij} - \frac{1}{N} D_i D_j N + R_{ij} \right. \\ &\quad \left. + K K_{ij} - 2K_i{}^l K_{lj} \right), \end{aligned} \quad (30b)$$

$$\mathcal{L}'^{(2)} = \frac{N\sqrt{q}}{2\kappa} [2s^{in} (D_i K - D_i K^l{}_l)], \quad (30c)$$

$$\mathcal{L}'^{(3)} = \frac{N\sqrt{q}}{2\kappa} s^{nn} \left(-\frac{1}{N} \mathcal{L}_m K + \frac{1}{N} D_i D^i N - K^{ij} K_{ij} \right), \quad (30d)$$

$$\begin{aligned} \mathcal{L}'^{(4)} &= -\frac{N\sqrt{q}}{2\kappa} u \left(\frac{2}{N} \mathcal{L}_m K - \frac{2}{N} D_i D^i N + R \right. \\ &\quad \left. + K^2 + K_{ij} K^{ij} \right). \end{aligned} \quad (30e)$$

Now, the associated canonical momentum density reads

$$(\pi')^{ij} \equiv \frac{\partial \mathcal{L}'}{\partial \dot{q}_{ij}} = \sum_{\alpha=1}^4 \pi^{(\alpha)ij}. \quad (31)$$

Starting from these relations, we will obtain the Hamilton density,

$$\mathcal{H}' = (\pi')^{ij} \dot{q}_{ij} - \mathcal{L}', \quad (32)$$

where calculational details will be relegated to Appendix B unless they are worthwhile to be mentioned in the main text. Note that the total Hamilton density simply follows from adding the modification to the EH part:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^{(0)} + \mathcal{H}' = (\pi_0)^{ij} \dot{q}_{ij} - \mathcal{L}^{(0)} + (\pi')^{ij} \dot{q}_{ij} - \mathcal{L}' \\ &= (\pi_0 + \pi')^{ij} \dot{q}_{ij} - (\mathcal{L}^{(0)} + \mathcal{L}') \equiv \pi^{ij} \dot{q}_{ij} - \mathcal{L}, \end{aligned} \quad (33)$$

where π^{ij} is the total canonical momentum density and \mathcal{L} the total Lagrange density.

1. Decoupling the sectors

Based on Eq. (27), $s^{\mu\nu}$ can be decomposed into three sectors. In what follows, we intend to analyze these sectors independently of each other. This can be accomplished by

choosing particular observer frames where only one of the three sets s^{ij} , s^{in} , and s^{nn} provides nonzero coefficients. For example, if one is interested in s^{nn} only, we would consider an observer frame where all components of $s^{\mu\nu}$ vanish except of s^{00} . We will precisely follow this strategy, as the three sectors have distinct characteristics that can be exploited in the computations. Hence, we will just focus on a single sector and turn off the remaining ones to simplify the analysis without losing generality.

2. Purely spacelike sector

Consider the purely spacelike modification given by Eq. (30b) (to be added to the EH Lagrangian later) with the remaining modifications turned off. In general, Lie derivatives with respect to m^μ involve time derivatives, which is why the term $\mathcal{L}_m K_{ij}$ implies time derivatives of the extrinsic curvature according to

$$\mathcal{L}_m K_{ij} = \dot{K}_{ij} - \mathcal{L}_N K_{ij}. \quad (34)$$

Such time derivatives give rise to additional second-order time derivatives of the induced metric. This finding would eventually force us to consider the Ostrogradsky formalism of higher-derivative theories [117–121]. To avoid this problem, we follow the method introduced in [122] and employ an identity that allows for shifting time derivatives from the extrinsic curvature to the Lorentz-violating background:

$$\begin{aligned} \frac{1}{N} s^{ij} \mathcal{L}_m K_{ij} &= \nabla_\mu (n^\mu K_{ij} s^{ij}) - K K_{ij} s^{ij} \\ &\quad - \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij}. \end{aligned} \quad (35)$$

Here, the Lie derivative of the purely spacelike background field with respect to m^μ reads

$$\mathcal{L}_m s^{ij} = \dot{s}^{ij} - \mathcal{L}_N s^{ij}, \quad (36a)$$

$$\mathcal{L}_N s^{ij} = N^k D_k s^{ij} - (D_k N^i) s^{kj} - (D_k N^j) s^{ik}. \quad (36b)$$

Let us now rewrite Eq. (30b):

$$\begin{aligned} \mathcal{L}'^{(1)} &= \frac{N\sqrt{q}}{2\kappa} \left[\nabla_\mu (n^\mu K_{ij} s^{ij}) - \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij} \right. \\ &\quad \left. - s^{ij} \left(\frac{1}{N} D_i D_j N - R_{ij} + 2K^l{}_i K_{lj} \right) \right]. \end{aligned} \quad (37)$$

Integrating over the first contribution above within the action leads to another boundary term [cf. Eq. (19)] that we discard. This step will turn out to be crucial to understand the results (see Sec. IV C). So we omit the covariant-derivative term and consider instead

$$\begin{aligned} \mathcal{L}^{(1)} &= \frac{\sqrt{q}}{2\kappa} [-K_{ij} \mathcal{L}_m s^{ij} - s^{ij} D_i D_j N \\ &\quad + N s^{ij} (R_{ij} - 2K^l{}_i K_{lj})]. \end{aligned} \quad (38)$$

After some calculation (for details we refer to Appendix B 1), the Hamiltonian associated with the Lagrange density is given by

$$H^{(1)} = - \int_{\Sigma_t} d^3x \frac{\sqrt{q}}{2\kappa} (N \mathcal{C}_0^{(1)} + N^i \mathcal{C}_i^{(1)}), \quad (39a)$$

with

$$\mathcal{C}_0^{(1)} = s^{ij} (R_{ij} + 2K^l{}_i K_{lj}) - D_i D_j s^{ij}, \quad (39b)$$

$$\mathcal{C}_i^{(1)} = -q_{ij} D_k \left[\frac{1}{N} \mathcal{L}_m s^{kj} + 2(s^{kl} K^j{}_l + s^{jl} K^k{}_l) \right]. \quad (39c)$$

We see that $H^{(1)}$ has a structure similar to that of the EH Hamiltonian in Eq. (24). Time derivatives of the lapse function and the shift vector do not occur. Note the presence of the problematic term $\mathcal{L}_m s^{ij}$ that appears in the modification of \mathcal{C}_i of Eq. (24c) and is proportional to the inverse of the lapse function. Therefore, it does not fit into the usual structure of the Hamiltonian. Understanding the physical implications of this contribution will turn out to be of paramount importance as a base for the internal consistency of the theory and we will return to this problem later.

The next step is to consider the total Hamiltonian $H = H^{(0)} + H^{(1)}$ with $H^{(0)}$ of Eq. (24) and $H^{(1)}$ stated in Eq. (39). Our goal now is to eliminate the extrinsic-curvature tensor in H in favor of the canonical momentum π^{ij} associated with H . To do so, we first need an expression for π^{ij} that is obtained in Eq. (B3). As K_{ij} is symmetric, the canonical momentum employed in the Legendre transformation can be symmetrized, whereupon we cast the latter result into the form

$$\begin{aligned} \pi^{ij} &= \frac{\sqrt{q}}{2\kappa} \left[K^{ij} - q^{ij} K - \frac{1}{2N} \mathcal{L}_m s^{ij} - (s^{il} K^j{}_l + s^{jl} K^i{}_l) \right] \\ &= \frac{\sqrt{q}}{2\kappa} \left(G^{ijab} K_{ab} - \frac{1}{2N} \mathcal{L}_m s^{ij} \right), \end{aligned} \quad (40a)$$

with the four-tensor (symmetrized in the first and second pair of indices):

$$\begin{aligned} G^{ijab} &= \frac{1}{2} (q^{ia} q^{jb} + q^{ib} q^{ja}) - q^{ij} q^{ab} \\ &\quad - \frac{1}{2} (s^{ia} q^{jb} + s^{ja} q^{ib} + s^{ib} q^{ja} + s^{jb} q^{ia}). \end{aligned} \quad (40b)$$

Inverting the canonical momentum for the extrinsic curvature and its trace gives

$$K_{ab} = G_{abij} \left(\frac{2\kappa}{\sqrt{q}} \pi^{ij} + \frac{1}{2N} \mathcal{L}_m s^{ij} \right), \quad (41a)$$

$$K = q^{ab} G_{abij} \left(\frac{2\kappa}{\sqrt{q}} \pi^{ij} + \frac{1}{2N} \mathcal{L}_m s^{ij} \right), \quad (41b)$$

with the inverse tensor G_{abij} satisfying

$$G_{cdij} G^{ijab} = \frac{1}{2} (\delta_c^a \delta_d^b + \delta_d^a \delta_c^b). \quad (42)$$

Note that the fourth-rank tensor in Eq. (40b) for $s^{ij} = 0$ is proportional to an object known as the Wheeler-DeWitt metric in the literature [cf., for example, Eq. (7.45) in [5]]. Its inverse in the diffeomorphism-invariant setting is readily found to be

$$G_{cdij}^{(0)} = \frac{1}{2} (q_{ci} q_{dj} + q_{di} q_{cj} - q_{cd} q_{ij}), \quad (43)$$

which is a valuable object in GR to invert Eq. (22) for the extrinsic curvature leading to Eq. (25).

For a generic s^{ij} it seems challenging to find an inverse in closed form. There are two possibilities of proceeding. First, a special case for s^{ij} could be considered, e.g., a decomposition into suitably chosen four-vectors. The exact inverse G_{cdij} can be computed for such cases. Second, we were able to determine the inverse G_{cdij} for a generic s^{ij} at first order in the controlling coefficients. To be as general as possible, we choose the second approach. It is reasonable to propose a suitable *Ansatz* for G_{cdij} that involves all fourth-rank tensors constructed from q^{ij} and s^{ij} at first order in s^{ij} . Contracting the latter with Eq. (40b) and requiring Eq. (42) at first order in the controlling coefficients implies a linear system of equations for the parameters. Solving this system provides the parameters of the *Ansatz*. The computation is performed best with computer algebra and the result reads

$$G_{cdij}^{(1)} = \frac{1}{2} [q_{ci} q_{dj} + q_{di} q_{cj} - (1 - s^l_l) q_{cd} q_{ij}] \\ + s_{cj} q_{di} + s_{di} q_{cj} - (s_{cd} q_{ij} + s_{ij} q_{cd}), \quad (44)$$

where the latter reproduces Eq. (43) for vanishing s^{ij} . This finding is one of the few places throughout the paper where we use lower-index controlling coefficients for convenience. We will now restrict our consideration to background fields that satisfy the requirement $\mathcal{L}_m s^{ij} = 0$ such that the constraint structure is standard. This particular choice is to be discussed at the end of Sec. IV B and will find substantial additional motivation for it in Sec. V.

By considering the Hamilton density $\mathcal{H}^{(1)}$ associated with $H^{(1)}$ and expressing K_{ij} in terms of π^{ij} in Eq. (39b), we deduce the total Hamilton density

$$\mathcal{H}^{(0)} + \mathcal{H}^{(1)} = N C_0^{(1)} + N^i C_i^{(1)}, \quad (45)$$

with the modified Hamiltonian constraint

$$C_0^{(1)} = C_0 - \delta C_0^{(1)}, \quad (46a)$$

$$\delta C_0^{(1)} = \frac{\kappa}{\sqrt{q}} [4s^{ij} (\pi_{ij} \pi - \pi^k_i \pi_{jk}) - s^i_i \pi^2] \\ + \frac{\sqrt{q}}{2\kappa} (s^{ij} R_{ij} - D_j D_i s^{ij}), \quad (46b)$$

where C_0 is given by Eq. (26b) with π_0 replaced by π . The latter result is valid at first order in the controlling coefficients and under the requirement that $\mathcal{L}_m s^{ij} = 0$. Performing the analogous replacement in Eq. (39c), we make an interesting discovery. The standard part implies

$$-\frac{\sqrt{q}}{\kappa} (D_k K^{ki} - D^i K) = -2 [D_k \pi^{ki} + s^{kl} D_k \pi^i_l \\ + s^{ik} (D_l \pi^l_k - D_k \pi) + \pi^{kl} D_k s^i_l \\ + \pi^{ik} D_l s^l_k - \pi D_k s^{ik}] + \dots, \quad (47)$$

where all contributions beyond linear order in the controlling coefficients have been dropped. In the modification given by Eq. (39c), it is sufficient to employ the standard relations of Eq. (25):

$$\frac{\sqrt{q}}{\kappa} D_k (s^{kl} K^i_l + s^{il} K^k_l) \\ = 2D_k \left[s^{kl} \left(\pi^i_l - \frac{\pi}{2} \delta^i_l \right) + s^{il} \left(\pi^k_l - \frac{\pi}{2} \delta^k_l \right) \right] \\ = 2D_k [s^{kl} \pi^i_l + s^{il} \pi^k_l - s^{ik} \pi]. \quad (48)$$

Thus, all diffeomorphism-violating contributions of Eqs. (47) and (48) cancel each other. Then, the momentum constraint remains unmodified at first order in diffeomorphism violation:

$$C_i^{(1)} = C_i = 2D_j \pi^j_i, \quad (49)$$

with C_i stated in Eq. (26c).

3. Mixed sector

In contrast to the Lagrange density of the purely space-like sector, $\mathcal{L}^{(2)}$ given by Eq. (30c) involves the covariant derivative defined on the spacelike hypersurface. The latter does not give rise to time derivatives of the extrinsic curvature. However, to be capable of shifting the spatial covariant derivatives to the vector-valued background field s^{in} , we add a suitable boundary term to the Lagrange density. Note that the latter is not integrated over the spacetime boundary (such as those to be considered in Sec. IV C), but over the boundary of the spacelike hypersurface Σ_t . Hence, the corresponding integral is two-dimensional and runs over the coordinates z employed

to parametrize the boundary $\partial\Sigma_t$. Then, the Lagrangian reads

$$\begin{aligned} L^{(2)} &= \frac{1}{\kappa} \left[\int_{\Sigma_t} d^3x \sqrt{q} N s^{in} (D_i K - D_l K^l{}_i) \right. \\ &\quad \left. - \oint_{\partial\Sigma_t} d^2z \sqrt{q} r_i N (K s^{in} - K^i{}_j s^{jn}) \right] \\ &= \frac{1}{\kappa} \int_{\Sigma_t} d^3x \sqrt{q} N \left\{ s^{in} (D_i K - D_l K^l{}_i) \right. \\ &\quad \left. - \frac{1}{N} D_i [N (K s^{in} - K^i{}_j s^{jn})] \right\}, \end{aligned} \quad (50)$$

with a properly normalized three-vector \mathbf{r} that is orthogonal to $\partial\Sigma_t$. The associated Lagrange density has the form

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{\sqrt{q}}{\kappa} \{ N s^{in} (D_i K - D_l K^l{}_i) - D_i [N (K s^{in} - K^i{}_j s^{jn})] \} \\ &= \frac{\sqrt{q}}{\kappa} N [K^i{}_j D_i s^{jn} - K D_i s^{in} + a_i (K^i{}_j s^{jn} - K s^{in})]. \end{aligned} \quad (51)$$

Based on the latter result, we can directly obtain the canonical momentum and the Hamilton density via a Legendre transformation. Details of the computation are relegated to Appendix B 2. The Hamiltonian is

$$H^{(2)} = - \int_{\Sigma_t} d^3x \frac{\sqrt{q}}{2\kappa} (N C_0^{(2)} + N^a C_a^{(2)}), \quad (52a)$$

with

$$C_0^{(2)} = 0, \quad (52b)$$

$$\begin{aligned} C_a^{(2)} &= q_{ab} [D_r (a^r + D^r) s^{bn} + D_r (a^b + D^b) s^{rn} \\ &\quad - 2D^b (a_i + D_i) s^{in}]. \end{aligned} \quad (52c)$$

Interestingly, C_0 in Eq. (24b) is not affected by the coefficients s^{in} and the modification of C_i of Eq. (24c) is independent of the extrinsic curvature. The form of the Lagrange density $\mathcal{L}^{(2)}$ suggests that it must be interpreted as a constraint that does not affect the dynamics. A possible explanation is given by the intriguing finding that $\mathcal{L}^{(2)}$ can be generated at first order in the controlling coefficients via a modified ADM decomposition (see Appendix C). The latter is characterized by an effective shift vector that includes the controlling coefficients s^{in} . Thus, we conclude that the mixed coefficients s^{in} are mere gauge degrees of freedom.

Despite this result, expressing the previous constraints in terms of the canonical momentum density π^{ij} may still provide further insight. Considering $\mathcal{H}^{(0)} + \mathcal{H}^{(2)}$ with the Hamilton density $\mathcal{H}^{(2)}$ of the mixed sector, the total

canonical momentum of Eq. (B5) can be inverted for the extrinsic curvature when we write

$$\pi^{ij} = \frac{\sqrt{q}}{2\kappa} (K^{ij} - q^{ij} K + \tilde{\pi}^{ij}), \quad (53a)$$

$$\begin{aligned} \tilde{\pi}^{ij} &= \frac{1}{2} [(a^r + D^r) s^{sn} + (a^s + D^s) s^{rn}] \\ &\quad - q^{rs} (a_i + D_i) s^{in}. \end{aligned} \quad (53b)$$

Note that $\tilde{\pi}^{ij}$ does not depend on the extrinsic curvature. By employing the inverse Wheeler-DeWitt metric of Eq. (43), we obtain

$$\begin{aligned} K_{ab} &= G_{abij}^{(0)} \left(\frac{2\kappa}{\sqrt{q}} \pi^{ij} - \tilde{\pi}^{ij} \right) \\ &= \frac{2\kappa}{\sqrt{q}} \left(\pi_{ab} - \frac{\pi}{2} q_{ab} \right) - \tilde{\pi}_{ab} + \frac{\tilde{\pi}}{2} q_{ab}. \end{aligned} \quad (54)$$

The total Hamilton density then has the form

$$\mathcal{H}^{(0)} + \mathcal{H}^{(2)} = N C_0^{(2)} + N^a C_a^{(2)}, \quad (55a)$$

with the Hamiltonian constraint

$$C_0^{(2)} = C_0 + \delta C_0^{(2)}, \quad (55b)$$

$$\delta C_0^{(2)} = \tilde{\pi}\pi - 2\tilde{\pi}_{ij}\pi^{ij} + \frac{\sqrt{q}}{2\kappa} \left(\tilde{\pi}^{ij}\tilde{\pi}_{ij} - \frac{\tilde{\pi}^2}{2} \right), \quad (55c)$$

where π_0 in C_0 must be replaced by π of Eq. (53a), and the momentum constraint

$$C_i^{(2)} = -2D_j P^j{}_i, \quad P^{ij} \equiv \pi^{ij} - \frac{\sqrt{q}}{2\kappa} \tilde{\pi}^{ij}. \quad (55d)$$

In principle, the latter can be interpreted as a redefined momentum constraint with the redefined momentum density P^{ij} . A short computation then also reveals that

$$C_0^{(2)} = \frac{2\kappa}{\sqrt{q}} \left(P^{ij} P_{ij} - \frac{P^2}{2} \right) - \frac{\sqrt{q}}{2\kappa} R. \quad (56)$$

These results are another indication for s^{in} not conveying any physical information. We were able to reproduce the Hamiltonian and momentum constraint of EH theory stated in Eqs. (26b) and (26c) simply by absorbing the controlling coefficients s^{in} into the momentum density. The only caveat is that we are putting nondynamical coefficients s^{in} into the canonical momentum P^{ij} , which is a dynamical entity. However, this procedure corresponds to performing a mere shift of the original canonical momentum π^{ij} that does not even depend on the extrinsic curvature [see $\tilde{\pi}^{ij}$ in Eq. (53b)].

4. Purely timelike sector

To obtain the Hamiltonian associated with the purely timelike sector based on Eq. (30d), we employ the relation

$$\frac{1}{N}s^{\text{nn}}(\dot{K} - \mathcal{L}_N K) = \nabla_\mu(n^\mu K s^{\text{nn}}) - K^2 s^{\text{nn}} - \frac{1}{N}K(\dot{s}^{\text{nn}} - \mathcal{L}_N s^{\text{nn}}), \quad (57)$$

which is similar to Eq. (35). Inserting the latter into $\mathcal{L}'^{(3)}$ leads to

$$\mathcal{L}'^{(3)} = \frac{N\sqrt{q}}{2\kappa} \left[-\nabla_\mu(n^\mu K s^{\text{nn}}) + \frac{1}{N}K\mathcal{L}_m s^{\text{nn}} + s^{\text{nn}} \left(\frac{1}{N}D_i D^i N - K^{ij}K_{ij} + K^2 \right) \right]. \quad (58)$$

Note the similarity to Eq. (37). Again, we discard the covariant-derivative contribution that would provide a boundary term in the action (see Sec. IV C). So we consider

$$\mathcal{L}^{(3)} = \frac{\sqrt{q}}{2\kappa} [s^{\text{nn}}(D_i D^i N - NK^{ij}K_{ij} + NK^2) + K\mathcal{L}_m s^{\text{nn}}], \quad (59a)$$

where the Lie derivative of the scalar background field s^{nn} can simply be understood as a directional derivative:

$$\frac{1}{N}\mathcal{L}_m s^{\text{nn}} = n^\mu \nabla_\mu s^{\text{nn}}. \quad (59b)$$

The Hamiltonian corresponding to $\mathcal{L}^{(3)}$ is derived in Appendix B 3. It reads

$$H^{(3)} = - \int_{\Sigma_t} d^3x \frac{\sqrt{q}}{2\kappa} (NC_0^{(3)} + N^a C_a^{(3)}), \quad (60a)$$

with

$$C_0^{(3)} = D_i D^i s^{\text{nn}} + s^{\text{nn}}(K^{ij}K_{ij} - K^2), \quad (60b)$$

$$C_i^{(3)} = D_i \left(\frac{1}{N}\mathcal{L}_m s^{\text{nn}} + 2s^{\text{nn}}K \right) - 2D_j(s^{\text{nn}}K^j_i). \quad (60c)$$

Note the similarities, but also the differences of the latter modifications of C_0, C_i in comparison to Eqs. (39b) and (39c). Such as for the purely spacelike sector, the contribution $\mathcal{L}_m s^{\text{nn}}$ deviates from the conventional constraint structure, as it is proportional to the inverse of the lapse function. This term will also play a pivotal role for the consistency of the purely timelike sector, as will become evident later.

Let us now introduce the total Hamiltonian $H = H^{(0)} + H^{(3)}$ with the total canonical momentum π^{ij}

associated. The modified π^{ij} is obtained by adding Eqs. (22) and (B7). Inverting the latter via the inverse of the Wheeler-DeWitt metric in Eq. (43) implies the extrinsic curvature in terms of the canonical momentum:

$$K_{ab} = \frac{G_{abij}^{(0)}}{1 - s^{\text{nn}}} \left(\frac{2\kappa}{\sqrt{q}}\pi^{ij} - \frac{\Xi}{2}q^{ij} \right) = \frac{1}{1 - s^{\text{nn}}} \left[\frac{2\kappa}{\sqrt{q}} \left(\pi_{ab} - \frac{\pi}{2}q_{ab} \right) + \frac{\Xi}{4}q_{ab} \right]. \quad (61a)$$

The covariant directional derivative of s^{nn} with respect to n^μ [cf. Eq. (59b)],

$$\Xi \equiv \frac{1}{N}\mathcal{L}_m s^{\text{nn}}, \quad (61b)$$

had to be isolated before computing the inverse. Therefore, the total Hamiltonian in canonical variables follows from adding Eqs. (24) and (60) and expressing the extrinsic curvature in terms of the canonical momentum via Eq. (61):

$$\mathcal{H}^{(0)} + \mathcal{H}^{(3)} = NC_0^{(3)} + N^a C_a^{(3)}, \quad (62a)$$

with the modified Hamiltonian constraint

$$C_0^{(3)} = \frac{2\kappa}{\sqrt{q}(1 - s^{\text{nn}})} \left(\pi^{ij}\pi_{ij} - \frac{\pi^2}{2} \right) - \frac{\sqrt{q}}{2\kappa} (R + D_a D^a s^{\text{nn}}) + \frac{\Xi}{2(1 - s^{\text{nn}})} \left(\pi - \frac{3\sqrt{q}}{8\kappa}\Xi \right), \quad (62b)$$

and the momentum constraint

$$C_a^{(3)} = C_a = -2D_b \pi_a^b. \quad (62c)$$

As before, the diffeomorphism-violating contributions in the momentum constraint, which follows from Eqs. (24c) and (60c), cancel when the latter is written as a function of the total canonical momentum [cf. Eq. (49)].

5. Scalar sector

According to Eq. (16b), the minimal gravitational SME also contains a scalar background field called u . In the context of spontaneous diffeomorphism violation, u can be eliminated in the PPN formalism by redefining the gravitational field, i.e., $1 - u$ is merely a scaling factor in this case. However, in the current section, we will demonstrate that the fate of u in the setting of explicit diffeomorphism violation is much more subtle. As u comes together with the Ricci scalar, the corresponding Lagrange density of Eq. (16b) can be decomposed by applying Eqs. (A12) and (A15). The result is given by Eq. (30e). Now, the identity

$$\frac{1}{N}u\mathcal{L}_m K = \nabla_\mu(n^\mu K u) - K^2 u - \frac{1}{N}K\mathcal{L}_m u, \quad (63)$$

which is analogous to Eqs. (35) and (57), allows us to move derivatives from the extrinsic curvature to the background scalar u modulo a total covariant derivative:

$$\begin{aligned} \frac{2\kappa}{N\sqrt{q}}\mathcal{L}'^{(4)} &= -2\nabla_\mu(n^\mu K u) + 2K^2 u + \frac{2}{N}K\mathcal{L}_m u \\ &\quad - \left(-\frac{2}{N}D_i D^i N + R + K^2 + K_{ij}K^{ij} \right) u \\ &= -2\nabla_\mu(n^\mu K u) + \frac{2}{N}(K\mathcal{L}_m u + uD_i D^i N) \\ &\quad - (R - K^2 + K_{ij}K^{ij})u. \end{aligned} \quad (64)$$

A suitable boundary term added to the action eliminates the total derivative (see Sec. IV C). Taking this boundary term into account implies the form of the Lagrange density that we are going to work with:

$$\begin{aligned} \mathcal{L}^{(4)} &= \frac{N\sqrt{q}}{2\kappa} \left[\frac{2}{N}(K\mathcal{L}_m u + uD_i D^i N) \right. \\ &\quad \left. - (R - K^2 + K_{ij}K^{ij})u \right]. \end{aligned} \quad (65)$$

A Legendre transformation (see Appendix B 4) leads to the Hamiltonian

$$H^{(4)} = - \int_{\Sigma_t} d^3x \frac{\sqrt{q}}{2\kappa} (\mathcal{C}_0^{(4)} N + \mathcal{C}_i^{(4)} N^i), \quad (66a)$$

where

$$\mathcal{C}_0^{(4)} = -(R + K^2 - K_{ij}K^{ij})u + 2D_i D^i u, \quad (66b)$$

$$\mathcal{C}_i^{(4)} = 2 \left[D_i \left(\frac{1}{N} \mathcal{L}_m u + uK \right) - D_j (uK^j_i) \right]. \quad (66c)$$

We now consider the theory based on the total Hamiltonian $H = H^{(0)} + H^{(4)}$. The total canonical momentum can be inverted for the extrinsic curvature via the inverse Wheeler-DeWitt metric of Eq. (43):

$$\begin{aligned} K_{ab} &= \frac{G_{abij}^{(0)}}{1-u} \left(\frac{2\kappa}{\sqrt{q}} \pi^{ij} - \Upsilon q^{ij} \right) \\ &= \frac{1}{1-u} \left[\frac{2\kappa}{\sqrt{q}} \left(\pi_{ab} - \frac{\pi}{2} q_{ab} \right) + \frac{\Upsilon}{2} q_{ab} \right], \end{aligned} \quad (67a)$$

where we introduced a symbol for the Lie derivative of the background field:

$$\Upsilon \equiv \frac{1}{N} \mathcal{L}_m u. \quad (67b)$$

Now, the total Hamilton density reads

$$\mathcal{H}^{(0)} + \mathcal{H}^{(4)} = N\mathcal{C}_0^{(4)} + N^i \mathcal{C}_i^{(4)}, \quad (68a)$$

with the modified Hamiltonian and momentum constraint:

$$\begin{aligned} \mathcal{C}_0^{(4)} &= \frac{2\kappa}{\sqrt{q}(1-u)} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2} \right) - \frac{\sqrt{q}}{2\kappa} [(1-u)R + 2D_i D^i u] \\ &\quad + \frac{\Upsilon}{1-u} \left(\pi - \frac{3\sqrt{q}}{4\kappa} \Upsilon \right), \end{aligned} \quad (68b)$$

$$\mathcal{C}_k^{(4)} = -2D_i \pi^i_k. \quad (68c)$$

The momentum constraint is unaffected by diffeomorphism violation such as for the purely spacelike and timelike sectors of $s^{\mu\nu}$; cf. Eqs. (49) and (62c). Note the parallels to Eq. (62b), although no curvature term is induced by s^{nn} in contrast to u .

A further interesting conclusion can be drawn from supposing that u arises from a nonvanishing trace of $s^{\mu\nu}$. In the case of spontaneous diffeomorphism violation, this argument is usually developed to disregard the trace of $s^{\mu\nu}$ as an unobservable contribution. We then choose $s^{\mu\nu} = u g^{\mu\nu}$ and use Gaussian normal coordinates where $s^{ij} = u q^{ij}$. Inserting the latter into the Lagrange densities of the purely spacelike and purely timelike sector of Eqs. (38) and (59a) results in

$$\begin{aligned} \frac{2\kappa}{N\sqrt{q}}\mathcal{L}^{(1)} &= -\frac{1}{N} [K_{ij}\mathcal{L}_m(uq^{ij}) + uq^{ij}D_i D_j N] \\ &\quad + uq^{ij}(R_{ij} - 2K^l_i K_{lj}) \\ &= 2uK_{ij}K^{ij} - \frac{1}{N}(K\mathcal{L}_m u + uD_i D^i N) \\ &\quad + uR - 2uK^{ij}K_{ij} \\ &= -\frac{1}{N}(K\mathcal{L}_m u + uD_i D^i N) + uR, \end{aligned} \quad (69a)$$

$$\begin{aligned} \frac{2\kappa}{N\sqrt{q}}\mathcal{L}^{(3)} &= -\frac{1}{N}(K\mathcal{L}_m u + uD_i D^i N) \\ &\quad + uK^{ij}K_{ij} - uK^2. \end{aligned} \quad (69b)$$

The sum of both corresponds to the negative of Eq. (65), as expected. In contrast, if we insert $s^{\mu\nu} = u g^{\mu\nu}$ in Gaussian normal coordinates into the modifications of \mathcal{C}_0 given by Eqs. (39b) and (60b), we obtain

$$\begin{aligned}
c_0^{(u)} &\equiv c_0^{(1)} + c_0^{(3)} \\
&= u(R + 2K^{ij}K_{ij}) - D_i D^i u - D_i D^i u \\
&\quad - u(K^{ij}K_{ij} - K^2) \\
&= u(R + K^2 + K^{ij}K_{ij}) - 2D_i D^i u, \quad (70)
\end{aligned}$$

which is off from the negative of Eq. (66b) by a term $2uK^{ij}K_{ij}$. The reason for this mismatch is found in the Lie derivative term of Eq. (38). The canonical momentum provides an additional contribution:

$$\pi^{kl} = \frac{\partial \mathcal{L}^{(1)}}{\partial \dot{q}_{kl}} \supset \frac{N\sqrt{q}}{2\kappa} \left(\frac{1}{N} u K_{ij} q^{ik} q^{jl} \right) = \frac{\sqrt{q}}{2\kappa} u K^{kl}, \quad (71)$$

where we can use that $\dot{q}^{lk} = -q^{li} \dot{q}_{ij} q^{jk}$. Thus, $c_0^{(u)}$ of Eq. (70) has to be endowed with a correction term given by

$$\pi^{kl} \dot{q}_{kl} \supset \frac{\sqrt{q}}{2\kappa} u K_{kl} 2N K^{kl} = \frac{N\sqrt{q}}{2\kappa} (2u K_{kl} K^{kl}). \quad (72)$$

Note that the prefactor and a global minus sign was extracted from Eqs. (39c) and (60b). So we reproduce Eq. (66b) only under these circumstances. This finding teaches us a crucial lesson. In the setting of explicit diffeomorphism violation, a statement like $s^{\mu\nu} = u g^{\mu\nu}$ is simply meaningless, as the background fields are non-dynamical, but the metric is a dynamical object [see the discussion below Eq. (17)].⁶

Hence, it is then also not possible to absorb the u term of Eq. (16b) into the gravitational field to eliminate it. The important message is that u becomes a physical object when explicit diffeomorphism violation is considered. One cannot get rid of it by a simple field redefinition.

C. Generalized Gibbons-Hawking-York boundary term

In what follows, we will comment on the time derivatives of the extrinsic curvature that occur in the ADM-decomposed EH Lagrange density of Eq. (14) as well as in the modifications of Eqs. (30b), (30d), and (30e) via the Lie derivative \mathcal{L}_m . These time derivatives imply that the Lagrangians contain second-order time derivatives of the metric, which is puzzling, as the (modified) Einstein equations themselves are of second order in time

⁶An analogous situation occurs when including the extended Chern-Simons term $\epsilon^{\mu\nu\alpha} \square A_\mu \partial_\nu A_\alpha$ into (1+2)-dimensional electrodynamics [123]. This effective term can be absorbed into the gauge field via the redefinition $\bar{A}_\mu \equiv A_\mu + \epsilon_{\mu\alpha} \partial^\nu A^\alpha$ in order to rewrite the Lagrange density in terms of a new field strength tensor as $-\frac{1}{4} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu}$ with $\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$. In spite of this form hiding the additional degrees of freedom, the latter still provides a parity-violating theory. The example presented demonstrates that field redefinitions have to be carried out and interpreted with care.

[cf. Eqs. (62) and (63) in [32] and Eqs. (6) and (7) in [33]). Although the EH action contains second-order time derivatives of the metric (see the definition of the Ricci scalar), the Einstein equations themselves do not involve time derivatives of the metric higher than 2. To gain a better understanding of this peculiar property, we consult Ref. [124] (see page 297) that provides a powerful decomposition of the EH action as follows:

$$\begin{aligned}
\int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} {}^{(4)}R &= \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} W \\
&\quad + \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \frac{\partial(\sqrt{-g} w^\lambda)}{\partial x^\lambda}, \quad (73a)
\end{aligned}$$

with the quantities W and w^μ given by

$$W = g^{\mu\nu} (\Gamma^{\rho}_{\mu\sigma} \Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu} \Gamma^{\rho}_{\sigma\rho}), \quad (73b)$$

$$w^\lambda = g^{\alpha\beta} \Gamma^{\lambda}_{\alpha\beta} - g^{\lambda\alpha} \Gamma^{\nu}_{\alpha\nu}, \quad (73c)$$

where $\Gamma^{\mu}_{\nu\rho}$ are the Christoffel symbols of four-dimensional spacetime. By following this procedure, $\sqrt{-g}W$ involves only first-order derivatives of the metric, whereas all second-order derivatives of the metric are put into $\partial_\lambda(\sqrt{-g}w^\lambda)$. This decomposition works, as the Ricci scalar is linear in the second-order time derivatives of the metric. Note that W is not a Lorentz scalar and w^λ is not a four-vector.

An explicit computation (for example, done with the powerful *Mathematica* package *xTensor* [125]) demonstrates that a variation of the first term on the right-hand side of Eq. (73a) with respect to the metric leads to $\sqrt{-g}G_{\mu\nu}$ with the Einstein tensor $G_{\mu\nu}$. Interestingly, [124] claims that a variation of the second term is zero, as it is a term on the boundary $\partial\mathcal{M}$ of the spacetime manifold \mathcal{M} . Therefore, it is not expected to contribute to the field equations. However, research done in the 1970s revealed that the situation is more subtle. As the second term on the right-hand side of Eq. (73a) depends on second-order derivatives of the metric, the corresponding surface term still contains first-order time derivatives of the dynamical field $g_{\mu\nu}$. In general, Hamilton's principle requires that variations of dynamical fields vanish on the boundary, which means $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$ for GR. However, the requirement that first-order derivatives of these variations also vanish on the boundary is too strong and should not be implemented, if one does not want to change Hamilton's principle. So $\partial_\nu(\delta g_{\mu\nu})|_{\partial\mathcal{M}} \neq 0$ must be assumed. Then the boundary term cannot simply be set to zero, which is a particular situation in GR, as the EH action already involves second-order derivatives of the dynamical fields.

Since there is a contribution on the boundary, the latter can only be canceled by subtracting a suitable term from the action. It is called the Gibbons-Hawking-York (GHY) term [44,126–128] and it has the form

$$S_{\text{GHY}} = \frac{\varepsilon}{\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} K, \quad (74)$$

where $\varepsilon = \mp 1$ for a spacelike (timelike) boundary and y are the coordinates on the boundary. As its derivation is not found in a number of GR books, we present the essential arguments and calculational steps in Appendix D 1. By doing so, the reader will also be able to understand better how the computation must be adapted to the settings of the $s^{\mu\nu}$ and u terms of the gravitational SME.

The ADM decomposition of the EH Lagrange density based on Eq. (14) provides an alternative explanation of the GHY boundary term. The Lagrange density involves a first-order time derivative of the metric within the extrinsic curvature tensor via Eqs. (20) and (21). Second-order time derivatives of the metric occur in the covariant-derivative term of Eq. (14). By considering this contribution inside the action with $N\sqrt{q} = \sqrt{-g}$, we have

$$\frac{1}{\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_{\mu} (n^{\mu} K - a^{\mu}) = \frac{\varepsilon}{\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} K, \quad (75)$$

where we employ $n \cdot a = 0$ and $n^2 = \varepsilon$.⁷ The result corresponds to Eq. (74). Therefore, the ADM formalism gives rise to the GHY boundary term automatically. The latter is needed to cancel the first-order derivatives of the metric on the boundary.

Let us now focus on the diffeomorphism-violating modification of the EH Lagrange density given by $\mathcal{L}'^{(s)}$ in Eq. (16c). We observed that the Lagrange density of the mixed sector of $\mathcal{L}'^{(s)}$ stated in Eq. (30c) is completely devoid of second-order time derivatives of the metric, as it only contains covariant derivatives defined in the spacelike hypersurface. In fact, we introduced a boundary term for this sector in Eq. (50). However, the latter lived on the boundary $\partial\Sigma_t$ of a spacelike hypersurface as opposed to the boundary $\partial\mathcal{M}$ of the spacetime manifold. Furthermore, the motivation for introducing this term was completely different and did not have any relation with second-order time derivatives of the metric.

In contrast, the situation is quite different for both the purely timelike and the purely spacelike sector whose Lagrange densities are given by Eqs. (30b) and (30d), respectively. They involve first-order time derivatives of K_{ij} and K via the Lie derivative along m^{μ} . Therefore, second-order time derivatives of q_{ij} are implied. Formally, a similar decomposition as that of Eq. (73) can still be carried out, since $\mathcal{L}^{(s)}$ is linear in the second-order time derivatives of the metric, as well:

⁷Note that the four-vector n^{μ} orthogonal to Σ_t can be employed as a vector normal to the boundary. The relevant parts of $\partial\mathcal{M}$ in this context are interpreted as hypersurfaces Σ_t being timelike or spacelike.

$$\int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} s^{\mu\nu(4)} R_{\mu\nu} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} W^{(s)} + \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \frac{\partial(\sqrt{-g} w^{(s)\lambda})}{\partial x^{\lambda}}, \quad (76a)$$

with modified quantities $W^{(s)}$ and $w^{(s)\mu}$:

$$W^{(s)} = s^{\mu\nu} \left\{ \Gamma^{\sigma}_{\mu\nu} \Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma} \Gamma^{\sigma}_{\nu\rho} + \frac{1}{2g} (\Gamma^{\rho}_{\mu\rho} \partial_{\nu} g - \Gamma^{\rho}_{\mu\nu} \partial_{\rho} g) \right\} + \Gamma^{\rho}_{\mu\rho} \partial_{\nu} s^{\mu\nu} - \Gamma^{\rho}_{\mu\nu} \partial_{\rho} s^{\mu\nu}, \quad (76b)$$

$$w^{(s)\lambda} = s^{\alpha\beta} \Gamma^{\lambda}_{\alpha\beta} - s^{\lambda\alpha} \Gamma^{\nu}_{\alpha\nu}. \quad (76c)$$

The second-order derivatives of the metric are absorbed in $\partial_{\lambda}(\sqrt{-g} w^{(s)\lambda})$, whereas $W^{(s)}$ only involves first-order time derivatives. A study analogous to that done before shows that there are nonvanishing contributions on the boundary originating from a variation of the second term on the right-hand side of Eq. (76a). The outcome is that modified GHY terms must be introduced to compensate these effects. Details of the procedure are presented in Appendix D 2. We also take into account $\mathcal{L}'^{(u)}$ in Eq. (16b), which is straightforward, since it has the same structure as the EH Lagrangian. The indispensable boundary terms are then found to be given by

$$S_{\text{GHY}}^{\text{mod}} = \frac{\varepsilon}{2\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} [s^{ij} K_{ij} - (s^{\text{nn}} + 2u)K]. \quad (77)$$

Hence, there is a boundary term for the purely spacelike part of $s^{\mu\nu}$ governed by the controlling coefficients s^{ij} , a second one for the purely timelike part parametrized by s^{nn} , and a third one for u . The mixed part of $s^{\mu\nu}$ does not have an associated boundary term of this form, though [cf. Eq. (50)]. Note that an additional global factor of 2 does not occur for s^{ij} as well as s^{nn} as opposed to the GHY term in Eq. (74).

At this point we may look at the ADM formalism again. Integrating the total-derivative contributions in Eqs. (37), (58), and (64) gives rise to exactly the same surface terms that we found above:

$$\frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_{\mu} (n^{\mu} K_{ij} s^{ij}) = \frac{\varepsilon}{2\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} K_{ij} s^{ij}, \quad (78a)$$

$$-\frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_{\mu} (n^{\mu} K s^{\text{nn}}) = -\frac{\varepsilon}{2\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} K s^{\text{nn}}, \quad (78b)$$

$$-\frac{1}{\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_{\mu} (n^{\mu} K u) = -\frac{\varepsilon}{\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} K u. \quad (78c)$$

In the setting of the u and $s^{\mu\nu}$ coefficients, the ADM formalism still correctly provides suitable surface terms that are generalizations of the GHY term. They are necessary to compensate the additional first-order time derivatives of the metric on the boundary that arise due to diffeomorphism violation. Note that modified GHY boundary terms associated with the minimal gravitational SME were also considered in [106].

We are interested in the boundary contributions introduced above to understand how to treat the second-order time derivatives of the metric properly that occur in Eqs. (30b), (30d), and (30e). Second- and higher-order time derivatives of the dynamical fields may lead to additional propagating degrees of freedom that are unphysical (such as ghosts). In the nonminimal SME such unphysical degrees of freedom are very common. They are often neglected in phenomenological analyses in the low-energy limit of quantum field theories based on the Lee-Wick procedure [129,130]. However, ghosts cannot simply be discarded when quantum corrections are taken into account and the internal consistency of such theories must be questioned and investigated (see, e.g., [131–142] where this list is not claimed to be exhaustive).

In the minimal SME, Lorentz-violating contributions can introduce additional first-order time derivatives of the dynamical fields. For example, additional time derivatives are known to occur in the Dirac fermion sector for certain choices of the dimensionless c , d , e , f , and g coefficients [26,27]. These time derivatives spoil the conventional time evolution of spinor solutions of the Dirac equation, but it is well-known that they can be removed by suitable field redefinitions in spinor space [143–145]. In spite of that, the Lagrange density of the minimal (nongravitational) SME does not exhibit second-order time derivatives of dynamical fields.

In general, second-order time derivatives of the dynamical fields must be treated with the method developed by Ostrogradsky [117–121]. This procedure is usually employed in the context of the nonminimal SME only. Hence, it must be considered as more than surprising that this approach should be necessary to deal with the second-order time derivatives of the metric in Eqs. (30b), (30d), and (30e), which are based on the minimal SME. The argument made via Eq. (76) is a justification for transferring the first-order time derivatives from K_{ij} to s^{ij} via Eq. (35), from K to s^{nn} with the help of Eq. (57), and from K to u by means of Eq. (63). In this process, generalized GHY boundary terms are introduced to cancel the first-order time derivatives of the metric on the boundary. Thus, the Ostrogradsky method is not needed.

As a consequence of our procedure, first-order time derivatives of the diffeomorphism-violating fields s^{ij} , s^{nn} , and u arise, which reveals two important properties of these contributions. First, the number of degrees of freedom must be conserved when applying Eqs. (35), (57), and (63).

In principle, the procedure transfers the degrees of freedom that come with the additional time derivatives of the extrinsic curvature to s^{ij} , s^{nn} , and u . For s^{ij} and u this means that the background fields must somehow absorb these degrees of freedom.

Recall that we work in the setting of explicit diffeomorphism violation, wherewith s^{ij} must be considered as a nondynamical tensor-valued function that is projected into Σ_t from an initially chosen $s^{\mu\nu}$. Hence, s^{ij} is not capable of absorbing any dynamical degrees of freedom, which indicates a mismatch. This behavior is how the well-known clash between explicit diffeomorphism violation and Riemannian geometry [32] manifests itself within the ADM formalism applied to this particular sector. The argument is similar for u . However, as will be discussed in Sec. V, this mismatch can possibly be resolved when taking into account a certain set of consistency conditions for the background fields.

Moreover, when the time derivatives are transferred from K to s^{nn} in the purely timelike sector [see Eq. (57)], they do not only act on s^{00} , but they imply time derivatives of the lapse function. An interpretation of this behavior is that some of the gauge degrees of freedom of GR can become dynamical in this sector. The significance of that observation is highly obscure. On the other hand, both the purely spacelike sector and the scalar sector do not involve any time derivatives of the lapse function or the shift vector, i.e., these gauge degrees of freedom remain nondynamical in the presence of s^{ij} and u , respectively. This additional problem specific to the purely timelike sector can also be tackled by introducing a suitable requirement for s^{nn} , as will become clear in Sec. V.

In short, based on the previous discussion as well as the form of the modifications of \mathcal{C}_i in Eqs. (39c) and (66c), one might be tempted to restrict the purely spacelike and the scalar sector to such background fields satisfying $\mathcal{L}_m s^{ij} = 0$ and $\mathcal{L}_m u = 0$, respectively. These conditions mean that s^{ij} and u , respectively, are generated by the flow defined by the four-vector m^μ and they are necessary requirements for the internal consistency of these sectors. The conclusion is that s^{ij} and u chosen suitably in this manner could, indeed, imply a base for constructing consistent sectors of the minimal gravitational SME that violate diffeomorphism invariance explicitly. Furthermore, a similar requirement $\mathcal{L}_m s^{\text{nn}} = 0$ could be employed [see also the modification of \mathcal{C}_i obtained in Eq. (60c)]. Note that we consider s^{nn} as a new degree of freedom independent of $s^{\mu\nu}$ and N , i.e., we will not think of it as a quantity that involves time derivatives \dot{s}^{00} and \dot{N} separately. The forthcoming section will provide further substance to these (preliminary) claims.

V. FIELD EQUATIONS AND CONSTRAINTS

It is remarkable that in the ADM formalism there are direct relationships between the Einstein equations and the

Hamiltonian constraint as well as the momentum constraint. These relations involve suitable contractions of the Einstein equations with the projector $q^\mu{}_\nu$ and the four-vector n^μ . We intend to give a brief summary of how this procedure works for GR. After that, we will be trying to tackle the diffeomorphism-violating modifications in an analogous way. Although the method applied to GR is rather unproblematic, its application to scenarios with explicit diffeomorphism violation has turned out to be a formidable task requiring tedious computations. A large part of those will be moved to Appendix F. However, the procedure will eventually imply several rewarding findings.

A. General relativity

The structure of GR allows us to relate the field equations with the Hamiltonian and momentum constraints. We consider the Einstein equations of GR,

$$(T_{\text{mat}})^{\alpha\beta} = {}^{(4)}G^{\alpha\beta}, \quad (79a)$$

where

$$\begin{aligned} {}^{(4)}G^{\alpha\beta} &= {}^{(4)}R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}{}^{(4)}R, \\ (T_{\text{mat}})^{\alpha\beta} &= \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g_{\alpha\beta}}, \end{aligned} \quad (79b)$$

with the Einstein tensor ${}^{(4)}G^{\alpha\beta}$ in four-dimensional space-time and the Belinfante energy-momentum tensor $(T_{\text{mat}})^{\alpha\beta}$ linked to a matter Lagrange density \mathcal{L}_{mat} . Since we are not taking matter into account, we will set $(T_{\text{mat}})^{\alpha\beta} = 0$.

Suitable projections of the Einstein equations imply the Hamiltonian and momentum constraints (expressed in terms of the extrinsic curvature). First, a total projection along the direction orthogonal to Σ_t leads to Eq. (24b):

$$\begin{aligned} 2n_\alpha n_\beta {}^{(4)}G^{\alpha\beta} &= 2n_\alpha n_\beta {}^{(4)}R^{\alpha\beta} - n^2 {}^{(4)}R \\ &= (R + K^2 - K_{ij}K^{ij} - 2n_\alpha n_\beta {}^{(4)}R^{\alpha\beta}) \\ &\quad + 2n_\alpha n_\beta {}^{(4)}R^{\alpha\beta} \\ &= \mathcal{C}_0, \end{aligned} \quad (80)$$

where we employed Eqs. (A12) and (A15). Second, a mixed projection parallel to Σ_t and along the direction orthogonal to Σ_t implies Eq. (24c):

$$\begin{aligned} 2q^i{}_\alpha n_\beta {}^{(4)}G^{\alpha\beta} &= 2q^i{}_\alpha n_\beta {}^{(4)}R^{\alpha\beta} - 2q^i{}_\alpha n^\alpha {}^{(4)}R \\ &= 2(D_j K^{ji} - D^i K) = \mathcal{C}^i. \end{aligned} \quad (81)$$

Here we used the contracted Codazzi-Mainardi relation of Eq. (A20b).

B. Minimal gravitational SME: $s^{\mu\nu}$ term

The modification of the Einstein equations follows from varying the action

$$S'^{(s)} = \int_{\mathcal{M}} d^4x \mathcal{L}'^{(s)}, \quad (82)$$

with $\mathcal{L}'^{(s)}$ given by Eq. (16c). The result is well-known and is stated in Eqs. (6) and (7) of [33]. Without a matter source and for a nonzero $s^{\mu\nu}$ only, the modified Einstein equations amount to

$$0 = {}^{(4)}G^{\alpha\beta} - (T^{Rs})^{\alpha\beta}, \quad (83a)$$

$$\begin{aligned} (T^{Rs})^{\alpha\beta} &= \frac{1}{2} [g^{\alpha\beta} s^{\mu\nu} R_{\mu\nu} + \nabla_\nu \nabla^\alpha s^{\nu\beta} + \nabla_\nu \nabla^\beta s^{\nu\alpha} \\ &\quad - \nabla^2 s^{\alpha\beta} - g^{\alpha\beta} \nabla_\mu \nabla_\nu s^{\mu\nu}]. \end{aligned} \quad (83b)$$

Note that the modified Einstein equations stated in Eqs. (62) and (63) of [32] have a slightly different form where the corresponding $(\tilde{T}^{Rs})^{\alpha\beta}$ has been reprinted in Eq. (F3) for completeness. The latter field equations are valid for a modification of GR given by the action

$$S''^{(s)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} s_{\mu\nu} {}^{(4)}R^{\mu\nu}, \quad (84)$$

i.e., for a background field with lower indices (although the field equations are expressed in terms of $s^{\mu\nu}$ with both indices raised by the metric). This finding shows that for explicit diffeomorphism violation, the field theories defined by the action $S'^{(s)}$ of Eq. (82) and $S''^{(s)}$ of Eq. (84) are not equivalent. More emphasis is put on this property in the recent work [39] (see also the remarks made in Sec. I). Since our setting is based on the action of Eq. (82), our analysis will be resting on the modified Einstein equations (83). For the purpose of clarification, a short derivation of the modified field equations is provided in Appendix E.

We now intend to find out whether there are possible connections between the modified Einstein equations and the constraints derived in Sec. IV B. To do so, we will have to compute suitable contractions of $(T^{Rs})^{\alpha\beta}$ in Eq. (83) with n_α and $q^\alpha{}_\beta$, respectively. The computations turned out to be challenging and revealed further interesting insights. Details are presented in Appendix F.

There is an additional peculiarity with respect to Eq. (27) that we have employed to decompose $s^{\mu\nu}$ into a purely spacelike, a mixed, and a purely timelike sector. The latter decomposition must be considered as an identity and the individual parts depend on the coordinates chosen for the ADM decomposition. The situation is most clear for the purely spacelike part. According to its definition as $s^{ij} \equiv q^i{}_\mu q^j{}_\nu s^{\mu\nu}$, the purely spacelike part definitely

involves coefficients of $s^{\mu\nu}$ with spacelike indices only—independently of the exact form of the projectors. However, the same does not hold true for the mixed and purely timelike sectors. The problem is best understood by looking at the following explicit example. At first, the only coefficient of $s^{\mu\nu}$ that contributes to the purely timelike sector is s^{00} , as $s^{\mathbf{nn}} = s^{\mu\nu}n_\mu n_\nu = N^2 s^{00}$. However, for a nonzero shift vector the purely timelike sector of the decomposition in Eq. (27) then reads

$$(s^{\alpha\beta})|_{\text{purely timelike}} = (n^\alpha n^\beta s^{\mathbf{nn}}) = \begin{pmatrix} 1 & -N^i \\ -N_j & N^i N_j \end{pmatrix} s^{00}, \quad (85)$$

which obviously involves coefficients other than s^{00} . The behavior is similar for the mixed sector. Thus, a nonzero shift vector implies that the purely timelike and mixed sector couple to each other as well as to the purely spacelike one. If one does not want to pick particular observer frames with specific forms of $s^{\mu\nu}$ (see Sec. IV B 1), such couplings between different sectors can be avoided by employing Gaussian normal coordinates [115,116] where $N = 1$ and $N^i = 0$. A slight generalization of Gaussian normal coordinates including a nontrivial lapse function N can also be considered. It is indispensable, though, to work with a zero shift vector if one wants to avoid that different sectors couple to each other.

1. Field equations for s^{ij}

In what follows, we will obtain the suitable projections of the modified Einstein equations. Computational details are shown in Appendix F 1. A complete projection of Eq. (83) along the direction orthogonal to Σ_t implies

$$2n_\alpha n_\beta (T^{Rs})^{\alpha\beta} = -(K^l{}_i K_{jl} s^{ij} + K K_{ij} s^{ij} + s^{ij} R_{ij} - q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \nabla_\nu s^{\alpha\beta}). \quad (86)$$

The final term is involved. Evaluating it with care gives rise to

$$q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \nabla_\nu s^{\alpha\beta} = D_i D_j s^{ij} - K^l{}_i K_{jl} s^{ij} + K K_{ij} s^{ij} - \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij}, \quad (87)$$

with the Lie derivative of the purely spacelike background given by Eq. (36b). Now we obtain the following intriguing result:

$$\begin{aligned} 2n_\alpha n_\beta (T^{Rs})^{\alpha\beta} &= - \left[s^{ij} (R_{ij} + 2K^l{}_i K_{lj}) - D_i D_j s^{ij} + \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij} \right] \\ &= - \left(\mathcal{C}_0^{(1)} + \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij} \right), \end{aligned} \quad (88)$$

with $\mathcal{C}_0^{(1)}$ given by Eq. (39b). Thus, the modification of the Einstein equations completely projected along the direction orthogonal to Σ_t almost equals the modification of \mathcal{C}_0 , but there is an additional contribution given by the Lie derivative of the purely spacelike background tensor with respect to m^μ .

Evaluating the mixed projection of Eq. (83b) is even more involved. An intermediate result reads

$$\begin{aligned} 2q^k{}_\alpha n_\beta (T^{Rs})^{\alpha\beta} &= -s^{ij} D_i K_j{}^k - K_{ij} D^k s^{ij} - a_i K^i{}_j s^{jk} \\ &\quad + 2K_{ij} D^i s^{jk} + K D_i s^{ik} \\ &\quad - (K \gamma^k{}_\lambda + K^k{}_\lambda) \nabla_\alpha s^{\alpha\lambda} \\ &\quad + n_\alpha q^\nu{}_\lambda q^k{}_\sigma \nabla_\nu \nabla^\alpha s^{\lambda\sigma} + s^{k\lambda} \nabla_\alpha K^\alpha{}_\lambda, \end{aligned} \quad (89)$$

and finally, we obtain

$$\begin{aligned} 2q^k{}_\alpha n_\beta (T^{Rs})^{\alpha\beta} &= - \left\{ -D_i \left[\frac{1}{N} \mathcal{L}_m s^{ik} + 2s^{lk} K^i{}_l \right] + K_{ij} D^k s^{ij} \right\} \\ &= -[q^{kl} \mathcal{C}_l^{(1)} + 2D_i (s^{il} K^k{}_l) + K_{ij} D^k s^{ij}], \end{aligned} \quad (90)$$

with the modification $\mathcal{C}_l^{(1)}$ of \mathcal{C}_l given by Eq. (39c).

2. Field equations in the mixed sector

In Sec. IV B 3 we have brought up convincing arguments for the mixed sector involving only gauge degrees of freedom. Therefore, we do not consider it worthwhile to compute projections of $(T^{Rs})^{\alpha\beta}$ for this particular sector.

3. Field equations for $s^{\mathbf{nn}}$

To avoid couplings with the other sectors, we will be working with coordinates such that $N^i = 0$. For the purely timelike sector there is the peculiarity that an additional contribution must be taken into account for the field equations that comes from varying the action. We have that

$$\begin{aligned} \delta(s^{\mathbf{nn}} n^\mu n^\nu R_{\mu\nu}) &\supset s^{\mathbf{nn}} R_{\mu\nu} \delta(n^\mu n^\nu) \\ &= s^{\mathbf{nn}} R_{\mu\nu} \delta(q^{\mu\nu} - g^{\mu\nu}) \\ &= -s^{\mathbf{nn}} R_{\mu\nu} \delta g^{\mu\nu}, \end{aligned} \quad (91)$$

according to Eq. (10). Thus, an extra term emerges within the diffeomorphism-violating modification of Eq. (83b) where the minus sign is extracted:

$$\begin{aligned} (T^{Rs})^{\alpha\beta} &\mapsto (T^{Rs})^{\alpha\beta} + s^{\mathbf{nn}} g^{\alpha\mu} g^{\beta\nu} R_{\mu\nu} \\ &= (T^{Rs})^{\alpha\beta} + s^{\mathbf{nn}} R^{\alpha\beta}. \end{aligned} \quad (92)$$

Then, a complete projection along the direction perpendicular to Σ_t implies

$$\begin{aligned}
2n_\alpha n_\beta (T^{Rs})^{\alpha\beta} &= -[D_i D^i s^{\text{nn}} + s^{\text{nn}}(K^{ij} K_{ij} - K^2) \\
&\quad + K n^\mu \nabla_\mu s^{\text{nn}}] \\
&= -\left(C_0^{(3)} - \frac{1}{N} K \mathcal{L}_m s^{\text{nn}} \right), \tag{93}
\end{aligned}$$

with $C_0^{(3)}$ stated in Eq. (60b). Carrying out a projection along Σ_t with respect to one index and a projection along n^μ for the second index results in

$$\begin{aligned}
2q^k_\alpha n_\beta (T^{Rs})^{\alpha\beta} &= -[D^k (n^\mu \nabla_\mu s^{\text{nn}} + 2s^{\text{nn}} K) \\
&\quad - 2D_j (s^{\text{nn}} K^{jk}) - K D^k s^{\text{nn}}] \\
&= -(q^{kl} C_l^{(3)} - K D^k s^{\text{nn}}), \tag{94}
\end{aligned}$$

where $C_l^{(3)}$ of Eq. (60c) can be employed here. Details on how to arrive at these results are relegated to Appendix F 2.

C. Minimal gravitational SME: u term

Without a matter source and only the u term present, the modification of the Einstein equations is obtained from varying

$$S^{(u)} = \int_{\mathcal{M}} d^4x \mathcal{L}'^{(u)}, \tag{95}$$

with $\mathcal{L}'^{(u)}$ given by Eq. (16b). Then,

$$\begin{aligned}
0 &= {}^{(4)}G^{\alpha\beta} - (T^{Ru})^{\alpha\beta}, \tag{96a} \\
(T^{Ru})^{\alpha\beta} &= -\frac{1}{2} (\nabla^\alpha \nabla^\beta u + \nabla^\beta \nabla^\alpha u) \\
&\quad + g^{\alpha\beta} \nabla^2 u + u {}^{(4)}G^{\alpha\beta}. \tag{96b}
\end{aligned}$$

A short derivation of this result is also presented in Appendix E. As we did before, we can compute suitable projections of the modified Einstein equations completely orthogonal to Σ_t and partially into Σ_t :

$$2n_\alpha n_\beta (T^{Ru})^{\alpha\beta} = -\left(C_0^{(4)} - \frac{2}{N} K \mathcal{L}_m u \right), \tag{97a}$$

$$2q^k_\alpha n_\beta (T^{Ru})^{\alpha\beta} = -(q^{kl} C_l^{(4)} - 2K D^k u), \tag{97b}$$

with $C_0^{(4)}$, $C_l^{(4)}$ given by Eqs. (66b) and (66c), respectively. Computational details are presented in Appendix F 3.

D. Concluding remarks

To summarize, projections of the diffeomorphism-violating modifications $(T^{Rs})^{\alpha\beta}$, $(T^{Ru})^{\alpha\beta}$ for the purely spacelike, the purely timelike, and the scalar sector with q^μ_ν and n^μ do not completely provide the Hamiltonian and momentum constraints (expressed in terms of the extrinsic curvature). In contrast to GR, there are correction terms. To substantiate these outcomes, we will be taking a deeper

look at additional properties of the ADM action in the forthcoming section.

E. Functional derivatives of ADM action

In the current section we intend to compute functional derivatives of the ADM action with respect to the lapse function and the shift vector. The first is expected to be connected to the Hamiltonian constraint, whereas the second is associated with the momentum constraint [146]. Computational details are shown in Appendix G. First, for the ADM-decomposed EH action of Eq. (19) we obtain

$$\frac{\delta S^{(0)}}{\delta N} = \frac{\sqrt{q}}{2\kappa} C_0, \tag{98a}$$

$$\frac{\delta S^{(0)}}{\delta N^k} = \frac{\sqrt{q}}{2\kappa} C_k, \tag{98b}$$

with C_0 , C_k given by Eqs. (24b) and (24c), respectively. To find out whether or not analogous relationships exist in the context of the background fields $s^{\mu\nu}$ and u , we consider the modifications of the GR action within the ADM formalism. The latter read

$$S^{(i)} = \int_{\mathcal{M}} d^4x \mathcal{L}'^{(i)}, \tag{99}$$

with the Lagrangians given by Eqs. (30b)–(30e) for $i \in \{1, 2, 3, 4\}$. Computing the functional derivatives implies

$$\frac{\delta S'^{(1)}}{\delta N} = \frac{\sqrt{q}}{2\kappa} \left(C_0^{(1)} + \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij} \right), \tag{100a}$$

$$\frac{\delta S'^{(1)}}{\delta N^k} = \frac{\sqrt{q}}{2\kappa} [C_k^{(1)} + K_{ij} D_k s^{ij} + 2D_i (s^{ij} K_{kj})], \tag{100b}$$

$$\frac{\delta S'^{(2)}}{\delta N} = \frac{\sqrt{q}}{2\kappa} \left\{ C_0^{(2)} - \frac{2}{N} D_i [N (s^{jn} K^i_j - s^{in} K)] \right\}, \tag{100c}$$

$$\frac{\delta S'^{(2)}}{\delta N^k} = \frac{\sqrt{q}}{2\kappa} C_k^{(2)}, \tag{100d}$$

$$\frac{\delta S'^{(3)}}{\delta N} = \frac{\sqrt{q}}{2\kappa} \left(C_0^{(3)} - \frac{1}{N} K \mathcal{L}_m s^{\text{nn}} \right), \tag{100e}$$

$$\frac{\delta S'^{(3)}}{\delta N^k} = \frac{\sqrt{q}}{2\kappa} (C_k^{(3)} - K D_k s^{\text{nn}}), \tag{100f}$$

$$\frac{\delta S'^{(4)}}{\delta N} = \frac{\sqrt{q}}{2\kappa} \left(C_0^{(4)} - \frac{2}{N} K \mathcal{L}_m u \right), \tag{100g}$$

$$\frac{\delta S'^{(4)}}{\delta N^k} = \frac{\sqrt{q}}{2\kappa} (C_k^{(4)} - 2K D_k u). \tag{100h}$$

Several observations are in order. First, in contrast to GR, there is no complete match between the functional derivatives and the constraints for the purely spacelike, the purely timelike, and the scalar sector. There are additional contributions that involve products of the extrinsic curvature and Lie derivatives or covariant derivatives of the background fields. Second, there is a match for the Hamiltonian constraint of the mixed sector modulo a boundary term on $\partial\Sigma_t$. Interestingly, the momentum constraint of the mixed sector matches perfectly without any additional contributions. Third, note the similarities between the purely timelike and the scalar sectors that are also evident in the boundary terms of Eqs. (78b) and (78c).

F. Consistency requirements

In what follows, we will draw some intriguing conclusions from the projections of the modified Einstein equations and the functional derivatives of the ADM actions. We do so for the purely spacelike and the purely timelike sector of $s^{\mu\nu}$ as well as for u . Since the mixed sector of $s^{\mu\nu}$ involves gauge degrees of freedom only, it will not be taken into consideration. For the purely spacelike sector we start by comparing Eqs. (88) and (90) with the corresponding functional derivatives of the ADM action:

$$\frac{\delta S'^{(1)}}{\delta N} = -\frac{\sqrt{q}}{\kappa} n_\alpha n_\beta (T^{Rs})^{\alpha\beta}, \quad (101a)$$

$$\frac{\delta S'^{(1)}}{\delta N_k} = -\frac{\sqrt{q}}{\kappa} q^k{}_\alpha n_\beta (T^{Rs})^{\alpha\beta}. \quad (101b)$$

For the purely timelike sector, we compare Eqs. (93) and (94) with the suitable functional derivatives:

$$\frac{\delta S'^{(3)}}{\delta N} = -\frac{\sqrt{q}}{\kappa} n_\alpha n_\beta (T^{Rs})^{\alpha\beta}, \quad (102a)$$

$$\frac{\delta S'^{(3)}}{\delta N_k} = -\frac{\sqrt{q}}{\kappa} q^k{}_\alpha n_\beta (T^{Rs})^{\alpha\beta}. \quad (102b)$$

Finally, we take Eqs. (97a) and (97b) from the scalar sector and compare those to the associated functional derivatives:

$$\frac{\delta S'^{(4)}}{\delta N} = -\frac{\sqrt{q}}{\kappa} n_\alpha n_\beta (T^{Ru})^{\alpha\beta}, \quad (103a)$$

$$\frac{\delta S'^{(4)}}{\delta N_k} = -\frac{\sqrt{q}}{\kappa} q^k{}_\alpha n_\beta (T^{Ru})^{\alpha\beta}. \quad (103b)$$

Thus, we conclude that the functional derivatives of the ADM-decomposed actions with respect to the lapse function are proportional to the associated modifications of the Einstein equations completely projected along the direction orthogonal to Σ_t . Furthermore, the functional derivatives for the shift covector are proportional to the mixed

projections of the modifications. In this context, discrepancies do not arise for the purely spacelike, the purely timelike, and the scalar sector.

Next, we compare the latter findings directly to the Hamiltonian and momentum constraints. For the purely spacelike sector we establish the correspondences

$$\frac{\delta}{\delta N} (S'^{(1)}|_{\mathcal{L}_m s^{ij}=0}) = \frac{\sqrt{q}}{2\kappa} C_0^{(1)}, \quad (104a)$$

$$\frac{\delta}{\delta N^k} (S'^{(1)}|_{\mathcal{L}_m s^{ij}=0}) = \frac{\sqrt{q}}{2\kappa} C_k^{(1)}|_{\mathcal{L}_m s^{ij}=0}, \quad (104b)$$

with $C_0^{(1)}$ and $C_k^{(1)}$ given by Eqs. (39b) and (39c). For the purely timelike sector it has to hold that

$$\frac{\delta}{\delta N} (S'^{(3)}|_{\mathcal{L}_m s^{nn}=0}) = \frac{\sqrt{q}}{2\kappa} C_0^{(3)}, \quad (105a)$$

$$\frac{\delta}{\delta N^k} (S'^{(3)}|_{\mathcal{L}_m s^{nn}=0}) = \frac{\sqrt{q}}{2\kappa} C_k^{(3)}|_{\mathcal{L}_m s^{nn}=0}, \quad (105b)$$

with $C_0^{(3)}$ and $C_k^{(3)}$ stated in Eqs. (60b) and (60c). Last but not least, for the scalar sector we deduce

$$\frac{\delta}{\delta N} (S'^{(4)}|_{\mathcal{L}_m u=0}) = \frac{\sqrt{q}}{2\kappa} C_0^{(4)}, \quad (106a)$$

$$\frac{\delta}{\delta N^k} (S'^{(4)}|_{\mathcal{L}_m u=0}) = \frac{\sqrt{q}}{2\kappa} C_k^{(4)}|_{\mathcal{L}_m u=0}, \quad (106b)$$

where $C_0^{(4)}$ and $C_k^{(4)}$ must be taken from Eqs. (66b) and (66c). Hence, we conclude that for these three sectors being internally consistent, the following necessary conditions must be required:

$$\mathcal{L}_m s^{ij} = 0, \quad (107a)$$

$$\mathcal{L}_m s^{nn} = 0, \quad (107b)$$

$$\mathcal{L}_m u = 0. \quad (107c)$$

In what follows, the latter will be denoted as consistency conditions. In the context of the ADM decomposition, a gravity theory endowed with s^{ij} -, s^{nn} - or u -type background fields, which violate diffeomorphism invariance explicitly, is likely to be internally consistent as long as the Lie derivatives of these backgrounds with respect to the vectorfield m^μ vanish. These consistency requirements are directly connected to diffeomorphisms acting on the underlying spacetime manifold. Diffeomorphisms affect tensorfields in the tangent bundle and are generated by Lie derivatives along arbitrary vectorfields. Thus, Eq. (107) means that the corresponding background fields must be invariant under diffeomorphisms generated by m^μ within

the ADM formalism such that the modified-gravity theory is consistent.

The recent findings demonstrate how a background field violating diffeomorphism invariance explicitly must be understood within gravity. A background field in the nongravitational SME is *defined* to not transform covariantly under particle Lorentz transformations, i.e., it is *defined* as fixed under such transformations. Note that diffeomorphisms in Minkowski spacetime are equivalent to translations induced by a constant four-vector ζ^μ . Then, the Lie derivative of a two-tensor-valued background field $k^{\mu\nu}$ simply corresponds to the directional derivative along ζ^μ . An analogous consistency requirement would amount to

$$\mathcal{L}_\zeta k^{\mu\nu} = \zeta^\rho \nabla_\rho k^{\mu\nu} = 0. \quad (108)$$

For an arbitrary ζ^μ this condition is satisfied for backgrounds $k^{\mu\nu}$ that do not depend on the spacetime position (at least when expressed in terms of Cartesian coordinates). In the context of the nongravitational SME, constant controlling coefficients are usually employed for two reasons. First, it is simpler to think of a background as being constant. More importantly, coefficients depending on the spacetime position violate translation invariance and, therefore, energy-momentum conservation due to Noether's theorem. This property would imply additional complications that are beyond studies of Lorentz violation.

However, in curved spacetime, a dependence of controlling coefficients on spacetime position must usually be assumed. For example, $\nabla_\rho s^{\mu\nu} = 0$ would only hold in spacetimes known as parallelizable [32]. The latter are very special choices and of lesser interest in the context of gravity. Within the ADM formalism in gravity, Eq. (107) can be interpreted as generalizations of Eq. (108) where the latter implies energy-momentum conservation in Minkowski spacetime.

A pictorial interpretation of the problems that arise with explicit diffeomorphism violation in gravity is as follows. The arguments to be made rely on test particles being present in the curved spacetime manifold. Note that we have not introduced a coupling term with matter in the action, so far. Hence, rigorous studies of the interplay between matter and diffeomorphism-violating background fields will be done elsewhere. Nevertheless, we can make some physical arguments to interpret the significance of Eq. (107).

We can consider a test particle propagating in a curved spacetime. The particle moving between two distinct points follows a geodesic along which it is in free fall, i.e., its acceleration vanishes. A background field giving rise to an explicit violation of diffeomorphism invariance modifies the geodesic equation, whereupon particle motion is

affected.⁸ Then, the momentum of the particle will change in a way that is not described by GR, but that has to be accounted for by the background field. However, as the latter is nondynamical, it is incapable of absorbing or emitting momentum [51].

Similar arguments can be developed for light rays propagating through a background field in curved spacetime. As long as the gravitational field is weak enough, the eikonal approximation is a suitable approach (see, e.g., [147,148]). Then, the curved spacetime manifold is approximately described by an inhomogeneous optical medium, i.e., its refractive index is position dependent (and may also depend on polarization). The presence of a diffeomorphism-violating background field leads to additional optical effects such as anisotropy, dispersion, and birefringence. In this context, a background field violating diffeomorphism symmetry explicitly corresponds to a nondynamical optical medium on top of the optical medium ascribed to the curved spacetime manifold \mathcal{M} . Whenever a light ray changes its propagation direction, its wave vector changes, whereupon this change must be transferred between the light ray and the diffeomorphism-violating medium. Furthermore, in the presence of an explicitly time-dependent medium, even energy is to be transferred between both entities. However, a nondynamical medium neither accomplishes the first nor the second. Similar arguments were already developed in [94].

To solve the aforementioned problems, Eq. (107) seems to include the necessary requirements that render a nondynamical background capable of incorporating energy-momentum transfer between a test particle and the background consistently. There may be a certain notion of energy-momentum that is conserved even for a nondynamical background satisfying Eq. (107). Whether or not these quantities correspond to the Killing energy and Killing momentum, which are associated with isometries of the underlying spacetime manifold, remains an intriguing open question to be studied in the future. Interestingly, an *ansatz* similar to Eq. (107) looks natural to be imposed in the presence of an AdS space in order to match isometries for tensor fields [149].

VI. ANALYSIS OF CONSTRAINTS AND HAMILTON FIELD EQUATIONS

Finally, let us analyze the structure of modified Hamiltonian and momentum constraints that we obtained for Eqs. (16b) and (16c). In general, our analysis has been based on a field theory described by a Lagrange density \mathcal{L} . A constraint is called primary when it follows directly from the form of \mathcal{L} . Such constraints occur for a certain

⁸Note that classical-particle analogs subject to certain types of Lorentz violation described by the (nongravitational) SME were shown to follow geodesics associated with Finsler geometries [85].

canonical variable, say Φ , when the Lagrange density does not involve the time derivative of the latter. The canonical momentum

$$\Pi_\Phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}}, \quad (109)$$

associated with Φ is then equal to zero. Therefore, it is not possible to express $\dot{\Phi}$ in terms of Π_Φ , which does not permit deriving a Hamilton density via the Legendre transform. Then a Hamilton density \mathcal{H} is obtained from \mathcal{L} without taking the constrained variable into consideration. Subsequently the constraint is added to \mathcal{H} via a Lagrange multiplier ξ to define an extended Hamiltonian of the form $\mathcal{H}^{(\text{ext})} \equiv \mathcal{H} + \xi \Pi_\Phi$. In the literature, a primary constraint is written as

$$\Pi_\Phi \approx 0, \quad (110)$$

where \approx means ‘‘weakly equal to zero.’’ This notation is used to indicate that Π_Φ is only taken to be zero when the constraint is satisfied, which is not necessarily assumed in computations right from the start. For example, setting $\Pi_\Phi = 0$ (‘‘strongly equal to zero’’) in $\mathcal{H}^{(\text{ext})}$ would imply that the term added with the Lagrange multiplier does not contribute at all, which is undesired.

In what follows, the constraint structure of GR shall be reviewed briefly (see, e.g., [5]). We start with a set of ten canonical variables $X_i = \{N, N^i, q_{ij}\}$. Note that the index of X_i is defined as a lower one although X_i involves the shift vector components N^i with an upper index. The canonical momenta follow from the generic Lagrange density \mathcal{L} via

$$\Pi^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}_i}, \quad (111)$$

leading to the set of ten canonical momenta $\Pi^i = \{\pi_N, \pi_i, \pi^{ij}\}$ given as

$$\pi_N \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}}, \quad \pi_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^i}, \quad \pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_{ij}}. \quad (112)$$

By employing the canonical variables and momenta, we define the Poisson bracket of two quantities $F = F(x)$, $G = G(x')$ via

$$\{F, G\} \equiv \int_{\Sigma} d^3y \left[\frac{\delta F}{\delta X_i(y)} \frac{\delta G}{\delta \Pi^i(y)} - \frac{\delta F}{\delta \Pi^i(y)} \frac{\delta G}{\delta X_i(y)} \right], \quad (113)$$

where $\delta/\delta\Phi$ denotes the variational derivative with respect to the variable Φ .

A. General relativity

As the EH Lagrange density of Eq. (15a), which is also contained in Eq. (19), does not involve time derivatives of both the lapse function and the shift vector, the associated canonical momenta vanish. Thus, according to the introductory explanations, we have already identified a set of four primary constraints in GR:

$$\pi_N \approx 0, \quad \pi_i \approx 0, \quad (114)$$

whereupon we define

$$\mathcal{H}^{(\text{ext})} \equiv \mathcal{H}^{(0)} + \eta \pi_N + \theta^i \pi_i, \quad (115)$$

with the four Lagrange multipliers η and θ^i . The time evolution of constraints is governed by suitable Poisson brackets with the extended Hamilton density. As the dependence of $\mathcal{H}^{(\text{ext})}$ on the lapse function and the shift vector is transparent, we quickly arrive at

$$\{\pi_N, H^{(\text{ext})}\} = -C_0, \quad (116a)$$

$$\{\pi_i, H^{(\text{ext})}\} = -C_i. \quad (116b)$$

Each primary constraint $\Pi_\Phi \approx 0$ should be preserved in time to not change the constraint structure. Its time evolution is governed by

$$\frac{d\Pi_\Phi}{dt} = \{\Pi_\Phi, \mathcal{H}^{(\text{ext})}\} + \frac{\partial \Pi_\Phi}{\partial t}. \quad (117)$$

The partial time derivative on the right-hand side is only needed when the primary constraint depends on time explicitly. To preserve this primary constraint in time, we must require that

$$\frac{d\Pi_\Phi}{dt} \approx 0, \quad (118)$$

which implies a further constraint that is called a secondary one. Note that we must again talk of this secondary constraint as being weakly equal to zero. Inserting $\Pi_\Phi = 0$ directly would trivially result in a vanishing time derivative. If the secondary constraint is not automatically weakly equal to zero when the primary constraint satisfies this property, this new constraint must be included into the Hamilton density via another Lagrange multiplier ζ :

$$\mathcal{H}^{(\text{ext})} \mapsto \tilde{\mathcal{H}}^{(\text{ext})} = \mathcal{H}^{(0)} + \xi \Pi_\Phi + \zeta \frac{d\Pi_\Phi}{dt}. \quad (119)$$

The procedure continues in this manner and may provide even further constraints. Hence, in the context of GR, for the primary constraints of Eq. (114) to be conserved, we must impose the following secondary constraints:

$$\{\pi_N, H^{(\text{ext})}\} = -C_0 \approx 0, \quad (120a)$$

$$\{\pi_i, H^{(\text{ext})}\} = -C_i \approx 0, \quad (120b)$$

with C_0 , C_i of Eqs. (26b) and (26c), respectively. The latter finding now clearly demonstrates why C_0 , C_i are considered as constraints in GR.

The last step and perhaps the most essential one to derive the number of degrees of freedom in a constrained theory is to identify the first- and second-class constraints [5,114]. Imagine that we have a set of n constraints $\{\phi_a\}$ with $a = 1, 2, \dots, n$. A constraint Q is called first-class if it has weakly vanishing Poisson brackets with each member of the set $\{\phi_a\}$, i.e., $\{Q, \phi_a\} \approx 0$ for all a . It is called second-class if at least one Poisson bracket is not weakly equal to zero: $\{Q, \phi_a\} \not\approx 0$ for not less than a single a . The Dirac bracket can be defined from the latter, which allows for imposing second-class constraints strongly equal to zero [5]. The total number of physical degrees of freedom then corresponds to (see, e.g., page 29 in [114])

$$N_{\text{dof}} = \frac{1}{2}(N_{\text{ph}} - 2N_1 - N_2), \quad (121)$$

where N_{ph} is the number of phase space variables and N_1 (N_2) is the number of first-class (second-class) constraints. Note that the number of first-class constraints has a weight factor of 2 showing that these contribute differently to the number of degrees of freedom than second-class constraints. In particular, GR involves 20 phase space variables in total (ten metric components and ten conjugate momenta). Equations (114) and (120) comprise a set of eight first-class constraints, whereas there are no second-class constraints. Then, $N_{\text{ph}} = 20$, $N_1 = 8$, and $N_2 = 0$, which implies $N_{\text{dof}} = 2$ corresponding to the correct number of physical, propagating degrees of freedom, as expected.

In gravity, the Hamiltonian and momentum constraints, which are first class, play an essential role in the context of diffeomorphisms. In general, diffeomorphisms are generated by vector fields ψ . A representation of the diffeomorphism algebra in the tangent bundle of the spacetime manifold, where tensor fields of arbitrary rank are defined, is given by the Lie derivative \mathcal{L}_ψ . In what follows, we will compute Poisson brackets of the canonical variables q_{ij} , π^{ij} with the Hamiltonian and momentum constraints according to Eq. (113). The constraints will be integrated over with the lapse function and the shift vector chosen as smearing functions. The following important results can then be derived within GR [150,151]:

$$\begin{aligned} & \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y C_i(y) N^i(y) \right\} \\ &= q_{jk}(x) D_j N^k(x) + q_{ik}(x) D_j N^k(x) \\ &= \mathcal{L}_N q_{ij}(x), \end{aligned} \quad (122a)$$

$$\begin{aligned} & \left\{ \pi_{ij}(x), \int_{\Sigma_t} d^3y C_i(y) N^i(y) \right\} \\ &= N^k(x) D_k \pi^{ij}(x) + \pi^{ij}(x) D_k N^k(x) \\ &\quad - \pi^{ik}(x) D_k N^j(x) - \pi^{jk}(x) D_k N^i(x) \\ &= \mathcal{L}_N \pi^{ij}(x). \end{aligned} \quad (122b)$$

For the second of these Poisson brackets it is crucial to take into account that π^{ij} transforms as a tensor density. These findings mean that the momentum constraint is the generator of spatial diffeomorphisms in the spacelike hypersurfaces Σ_t , as these are connected to the shift vector N [150]. Furthermore, we obtain

$$\begin{aligned} \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y C_0(y) N(y) \right\} &= 2N(x) K_{ij}(x) \\ &= \dot{q}_{ij}(x) - \mathcal{L}_N q_{ij}(x) \\ &= \mathcal{L}_m q_{ij}(x), \end{aligned} \quad (123)$$

where Eqs. (20) and (25) are understood to be used here. This relation means that the Hamiltonian constraint generates spacetime diffeomorphisms connected to the four-vector m^μ [150]. Finally, we can confirm the validity of the first set of Hamilton's field equations:

$$\dot{q}_{ij}(x) = \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y \mathcal{H}^{(0)}(y) \right\}, \quad (124)$$

with $\mathcal{H}^{(0)}$ given by Eq. (26a). The latter Hamilton equations are interpreted as geometrical identities and should not be modified as long as Riemannian geometry is taken as the foundation of a modified-gravity theory. The second of Hamilton's field equations involves the canonical momentum π^{ij} and conveys information on the dynamics.

B. Minimal gravitational SME: $s^{\mu\nu}$ term

By adding suitable modified GHY boundary terms to the action (see Sec. IV C for their construction), we were able to move all additional time derivatives acting on the extrinsic curvature to the diffeomorphism-violating background fields. Now, a further support of the consistency conditions in Eq. (107) is provided by the following argument. As long as these requirements are satisfied, the modifications $\mathcal{H}^{(1,3,4)}$ of the Hamilton densities provided by Eqs. (39), (60), and (66) neither involve time derivatives of the lapse function nor of the shift vector. Hence, the constraints provided by Eq. (114) remain

primary ones in a gravity theory modified by the presence of s^{ij} , s^{nn} , and u . Furthermore, the structure of the modified Hamilton densities in terms of N and N^i remains standard, whereupon Eq. (116) still holds with the corresponding $C_0^{(1,3,4)}$ of Eqs. (46), (62b), and (68b) and $C_k^{(1,3,4)}$ of Eqs. (49), (62c), and (68c) inserted. Then the Hamiltonian and momentum constraints remain secondary constraints and the structure of these constraints is not modified by diffeomorphism violation, as proposed right at the beginning of the paper. To be able to make a statement on the number of degrees of freedom based on Eq. (121), we would need to check whether or not the modified constraints remain first-class and what is the constraint algebra. Such investigations will be pursued in a future paper. Let us now compute the Poisson brackets previously considered for EH theory in the diffeomorphism-violating setting under consideration.

1. Hamilton equation in the purely spacelike sector

For the purely spacelike sector, the Poisson bracket of Eq. (46) with the induced metric implies

$$\begin{aligned} & \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y C_0^{(1)}(y) N(y) \right\} \\ &= N \frac{2\kappa}{\sqrt{q}} [2(\pi_{ij} + s_i^k \pi_{jk} + s_j^k \pi_{ik} - s_{ij} \pi) \\ & \quad - (\pi - s^k_k \pi + 2s^{kl} \pi_{kl}) q_{ij}] \\ &= 2NG_{ijab}^{(1)} \left(\frac{2\kappa}{\sqrt{q}} \pi^{ab} \right) = 2NK_{ij}, \end{aligned} \quad (125)$$

with the modified inverse Wheeler-deWitt metric of Eq. (44). The variables on the right-hand side of the first equality sign in the latter relation are understood to depend on the coordinates x . From now on, such dependencies are omitted for brevity. As the momentum constraint remains unmodified at first order in the controlling coefficients, Eq. (122a) can be taken over. Therefore, the first of Hamilton's field equations of Eq. (124) remains valid for the purely spacelike sector—at least at first order in diffeomorphism violation.

2. Hamilton equation in the mixed sector

We are already aware of the mixed sector involving mere gauge degrees of freedom. Nevertheless, we will take a brief look at the constraint structure that has to reduce to that of EH theory when the analysis of Sec. IV B 3 is correct. We will consider the Poisson bracket of the induced metric with the smeared Hamiltonian constraint $C_0^{(2)}$ given by Eq. (55b):

$$\begin{aligned} \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y C_0^{(2)}(y) N(y) \right\} &= \frac{4N\kappa}{\sqrt{q}} \left(\pi_{ij} - \frac{\pi}{2} q_{ij} \right) \\ & \quad + N(\tilde{\pi} q_{ij} - 2\tilde{\pi}_{ij}) \\ &= 2NK_{ij}. \end{aligned} \quad (126)$$

For the momentum constraint we can directly reproduce Eq. (122) with C_0 replaced by $C_0^{(2)}$. Clearly, these outcomes are expected when employing the redefined momentum density P^{ij} of Eq. (55d).

3. Hamilton equation in the purely timelike sector

Let us now evaluate the Poisson bracket of the smeared Hamiltonian constraint $C_0^{(3)}$ of Eq. (62b) with the induced metric according to Eq. (123):

$$\begin{aligned} & \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y C_0^{(3)}(y) N(y) \right\} \\ &= \frac{2N}{1-s^{nn}} \left[\frac{2\kappa}{\sqrt{q}} \left(\pi_{ij} - \frac{\pi}{2} q_{ij} \right) + \frac{\Xi}{4} q_{ij} \right] = 2NK_{ij}, \end{aligned} \quad (127)$$

which corresponds to the expected result when the extrinsic curvature of Eq. (61) is taken into account. As the momentum constraint remains unmodified when expressed in terms of the canonical momentum density, Eq. (122) remains valid when C_0 is substituted by $C_0^{(3)}$. Furthermore, the first of Hamilton's field equations given by Eq. (124) still applies.

C. Minimal gravitational SME: u term

Repeating the procedure employed for the three sectors of $s^{\mu\nu}$ for Eq. (68b) implies

$$\begin{aligned} & \left\{ q_{ij}(x), \int_{\Sigma_t} d^3y C_0^{(4)}(y) N(y) \right\} \\ &= \frac{2N}{1-u} \left[\frac{2\kappa}{\sqrt{q}} \left(\pi_{ij} - \frac{\pi}{2} q_{ij} \right) + \frac{\Upsilon}{2} q_{ij} \right] = 2NK_{ij}, \end{aligned} \quad (128)$$

as expected, when Eq. (67a) is employed. The momentum constraint remains again unmodified, as reported in Eq. (68c). Under these conditions, both Eqs. (122) and (124) are not in conflict with the current setting.

D. Final remarks

Thus, we conclude that the first set of Hamilton's field equations (124) remains valid for the canonical Hamiltonians in the presence of the diffeomorphism-violating contributions that we have been focusing on in this article. This finding is reasonable, as we do not modify the geometrical setting. The second set of Hamilton's field equations is linked to the modified Einstein equations [Eqs. (83) and (96)] and is expected to involve modifications in comparison to GR. We will delve into this problem in a future work.

VII. CONCLUSIONS AND OUTLOOK

In the current work we employed the ADM formalism [3,4] to investigate a modified theory of gravity characterized by the observer Lorentz tensor $s^{\mu\nu}$ as well the observer scalar u of the minimal gravitational SME [32]. We worked in the setting of explicit diffeomorphism violation, i.e., the background fields $s^{\mu\nu}$ and u did not arise dynamically, but they were put into the action by hand. The main objective was to understand within the ADM formalism what kind of restrictions Riemannian geometry poses on $s^{\mu\nu}$ and u , in other words, how the no-go result in the context of the gravitational SME [32] becomes manifest in this formalism.

To do so, we decomposed the diffeomorphism-violating background field $s^{\mu\nu}$ into three sectors. The first was formed from the subset of six independent purely spacelike coefficients s^{ij} . The second contained the vector-valued object s^{in} endowed with a single spacelike index only and the third involved the single Lorentz scalar s^{nn} without spacelike indices. We obtained the Hamiltonians associated with each of the three sectors of $s^{\mu\nu}$ as well as for u and were able to identify modified Hamiltonian and momentum constraints. To derive the Hamiltonians, it turned out to be crucial to include modified GHY boundary terms in the action that allowed us to move time derivatives from the extrinsic-curvature tensor to the diffeomorphism-violating background field in an unambiguous manner.

One interesting outcome is that the mixed sector is governed by mere gauge degrees of freedom, i.e., the coefficients s^{in} are unphysical, as they can be absorbed into a redefinition of the shift vector. In other words, diffeomorphism invariance is restored in this sector, since the would-be diffeomorphism-violating coefficients s^{in} are simply unobservable. This observation may even have implications for phenomenology. If s^{in} is comprised, indeed, by gauge degrees of freedom, it is meaningless to constrain these coefficients by experiment. Instead, they should be disregarded in any phenomenological study of explicit diffeomorphism violation in gravity. Note that this finding can most probably not be taken over to spontaneous diffeomorphism violation, as then s^{in} would be dynamical and it does not make sense to say that they are absorbed into the nondynamical shift vector. Another remarkable property is that the scalar background u in the context of explicit diffeomorphism violation cannot be removed by a redefinition of the gravitational field. Hence, the latter is physical, in fact, and could be searched for in experiments. Therefore, we conclude that explicit and spontaneous diffeomorphism violation can be distinguished from each other in experimental searches.

We also tried to connect the Hamiltonian and momentum constraints to suitable functional derivatives of the ADM actions as well as to projections of the modified Einstein equations along directions orthogonal and parallel to the spacelike hypersurfaces. For the spacelike, timelike, and

scalar sectors we found a match under a set of consistency requirements given by Eq. (107) in the text: $\mathcal{L}_m s^{ij} = 0$, $\mathcal{L}_m s^{nn} = 0$, and $\mathcal{L}_m u = 0$. The latter are considered as the central results of this work. These conditions are interpreted as consequences of the no-go result [32] applied to the pure-gravity sector.

However, Eq. (107) can also be interpreted as the very base of a setting where the no-go result might be avoided as long as the background fields satisfy these conditions. In general, a violation of diffeomorphism invariance via the nondynamical background fields u and $s^{\mu\nu}$ changes the constraint structure of GR, as expected. The parts of the Hamiltonian proportional to N and N^i lose their fundamental property of being constraints, whereupon problems of technical nature arise. Requiring that the secondary constraints be stationary is likely to imply further sets of new constraints whose time evolution has to be studied again. This procedure may quickly get out of control if new constraints arise steadily. By taking into account Eq. (107), modifications of GR are considered that are still close enough to GR. However, only a derivation of the constraint algebra can clarify whether or not additional constraints arise. The latter is a worthwhile project that merits further analysis. Furthermore, whether or not the conditions of Eq. (107) can be disregarded without rendering the constraint analysis unfeasible, will also be subject to future studies.

In Gaussian normal coordinates, in particular, the first relation of Eq. (107) requires that the background field s^{ij} does not exhibit an explicit dependence on the time coordinate. An arbitrary dependence on the spatial coordinates does not seem to be in conflict with Riemannian geometry, though. In summary, the benefits of using the ADM formalism in a setting of diffeomorphism violation in gravity are apparent. One has additional control over diffeomorphism violation and understands better whether or not there are unphysical sets of coefficients. A future analysis based on a covariant canonical formulation (see [152] for a review) could be a worthwhile task to do.

The analysis performed in the current article may pave new pathways of exciting research in the context of explicit diffeomorphism violation and/or local Lorentz violation in gravity. First, we intend to better understand the connection between the consistency conditions and the no-go result. To do so, it will be necessary to include matter fields into the pure-gravity sector and to treat them within the ADM formalism. Second, performing an analogous study for the coefficients $t^{\mu\nu\rho\sigma}$ [32] of the minimal gravitational SME will be reasonable. Such an investigation could provide further insights into the problem known as the “t puzzle” [106,153]. Third, our expectation is that the ADM formalism will also be valuable in the context of the nonminimal gravitational SME [39]. An intriguing question is whether nonminimal diffeomorphism violation in gravity implies additional consistency requirements linked to the higher

derivatives in the action. Last but not least, the obtained results are highly promising to find applications in phenomenological studies of explicit diffeomorphism and local Lorentz violation in the context of cosmology or scenarios of strong gravitational fields such as black holes.

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APPENDIX A: MATHEMATICAL APPENDIX

Here we intend to provide a compilation of the essential geometrical formulas that our work rests on. The books and papers [112,113,115,116] serve as primary references for these results.

1. Extrinsic curvature

The covariant derivative in the spacetime manifold \mathcal{M} is denoted as ∇_μ , whereas the covariant derivative on a spacelike hypersurface Σ_t is called D_μ . The four-acceleration a_μ associated with a foliation via spacelike hypersurfaces is a measure for how n_μ changes covariantly along itself [115,116]. It holds that

$$a_\mu \equiv n^\nu \nabla_\nu n_\mu = \frac{D_\mu N}{N} = D_\mu \ln N, \quad (\text{A1})$$

i.e., it can be expressed via the covariant derivative D_μ linked to the induced metric of Eq. (10) and the lapse function N . This acceleration is tangent to the hypersurface:

$$n \cdot a = 0. \quad (\text{A2})$$

Note that $a_\mu = 0$ in Gaussian normal coordinates. Next, we define the extrinsic-curvature tensor as

$$K_{\mu\nu} \equiv q^\rho{}_\mu q^\sigma{}_\nu \nabla_\rho n_\sigma. \quad (\text{A3})$$

Due to the property

$$n^\mu K_{\mu\nu} = K_{\mu\nu} n^\nu = 0, \quad (\text{A4})$$

the extrinsic curvature lives completely in Σ_t . The extrinsic curvature is symmetric and can be expressed via the Lie derivative of $q_{\mu\nu}$ with respect to $m^\mu \equiv N n^\mu$ [108]:

$$K_{\mu\nu} = q^\rho{}_\mu q^\alpha{}_\nu \nabla_{(q)n_{\sigma)} = \frac{1}{2N} \mathcal{L}_m q_{\mu\nu}, \quad (\text{A5})$$

where a pair of parentheses enclosing a set of indices indicates symmetrization. The latter quantity is a measure for the curvature of a hypersurface Σ_t due to its embedding in \mathcal{M} . It is in stark contrast to the intrinsic curvature of a manifold that is given by the Riemann curvature tensor and does not require an embedding into an ambient (higher-dimensional) manifold. By considering the spacelike part of Eq. (A3), we have

$$K_{ij} = D_i n_j = \partial_i n_j - \Gamma^\lambda{}_{ij} n_\lambda. \quad (\text{A6})$$

As the first term vanishes in the latter, we obtain Eq. (20), which is a highly valuable result in the Hamiltonian description of GR.

Both the acceleration of Eq. (A1) and the extrinsic curvature of Eq. (20) play a pivotal role in projecting the intrinsic curvature of \mathcal{M} into the hypersurface Σ_t . Tensors defined in the ambient manifold \mathcal{M} will be denoted by a label “(4).” For brevity, the analogous label “(3)” for quantities defined in Σ_t will be dropped. We take the commonly used viewpoint that tensors defined on a spacelike hypersurface Σ_t can be extended into the space-time \mathcal{M} via suitable pull-back and push-forward operations [115,116]. By doing so, the extrinsic curvature of Eq. (20) is extended into \mathcal{M} via [115,116]

$$K_{\rho\sigma} \equiv \nabla_\sigma n_\rho + a_\rho n_\sigma, \quad (\text{A7})$$

where, for brevity, we omit the index (4). Whenever the extrinsic curvature occurs with spatial indices, Eq. (A6) is understood to be employed. Note that we take over the sign convention of [108], but different sign conventions are also common (see, e.g., [1,47,115,116]). In addition, Eq. (A4) also holds for Eq. (A7), which follows from

$$0 = \nabla_\nu (n_\mu n^\mu) = 2n^\mu \nabla_\nu n_\mu = 2n^\mu K_{\mu\nu}. \quad (\text{A8})$$

This property emphasizes again that the extrinsic curvature lives in the spatial hypersurface Σ_t entirely. The trace K of the extrinsic curvature is defined via the contraction of Eq. (A7) with the spacetime metric:

$$K \equiv g^{\rho\sigma} K_{\rho\sigma} = \nabla_\mu n^\mu. \quad (\text{A9})$$

Alternatively, Eq. (A6) can be contracted with q_{ij} .

2. Decomposition formula for curvature tensors

As a starting point, we quote the Gauss relation (sometimes also called the Gauss-Codazzi equation) that gives the projection of the Riemann curvature tensor into Σ_t :

$$q^\mu{}_\alpha q^\nu{}_\beta q^\rho{}_\gamma q^\sigma{}_\delta {}^{(4)}R_{\mu\nu\rho\sigma} = R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma} K_{\beta\delta} - K_{\beta\gamma} K_{\alpha\delta}. \quad (\text{A10})$$

Thus, a projection of the Riemann curvature tensor in \mathcal{M} into the spacelike hypersurface induces the curvature tensor in this hypersurface plus correction terms that involve the extrinsic curvature. This property is not a surprise, as the hypersurface Σ_t is embedded into \mathcal{M} .

The Gauss relation of Eq. (A10) can be contracted once to provide a valuable result for the projected Ricci curvature into the hypersurface:

$$\begin{aligned} q^\mu{}_\alpha q^\nu{}_\beta {}^{(4)}R_{\mu\nu} + q_{\alpha\mu} n^\nu q^\rho{}_\beta n^\sigma {}^{(4)}R^\mu{}_{\nu\rho\sigma} \\ = R_{\alpha\beta} + KK_{\alpha\beta} - K_{\alpha\mu} K^\mu{}_\beta, \end{aligned} \quad (\text{A11})$$

with the trace K of the extrinsic curvature stated in Eq. (A9). In this relation, the Riemann curvature tensor cannot be eliminated, but it must be kept. Another contraction results in the scalar Gauss relation that has the form

$${}^{(4)}R + 2{}^{(4)}R_{\mu\nu} n^\mu n^\nu = R + K^2 - K_{\mu\nu} K^{\mu\nu}. \quad (\text{A12})$$

A highly valuable equation links the spatial projection of the Ricci tensor to the extrinsic curvature as follows:

$$\begin{aligned} q^\nu{}_\beta q^\sigma{}_\delta {}^{(4)}R_{\nu\sigma} = \frac{1}{N} \mathcal{L}_m K_{\beta\delta} - \frac{1}{N} D_\beta D_\delta N \\ + R_{\beta\delta} + KK_{\beta\delta} - 2K^\mu{}_\beta K_{\mu\delta}. \end{aligned} \quad (\text{A13})$$

By using Eq. (A11) together with

$$q^{\beta\delta} \mathcal{L}_m K_{\beta\delta} = \mathcal{L}_m K + 2NK^{\mu\nu} K_{\mu\nu}, \quad (\text{A14})$$

we obtain

$$n^\nu n^\sigma {}^{(4)}R_{\nu\sigma} = -\frac{1}{N} \mathcal{L}_m K + \frac{1}{N} D_\beta D^\beta N - K^{\mu\nu} K_{\mu\nu}. \quad (\text{A15})$$

The Ricci scalar can be decomposed by applying Eqs. (A12) and (A15):

$$\begin{aligned} {}^{(4)}R = R + K^2 - K_{ij} K^{ij} - 2{}^{(4)}R_{\mu\nu} n^\mu n^\nu \\ = \frac{2}{N} \mathcal{L}_m K - \frac{2}{N} D_i D^i N + R + K^2 + K_{ij} K^{ij}. \end{aligned} \quad (\text{A16})$$

Also, there are some helpful results involving the acceleration:

$$\nabla_\mu a^\mu = D_i a^i + a_i a^i, \quad (\text{A17a})$$

$$D_\beta a_\sigma + a_\beta a_\sigma = \frac{1}{N} D_\beta D_\sigma N. \quad (\text{A17b})$$

Now, by using

$$\frac{1}{N} \mathcal{L}_m K = n^\mu \nabla_\mu K = \nabla_\mu (n^\mu K) - K^2, \quad (\text{A18a})$$

$$D_i D^i N = D_i (N a^i) = N (a_i a^i + D_i a^i) = N \nabla_\mu a^\mu, \quad (\text{A18b})$$

it is clear that Eq. (A16) can be brought into the form occurring in the ADM-decomposed EH action of Eq. (19):

$$\begin{aligned} {}^{(4)}R = 2\nabla_\mu (n^\mu K) - \frac{2}{N} D_i D^i N + R - K^2 + K_{ij} K^{ij} \\ = R - K^2 + K_{ab} K^{ab} + 2\nabla_\mu (n^\mu K - a^\mu). \end{aligned} \quad (\text{A19})$$

Finally, it is possible to project the Riemann curvature tensor partially into the hypersurface Σ_t [115,116]. The result involves covariant derivatives of the extrinsic curvature defined within the hypersurface:

$$q^\gamma{}_\rho n^\sigma q^\mu{}_\alpha q^\nu{}_\beta {}^{(4)}R^\rho{}_{\sigma\mu\nu} = D_\alpha K^\gamma{}_\beta - D_\beta K^\gamma{}_\alpha. \quad (\text{A20a})$$

The latter bears the name Codazzi-Mainardi relation. Contracting it once implies

$$q^\mu{}_\beta n^\nu {}^{(4)}R_{\mu\nu} = D_\mu K^\mu{}_\beta - D_\beta K. \quad (\text{A20b})$$

Note that all relations derived before are identities, as they stand. In principle, tensors (or parts of tensors) live in the spacelike hypersurface, if contractions of the corresponding Lorentz indices with n^μ give zero. These indices can, in principle, be interpreted as spatial ones. In particular, if $n^\mu \Psi_{\mu\nu\dots} = 0$ of a spacetime tensor $\Psi_{\mu\nu\dots}$, we do not lose any information by considering $\Psi_{i\nu\dots}$. For example, this holds for the extrinsic curvature $K_{\mu\nu}$, the acceleration a_μ , and the covariant derivative D_μ on the hypersurface.

APPENDIX B: HAMILTONIAN FORMULATION

The current section will provide computational details on how to perform Legendre transformations to obtain Hamiltonians from the Lagrange densities of the modified-gravity theory under consideration. In particular, we derive the Hamiltonians given by Eqs. (39), (52), (60), and (66).

1. Purely spacelike sector

We employ the Lie derivative stated in Eq. (36a). As the latter result only involves quantities and derivatives defined in the spatial hypersurface, it is clear that the Lie derivative $\mathcal{L}_m s^{ij}$ does not depend on \dot{q}_{kl} . Based on Eqs. (37) and (38), the canonical momentum of $\mathcal{L}^{(1)}$ is given by

$$\pi^{(1)rs} = \frac{N\sqrt{q} \partial f^{(1)}}{2\kappa \partial \dot{q}_{rs}}, \quad (\text{B1a})$$

with

$$\frac{\partial f^{(1)}}{\partial \dot{q}_{rs}} = -\frac{\partial}{\partial \dot{q}_{rs}} \left\{ \frac{1}{N} [K_{ij}(\dot{s}^{ij} - \mathcal{L}_N s^{ij}) + s^{ij} D_i D_j N] + s^{ij} (2K_i^l K_{lj} - R_{ij}) \right\} \quad \frac{\partial K}{\partial \dot{q}_{ij}} = \frac{\partial K^a_a}{\partial \dot{q}_{ij}} = \frac{q^{ai} \delta^j_a}{2N} = \frac{1}{2N} q^{ij}. \quad (\text{B2b})$$

$$= -\frac{1}{N} \frac{\partial K_{ij}}{\partial \dot{q}_{rs}} (\dot{s}^{ij} - \mathcal{L}_N s^{ij}) - 2s^{ij} \frac{K_i^l K_{lj}}{\partial \dot{q}_{rs}}. \quad (\text{B1b})$$

At this point we will benefit from Eq. (21) that is still valid in the presence of explicit diffeomorphism violation as long as Riemannian geometry is imposed. In a more general Finsler setting, this relation would probably be subject to modifications. In addition, we take advantage of

$$K^{ab} \frac{\partial K_{ab}}{\partial \dot{q}_{ij}} = K^{ab} \frac{\delta^i_a \delta^j_b}{2N} = \frac{1}{2N} K^{ij}, \quad (\text{B2a})$$

Thus,

$$\frac{\partial f^{(1)}}{\partial \dot{q}_{rs}} = -\frac{\delta^r_i \delta^s_j}{2N^2} \mathcal{L}_m s^{ij} - 2s^{ij} \left(\frac{\delta^r_i q^{ls}}{2N} K_{lj} + K_i^l \frac{\delta^r_l \delta^s_j}{2N} \right) = -\frac{1}{N} \left(\frac{1}{2N} \mathcal{L}_m s^{rs} + s^{rj} K^s_j + s^{is} K^r_i \right). \quad (\text{B3})$$

To obtain the Hamilton density $\mathcal{H}^{(1)}$, we perform a Legendre transformation. Note that the presence of the integral measure $d^3x \sqrt{q}$ allows us to carry out partial integrations of the spatial covariant derivative D_i . By doing so, we arrive at

$$\begin{aligned} \mathcal{H}^{(1)} &= \pi^{(1)ij} \dot{q}_{ij} - \mathcal{L}^{(1)} \\ &= -\frac{\sqrt{q}}{2\kappa} \left[\frac{1}{2N} \mathcal{L}_m s^{ij} + s^{il} K^j_l + s^{jl} K^i_l \right] (2N K_{ij} + D_i N_j + D_j N_i) \\ &\quad + \frac{\sqrt{q}}{2\kappa} [K_{ij} \mathcal{L}_m s^{ij} + s^{ij} (D_i D_j N - N R_{ij} + 2N K^l_i K_{lj})] \\ &= \frac{\sqrt{q}}{2\kappa} \left\{ -\left[\frac{1}{2N} \mathcal{L}_m s^{ij} + s^{il} K^j_l + s^{jl} K^i_l \right] (D_i N_j + D_j N_i) + s^{ij} (D_i D_j N - N R_{ij} - 2N K^l_i K_{lj}) \right\} \\ &= \frac{\sqrt{q}}{2\kappa} \left\{ -\left[\frac{1}{N} \mathcal{L}_m s^{ij} + 2(s^{il} K^j_l + s^{jl} K^i_l) \right] D_i N_j + s^{ij} (D_i D_j N - N R_{ij} - 2N K^l_i K_{lj}) \right\} \\ &\stackrel{\text{p.i.}}{=} \frac{\sqrt{q}}{2\kappa} \left\{ D_i \left[\frac{1}{N} \mathcal{L}_m s^{ij} + 2(s^{il} K^j_l + s^{jl} K^i_l) \right] N_j + [D_i D_j s^{ij} - s^{ij} (R_{ij} + 2K^l_i K_{lj})] N \right\}, \quad (\text{B4}) \end{aligned}$$

where ‘‘p.i.’’ stands for partial integration. The latter result implies Eq. (39).

2. Mixed sector

We derive the canonical momentum density from $\mathcal{L}^{(2)}$ stated in Eq. (51):

$$\begin{aligned} \pi^{(2)rs} &= \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{q}_{rs}} \\ &= \frac{\sqrt{q}}{\kappa} N \left[\frac{q^{ir} \delta_j^s}{2N} D_i s^{jn} - \frac{q^{rs}}{2N} D_i s^{in} + a_i \left(\frac{q^{ir} \delta_j^s}{2N} s^{jn} - \frac{q^{rs}}{2N} s^{in} \right) \right] \\ &= \frac{\sqrt{q}}{2\kappa} [(a^r + D^r) s^{sn} - q^{rs} (a_i + D_i) s^{in}] \\ &\stackrel{\text{sym}}{=} \frac{\sqrt{q}}{4\kappa} [(a^r + D^r) s^{sn} + (a^s + D^s) s^{rn} - 2q^{rs} (a_i + D_i) s^{in}], \quad (\text{B5}) \end{aligned}$$

where the latter has been symmetrized in the last step. A Legendre transformation provides the corresponding Hamilton density:

$$\begin{aligned}
\mathcal{H}^{(2)} &= \pi^{(2)rs} \dot{q}_{rs} - \mathcal{L}^{(2)} \\
&= \frac{\sqrt{q}}{4\kappa} [(a^r + D^r) s^{sn} + (a^s + D^s) s^{rn} - 2q^{rs} (a_i + D_i) s^{in}] (2NK_{rs} + D_r N_s + D_s N_r) \\
&\quad - \frac{\sqrt{q}}{\kappa} N [K^i_j (a_i + D_i) s^{jn} - K (a_i + D_i) s^{in}] \\
&= \frac{\sqrt{q}}{2\kappa} [(a^r + D^r) s^{sn} + (a^s + D^s) s^{rn} - 2q^{rs} (a_i + D_i) s^{in}] D_r N_s \\
&\stackrel{\text{p.i.}}{=} \frac{\sqrt{q}}{2\kappa} [2(D^s a_i + D^s D_i) s^{in} - (D_r a^r + D_r D^r) s^{sn} - (D_r a^s + D_r D^s) s^{rn}] N_s.
\end{aligned} \tag{B6}$$

This finding leads to the result quoted in Eq. (52).

3. Purely timelike sector

We compute the canonical momentum of $\mathcal{L}^{(3)}$ by using Eqs. (58) and (59a):

$$\pi^{(3)rs} = \frac{N\sqrt{q}}{2\kappa} \frac{\partial f^{(3)}}{\partial \dot{q}_{rs}}, \tag{B7a}$$

where

$$\begin{aligned}
\frac{\partial f^{(3)}}{\partial \dot{q}_{rs}} &= \frac{\partial K}{\partial \dot{q}_{rs}} n^\mu \nabla_\mu s^{nn} + s^{nn} \frac{\partial}{\partial \dot{q}_{rs}} (K^2 - K^{ij} K_{ij}) \\
&= \frac{1}{N} \left[\frac{q^{rs}}{2N} \mathcal{L}_m s^{nn} + s^{nn} (q^{rs} K - K^{rs}) \right].
\end{aligned} \tag{B7b}$$

A suitable Legendre transformation results in

$$\begin{aligned}
\mathcal{H}^{(3)} &= \pi^{(3)ij} \dot{q}_{ij} - \mathcal{L}^{(3)} \\
&= \frac{\sqrt{q}}{2\kappa} \left[\frac{1}{2N} q^{ij} \mathcal{L}_m s^{nn} + s^{nn} (q^{ij} K - K^{ij}) \right] (2NK_{ij} + D_i N_j + D_j N_i) - \frac{\sqrt{q}}{2\kappa} [K \mathcal{L}_m s^{nn} + s^{nn} (D_i D^i N - NK^{ij} K_{ij} + NK^2)] \\
&= \frac{\sqrt{q}}{2\kappa} \left[\left(\frac{1}{N} \mathcal{L}_m s^{nn} + 2s^{nn} K \right) D_i N^i + 2s^{nn} (NK^2 - NK^{ij} K_{ij} - K^{ij} D_i N_j) - s^{nn} (D_i D^i N - NK^{ij} K_{ij} + NK^2) \right] \\
&\stackrel{\text{p.i.}}{=} \frac{\sqrt{q}}{2\kappa} \left\{ -D_i \left(\frac{1}{N} \mathcal{L}_m s^{nn} + 2s^{nn} K \right) N^i + 2D_i (s^{nn} K^{ij}) N_j - N [D_i D^i s^{nn} + s^{nn} (K^{ij} K_{ij} - K^2)] \right\}.
\end{aligned} \tag{B8}$$

The Hamiltonian of Eq. (60) is a direct implication of the latter result.

4. Scalar sector

Finally, the canonical momentum density associated with the Lagrange density of Eq. (65) reads

$$\pi^{(4)ij} = \frac{\partial \mathcal{L}^{(4)}}{\partial \dot{q}_{ij}} = \frac{\sqrt{q}}{2\kappa} \left[\frac{q^{ij}}{N} \mathcal{L}_m u + (q^{ij} K - K^{ij}) u \right], \tag{B9}$$

whereupon we can compute the Hamilton density:

$$\begin{aligned}
\mathcal{H}^{(4)} &= \pi^{(4)kl} \dot{q}_{kl} - \mathcal{L}^{(u)} \\
&= \frac{\sqrt{q}}{2\kappa} \left[\frac{1}{N} q^{kl} \mathcal{L}_m u + (q^{kl} K - K^{kl}) u \right] (2NK_{kl} + D_k N_l + D_l N_k) - \frac{\sqrt{q}}{2\kappa} [2(K\mathcal{L}_m u + u D_i D^i N) - N(R - K^2 + K_{ij} K^{ij}) u] \\
&= \frac{\sqrt{q}}{2\kappa} \left[(R + K^2 - K_{ij} K^{ij}) u N - 2u D_i D^i N + 2 \left(\frac{1}{N} \mathcal{L}_m u + u K \right) D_l N^l - 2K^{kl} u D_k N_l \right]. \tag{B10}
\end{aligned}$$

Carrying out suitable partial integrations implies Eq. (66).

APPENDIX C: MODIFIED ADM DECOMPOSITION

To understand the mixed sector based on Eq. (30c) better, we perform an ADM decomposition with an effective shift vector

$$\tilde{N}^i \equiv N^i - N s^{in}, \tag{C1}$$

i.e., the diffeomorphism-violating degrees of freedom of this sector are put into the shift vector. In analogy to Eq. (20), the corresponding effective extrinsic-curvature tensor is now defined as

$$\tilde{K}_{ij} \equiv \frac{1}{2N} (\dot{q}_{ij} - D_i \tilde{N}_j - D_j \tilde{N}_i). \tag{C2}$$

Besides, we define a Lagrange density that has a form analogous to that of the ADM-decomposed EH Lagrange density (19):

$$\tilde{\mathcal{L}}^{(0)} \equiv \frac{N\sqrt{q}}{2\kappa} (R - \tilde{K}^2 + \tilde{K}_{ij} \tilde{K}^{ij}), \tag{C3}$$

where the conventional shift vector is replaced by the effective one in Eq. (C1). Boundary terms are disregarded. We then evaluate

$$\begin{aligned}
\tilde{K}_{ij} \tilde{K}^{ij} &= \frac{1}{4N^2} \{ \dot{q}_{ij} \dot{q}^{ij} - 2\dot{q}_{ij} [D^i (N^j - N s^{jn}) + D^j (N^i - N s^{in})] \\
&\quad + [D_i (N_j - N s_j^n) + D_j (N_i - N s_i^n)] [D^i (N^j - N s^{jn}) + D^j (N^i - N s^{in})] \} \\
&\simeq \frac{1}{4N^2} \{ \dot{q}_{ij} \dot{q}^{ij} - 2\dot{q}_{ij} [D^i (N^j - N s^{jn}) + D^j (N^i - N s^{in})] - 2(D_i N_j + D_j N_i) [D^i (N s^{jn}) + D^j (N s^{in})], \tag{C4}
\end{aligned}$$

where we dropped terms beyond linear order in the controlling coefficients. Thereupon,

$$\begin{aligned}
\tilde{K}_{ij} \tilde{K}^{ij} &= K_{ij} K^{ij} + \frac{1}{2N^2} (\dot{q}_{ij} - D_i N_j - D_j N_i) \\
&\quad \times [D^i (N s^{jn}) + D^j (N s^{in})] \\
&= K_{ij} K^{ij} + \frac{1}{N} K_{ij} [D^i (N s^{jn}) + D^j (N s^{in})] \\
&= K_{ij} K^{ij} + \frac{2}{N} K_{ij} D^i (N s^{jn}). \tag{C5}
\end{aligned}$$

In an analogous manner we get

$$\tilde{K}^2 = K^2 + \frac{2}{N} K D_i (N s^{in}). \tag{C6}$$

Hence, it is possible to write

$$\begin{aligned}
N(\tilde{K}^2 - \tilde{K}_{ij} \tilde{K}^{ij}) &\simeq N(K^2 - K_{ij} K^{ij}) \\
&\quad + 2[K D_i (N s^{in}) - K_{ij} D^i (N s^{jn})] \\
&\stackrel{\text{p.i.}}{=} N(K^2 - K_{ij} K^{ij}) \\
&\quad - 2N s^{in} (D_i K - D_j K^j_i), \tag{C7}
\end{aligned}$$

after suitable partial integrations where the surface terms are discarded again. Therefore, at first order in diffeomorphism violation, the following correspondence holds:

$$\tilde{\mathcal{L}}^{(0)} = \mathcal{L}^{(0)} - \mathcal{L}^{(2)}, \tag{C8}$$

with $\mathcal{L}^{(0)}$ of Eq. (19) and the diffeomorphism-violating piece $\mathcal{L}^{(2)}$ of Eq. (30c). Hence, we have shown that at first order in diffeomorphism violation, the coefficients of the mixed sector can be absorbed into a redefined shift vector. This demonstration is another argument for s^{in} being gauge degrees of freedom (see the discussion in Sec. IV B 3).

APPENDIX D: BOUNDARY TERMS IN THE ACTION

In this section we present detailed computations on how to obtain the (modified) GHY boundary terms that play a crucial role when moving time derivatives from the extrinsic curvature to the background fields in the Lagrange densities of Eqs. (30b), (30d), and (30e). The corresponding results are presented and interpreted in Sec. IV C.

1. General relativity: Gibbons-Hawking-York boundary term

We would like to compute the variation of the second term on the right-hand side of Eq. (73a). In local-frame coordinates we have that $g_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\rho g_{\mu\nu} = 0$, and $\Gamma^\mu{}_{\nu\rho} = 0$, but $\partial_\rho \partial_\sigma g_{\mu\nu} \neq 0$ and $\partial_\sigma \Gamma^\mu{}_{\nu\rho} \neq 0$. Therefore, in these coordinates we can express the variation of the second term as

$$\begin{aligned} \delta \int_{\mathcal{M}} d^4x \frac{\partial(\sqrt{-\eta}w^\lambda)}{\partial x^\lambda} \\ = \int_{\mathcal{M}} d^4x \frac{\partial}{\partial x^\lambda} [\sqrt{-\eta}(\eta^{\alpha\beta} \delta\Gamma^\lambda{}_{\alpha\beta} - \eta^{\lambda\alpha} \delta\Gamma^\nu{}_{\alpha\nu})]. \end{aligned} \quad (\text{D1})$$

Note that the Minkowski metric is a nondynamical object. The contributions that transform $\Gamma^\mu{}_{\nu\rho}$ nonlinearly under general coordinate transformations cancel when the variation of the Christoffel symbols is considered, whereupon $\delta\Gamma^\mu{}_{\nu\rho}$ transforms as a tensor. Then the above integrand is a Lorentz scalar in a local frame, which means that it is a Lorentz scalar in an arbitrary frame. Hence, we can generalize this expression to arbitrary coordinates and obtain

$$\delta \int_{\mathcal{M}} d^4x \frac{\partial(\sqrt{-g}w^\lambda)}{\partial x^\lambda} = \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_\lambda V^\lambda, \quad (\text{D2a})$$

$$V^\lambda = g^{\alpha\beta} \delta\Gamma^\lambda{}_{\alpha\beta} - g^{\lambda\alpha} \delta\Gamma^\nu{}_{\alpha\nu}. \quad (\text{D2b})$$

Inserting the variation of the Christoffel symbols expressed in terms of covariant derivatives leads to

$$V^\lambda = g^{\alpha\beta} g^{\lambda\rho} (\nabla_\alpha \delta g_{\rho\beta} - \nabla_\rho \delta g_{\alpha\beta}). \quad (\text{D3})$$

To apply Gauss's theorem to the right-hand side of Eq. (D2a), the integrand must be contracted with the normal vector n^μ associated with the boundary. Therefore,

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_\lambda V^\lambda = \oint_{\partial\mathcal{M}} d^3y \sqrt{q} \varepsilon n_\lambda V^\lambda, \quad (\text{D4})$$

where $\varepsilon = -1$ for timelike n_μ (spacelike boundary) and $\varepsilon = 1$ for spacelike n_μ (timelike boundary). Coordinates denoted as y are used on the boundary. We now decompose the four-metric on the boundary into the induced metric and

a combination of normal vectors according to Eq. (10). Then,

$$\begin{aligned} n_\lambda V^\lambda &= g^{\alpha\beta} n^\rho (\nabla_\alpha \delta g_{\rho\beta} - \nabla_\rho \delta g_{\alpha\beta}) \\ &= (q^{\alpha\beta} n^\rho \mp n^\alpha n^\beta n^\rho) (\nabla_\alpha \delta g_{\rho\beta} - \nabla_\rho \delta g_{\alpha\beta}) \\ &= q^{\alpha\beta} n^\rho (\nabla_\alpha \delta g_{\rho\beta} - \nabla_\rho \delta g_{\alpha\beta}) \\ &= -q^{\alpha\beta} n^\rho \nabla_\rho \delta g_{\alpha\beta}. \end{aligned} \quad (\text{D5})$$

The contribution involving three normal vectors is eliminated, as it is contracted with an antisymmetric term. In the last step we took into account that the induced metric is fixed on the boundary as is $g_{\mu\nu}|_{\partial\mathcal{M}} = 0$. Therefore, directional derivatives of the variation within the boundary can safely be set to zero, which eliminates the first term. However, taking assumptions on the derivative of the variation along directions perpendicular to the boundary is beyond Hamilton's principle applied to field theory. Thus, the remaining term provides a nonvanishing contribution on the boundary that reads

$$-\varepsilon \oint_{\partial\mathcal{M}} d^3y \sqrt{q} q^{\alpha\beta} n^\rho \nabla_\rho \delta g_{\alpha\beta}. \quad (\text{D6})$$

In fact, this term is canceled by the Gibbons-Hawking-York (GHY) boundary term added to the action. We employ the extrinsic curvature defined in Eq. (20) to obtain

$$\begin{aligned} \delta K &= -q^{\alpha\beta} \delta\Gamma^\lambda{}_{\alpha\beta} n_\lambda \\ &= -q^{\alpha\beta} \frac{1}{2} g^{\lambda\rho} (\nabla_\alpha \delta g_{\rho\beta} + \nabla_\beta \delta g_{\rho\alpha} - \nabla_\rho \delta g_{\alpha\beta}) n_\lambda \\ &= -q^{\alpha\beta} \frac{1}{2} n^\rho (\nabla_\alpha \delta g_{\rho\beta} + \nabla_\beta \delta g_{\rho\alpha} - \nabla_\rho \delta g_{\alpha\beta}) \\ &= \frac{1}{2} q^{\alpha\beta} n^\rho \nabla_\rho \delta g_{\alpha\beta}, \end{aligned} \quad (\text{D7})$$

where we again used that derivatives of variations along the boundary vanish. So we identify $n_\lambda V^\lambda = -\delta K/2$ based on Eq. (D5). Therefore, to cancel Eq. (D6), we must add the GHY boundary term of Eq. (74) to the EH action.

2. Gravitational SME: Modified boundary terms

Now we would like to evaluate the variation of the second term on the right-hand side of Eq. (76a). As before, we employ local coordinates:

$$\begin{aligned} \delta \int_{\mathcal{M}} d^4x \frac{\partial(\sqrt{-\eta}w^{(s)\lambda})}{\partial x^\lambda} \\ = \int_{\mathcal{M}} d^4x \frac{\partial}{\partial x^\lambda} [\sqrt{-\eta}(s^{\alpha\beta} \delta\Gamma^\lambda{}_{\alpha\beta} - s^{\lambda\alpha} \delta\Gamma^\nu{}_{\alpha\nu})], \end{aligned} \quad (\text{D8})$$

which is a scalar with respect to general coordinate transformations. Thus, in general coordinates it can be written as

$$\delta \int_{\mathcal{M}} d^4x \frac{\partial(\sqrt{-g}w^{(s)\lambda})}{\partial x^\lambda} = \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_\lambda Q^\lambda, \quad (\text{D9a})$$

$$Q^\lambda = s^{\alpha\beta} \delta \Gamma^\lambda_{\alpha\beta} - s^{\lambda\alpha} \delta \Gamma^\nu_{\alpha\nu}. \quad (\text{D9b})$$

Inserting the variations of the Christoffel symbols results in

$$Q^\lambda = s^{\alpha\beta} g^{\lambda\epsilon} \nabla_\alpha \delta g_{\epsilon\beta} - \frac{1}{2} (s^{\alpha\beta} g^{\lambda\epsilon} \nabla_\epsilon \delta g_{\alpha\beta} + s^{\alpha\lambda} g^{\nu\gamma} \nabla_\alpha \delta g_{\gamma\nu}), \quad (\text{D10})$$

and a subsequent contraction with n_μ implies

$$n_\lambda Q^\lambda = s^{\alpha\beta} n^\epsilon \nabla_\alpha \delta g_{\epsilon\beta} - \frac{1}{2} (s^{\alpha\beta} n^\epsilon \nabla_\epsilon \delta g_{\alpha\beta} + n_\lambda s^{\alpha\lambda} g^{\nu\gamma} \nabla_\alpha \delta g_{\gamma\nu}). \quad (\text{D11})$$

We can now benefit again from the decomposition of $s^{\mu\nu}$ stated in Eq. (27). For each of the three terms this decomposition gives rise to

$$s^{\alpha\beta} n^\epsilon \nabla_\alpha \delta g_{\epsilon\beta} = [q^\alpha_\mu q^\beta_\nu s^{\mu\nu} - (q^\alpha_\nu n^\beta + q^\beta_\nu n^\alpha) s^{\nu\alpha}] n^\epsilon \nabla_\alpha \delta g_{\epsilon\beta} + n^\alpha n^\beta s^{\alpha\gamma} n^\epsilon \nabla_\alpha \delta g_{\epsilon\beta} = (n^\alpha n^\beta s^{\alpha\gamma} n^\epsilon - q^\beta_\nu n^\alpha s^{\nu\alpha}) n^\epsilon \nabla_\alpha \delta g_{\epsilon\beta}, \quad (\text{D12a})$$

$$s^{\alpha\beta} n^\epsilon \nabla_\epsilon \delta g_{\alpha\beta} = [q^\alpha_\mu q^\beta_\nu s^{\mu\nu} - 2q^\alpha_\nu n^\beta s^{\nu\alpha}] n^\epsilon \nabla_\epsilon \delta g_{\alpha\beta} + n^\alpha n^\beta s^{\alpha\gamma} n^\epsilon \nabla_\epsilon \delta g_{\alpha\beta}, \quad (\text{D12b})$$

$$n_\lambda s^{\alpha\lambda} g^{\nu\gamma} \nabla_\alpha \delta g_{\gamma\nu} = n_\lambda [q^\alpha_\mu q^\lambda_\nu s^{\mu\nu} - (q^\alpha_\nu n^\lambda + q^\lambda_\nu n^\alpha) s^{\nu\alpha}] n^\epsilon \nabla_\alpha \delta g_{\gamma\nu} + n^\alpha n^\lambda s^{\alpha\gamma} n^\epsilon \nabla_\alpha \delta g_{\gamma\nu} = -n^\alpha (q^{\nu\gamma} - n^\nu n^\gamma) s^{\alpha\gamma} n^\epsilon \nabla_\alpha \delta g_{\gamma\nu} = (n^\nu n^\gamma - q^{\nu\gamma}) n^\alpha s^{\alpha\gamma} n^\epsilon \nabla_\alpha \delta g_{\gamma\nu}. \quad (\text{D12c})$$

Computing the linear combination of these terms that forms $n_\lambda Q^\lambda$, many contributions cancel each other. In particular, cancelations occur for all terms involving the mixed coefficients $s^{\nu\alpha}$ and for the purely timelike ones $s^{\alpha\gamma}$ multiplied by $n^\alpha n^\beta \nabla_\epsilon \delta g_{\alpha\beta}$. What remains is

$$n_\lambda Q^\lambda = \frac{1}{2} (q^{\nu\gamma} s^{\alpha\gamma} n^\alpha \nabla_\alpha \delta g_{\gamma\nu} - q^\alpha_\mu q^\beta_\nu s^{\mu\nu} n^\epsilon \nabla_\epsilon \delta g_{\alpha\beta}). \quad (\text{D13})$$

So there is a nonvanishing contribution on the boundary given by

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_\lambda Q^\lambda = \oint_{\partial\mathcal{M}} d^3y \sqrt{\bar{q}} \epsilon n_\lambda Q^\lambda. \quad (\text{D14})$$

As in the case of GR, we try to reproduce this contribution via variations of suitable coordinate scalars on the boundary that are formed from the controlling coefficients and the extrinsic-curvature tensor. There are not too many

possibilities, but we can consider $s^{ij} K_{ij}$ and the trace K . The variation of the first contraction gives

$$\begin{aligned} \delta(s^{ij} K_{ij}) &= s^{ij} \delta K_{ij} = s^{ij} q^\alpha_i q^\beta_j \delta(\nabla_{(\alpha} n_{\beta)}) \\ &= s^{ij} q^\alpha_i q^\beta_j (-\delta \Gamma^\lambda_{\alpha\beta} n_\lambda) \\ &= -\frac{1}{2} s^{ij} q^\alpha_i q^\beta_j (q^{\lambda\delta} - n^\lambda n^\delta) \\ &\quad \times (\nabla_\alpha \delta g_{\beta\delta} + \nabla_\beta \delta g_{\alpha\delta} - \nabla_\delta \delta g_{\alpha\beta}) n_\lambda \\ &= \frac{1}{2} q^\alpha_i q^\beta_j s^{ij} n^\delta \nabla_\delta \delta g_{\alpha\beta}. \end{aligned} \quad (\text{D15})$$

Therefore, by employing the previous result of Eq. (D7), we deduce that

$$n_\lambda Q^\lambda = s^{\alpha\beta} \delta K - \delta(s^{\mu\nu} K_{\mu\nu}), \quad (\text{D16})$$

which implies the boundary terms stated in Eq. (77). Note that the boundary contribution associated with the u term simply follows from scaling the GHY boundary contribution by the factor of $(1-u)$.

APPENDIX E: DERIVATION OF MODIFIED EINSTEIN EQUATIONS

For clarification, we will provide a brief derivation of the modified field equations stated in Eqs. (83) and (96). All quantities are defined in the spacetime manifold \mathcal{M} and, for brevity, the superscript “(4)” is omitted throughout this section. In what follows, we will benefit from the variation of the metric determinant:

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta}. \quad (\text{E1})$$

Furthermore, we need the variation of the Ricci tensor given by the Palatini identity:

$$\delta R_{\alpha\beta} = \nabla_\lambda \delta \Gamma^\lambda_{\alpha\beta} - \nabla_\beta \delta \Gamma^\lambda_{\alpha\lambda}, \quad (\text{E2a})$$

$$\delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}). \quad (\text{E2b})$$

Last but not least, the contracted Palatini identity

$$g^{\alpha\beta} \delta R_{\alpha\beta} = (\nabla^\alpha \nabla^\beta - g^{\alpha\beta} \nabla^2) \delta g_{\alpha\beta}, \quad (\text{E3})$$

as well as the variations involving the inverse metric,

$$\delta g^{\alpha\beta} = -g^{\alpha\mu} \delta g_{\mu\nu} g^{\nu\beta}, \quad (\text{E4a})$$

$$g_{\alpha\beta} \delta g^{\alpha\beta} = -g^{\alpha\beta} \delta g_{\alpha\beta}, \quad (\text{E4b})$$

will also turn out to be valuable. Now, a variation of the EH action implies

$$\delta S^{(0)} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \left[\left(\frac{1}{2} \sqrt{-g} g^{\alpha\beta} \delta g_{\alpha\beta} \right) R + \sqrt{-g} (\nabla^\alpha \nabla^\beta - g^{\alpha\beta} \nabla^2) \delta g_{\alpha\beta} - \sqrt{-g} R^{\alpha\beta} \delta g_{\alpha\beta} \right]. \quad (\text{E5})$$

The second term is a total derivative. However, its treatment is subtle, as it involves second-order derivatives of the metric variation. In accordance with Hamilton's principle, this contribution can only be discarded when taking into account a boundary term as described in Sec. IV C. Then,

$$\delta S^{(0)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} (-G^{\alpha\beta}) \delta g_{\alpha\beta}, \quad (\text{E6a})$$

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R, \quad (\text{E6b})$$

with the Einstein tensor $G^{\mu\nu}$. The latter finding implies the Einstein equations without matter

$$G^{\alpha\beta} = 0. \quad (\text{E7})$$

Varying the diffeomorphism-violating contribution of the action given by Eqs. (16b) and (95) leads to

$$\delta S^{(u)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} [-u (\nabla^\alpha \nabla^\beta - g^{\alpha\beta} \nabla^2) \delta g_{\alpha\beta} + u G^{\alpha\beta} \delta g_{\alpha\beta}], \quad (\text{E8})$$

where fluctuations δu of the nondynamical background do not occur. A double partial integration with suitable boundary terms taken into account implies

$$\delta S^{(u)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} (u G^{\alpha\beta} - \nabla^\alpha \nabla^\beta u + g^{\alpha\beta} \nabla^2 u) \delta g_{\alpha\beta}. \quad (\text{E9})$$

Hence, we arrive at Eq. (96b). Finally, the variation of the action given by Eqs. (16c) and (82) is considered:

$$\delta S^{(s)} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x [(\delta \sqrt{-g}) s^{\mu\nu} R_{\mu\nu} + \sqrt{-g} s^{\mu\nu} (\delta R_{\mu\nu}) + \sqrt{-g} (\delta s^{\mu\nu}) R_{\mu\nu}]. \quad (\text{E10})$$

The last term vanishes in the setting of explicit diffeomorphism violation, because the background field does not exhibit fluctuations in this case: $\delta s^{\mu\nu} = 0$. We then employ Eq. (E4b) to obtain

$$\delta S^{(s)} = \sum_{i=1}^3 \delta S_i^{(s)}, \quad (\text{E11a})$$

$$\delta S_1^{(s)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} \left(\frac{1}{2} g^{\alpha\beta} s^{\mu\nu} R_{\mu\nu} \right) \delta g_{\alpha\beta}, \quad (\text{E11b})$$

$$\delta S_2^{(s)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} s^{\mu\nu} (\nabla_\lambda \delta \Gamma^\lambda_{\mu\nu}), \quad (\text{E11c})$$

$$\delta S_3^{(s)} = - \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} s^{\mu\nu} (\nabla_\nu \delta \Gamma^\lambda_{\mu\lambda}). \quad (\text{E11d})$$

To treat Eq. (E11c), we perform integrations by parts, use Eq. (E2b), and rename some indices to arrive at

$$\delta S_2^{(s)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} \frac{1}{2} (\nabla_\mu \nabla^\alpha s^{\mu\beta} + \nabla_\nu \nabla^\alpha s^{\beta\nu} - \nabla^2 s^{\alpha\beta}) \delta g_{\alpha\beta}. \quad (\text{E12})$$

We now apply the same procedure to $\delta S_3^{(s)}$:

$$\delta S_3^{(s)} = \int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} \frac{1}{2} (-\nabla_\mu \nabla_\nu s^{\mu\nu} g^{\alpha\beta}) \delta g_{\alpha\beta}. \quad (\text{E13})$$

Summing all the contributions and performing a symmetrization leads to Eq. (83b).

APPENDIX F: PROJECTIONS OF MODIFIED EINSTEIN EQUATIONS

In the current section we are going to present detailed computations showing how to arrive at the results presented in Sec. V B. For brevity, we will introduce the following observer two-tensors:

$$(T_1^{Rs})^{\alpha\beta} = g^{\alpha\beta} s^{\mu\nu(4)} R_{\mu\nu}, \quad (\text{F1a})$$

$$(T_2^{Rs})^{\alpha\beta} = s^{\alpha\mu(4)} R^\beta_{\mu}, \quad (\text{F1b})$$

$$(T_3^{Rs})^{\alpha\beta} = s^{\beta\mu(4)} R^\alpha_{\mu}, \quad (\text{F1c})$$

$$(T_4^{Rs})^{\alpha\beta} = \nabla_\mu \nabla^\alpha s^{\mu\beta}, \quad (T_5^{Rs})^{\alpha\beta} = \nabla_\mu \nabla^\beta s^{\mu\alpha}, \quad (\text{F1d})$$

$$(T_6^{Rs})^{\alpha\beta} = \nabla_\mu \nabla^\mu s^{\alpha\beta}, \quad (T_7^{Rs})^{\alpha\beta} = g^{\alpha\beta} \nabla_\mu \nabla_\nu s^{\mu\nu}. \quad (\text{F1e})$$

Now, the tensor $(T^{Rs})^{\alpha\beta}$ given in Eq. (83b) is expressed in terms of these quantities as follows:

$$(T^{Rs})^{\alpha\beta} = \frac{1}{2} [(T_4^{Rs})^{\alpha\beta} + (T_5^{Rs})^{\alpha\beta} - (T_6^{Rs})^{\alpha\beta} - (T_7^{Rs})^{\alpha\beta}] + \frac{1}{2} (T_1^{Rs})^{\alpha\beta}. \quad (\text{F2})$$

For completeness, note that the analogous tensor occurring in the modified Einstein equations in [32] reads

$$\begin{aligned}
(\tilde{T}^{Rs})^{\alpha\beta} &= \frac{1}{2}(T_1^{Rs})^{\alpha\beta} - (T_2^{Rs})^{\alpha\beta} - (T_3^{Rs})^{\alpha\beta} \\
&+ \frac{1}{2}[(T_4^{Rs})^{\alpha\beta} + (T_5^{Rs})^{\alpha\beta} \\
&- (T_6^{Rs})^{\alpha\beta} - (T_7^{Rs})^{\alpha\beta}]. \quad (F3)
\end{aligned}$$

Obviously, there are two additional terms involving the tensors $(T_{2,3}^{Rs})^{\alpha\beta}$ showing that the theory based on the lower-index background field $s_{\mu\nu}$ in the setting of explicit diffeomorphism violation is different from the theory described by (16c) that this paper rests on. Hence, we will be working with Eq. (F2) instead of Eq. (F3).

1. Purely spacelike sector

For this sector, it is paramount to employ the key equation:

$$\begin{aligned}
q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \nabla_\nu s^{\alpha\beta} &= D_i D_j s^{ij} - K^l{}_i K_j{}_l s^{ij} + K K_{ij} s^{ij} \\
&- \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij}, \quad (F4a)
\end{aligned}$$

$$\mathcal{L}_m s^{ij} = m^\alpha \partial_\alpha s^{ij} + s^{il} \partial_l N^j + s^{jl} \partial_l N^i, \quad (F4b)$$

which is to be derived as follows. We start by expressing the covariant derivatives in Σ_t in terms of projected covariant derivatives of the four-dimensional spacetime manifold:

$$\begin{aligned}
D_\alpha (D_\beta s^{\alpha\beta}) &= q^\lambda{}_\alpha q^\mu{}_\beta q^\alpha{}_\nu q^\beta{}_\sigma \nabla_\lambda (D_\mu s^{\nu\sigma}) \\
&= q^\lambda{}_\nu q^\mu{}_\sigma \nabla_\lambda (q^\alpha{}_\mu q^\nu{}_\beta q^\sigma{}_\gamma \nabla_\alpha s^{\beta\gamma}). \quad (F5)
\end{aligned}$$

Now we apply the outer covariant derivative providing four terms:

$$\begin{aligned}
\nabla_\lambda (q^\alpha{}_\mu q^\nu{}_\beta q^\sigma{}_\gamma \nabla_\alpha s^{\beta\gamma}) &= (\nabla_\lambda q^\alpha{}_\mu) q^\nu{}_\beta q^\sigma{}_\gamma \nabla_\alpha s^{\beta\gamma} \\
&+ q^\alpha{}_\mu (\nabla_\lambda q^\nu{}_\beta) q^\sigma{}_\gamma \nabla_\alpha s^{\beta\gamma} \\
&+ q^\alpha{}_\mu q^\nu{}_\beta (\nabla_\lambda q^\sigma{}_\gamma) \nabla_\alpha s^{\beta\gamma} \\
&+ q^\alpha{}_\mu q^\nu{}_\beta q^\sigma{}_\gamma \nabla_\lambda \nabla_\alpha s^{\beta\gamma}. \quad (F6)
\end{aligned}$$

Each of these four contributions must be evaluated:

$$\begin{aligned}
r_1 &= q^\lambda{}_\nu q^\mu{}_\sigma [(\nabla_\lambda q^\alpha{}_\mu) q^\nu{}_\beta q^\sigma{}_\gamma \nabla_\alpha s^{\beta\gamma}] \\
&= q^\lambda{}_\beta q^\mu{}_\gamma \nabla_\lambda (n^\alpha n_\mu) \nabla_\alpha s^{\beta\gamma} \\
&= q^\lambda{}_\beta q^\mu{}_\gamma n^\alpha (\nabla_\lambda n_\mu) \nabla_\alpha s^{\beta\gamma} \\
&= q^\lambda{}_\beta q^\mu{}_\gamma n^\alpha K_{\lambda\mu} \nabla_\alpha s^{\beta\gamma} = K_{\beta\gamma} n^\alpha \nabla_\alpha s^{\beta\gamma}, \quad (F7a)
\end{aligned}$$

$$\begin{aligned}
r_2 &= q^\lambda{}_\nu q^\mu{}_\sigma [q^\alpha{}_\mu (\nabla_\lambda q^\nu{}_\beta) q^\sigma{}_\gamma \nabla_\alpha s^{\beta\gamma}] \\
&= q^\lambda{}_\nu q^\alpha{}_\gamma \nabla_\lambda (n^\nu n_\beta) \nabla_\alpha s^{\beta\gamma} \\
&= q^\lambda{}_\nu q^\alpha{}_\gamma (\nabla_\lambda n^\nu) n_\beta \nabla_\alpha s^{\beta\gamma} \\
&= q^\lambda{}_\nu q^\alpha{}_\gamma K_{\lambda\nu} n_\beta \nabla_\alpha s^{\beta\gamma} = K_{\nu\beta} q^\alpha{}_\gamma n_\beta \nabla_\alpha s^{\beta\gamma} \\
&= K q^\alpha{}_\gamma n_\beta \nabla_\alpha s^{\beta\gamma}, \quad (F7b)
\end{aligned}$$

$$\begin{aligned}
r_3 &= q^\lambda{}_\nu q^\mu{}_\sigma [q^\alpha{}_\mu q^\nu{}_\beta (\nabla_\lambda q^\sigma{}_\gamma) \nabla_\alpha s^{\beta\gamma}] \\
&= q^\lambda{}_\beta q^\alpha{}_\sigma \nabla_\lambda (n^\sigma n_\gamma) \nabla_\alpha s^{\beta\gamma} \\
&= q^\lambda{}_\beta q^\alpha{}_\sigma (\nabla_\lambda n^\sigma) n_\gamma \nabla_\alpha s^{\beta\gamma} \\
&= q^\lambda{}_\beta q^\alpha{}_\sigma K_{\lambda\sigma} n_\gamma \nabla_\alpha s^{\beta\gamma} = K_{\beta\gamma} n_\alpha \nabla_\alpha s^{\beta\gamma}, \quad (F7c)
\end{aligned}$$

$$\begin{aligned}
r_4 &= q^\lambda{}_\nu q^\mu{}_\sigma [q^\alpha{}_\mu q^\nu{}_\beta q^\sigma{}_\gamma \nabla_\lambda \nabla_\alpha s^{\beta\gamma}] \\
&= q^\lambda{}_\beta q^\alpha{}_\gamma \nabla_\lambda \nabla_\alpha s^{\beta\gamma}. \quad (F7d)
\end{aligned}$$

Now, considering a purely spacelike $s^{\mu\nu}$, we employ

$$\begin{aligned}
0 &= \nabla_\alpha (K q^\alpha{}_\gamma n_\beta s^{\beta\gamma}) \\
&= K q^\alpha{}_\gamma (\nabla_\alpha n_\beta) s^{\beta\gamma} + K q^\alpha{}_\gamma n_\beta \nabla_\alpha s^{\beta\gamma}, \quad (F8a)
\end{aligned}$$

$$\begin{aligned}
0 &= \nabla_\alpha (K_\beta{}^\alpha n_\gamma s^{\beta\gamma}) \\
&= K_\beta{}^\alpha (\nabla_\alpha n_\gamma) s^{\beta\gamma} + K_\beta{}^\alpha n_\gamma \nabla_\alpha s^{\beta\gamma}, \quad (F8b)
\end{aligned}$$

to reformulate the second and third term:

$$\begin{aligned}
r_2 &= -K q^\alpha{}_\gamma (\nabla_\alpha n_\beta) s^{\beta\gamma} = -K q^\alpha{}_\gamma K_{\alpha\beta} s^{\beta\gamma} \\
&= -K K_{\gamma\beta} s^{\beta\gamma}, \quad (F9a)
\end{aligned}$$

$$r_3 = -K_\beta{}^\alpha (\nabla_\alpha n_\gamma) s^{\beta\gamma} = -K_\beta{}^\alpha K_{\alpha\gamma} s^{\beta\gamma}. \quad (F9b)$$

Finally, by organizing all contributions, we have

$$\begin{aligned}
q^\lambda{}_\beta q^\alpha{}_\gamma \nabla_\lambda \nabla_\alpha s^{\beta\gamma} &= D_\alpha D_\beta s^{\alpha\beta} + K^\alpha{}_\beta K_{\alpha\gamma} s^{\beta\gamma} \\
&+ K K_{\beta\gamma} s^{\beta\gamma} - K_{\beta\gamma} n^\alpha \nabla_\alpha s^{\beta\gamma}. \quad (F10)
\end{aligned}$$

Introducing the Lie derivative on s^{ij} and considering only spacelike indices on the right-hand side of the latter relation implies Eq. (F4).

a. Orthogonal projection

First of all, based on Eqs. (35) and (A13), we obtain the following contraction of the background tensor with the projected Ricci tensor into Σ_t :

$$\begin{aligned}
s^{\lambda\sigma} q^\mu{}_\lambda q^\nu{}_\sigma {}^{(4)}R_{\mu\nu} &= -\frac{1}{N} K_{ij} \mathcal{L}_m s^{ij} + \nabla_\mu (n^\mu K_{ij} s^{ij}) \\
&- s^{ij} \frac{1}{N} D_i D_j N + s^{ij} R_{ij} \\
&- 2s^{ij} K_{il} K^l{}_j. \quad (F11)
\end{aligned}$$

We also employ

$$\frac{1}{N} K_{\mu\nu} \mathcal{L}_m s^{\mu\nu} = K_{\mu\nu} n^\alpha \nabla_\alpha s^{\mu\nu} - 2K_{\mu\nu} K^\nu{}_\lambda s^{\mu\lambda}. \quad (\text{F12})$$

Then,

$$\begin{aligned} n_\alpha n_\beta (T_1^{Rs})^{\alpha\beta} &= n_\alpha n_\beta g^{\alpha\beta} q^\mu{}_\lambda q^\nu{}_\sigma s^{\lambda\sigma(4)} R_{\mu\nu} \\ &= - \left[\nabla_\mu (n^\mu K_{\alpha\beta} s^{\alpha\beta}) - K_{\mu\nu} n^\alpha \nabla_\alpha s^{\mu\nu} - s^{ij} \frac{1}{N} D_i D_j N + s^{ij} R_{ij} \right], \end{aligned} \quad (\text{F13a})$$

$$\begin{aligned} n_\alpha n_\beta (T_4^{Rs})^{\alpha\beta} &= n_\alpha n_\beta \nabla_\nu \nabla^\alpha (q^\nu{}_\lambda q^\beta{}_\sigma s^{\lambda\sigma}) \\ &= -a_\lambda a_\sigma s^{\lambda\sigma} + K_{\nu\alpha} s^{\nu\sigma} K^\alpha{}_\sigma - s^{\nu\sigma} \nabla_\nu a_\sigma - a_\sigma \nabla_\lambda s^{\lambda\sigma} - n^\alpha K_{\lambda\sigma} \nabla_\alpha s^{\lambda\sigma} \\ &= n_\alpha n_\beta (T_5^{Rs})^{\alpha\beta}, \end{aligned} \quad (\text{F13b})$$

$$n_\alpha n_\beta (T_6^{Rs})^{\alpha\beta} = n_\alpha n_\beta \nabla^2 (q^\alpha{}_\mu q^\beta{}_\nu s^{\mu\nu}) = 2(K^\mu{}_\lambda K_{\mu\sigma} s^{\lambda\sigma} - a_\lambda a_\sigma s^{\lambda\sigma}), \quad (\text{F13c})$$

$$\begin{aligned} n_\alpha n_\beta (T_7^{Rs})^{\alpha\beta} &= n_\alpha n_\beta g^{\alpha\beta} \nabla_\mu \nabla_\nu (q^\mu{}_\kappa q^\nu{}_\lambda s^{\kappa\lambda}) \\ &= -[\nabla_\mu (n^\mu K_{\alpha\beta} s^{\alpha\beta}) + (\nabla_\mu a_\beta) s^{\mu\beta} + a_\beta \nabla_\mu s^{\mu\beta} - K K_{\alpha\beta} s^{\alpha\beta} \\ &\quad + a_\alpha \nabla_\beta s^{\alpha\beta} - a_\alpha a_\beta s^{\alpha\beta} - K^\nu{}_\alpha K_{\beta\nu} s^{\alpha\beta} + n^\nu K_{\alpha\beta} \nabla_\nu s^{\alpha\beta} + q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \nabla_\nu s^{\alpha\beta}]. \end{aligned} \quad (\text{F13d})$$

Summing all these contributions implies

$$\begin{aligned} 2n_\alpha n_\beta (T^{Rs})^{\alpha\beta} &= n_\alpha n_\beta [(T_1^{Rs})^{\alpha\beta} + 2(T_4^{Rs})^{\alpha\beta} - (T_6^{Rs})^{\alpha\beta} - (T_7^{Rs})^{\alpha\beta}] \\ &= -a_\alpha a_\beta s^{\alpha\beta} - K^\mu{}_\lambda K_{\mu\sigma} s^{\lambda\sigma} - K K_{\alpha\beta} s^{\alpha\beta} - s^{\lambda\sigma} \nabla_\lambda a_\sigma + s^{ij} \frac{1}{N} D_i D_j N - s^{ij} R_{ij} + q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \nabla_\nu s^{\alpha\beta}. \end{aligned} \quad (\text{F14})$$

We now use that

$$s^{\lambda\sigma} D_\lambda a_\sigma = s^{\lambda\sigma} q^\alpha{}_\lambda q^\beta{}_\sigma \nabla_\alpha a_\beta = s^{\alpha\beta} \nabla_\alpha a_\beta, \quad (\text{F15})$$

in combination with Eq. (A17b) to obtain

$$s^{\alpha\beta} \nabla_\alpha a_\beta = s^{\lambda\sigma} \frac{1}{N} D_\lambda D_\sigma N - s^{\lambda\sigma} a_\lambda a_\sigma. \quad (\text{F16})$$

Finally, by employing Eq. (F4), we deduce

$$2n_\alpha n_\beta (T^{Rs})^{\alpha\beta} = \frac{1}{2} \left[-2K^j{}_i K_{jk} s^{ik} - s^{ij} R_{ij} + D_i D_j s^{ij} - \frac{1}{N} K_{ij} \mathcal{L}_m s^{ij} \right]. \quad (\text{F17})$$

The latter corresponds to Eq. (88).

b. Mixed projection

Here it is reasonable to employ the following form of the Lie derivative of the background:

$$\frac{1}{N} \mathcal{L}_m s^{\lambda\sigma} = n^\alpha \nabla_\alpha s^{\lambda\sigma} - (a_\mu n^\lambda + K^\lambda{}_\mu) s^{\mu\sigma} - (a_\mu n^\sigma + K^\sigma{}_\mu) s^{\lambda\mu}. \quad (\text{F18})$$

For any tensor $T_{\lambda\sigma}$ with spacelike lower indices we have

$$T_{\lambda\sigma} n^\alpha \nabla_\alpha s^{\lambda\sigma} = T_{\lambda\sigma} (K^\lambda{}_\mu s^{\mu\sigma} + K^\sigma{}_\mu s^{\lambda\mu}) + T_{\lambda\sigma} \frac{1}{N} \mathcal{L}_m s^{\lambda\sigma}. \quad (\text{F19})$$

In the forthcoming analysis we only consider those terms that are nonzero when contracted with $q^k{}_\alpha n_\beta$. We also take $s^{\lambda\sigma}$ as purely spacelike, which tells us that $n_\lambda s^{\lambda\sigma} = 0$. Then,

$$\begin{aligned}
q^k{}_\alpha n_\beta (T_4^{Rs})^{\alpha\beta} &= q^k{}_\alpha n_\beta \nabla_\nu \nabla^\alpha (q^\nu{}_\lambda q^\beta{}_\sigma s^{\lambda\sigma}) \\
&= q^k{}_\alpha n_\beta \nabla_\nu [(\nabla^\alpha q^\nu{}_\lambda) q^\beta{}_\sigma s^{\lambda\sigma} + q^\nu{}_\lambda (\nabla^\alpha q^\beta{}_\sigma) s^{\lambda\sigma} + q^\nu{}_\lambda q^\beta{}_\sigma (\nabla^\alpha s^{\lambda\sigma})] \\
&= q^k{}_\alpha n_\beta [(\nabla^\alpha q^\nu{}_\lambda) (\nabla_\nu q^\beta{}_\sigma) s^{\lambda\sigma} + (\nabla_\nu q^\nu{}_\lambda) (\nabla^\alpha q^\beta{}_\sigma) s^{\lambda\sigma} \\
&\quad + q^\nu{}_\lambda (\nabla_\nu \nabla^\alpha q^\beta{}_\sigma) s^{\lambda\sigma} + q^\nu{}_\lambda (\nabla^\alpha q^\beta{}_\sigma) (\nabla_\nu s^{\lambda\sigma}) + q^\nu{}_\lambda (\nabla_\nu q^\beta{}_\sigma) (\nabla^\alpha s^{\lambda\sigma})] \\
&= q^k{}_\alpha n_\beta [K^\alpha{}_\lambda n^\beta a_\sigma s^{\lambda\sigma} + a_\lambda n^\beta K^\alpha{}_\sigma s^{\lambda\sigma} + q^\nu{}_\lambda n^\beta (\nabla_\nu K^\alpha{}_\sigma - K^\alpha{}_\nu a_\sigma) s^{\lambda\sigma} + q^\nu{}_\lambda n^\beta K^\alpha{}_\sigma \nabla_\nu s^{\lambda\sigma} + q^\nu{}_\lambda n^\beta K_{\nu\sigma} \nabla^\alpha s^{\lambda\sigma}] \\
&= -K^k{}_\lambda a_\sigma s^{\lambda\sigma} - D_\lambda (K^k{}_\sigma s^{\lambda\sigma}) - K_{\lambda\sigma} D^k s^{\lambda\sigma}.
\end{aligned} \tag{F20}$$

In an analogous manner, we obtain

$$\begin{aligned}
q^k{}_\alpha n_\beta (T_5^{Rs})^{\alpha\beta} &= q^k{}_\beta n_\alpha \nabla_\nu \nabla^\alpha (q^\nu{}_\lambda q^\beta{}_\sigma s^{\lambda\sigma}) \\
&= K a_\lambda s^{\lambda k} - a_\alpha K^\alpha{}_\lambda s^{\lambda k} + n^\nu \nabla_\nu a_\lambda s^{\lambda k} + n^\nu q^k{}_\beta a_\lambda \nabla_\nu s^{\lambda\beta} + K^k{}_\nu a_\sigma s^{\nu\sigma} \\
&\quad + n_\alpha q^k{}_\sigma (K n_\lambda + a_\lambda) \nabla^\alpha s^{\lambda\sigma} + n_\alpha K^k{}_\lambda n_\sigma \nabla^\alpha s^{\lambda\sigma} + n_\alpha q^\nu{}_\lambda q^k{}_\sigma \nabla_\nu \nabla^\alpha s^{\lambda\sigma},
\end{aligned} \tag{F21}$$

as well as

$$\begin{aligned}
q^k{}_\alpha n_\beta (T_6^{Rs})^{\alpha\beta} &= q^k{}_\alpha n_\beta \nabla^2 (q^\alpha{}_\lambda q^\beta{}_\sigma s^{\lambda\sigma}) \\
&= -q^k{}_\lambda [\nabla_\mu K^\mu{}_\sigma - \nabla_\mu (n^\mu a_\sigma)] s^{\lambda\sigma} - 2q^k{}_\lambda (K^\mu{}_\sigma - n^\mu a_\sigma) \nabla_\mu s^{\lambda\sigma},
\end{aligned} \tag{F22}$$

whereby

$$q^k{}_\alpha n_\beta (T_1^{Rs})^{\alpha\beta} = q^k{}_\alpha n_\beta (T_7^{Rs})^{\alpha\beta} = 0. \tag{F23}$$

Performing the sum of Eqs. (F20)–(F23), leads to Eq. (89). To simplify this result, it is important to find

$$n_\alpha q^\nu{}_\lambda q^k{}_\sigma \nabla_\nu \nabla^\alpha s^{\lambda\sigma} = \nabla_\nu (n_\alpha q^\nu{}_\lambda q^k{}_\sigma \nabla^\alpha s^{\lambda\sigma}) - \nabla_\nu (n_\alpha q^\nu{}_\lambda q^k{}_\sigma) (\nabla^\alpha s^{\lambda\sigma}). \tag{F24}$$

We need to consider the two pieces:

$$n_\alpha q^\nu{}_\lambda q^k{}_\sigma \nabla^\alpha s^{\lambda\sigma} = q^\nu{}_\lambda q^k{}_\sigma \left(\frac{1}{N} \mathcal{L}_m s^{\lambda\sigma} + K^\lambda{}_\mu s^{\mu\sigma} + K^\sigma{}_\mu s^{\lambda\mu} \right), \tag{F25a}$$

$$\nabla_\nu (n_\alpha q^\nu{}_\lambda q^k{}_\sigma) (\nabla^\alpha s^{\lambda\sigma}) = [K_{\lambda\alpha} q^k{}_\sigma + K n_\alpha n_\lambda q^k{}_\sigma + n_\alpha a_\lambda q^k{}_\sigma + n_\alpha K^k{}_\lambda n_\sigma + n_\alpha n^k K_{\lambda\sigma}] \nabla^\alpha s^{\lambda\sigma}. \tag{F25b}$$

Finally, some algebra gives rise to

$$\begin{aligned}
n_\alpha q^\nu{}_\lambda q^k{}_\sigma \nabla_\nu \nabla^\alpha s^{\lambda\sigma} &= \nabla_\nu (K^\nu{}_\mu s^{\mu k} + K^k{}_\mu s^{\nu\mu}) - K_{\lambda\alpha} q^k{}_\sigma \nabla^\alpha s^{\lambda\sigma} + K a_\mu s^{\mu k} - a_\lambda K^\lambda{}_\mu s^{\mu k} - n^k (K_{\lambda\sigma} K^\lambda{}_\mu s^{\mu\sigma} + K_{\lambda\sigma} K^\sigma{}_\mu s^{\lambda\mu}) \\
&\quad + D_i \left(\frac{1}{N} \mathcal{L}_m s^{ik} \right).
\end{aligned} \tag{F26}$$

2. Purely timelike sector

Here we must compute the projections of $(T^{Rs})^{\alpha\beta} + s^{\text{nn}} R^{\alpha\beta}$ with $(T^{Rs})^{\alpha\beta}$ given by Eq. (F2).

a. Orthogonal projection

We start by deriving the complete projection along the direction orthogonal to Σ_t [see also Eq. (92)]. The individual contributions amount to

$$n_\alpha n_\beta (T_1^{Rs})^{\alpha\beta} = s^{\text{nn}} n_\alpha n_\beta R^{\alpha\beta} = s^{\text{nn}} (K_{ij} K^{ij} + n^\mu \nabla_\mu K - \nabla_\mu a^\mu), \quad (\text{F27a})$$

$$\begin{aligned} n_\alpha n_\beta (T_4^{Rs})^{\alpha\beta} &= n_\alpha n_\beta [(\nabla_\nu \nabla^\alpha s^{\text{nn}}) n^\nu n^\beta + (\nabla^\alpha s^{\text{nn}}) (\nabla_\nu n^\nu) n^\beta + (\nabla_\nu s^{\text{nn}}) (\nabla^\alpha n^\nu) n^\beta + s^{\text{nn}} (\nabla_\nu \nabla^\alpha n^\nu) n^\beta + s^{\text{nn}} n^\nu \nabla_\nu \nabla^\alpha n^\beta] \\ &= n_\alpha n_\beta [(\nabla_\nu \nabla^\alpha s^{\text{nn}}) n^\nu n^\beta + (\nabla^\alpha s^{\text{nn}}) K n^\beta + \nabla_\nu s^{\text{nn}} (K^{\alpha\nu} - n^\alpha a^\nu) n^\beta \\ &\quad + s^{\text{nn}} (\nabla_\nu K^{\alpha\nu} - n^\alpha \nabla_\nu a^\nu) n^\beta + s^{\text{nn}} n^\nu (\nabla_\nu K^{\alpha\beta} - n^\alpha \nabla_\nu a^\beta)] \\ &= -(n_\alpha n^\nu \nabla_\nu \nabla^\alpha s^{\text{nn}} + n_\alpha \nabla^\alpha s^{\text{nn}} K + a^\nu \nabla_\nu s^{\text{nn}} - s^{\text{nn}} K_{\alpha\nu} K^{\alpha\nu} + s^{\text{nn}} \nabla_\nu a^\nu + s^{\text{nn}} a_\beta a^\beta), \end{aligned} \quad (\text{F27b})$$

$$\begin{aligned} n_\alpha n_\beta (T_6^{Rs})^{\alpha\beta} &= n_\alpha n_\beta [(\nabla_\mu \nabla^\mu n^\alpha) n^\beta s^{\text{nn}} + n^\alpha (\nabla_\mu \nabla^\mu n^\beta) s^{\text{nn}} + n^\alpha n^\beta \nabla_\mu \nabla^\mu s^{\text{nn}}] \\ &= n_\alpha n_\beta [(\nabla_\mu K^{\mu\alpha}) n^\beta s^{\text{nn}} - n^\mu (\nabla_\mu a^\alpha) n^\beta s^{\text{nn}} + n^\alpha (\nabla_\mu K^{\mu\beta}) s^{\text{nn}} - n^\alpha (\nabla_\mu a^\beta) n^\mu s^{\text{nn}} + n^\alpha n^\beta \nabla_\mu \nabla^\mu s^{\text{nn}}] \\ &= -n_\alpha (\nabla_\mu K^{\mu\alpha}) s^{\text{nn}} + n^\mu n_\alpha (\nabla_\mu a^\alpha) s^{\text{nn}} - n_\beta (\nabla_\mu K^{\mu\beta}) s^{\text{nn}} + n_\beta n^\mu (\nabla_\mu a^\beta) s^{\text{nn}} + \nabla_\mu \nabla^\mu s^{\text{nn}} \\ &= 2s^{\text{nn}} (K_{\mu\alpha} K^{\mu\alpha} - a_\alpha a^\alpha) + \nabla_\mu \nabla^\mu s^{\text{nn}}, \end{aligned} \quad (\text{F27c})$$

$$\begin{aligned} n_\alpha n_\beta (T_7^{Rs})^{\alpha\beta} &= -\nabla_\mu [(a^\mu + n^\mu K) s^{\text{nn}} + n^\mu n^\nu \nabla_\nu s^{\text{nn}}] \\ &= -[(\nabla_\mu a^\mu) s^{\text{nn}} + a^\mu \nabla_\mu s^{\text{nn}} + (\nabla_\mu n^\mu) K s^{\text{nn}} + n^\mu (\nabla_\mu K) s^{\text{nn}} + n^\mu K \nabla_\mu s^{\text{nn}} \\ &\quad + (\nabla_\mu n^\mu) n^\nu \nabla_\nu s^{\text{nn}} + n^\mu (\nabla_\mu n^\nu) \nabla_\nu s^{\text{nn}} + n^\mu n^\nu \nabla_\mu \nabla_\nu s^{\text{nn}}] \\ &= -[s^{\text{nn}} \nabla_\nu a^\nu + 2a^\mu \nabla_\mu s^{\text{nn}} + s^{\text{nn}} \nabla_\mu (n^\mu K) + 2K n^\nu \nabla_\nu s^{\text{nn}} + n^\mu n^\nu \nabla_\mu \nabla_\nu s^{\text{nn}}]. \end{aligned} \quad (\text{F27d})$$

Here we used that

$$0 = \nabla_\nu (n_\alpha n_\beta s^{\text{nn}} n^\nu K^{\alpha\beta}) = n_\alpha n_\beta s^{\text{nn}} n^\nu \nabla_\nu K^{\alpha\beta}, \quad (\text{F28a})$$

$$0 = \nabla_\nu (n_\alpha s^{\text{nn}} K^{\alpha\nu}) = (\nabla_\nu n_\alpha) s^{\text{nn}} K^{\alpha\nu} + n_\alpha s^{\text{nn}} \nabla_\nu K^{\alpha\nu} = s^{\text{nn}} K_{\alpha\nu} K^{\alpha\nu} + n_\alpha s^{\text{nn}} \nabla_\nu K^{\alpha\nu}, \quad (\text{F28b})$$

$$0 = \nabla_\nu (s^{\text{nn}} n^\nu n_\beta a^\beta) = s^{\text{nn}} n^\nu (\nabla_\nu n_\beta) a^\beta + s^{\text{nn}} n^\nu n_\beta \nabla_\nu a^\beta = s^{\text{nn}} a_\beta a^\beta + s^{\text{nn}} n^\nu n_\beta \nabla_\nu a^\beta. \quad (\text{F28c})$$

Summing the individual terms implies

$$2n_\alpha n_\beta [(T^{Rs})^{\alpha\beta} + s^{\text{nn}} R^{\alpha\beta}] = s^{\text{nn}} (K^2 - K_{ij} K^{ij}) - q^\nu{}_\alpha \nabla_\nu \nabla^\alpha s^{\text{nn}}. \quad (\text{F29})$$

By employing

$$q^\lambda{}_\sigma \nabla_\lambda \nabla^\sigma s^{\text{nn}} = q^\lambda{}_\sigma q_\sigma{}^\alpha \nabla_\lambda \nabla^\alpha s^{\text{nn}} = D_\alpha D^\alpha s^{\text{nn}} - \frac{1}{N} K \mathcal{L}_m s^{\text{nn}}, \quad (\text{F30})$$

we get

$$2n_\alpha n_\beta [(T^{Rs})^{\alpha\beta} + s^{\text{nn}} R^{\alpha\beta}] = s^{\text{nn}} (K^2 - K_{ij} K^{ij}) - D_i D^i s^{\text{nn}} + \frac{1}{N} K \mathcal{L}_m s^{\text{nn}}, \quad (\text{F31})$$

which corresponds to Eq. (93). We still have to prove Eq. (F30). It is reasonable to proceed as for Eq. (F4), although it turns out that the current computations are much less involved:

$$D_\alpha D^\alpha s^{\text{nn}} = q^\beta{}_\alpha \nabla_\beta (D^\alpha s^{\text{nn}}) = q^\beta{}_\alpha \nabla_\beta (q^\alpha{}_\gamma \nabla^\gamma s^{\text{nn}}) = q^\beta{}_\alpha (\nabla_\beta q^\alpha{}_\gamma) \nabla^\gamma s^{\text{nn}} + q^\beta{}_\alpha q^\alpha{}_\gamma \nabla_\beta \nabla^\gamma s^{\text{nn}}. \quad (\text{F32})$$

The second term just corresponds to the left-hand side of Eq. (F30). The first contribution can be reformulated as

$$\begin{aligned} q^\beta{}_\alpha (\nabla_\beta q^\alpha{}_\gamma) \nabla^\gamma s^{\text{nn}} &= q^\beta{}_\alpha (\nabla_\beta n^\alpha n_\gamma) \nabla^\gamma s^{\text{nn}} = q^\beta{}_\alpha (\nabla_\beta n^\alpha) n_\gamma \nabla^\gamma s^{\text{nn}} \\ &= q^\beta{}_\alpha (K^\alpha{}_\beta - a^\alpha n_\beta) n_\gamma \nabla^\gamma s^{\text{nn}} = K n^\gamma \nabla_\gamma s^{\text{nn}} = \frac{1}{N} K \mathcal{L}_m s^{\text{nn}}, \end{aligned} \quad (\text{F33})$$

which directly implies Eq. (F30).

b. Mixed projection

The mixed projections are given as follows:

$$n_\alpha q^k_\beta (T_1^{Rs})^{\alpha\beta} = n_\alpha q^k_\beta (T_7^{Rs})^{\alpha\beta} = 0, \quad (\text{F34a})$$

$$\begin{aligned} n_\alpha q^k_\beta (T_4^{Rs})^{\alpha\beta} &= n_\alpha q^k_\beta \nabla_\nu [(\nabla^\alpha n^\nu) n^\beta s^{\text{nn}} + n^\nu (\nabla^\alpha n^\beta) s^{\text{nn}} + n^\nu n^\beta (\nabla^\alpha s^{\text{nn}})] \\ &= n_\alpha q^k_\beta [(\nabla^\alpha n^\nu) (\nabla_\nu n^\beta) s^{\text{nn}} + (\nabla_\nu n^\nu) (\nabla^\alpha n^\beta) s^{\text{nn}} + n^\nu (\nabla_\nu \nabla^\alpha n^\beta) s^{\text{nn}} + n^\nu (\nabla^\alpha n^\beta) \nabla_\nu s^{\text{nn}} + n^\nu (\nabla_\nu n^\beta) \nabla^\alpha s^{\text{nn}}] \\ &= n_\alpha q^k_\beta [-n^\alpha a^\nu K_\nu^\beta s^{\text{nn}} - K n^\alpha a^\beta s^{\text{nn}} + n^\nu (\nabla_\nu K^{\alpha\beta}) s^{\text{nn}} - n^\alpha n^\nu (\nabla_\nu a^\beta) s^{\text{nn}} - n^\alpha n^\nu a^\beta \nabla_\nu s^{\text{nn}} + a^\beta \nabla^\alpha s^{\text{nn}}] \\ &= a^k (s^{\text{nn}} K + 2n^\nu \nabla_\nu s^{\text{nn}}) + s^{\text{nn}} n^\nu \nabla_\nu a^k, \end{aligned} \quad (\text{F34b})$$

$$\begin{aligned} n_\alpha q^k_\beta (T_5^{Rs})^{\alpha\beta} &= n_\alpha q^k_\beta \nabla_\nu [(\nabla^\beta n^\nu) n^\alpha s^{\text{nn}} + n^\nu (\nabla^\beta n^\alpha) s^{\text{nn}} + n^\nu n^\alpha (\nabla^\beta s^{\text{nn}})] \\ &= n_\alpha q^k_\beta [(\nabla_\nu \nabla^\beta n^\nu) n^\alpha s^{\text{nn}} + (\nabla^\beta n^\nu) n^\alpha \nabla_\nu s^{\text{nn}} + n^\nu (\nabla_\nu \nabla^\beta n^\alpha) s^{\text{nn}} + (\nabla_\nu n^\nu) n^\alpha \nabla^\beta s^{\text{nn}} + n^\nu n^\alpha \nabla_\nu \nabla^\beta s^{\text{nn}}] \\ &= n_\alpha q^k_\beta [(\nabla_\nu K^{\beta\nu} - K_\nu^\beta a^\nu) n^\alpha s^{\text{nn}} + K^{\beta\nu} n^\alpha \nabla_\nu s^{\text{nn}} + n^\nu \nabla_\nu K^{\beta\alpha} s^{\text{nn}} + K n^\alpha \nabla^\beta s^{\text{nn}} + n^\nu n^\alpha \nabla_\nu \nabla^\beta s^{\text{nn}}] \\ &= -s^{\text{nn}} q^k_\beta \nabla_\nu K^{\beta\nu} + s^{\text{nn}} a^\nu K_\nu^k - K^{k\nu} \nabla_\nu s^{\text{nn}} + s^{\text{nn}} n_\alpha q^k_\beta n^\nu \nabla_\nu K^{\alpha\beta} - K q^k_\beta \nabla^\beta s^{\text{nn}} - n^\nu q^k_\beta \nabla_\nu \nabla^\beta s^{\text{nn}} \\ &= -s^{\text{nn}} \nabla_\nu K^{k\nu} + s^{\text{nn}} a_i K^{ik} - K^{ki} D_i s^{\text{nn}} + s^{\text{nn}} n_\alpha n^\nu \nabla_\nu K^{\alpha k} - K D^k s^{\text{nn}} - n^\nu q^k_\beta \nabla_\nu \nabla^\beta s^{\text{nn}}, \end{aligned} \quad (\text{F34c})$$

$$\begin{aligned} n_\alpha q^k_\beta (T_6^{Rs})^{\alpha\beta} &= n_\alpha q^k_\beta \nabla_\mu [(\nabla^\mu n^\alpha) n^\beta s^{\text{nn}} + n^\alpha (\nabla^\mu n^\beta) s^{\text{nn}} + n^\alpha n^\beta \nabla^\mu s^{\text{nn}}] \\ &= n_\alpha q^k_\beta [n^\alpha (\nabla_\mu \nabla^\mu n^\beta) s^{\text{nn}} + 2n^\alpha (\nabla^\mu n^\beta) \nabla_\mu s^{\text{nn}}] \\ &= n_\alpha q^k_\beta [n^\alpha (\nabla_\mu K^{\mu\beta} - K a^\beta - n^\mu \nabla_\mu a^\beta) s^{\text{nn}} + 2n^\alpha (K^{\mu\beta} - n^\mu a^\beta) \nabla_\mu s^{\text{nn}}] \\ &= s^{\text{nn}} (-q^k_\beta \nabla_\mu K^{\mu\beta} + K a^k + q^k_\beta n^\mu \nabla_\mu a^\beta) + 2(a^k n^\mu - K^{\mu k}) \nabla_\mu s^{\text{nn}}. \end{aligned} \quad (\text{F34d})$$

By summing all the terms, we obtain

$$2n_\alpha q^k_\beta (T^{Rs})^{\alpha\beta} = s^{\text{nn}} a_i K^{ik} + K^{ki} D_i s^{\text{nn}} + s^{\text{nn}} n_\alpha n^\nu \nabla_\nu K^{\alpha k} - K D^k s^{\text{nn}} - n^\nu q^k_\beta \nabla_\nu \nabla^\beta s^{\text{nn}}. \quad (\text{F35})$$

To evaluate the contraction of the extra contribution given by Eq. (92), we employ the contracted Codazzi-Mainardi relation stated in Eq. (A20b):

$$n_\alpha q^k_\beta (s^{\text{nn}} R^{\alpha\beta}) = s^{\text{nn}} (D_i K^{ik} - D^k K). \quad (\text{F36})$$

Furthermore, we use

$$0 = s^{\text{nn}} n^\nu \nabla_\nu (n_\alpha K^{\alpha k}) = s^{\text{nn}} a_i K^{\alpha i} + s^{\text{nn}} n_\alpha n^\nu \nabla_\nu K^{\alpha k}, \quad (\text{F37})$$

as well as

$$n^\nu q^k_\beta \nabla_\nu \nabla^\beta s^{\text{nn}} = D^k \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right) - K^{ki} D_i s^{\text{nn}}, \quad (\text{F38})$$

to cast the mixed projection into its final form:

$$\begin{aligned} 2n_\alpha q^k_\beta [(T^{Rs})^{\alpha\beta} + s^{\text{nn}} R^{\alpha\beta}] &= -D^k \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right) + 2s^{\text{nn}} (D_i K^{ik} - D^k K) + 2K^{ki} D_i s^{\text{nn}} - K D^k s^{\text{nn}} \\ &= -D^k \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} + 2s^{\text{nn}} K \right) + 2D_j (s^{\text{nn}} K^i_k) + K D^k s^{\text{nn}}. \end{aligned} \quad (\text{F39})$$

The latter result corresponds to Eq. (94). The validity of Eq. (F38) remains to be shown. We start by considering the covariant derivative of the Lie derivative:

$$\begin{aligned}
D^k \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right) &= D^k (n^\mu \nabla_\mu s^{\text{nn}}) = q^k{}_\nu \nabla^\nu (n^\mu \nabla_\mu s^{\text{nn}}) \\
&= q^k{}_\nu (\nabla^\nu n^\mu) \nabla_\mu s^{\text{nn}} + q^k{}_\nu n^\mu \nabla^\nu \nabla_\mu s^{\text{nn}} \\
&= q^k{}_\nu (K^{\mu\nu} - a^\mu n^\nu) \nabla_\mu s^{\text{nn}} + q^k{}_\nu n^\mu \nabla^\nu \nabla_\mu s^{\text{nn}} \\
&= K^{ki} D_i s^{\text{nn}} + q^k{}_\nu n^\mu \nabla_\mu \nabla^\nu s^{\text{nn}}, \tag{F40}
\end{aligned}$$

as the indices of the double covariant derivative in the last term can be switched:

$$\nabla_\nu \nabla_\mu s^{\text{nn}} = \nabla_\nu \partial_\mu s^{\text{nn}} = \partial_\nu \partial_\mu s^{\text{nn}} - \Gamma^\lambda{}_{\nu\mu} \partial_\lambda s^{\text{nn}} = \partial_\mu \partial_\nu s^{\text{nn}} - \Gamma^\lambda{}_{\mu\nu} \partial_\lambda s^{\text{nn}} = \nabla_\mu \nabla_\nu s^{\text{nn}}. \tag{F41}$$

Thus, Eq. (F38) is confirmed.

3. Scalar sector

The scalar sector rests on the modification $(T^{Ru})^{\alpha\beta}$ of the Einstein equations given by Eq. (96b). Its purely orthogonal projection follows from

$$n_\alpha n_\beta (T^{Ru})^{\alpha\beta} = -n_\alpha n_\beta \nabla^\alpha \nabla^\beta u - g_{\alpha\beta} \nabla^\alpha \nabla^\beta u + u \left(R_{\text{nn}} + \frac{1}{2} {}^{(4)}R \right) = -q_{\alpha\beta} \nabla^\alpha \nabla^\beta u + u \left(R_{\text{nn}} + \frac{1}{2} {}^{(4)}R \right). \tag{F42}$$

Here we can use Eqs. (A15), (A16), and (F30) to derive

$$\begin{aligned}
2n_\alpha n_\beta (T^{Ru})^{\alpha\beta} &= -2D_i D^i u + \frac{2}{N} K \mathcal{L}_m u + 2u \left[-\frac{1}{N} \mathcal{L}_m K + \frac{1}{N} D_i D^i N - K^{ij} K_{ij} + \frac{1}{N} \mathcal{L}_m K - \frac{1}{N} D_i D^i N + \frac{1}{2} (R + K^2 + K_{ij} K^{ij}) \right] \\
&= (R + K^2 - K_{ij} K^{ij}) u - 2D_i D^i u + \frac{2}{N} K \mathcal{L}_m u, \tag{F43}
\end{aligned}$$

which results in Eq. (97a). Furthermore, we compute the mixed projection of the modification in benefiting from Eq. (F38) as well as the contracted Codazzi-Mainardi relation of Eq. (A20b). Then,

$$\begin{aligned}
2n_\alpha q^k{}_\beta (T^{Ru})^{\alpha\beta} &= -2n_\alpha q^k{}_\beta (\nabla^\alpha \nabla^\beta u + \nabla^\beta \nabla^\alpha u) + 2u n_\alpha q^k{}_\beta R^{\alpha\beta} \\
&= 2K^{ki} D_i u - 2D^k \left(\frac{1}{N} \mathcal{L}_m u \right) + 2u (D_i K^{ik} - D^k K) \\
&= 2 \left[-D^k \left(\frac{1}{N} \mathcal{L}_m u \right) + D_i (u K^{ik}) - u D^k K \right]. \tag{F44}
\end{aligned}$$

The latter corresponds to Eq. (97b).

APPENDIX G: FUNCTIONAL DERIVATIVES OF ADM ACTION

In the current section we intend to present some details on the computation of functional derivatives of ADM-decomposed actions with respect to the lapse function and the shift vector. The corresponding results serve as a base for Sec. V E.

1. General relativity

To compute functional derivatives of the ADM action for the lapse function, we benefit from the result that K and K_{ij} scale with $1/N$. Therefore,

$$\frac{\delta(NK_{ij}K^{ij})}{\delta N} = -K_{ij}K^{ij}, \quad \frac{\delta(NK^2)}{\delta N} = -K^2. \tag{G1}$$

Hence, Eq. (98a) immediately follows. The derivatives of the shift vector are a bit more involved. To evaluate them, implicit partial integrations with respect to the measure $d^3x\sqrt{q}$ are performed where surface terms are discarded. A partial integration having been carried out is indicated by a covariant derivative acting to the left. Then we obtain the following general result:

$$Nf \frac{\delta}{\delta N^k} K^{ij} = \frac{f}{2} (\delta_k^j \tilde{D}^i + \delta_k^i \tilde{D}^j) = \frac{1}{2} (\delta_k^j D^i f + \delta_k^i D^j f), \quad \frac{\delta}{\delta N} (N s^{ij} R_{ij}) = s^{ij} R_{ij}, \quad (\text{G4c})$$

with a generic spacetime coordinate-dependent tensor f (with its indices omitted for brevity). Then,

$$\frac{\delta(NK_{ij}K^{ij})}{\delta N^k} = 2D^i K_{ik}, \quad \frac{\delta(NK^2)}{\delta N^k} = 2D_k K. \quad (\text{G3})$$

A combination of the latter findings implies Eq. (98b).

2. Standard-Model extension: $s^{\mu\nu}$ term

We perform the analogous computations for the ADM-decomposed actions based on the three Lagrange densities of Eqs. (30b)–(30d).

a. Purely spacelike sector

We employ the following results:

$$-\frac{\delta}{\delta N} (K_{ij} \mathcal{L}_m s^{ij}) = \frac{1}{N} (K_{ij} \mathcal{L}_m s^{ij}), \quad (\text{G4a})$$

$$\frac{\delta}{\delta N} (-s^{ij} D_i D_j N) = -D_j D_i s^{ij}, \quad (\text{G4b})$$

$$\frac{\delta}{\delta N} (-2N s^{ij} K_{il} K^l_j) = 2s^{ij} K_{il} K^l_j, \quad (\text{G4d})$$

leading to Eq. (100a). For the derivatives with respect to the shift vector we use

$$\frac{\delta}{\delta N^k} (K_{ij} \mathcal{L}_m s^{ij}) = \frac{\delta(K_{ij})}{\delta N^k} \mathcal{L}_m s^{ij} + K_{ij} \frac{\delta}{\delta N^k} (\mathcal{L}_m s^{ij}), \quad (\text{G5a})$$

$$\frac{\delta}{\delta N^k} (2N s^{ij} K_{il} K^l_j) = (2N s^{ij}) \frac{\delta}{\delta N^k} (K_{il} K^l_j). \quad (\text{G5b})$$

From Eq. (G2) we deduce that

$$\begin{aligned} \frac{\delta(K_{ij})}{\delta N^k} \mathcal{L}_m s^{ij} &= N \left(\frac{1}{N} \mathcal{L}_m s^{ij} \right) \frac{\delta}{\delta N^k} K_{ij} \\ &= \frac{1}{2} (q_{jk} D_i + q_{ik} D_j) \left(\frac{1}{N} \mathcal{L}_m s^{ij} \right) \\ &= q_{jk} D_i \left(\frac{1}{N} \mathcal{L}_m s^{ij} \right). \end{aligned} \quad (\text{G6})$$

We now evaluate the functional derivative of the Lie derivative:

$$\begin{aligned} K_{ij} \frac{\delta}{\delta N^k} (\mathcal{L}_m s^{ij}) &= -K_{ij} \frac{\delta}{\delta N^k} (\mathcal{L}_N s^{ij}) \\ &= -K_{ij} \frac{\delta}{\delta N^k} [N^m D_m s^{ij} - (D_p N^i) s^{pj} - (D_p N^j) s^{ip}] \\ &= -K_{ij} (D_k s^{ij} + \delta_k^i s^{pj} \tilde{D}_p + \delta_k^j s^{ip} \tilde{D}_p) \\ &= -(K_{ij} D_k s^{ij} + K_{kj} s^{pj} \tilde{D}_p + K_{ik} s^{ip} \tilde{D}_p) \\ &= -[K_{ij} D_k s^{ij} + 2D_p (K_{kj} s^{pj})]. \end{aligned} \quad (\text{G7})$$

Finally, applying Eq. (G2) again results in

$$\begin{aligned} (2N s^{ij}) \frac{\delta}{\delta N^k} (K_{il} K^l_j) &= 2N s^{ij} \left(K^l_j \frac{\delta}{\delta N^k} K_{il} + K_{il} \frac{\delta}{\delta N^k} K^l_j \right) \\ &= (q_{kl} D_i + q_{ik} D_l) (s^{ij} K^l_j) + (q_{kj} D^l + \delta_k^l D_j) (s^{ij} K_{il}) \\ &= 2[D_i (s^{ij} K_{jk}) + q_{ik} D_l (s^{ij} K^l_j)]. \end{aligned} \quad (\text{G8})$$

Using these findings provides Eq. (100b).

3. Mixed sector

Based on Eq. (G1), we perform implicit partial integrations to obtain

$$\begin{aligned} N s^{in} \frac{\delta}{\delta N} (D_i K - D_j K^j_i) &= -D_i (N s^{in}) \frac{\delta}{\delta N} K + D_j (N s^{in}) \frac{\delta}{\delta N} K^j_i \\ &= \frac{1}{N} D_i (N s^{in}) K - \frac{1}{N} D_j (N s^{in}) K^j_i \\ &= \frac{1}{N} D_i [N (s^{in} K - s^{jn} K^i_j)]. \end{aligned} \quad (\text{G9})$$

Adding the other contribution with the lapse function eliminated leads to

$$\begin{aligned}\frac{\delta S^{(2)}}{\delta N} &= 2 \left[s^{in} D_i K - s^{jn} D_j K^i{}_j + \frac{1}{N} D_i (N s^{in}) K - \frac{1}{N} D_i (N s^{jn}) K^i{}_j \right] \\ &= \frac{2}{N} D_i [N (s^{in} K - s^{jn} K^i{}_j)],\end{aligned}\quad (\text{G10})$$

which implies Eq. (100c). Furthermore, we employ Eq. (G2) again to derive

$$\begin{aligned}2N s^{in} \frac{\delta}{\delta N^k} D_i K &= -2N \left[\frac{1}{N} D_i (N s^{in}) \right] \frac{\delta}{\delta N^k} K^j{}_j \\ &= - \left\{ q_{kj} D^j \left[\frac{1}{N} D_i (N s^{in}) \right] + \delta_k^j D_j \left[\frac{1}{N} D_i (N s^{in}) \right] \right\} \\ &= -2D_k \left[\frac{1}{N} D_i (N s^{in}) \right] = -2D_k (a_i s^{in} + D_i s^{in}),\end{aligned}\quad (\text{G11a})$$

as well as

$$\begin{aligned}2N s^{in} \frac{\delta}{\delta N^k} (D_j K^i{}_j) &= -2N \left[\frac{1}{N} D_j (N s^{in}) \right] \frac{\delta}{\delta N^k} K^i{}_j \\ &= - \left\{ \delta_k^j D_i \left[\frac{1}{N} D_j (N s^{in}) \right] + q_{ki} D^j \left[\frac{1}{N} D_j (N s^{in}) \right] \right\} \\ &= - \left\{ D_i \left[\frac{1}{N} D_k (N s^{in}) \right] + q_{ki} D^j \left[\frac{1}{N} D_j (N s^{in}) \right] \right\} \\ &= - [D_i (a_k s^{in} + D_k s^{in}) + q_{ki} D^j (a_j s^{in} + D_j s^{in})].\end{aligned}\quad (\text{G11b})$$

These findings result in Eq. (100d).

4. Purely timelike sector

Here we use that

$$\frac{\delta}{\delta N} (K \mathcal{L}_m s^{\text{nn}}) = -\frac{K}{N} \mathcal{L}_m s^{\text{nn}}, \quad (\text{G12})$$

which is a consequence of Eq. (G1). In addition,

$$\frac{\delta}{\delta N} (s^{\text{nn}} D_i D^i N) = D_i D^i s^{\text{nn}}. \quad (\text{G13})$$

These results immediately provide Eq. (100e). To compute the derivatives for the shift vector, we need

$$\frac{\delta}{\delta N^k} (\mathcal{L}_m s^{\text{nn}}) = -\nabla_k s^{\text{nn}}, \quad (\text{G14})$$

as well as

$$\begin{aligned}\mathcal{L}_m s^{\text{nn}} \frac{\delta}{\delta N^k} K &= N \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right) \frac{\delta}{\delta N^k} K^i{}_i \\ &= \frac{1}{2} \left[q_{ki} D^i \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right) + \delta_k^i D_i \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right) \right] \\ &= D_k \left(\frac{1}{N} \mathcal{L}_m s^{\text{nn}} \right),\end{aligned}\quad (\text{G15})$$

and

$$\begin{aligned}N s^{\text{nn}} \frac{\delta}{\delta N^k} (K^2 - K_{ij} K^{ij}) &= 2D_k (K s^{\text{nn}}) \\ &\quad - 2D^i (s^{\text{nn}} K_{ik}),\end{aligned}\quad (\text{G16})$$

which follow from Eq. (G2). Thus, we arrive at Eq. (100f).

APPENDIX H: COUNTING SCHEME

To check the consistency of expressions in the context of the ADM formalism, it turned out to be valuable to associate a set of “units” to the various quantities that play a role in this paper. These units count how often the induced metric q_{ij} (or the four-dimensional spacetime metric $g_{\mu\nu}$) occurs in a certain expression. Hence, we start with

$$[q_{ij}] = 1, \quad [q^i_j] = [q_i^j] = 0, \quad [q^{ij}] = -1, \quad (\text{H1a})$$

$$[\sqrt{q}] = \frac{3}{2}, \quad (\text{H1b})$$

$$[g_{\mu\nu}] = 1, \quad [g^{\mu\nu}] = -1, \quad [\sqrt{-g}] = 2. \quad (\text{H1c})$$

Then, from Eqs. (5) and (8) we immediately deduce that

$$[N_i] = 1, \quad [N^i] = 0, \quad [N] = \frac{1}{2}, \quad (\text{H2a})$$

$$[n_\mu] = \frac{1}{2}, \quad [n^\mu] = -\frac{1}{2}. \quad (\text{H2b})$$

We define both partial derivatives and covariant derivatives with lower indices as “dimensionless,” which is why

$$[\dot{q}_{ij}] = 1, \quad [\dot{N}] = \frac{1}{2}, \quad (\text{H3a})$$

$$[D_i N_j] = 1, \quad [\dot{N}^i] = 0, \quad [\dot{N}_i] = 1, \quad (\text{H3b})$$

$$[a_\mu] = 0, \quad [a^\mu] = -1. \quad (\text{H3c})$$

The latter follow from Eq. (A1). From the definition of the Christoffel symbol, the Riemann tensor and its contractions as well as from Eqs. (H1a), we quickly obtain

$$[\Gamma^i_{jk}] = 0, \quad [R^i_{jkl}] = 0, \quad [R_{ijkl}] = 1, \quad (\text{H4a})$$

$$[R_{ij}] = 0, \quad [R] = -1, \quad [R^{ij}] = -2. \quad (\text{H4b})$$

The form of the EH Lagrange density (14) and the Hamilton density (23) implies immediately

$$[\mathcal{L}] = [\mathcal{H}] = 1. \quad (\text{H5})$$

Based on Eq. (A9), we choose

$$[K_{ij}] = \frac{1}{2}, \quad [K] = -\frac{1}{2}, \quad [K^{ij}] = -\frac{3}{2}, \quad (\text{H6})$$

which is consistent with Eq. (H5). From the definitions of the canonical momenta we get

$$[\pi^{ij}] = 0, \quad [\pi] = 1, \quad [\pi_{ij}] = 2, \quad (\text{H7a})$$

$$[\pi_i] = 1, \quad [\pi^i] = 0, \quad [\pi_N] = \frac{1}{2}. \quad (\text{H7b})$$

Finally, it is possible to assign the same units to the diffeomorphism-violating background fields. Equation (16) provides

$$[s_{\mu\nu}] = 1, \quad [s_\mu{}^\nu] = [s^\mu{}_\nu] = 0, \quad (\text{H8a})$$

$$[s^{\mu\nu}] = -1, \quad [u] = 0, \quad [t^{\mu\nu\rho\sigma}] = -2. \quad (\text{H8b})$$

The reason for these results is that $s_{\mu\nu}$ plays an analogous role as the metric, i.e., it is contracted with $R^{\mu\nu}$. Also,

$$s^{nn} = 0, \quad s^{ni} = -\frac{1}{2}, \quad (\text{H9})$$

which follows from their definitions under Eq. (28) as well as Eq. (H2b). A generic rule is that each lower Lorentz index leads to a dimension of 1/2, whereas each upper one implies a dimension of -1/2. For consistency, each term in a sum of contributions must have the same dimension.

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