New approach to the thermodynamics of scalar-tensor gravity

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We discuss and expand a new approach to the thermodynamics of scalar-tensor gravity and its diffusion toward general relativity (seen as an equilibrium state) proposed in a previous paper [Phys. Rev. D **103**, L121501 (2021)], upon which we build. We describe scalar-tensor gravity as an effective dissipative fluid and apply Eckart's first order thermodynamics to it, obtaining explicitly effective quantities such as heat flux, "temperature of gravity," viscosities, entropy density, plus an equation describing the "diffusion" to Einstein gravity. These quantities, still missing in the usual thermodynamics of spacetime, are obtained with minimal assumptions. Furthermore, we examine certain exact solutions of scalar-tensor gravity to test the proposed formalism and gain some physical insight on the "approach to equilibrium" for this class of theories.

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I. INTRODUCTION

There seems to be a deep connection between thermodynamics and gravity, first observed in black hole thermodynamics, which produced an unexpected relation between the entropy and horizon area for stationary black holes and the temperatures of black hole and Rindler horizons. This connection took on a new meaning with Jacobson's seminal work [1] obtaining the Einstein field equation of general relativity (GR) as an equation of state, based only on thermodynamical considerations. This derivation of the Einstein equation and the ensuing "thermodynamics of spacetime" picture carry deep implications for gravity. Their main consequence would be that classical gravity is an emergent phenomenon instead of having a fundamental nature. If confirmed, this property would have radical consequences for quantum gravity as well. In the thermodynamics of spacetime picture, quantizing the Einstein equation would not be more meaningful than quantizing the macroscopic ideal gas equation of state, which cannot produce fundamental quantum results such as the energy spectrum and eigenfunctions of the hydrogen atom. In quantum gravity, entities such as the "atoms of spacetime" may not even exist or, if they do, they may have to be found using approaches radically different from the quantization of the Einstein equation.

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A second idea,which is probably equally important, was proposed in Ref. [2], in which the authors derived the field equation of fourth order metric $f(\mathcal{R})$ gravity using only thermodynamics. This modification of GR, which contains an extra scalar degree of freedom $f'(\mathcal{R})$ (see [3–5] for reviews) would correspond to dissipative nonequilibrium "thermodynamics of gravitational theories" in which a "bulk viscosity of spacetime" was introduced to explain dissipation [2]. By contrast, GR would correspond to a state of thermodynamic equilibrium [2].

The works [1,2] have generated a huge literature. In view of our new approach to this paradigm, it is useful to note that Ref. [6] has stressed the essential role of shear viscosity, while removing altogether bulk viscosity from the thermodynamical picture of $f(\mathcal{R})$ gravity. In spite of the large literature, the equation(s) ruling the approach of modified gravity to the GR equilibrium state remain a mystery. Furthermore, the order parameter (presumably, the temperature) regulating this dissipative phenomenon has not yet been identified. Here we discuss in detail and expand a new approach proposed in Refs. [7,8] to the last two problems in the spirit of the thermodynamics of spacetime, but in a very different context. We consider the larger class of scalar-tensor theories of gravity [9–12] which contains $f(\mathcal{R})$ gravity as a subclass [3–5]. Scalar-tensor gravity is a minimal modification of GR obtained by adding a massive scalar degree of freedom ϕ to the usual two massless spin two modes of GR contained in the metric tensor g_{ab}. The contribution of ϕ to the field equations can be described as an effective relativistic dissipative fluid

[13,14]. Using this description, we apply Eckart's first order thermodynamics [15] to this effective fluid and extract explicit expressions for the relevant effective thermodynamic quantities, including the heat current density, the "temperature of modified gravity," the viscosity coefficients, and the entropy density.

To anticipate our findings: the product between the effective temperature $\mathcal{T}$ and the thermal conductivity $\mathcal{K}$ is positive-definite and the GR equilibrium state corresponds to $\mathcal{K}\mathcal{T} = 0$; the bulk viscosity vanishes, and the shear viscosity η is negative. This unexpected sign could allow the entropy density s to decrease, which is consistent with the fact that the ϕ-fluid, seen as a thermodynamic system, is neither an isolated fluid—it exchanges energy with its "surroundings"—nor a real fluid. To proceed, we describe explicitly the approach of scalar-tensor gravity to the GR equilibrium state. In a sense, it is remarkable that, in spite of the well-known limitations of Eckart's first order thermodynamics, these explicit expressions and effective diffusion equation emerge from the formal identification of an effective fluid with a thermodynamic system, which would seem very unlikely *a priori*. The simplicity and minimality of assumptions of this new approach point again to some deeper connection between thermodynamics and gravity. Furthermore, in order to properly illustrate the physical implications of the proposed approach, we study the thermodynamical behavior of certain exact solutions of scalar-tensor gravity.

Let us review the basics of scalar-tensor gravity. The scalar-tensor action in the Jordan frame is[1]

$$S_{\rm ST} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi\mathcal{R} - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(m)}, \tag{1.1}$$

where $\mathcal{R}$ is the Ricci scalar, the Brans-Dicke scalar $\phi > 0$ is approximately the inverse of the effective gravitational coupling, $\omega(\phi)$ is the "Brans-Dicke coupling", $V(\phi)$ is a potential for the scalar field, and $S^{(m)} = \int d^4x\sqrt{-g}\mathcal{L}^{(m)}$ is the matter action.

By varying the action (1.1) with respect to the inverse metric g^{ab} and to ϕ, one obtains the (Jordan frame) field equations [9–12]

$$\begin{aligned} G_{ab} &\equiv \mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R} \\ &= \frac{8\pi}{\phi}T^{(m)}_{ab} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) \\ &\quad + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}, \end{aligned} \tag{1.2}$$

$$\Box\phi = \frac{1}{2\omega+3}\left(\frac{8\pi T^{(m)}}{\phi} + \phi\frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi}\nabla^c\phi\nabla_c\phi\right), \tag{1.3}$$

where $\mathcal{R}_{ab}$ is the Ricci tensor and $T^{(m)} \equiv g^{ab}T^{(m)}_{ab}$ is the trace of the matter stress-energy tensor $T^{(m)}_{ab}$.

II. EFFECTIVE SCALAR FIELD FLUID

Here we summarize the formulas for the effective field fluid derived in [13,14] that are needed in the calculations of this paper.

A. Kinematic quantities

Let us begin with the kinematic quantities of the effective ϕ-fluid [17] (here we provide extra expressions for the kinematic quantities and for the effective fluid quantities which were not given in [7], but are handy for calculations). The ϕ-fluid description is natural when the gradient $\nabla^a\phi$ is timelike and can be used to construct the effective fluid four-velocity

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}} \tag{2.1}$$

normalized to $u^cu_c = -1$. The $3+1$ splitting of spacetime into the time direction u^c and the 3-dimensional space of the comoving observers of this effective fluid (with four-tangent u^c) follows. Their 3-space is endowed with the Riemannian metric

$$h_{ab} \equiv g_{ab} + u_au_b \tag{2.2}$$

and ${h_a}^b$ is the projection operator on this 3-space, then

$$h_{ab}u^a = h_{ab}u^b = 0, \tag{2.3}$$

$$h^a{}_b h^b{}_c = h^a{}_c, \qquad h^a{}_a = 3. \tag{2.4}$$

The effective fluid four-acceleration is $\dot{u}^a \equiv u^b\nabla_bu^a$ and is orthogonal to the four-velocity, $\dot{u}^cu_c = 0$ (exceptions to this rule, which include Friedmann-Lemaître-Robertson-Walker spaces, particles with variable mass, *etc.* [18] will not be considered in this work).

The projection of the velocity gradient onto the 3-space of the comoving observers is the purely spatial tensor

$$V_{ab} \equiv {h_a}^c{h_b}^d\nabla_du_c. \tag{2.5}$$

It splits into symmetric and antisymmetric parts, with the symmetric part further decomposed into tracefree part and pure trace as

[1]We follow the notation of Ref. [16] and we use units in which Newton's constant G and the speed of light c are unity.

$$V_{ab} = \theta_{ab} + \omega_{ab} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab}, \quad (2.6)$$

with $\theta_{ab} = V_{(ab)}$ the expansion tensor (symmetric part of V_{ab}) with trace $\theta \equiv \theta^c{}_c = \nabla^c u_c$; the vorticity tensor $\omega_{ab} = V_{[ab]}$ is its antisymmetric part, while the tracefree shear tensor is

$$\sigma_{ab} \equiv \theta_{ab} - \frac{\theta}{3} h_{ab}. \quad (2.7)$$

V_{ab}, θ_{ab}, σ_{ab}, and ω_{ab} are purely spatial tensors,

$$\theta_{ab} u^a = \theta_{ab} u^b = \omega_{ab} u^a = \omega_{ab} u^b = \sigma_{ab} u^a = \sigma_{ab} u^b = 0, \quad (2.8)$$

while $\sigma^a{}_a = \omega^a{}_a = 0$. The shear scalar σ and vorticity[2] scalar ω are

$$\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab} \geq 0, \quad (2.9)$$

$$\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^{ab} \geq 0. \quad (2.10)$$

The velocity gradient decomposes as [17]

$$\nabla_b u_a = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} - \dot{u}_a u_b = V_{ab} - \dot{u}_a u_b. \quad (2.11)$$

Projecting Eq. (2.11) onto u^c produces $\dot{u}_a$, while projecting it onto the 3-space orthogonal to u^a yields V_{ab}.

When these general definitions [16,17] are specialized to our effective ϕ-fluid, one obtains [14]

$$h_{ab} = g_{ab} - \frac{\nabla_a \phi \nabla_b \phi}{\nabla^e \phi \nabla_e \phi}, \quad (2.12)$$

$$\nabla_b u_a = \frac{1}{\sqrt{-\nabla^e \phi \nabla_e \phi}} \left(\nabla_a \nabla_b \phi - \frac{\nabla_a \phi \nabla^c \phi \nabla_b \nabla_c \phi}{\nabla^e \phi \nabla_e \phi} \right). \quad (2.13)$$

The 4-acceleration then reads[3]

$$\begin{aligned} \dot{u}_a &= u^c \nabla_c u_a \\ &= (-\nabla^e \phi \nabla_e \phi)^{-2} \nabla^b \phi [(-\nabla^e \phi \nabla_e \phi) \nabla_a \nabla_b \phi \\ &\quad + \nabla^c \phi \nabla_b \nabla_c \phi \nabla_a \phi]. \end{aligned} \quad (2.14)$$

The (timelike) worldlines of the ϕ-fluid are geodesics (equivalently, this fluid is a dust) if and only if

$$\nabla^e \phi \nabla_{[e} \phi \nabla_{a]} \nabla_b \phi \nabla^b \phi = 0, \quad (2.15)$$

from which it follows that

$$\nabla^b \nabla^c \phi \nabla_b \nabla_c \phi = -\frac{\nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \quad (2.16)$$

for a geodesic flow. In this case, V_{ab} reduces to

$$\begin{aligned} V_{ab} = &\frac{\nabla_a \nabla_b \phi}{(-\nabla^e \phi \nabla_e \phi)^{1/2}} + \frac{(\nabla_a \phi \nabla_b \nabla_c \phi + \nabla_b \phi \nabla_a \nabla_c \phi) \nabla^c \phi}{(-\nabla^e \phi \nabla_e \phi)^{3/2}} \\ &+ \frac{\nabla_d \nabla_c \phi \nabla^c \phi \nabla^d \phi}{(-\nabla^e \phi \nabla_e \phi)^{5/2}} \nabla_a \phi \nabla_b \phi. \end{aligned} \quad (2.17)$$

Since the ϕ-fluid four-velocity u^c is derived from a scalar field gradient, this fluid is irrotational, $\omega_{ab} = \omega^2 = 0$, leaving

$$V_{ab} = \theta_{ab}, \qquad \nabla_b u_a = \theta_{ab} - \dot{u}_a u_b. \quad (2.18)$$

Moreover, the vector field u^a is hypersurface-orthogonal and the line element becomes diagonal in adapted coordinates. In other words, the existence of a foliation of 3-dimensional hypersurfaces Σ with Riemannian metric h_{ab} orthogonal to u^a is guaranteed [16,17].

Since $u_a \dot{u}^a = 0$, the expansion scalar (2.11) reduces to

$$\theta = \nabla_a u^a = \frac{\Box \phi}{(-\nabla^e \phi \nabla_e \phi)^{1/2}} + \frac{\nabla_a \nabla_b \phi \nabla^a \phi \nabla^b \phi}{(-\nabla^e \phi \nabla_e \phi)^{3/2}}, \quad (2.19)$$

while

$$\begin{aligned} \sigma_{ab} = (-\nabla^e \phi \nabla_e \phi)^{-3/2} \Big[&-(\nabla^e \phi \nabla_e \phi) \nabla_a \nabla_b \phi \\ &- \frac{1}{3} (\nabla_a \phi \nabla_b \phi - g_{ab} \nabla^c \phi \nabla_c \phi) \Box \phi \\ &- \frac{1}{3} \left(g_{ab} + \frac{2 \nabla_a \phi \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \right) \nabla_c \nabla_d \phi \nabla^d \phi \nabla^c \phi \\ &+ (\nabla_a \phi \nabla_c \nabla_b \phi + \nabla_b \phi \nabla_c \nabla_a \phi) \nabla^c \phi \Big], \end{aligned} \quad (2.20)$$

$$\begin{aligned} \sigma \equiv \left(\frac{1}{2} \sigma^{ab} \sigma_{ab} \right)^{1/2} = &(-\nabla^e \phi \nabla_e \phi)^{-3/2} \Big\{ \frac{1}{2} (\nabla^e \phi \nabla_e \phi)^2 \\ &\times \left[\nabla^a \nabla^b \phi \nabla_a \nabla_b \phi - \frac{1}{3} (\Box \phi)^2 \right] + \frac{1}{3} (\nabla_a \nabla_b \phi \nabla^a \phi \nabla^b \phi)^2 \\ &- (\nabla^e \phi \nabla_e \phi) \Big(\nabla_a \nabla_b \phi \nabla^b \nabla_c \phi \\ &- \frac{1}{3} \Box \phi \nabla_a \nabla_c \phi \Big) \nabla^a \phi \nabla^c \phi \Big\}^{1/2}. \end{aligned} \quad (2.21)$$

[2]Here there is no risk of confusing the vorticity scalar with the Brans-Dicke coupling because the former is always zero as the effective ϕ-fluid is irrotational.

[3]It is straightforward to check that $\dot{u}_c u^c = 0$ using Eqs. (2.14) and (2.1).

B. Effective stress-energy tensor of the ϕ-fluid

The effective stress-energy tensor of the Brans-Dicke-like field that one reads off the right-hand side of Eq. (1.2) is

$$8\pi T^{(\phi)}_{ab} = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}. \tag{2.22}$$

$T^{(\phi)}_{ab}$ takes the form of an imperfect fluid energy-momentum tensor [13,14]

$$T_{ab} = \rho u_a u_b + q_a u_b + q_b u_a + \Pi_{ab}, \tag{2.23}$$

where the effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stresses (the tracefree part π_{ab} of the stress tensor Π_{ab}) in the comoving frame are

$$\rho = T_{ab}u^a u^b, \tag{2.24}$$

$$q_a = -T_{cd}u^c h_a{}^d, \tag{2.25}$$

$$\Pi_{ab} = Ph_{ab} + \pi_{ab} = T_{cd}h_a{}^c h_b{}^d, \tag{2.26}$$

$$P = \frac{1}{3}g^{ab}\Pi_{ab} = \frac{1}{3}h^{ab}T_{ab}, \tag{2.27}$$

$$\pi_{ab} = \Pi_{ab} - Ph_{ab}, \tag{2.28}$$

respectively. Here we have set the bulk viscous pressure to zero in the most economical interpretation of the effective ϕ-fluid (we refer the reader to [7,14] for details). The heat flux density is purely spatial,

$$q_c u^c = 0 \tag{2.29}$$

and

$$\Pi_{ab}u^b = \pi_{ab}u^b = \Pi_{ab}u^a = \pi_{ab}u^a = 0, \qquad \pi^a{}_a = 0. \tag{2.30}$$

The explicit expressions of the effective stress-energy quantities are [13,14]

$$8\pi\rho^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi} + \frac{1}{\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.31}$$

$$\begin{aligned} 8\pi q^{(\phi)}_a &= \frac{\nabla^c\phi\nabla^d\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{3/2}}(\nabla_d\phi\nabla_c\nabla_a\phi - \nabla_a\phi\nabla_c\nabla_d\phi) \\ &= -\frac{\nabla^c\phi\nabla_a\nabla_c\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{1/2}} - \frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{3/2}}\nabla_a\phi, \end{aligned} \tag{2.32}$$

$$\begin{aligned} 8\pi\Pi^{(\phi)}_{ab} &= (-\nabla^e\phi\nabla_e\phi)^{-1}\left[\left(-\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^e\phi\nabla_e\phi)\right. \\ &\quad \left. - \frac{\nabla^d\phi}{\phi}\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\right] \\ &= \left(-\frac{\omega}{2\phi^2}\nabla^c\phi\nabla_c\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)h_{ab} + \frac{1}{\phi}h_a{}^c h_b{}^d\nabla_c\nabla_d\phi, \end{aligned} \tag{2.33}$$

$$8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.34}$$

$$\begin{aligned} 8\pi\pi^{(\phi)}_{ab} &= \frac{1}{\phi\nabla^e\phi\nabla_e\phi}\left[\frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\left(\Box\phi - \frac{\nabla^c\phi\nabla^d\phi\nabla_d\nabla_c\phi}{\nabla^e\phi\nabla_e\phi}\right)\right. \\ &\quad \left. + \nabla^d\phi\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\right], \end{aligned} \tag{2.35}$$

and

$$8\pi T^{(\phi)} \equiv 8\pi g^{ab}T^{(\phi)}_{ab} = -\frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi - \frac{3\Box\phi}{\phi} - \frac{2V}{\phi}. \tag{2.36}$$

Alternative expressions for $\rho^{(\phi)}$ and $P^{(\phi)}$ are obtained by replacing $\Box\phi$ with the help of Eq. (1.3):

$$8\pi\rho^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi}\left(\frac{2\omega-1}{2\omega+3}\right) + \frac{1}{\phi}\left[\frac{1}{2\omega+3}\left(\phi\frac{dV}{d\phi} - \nabla^e\phi\nabla_e\phi\frac{d\omega}{d\phi}\right) - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right], \tag{2.37}$$

$$8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{6\phi}\frac{(6\omega+1)}{(2\omega+3)} - \frac{1}{3\phi}\left[\frac{2}{2\omega+3}\left(\phi\frac{dV}{d\phi} - \nabla^e\phi\nabla_e\phi\frac{d\omega}{d\phi}\right) + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right]. \tag{2.38}$$

In general, the effective fluid stress-energy tensor $T_{ab}^{(\phi)}$ does not satisfy any energy condition because it contains second derivatives of ϕ (which can have either sign) together with the more standard squares of first derivatives (which are instead positive-definite), preventing conclusions. For reference, we list the energy conditions, although they will be violated in general by the ϕ-fluid[4] [14,21,22]. The weak energy condition ($T_{ab}t^a t^b \geq 0$ for all timelike vectors t^a [16]) becomes

$$T_{ab}^{(\phi)}u^a u^b = -\frac{\omega}{2\phi}\nabla^e\phi\nabla_e\phi + \frac{V}{2} + \Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi} \geq 0, \tag{2.39}$$

while the strong energy condition ($(T_{ab} - Tg_{ab}/2)t^a t^b \geq 0$ for all timelike vectors t^a [16]) is

$$\left(T_{ab}^{(\phi)} - \frac{1}{2}T^{(\phi)}g_{ab}\right)u^a u^b = \frac{1}{2}(\rho^{(\phi)} + 3P^{(\phi)}) = -\frac{\omega}{\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} + \frac{1}{\phi}\left[-\frac{1}{2}\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right] \geq 0. \tag{2.40}$$

III. ECKART'S THERMODYNAMICS FOR SCALAR-TENSOR GRAVITY

In Eckart's thermodynamics [15] (see also Refs. [23,24] for a pedagogical exposition and [25] for relativistic fluids in general), the dissipative quantities (i.e., viscous pressure P_{vis}, heat current density q^c, and anisotropic stresses π_{ab}) are related to the expansion θ, temperature $\mathcal{T}$, and shear tensor σ_{ab} by the constitutive equations [15]

$$P_{\text{vis}} = -\zeta\theta, \tag{3.1}$$

$$q_a = -\mathcal{K}(h_{ab}\nabla^b\mathcal{T} + \mathcal{T}\dot{u}_a), \tag{3.2}$$

$$\pi_{ab} = -2\eta\sigma_{ab}, \tag{3.3}$$

where ζ is the bulk viscosity, $\mathcal{K}$ is the thermal conductivity, and η is the shear viscosity.

The comparison of Eqs. (2.32) and (2.14) yields

$$q_a^{(\phi)} = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\dot{u}_a. \tag{3.4}$$

Not only is this vector purely spatial, which we already know, but it is proportional to the 4-acceleration of the effective fluid "elements."

A. The temperature of scalar-tensor gravity

Equation (3.4), already obtained in Ref. [14], is interpreted in the context of Eckart's first order (noncausal) thermodynamics [15], in which the constitutive relation for the heat flux density (generalized Fourier law) is given by Eq. (3.2). Then, Eq. (3.4) allows us to make the identifications

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}, \tag{3.5}$$

and

$$h_{ab}\nabla^b\mathcal{T} = 0, \tag{3.6}$$

i.e., the spatial temperature gradient vanishes identically and the heat flow arises solely from the inertia of energy (a possible contribution to the heat flux first discovered by Eckart [15]).

Equation (3.5) can then be used to identify the Eckart temperature of the ϕ-fluid [14], which we dub "temperature of scalar-tensor gravity." It is reassuring that $\mathcal{K}\mathcal{T}$ is positive

[4] See Refs. [19,20] for a discussion of the energy conditions for an imperfect fluid with respect to arbitrary (timelike) observers.

definite, which could not have been taken for granted in a formal identification of quantities which, *a priori*, could generate either sign. Furthermore, $\mathcal{K}\mathcal{T}$ formally vanishes when $\phi = \text{const.}$, which corresponds to GR and to the disappearance of the ϕ-fluid.

B. The effective viscosity of scalar-tensor gravity

The structure of the effective imperfect fluid of the Brans-Dicke-like scalar field ϕ was chosen so that it does not contain a bulk viscosity term. Therefore, the bulk viscosity $\zeta = 0$, but there is shear viscosity. This choice of splitting between the isotropic pressure term and the viscous one does not affect the generalized Fourier law and the definition of temperature of the ϕ-fluid.

In order to calculate the shear viscosity η, it is sufficient to compare the anisotropic stress tensor (2.35) with the shear tensor (2.20). Rather surprisingly, these two expressions match term to term and are proportional to each other. Eckart's constitutive relation $\pi_{ab} = -2\eta\sigma_{ab}$ is satisfied if

$$\eta = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{16\pi\phi} \tag{3.7}$$

or, using the expression (3.5),

$$\eta = -\frac{\mathcal{K}\mathcal{T}}{2}. \tag{3.8}$$

The effective shear viscosity of scalar-tensor gravity is negative and vanishes at $\mathcal{K}\mathcal{T} = 0$, the GR case corresponding to equilibrium, to $\phi = \text{const.}$, and to the disappearance of the effective ϕ-fluid. Negative viscosities are common in fluid mechanics, including jet streams, ocean currents, liquid crystals, and many other phenomena. They are turbulent (as opposed to molecular) viscosities and appear in systems into which energy is fed from the outside (see, e.g., [26–29]). In our case, there is no obvious turbulent interpretation (indeed, the ϕ-fluid is irrotational) but, as a thermodynamical system, the ϕ-fluid is not isolated. In the action (1.1), ϕ couples explicitly to gravity through the term $\phi\mathcal{R}$ mixing scalar and tensor degrees of freedom and exchanges energy with its thermodynamical "surroundings."

In the different context of spacetime thermodynamics, Ref. [6] identified shear viscosity as the source of dissipation and pointed to the absence of bulk viscosity in $f(\mathcal{R})$ gravity, contrary to the previous proposal of Ref. [2]. The results of [6] are echoed in our different approach, in the wider class of scalar-tensor theories.

C. Entropy generation and the second law

In Eckart's thermodynamics, the entropy due to the heat flux is $R^a = q^a/\mathcal{T}$ which, in the comoving frame, has components [15,23,25]

$$R_\mu = \frac{q_\mu}{\mathcal{T}} = \left(0, \frac{\vec{q}}{\mathcal{T}}\right). \tag{3.9}$$

The entropy current density in a fluid with particle density n and entropy density s is

$$s^a = snu^a + R^a = snu^a + \frac{q^a}{\mathcal{T}}, \tag{3.10}$$

where R^a describes entropy generation due to dissipative processes. While, for an isolated system, entropy is conserved in a nondissipative fluid ($\nabla_c s^c = 0$), in a dissipative one it is $\nabla_c s^c > 0$ due to the entropy generation described by the vector R^a.

The entropy density is obtained from the first law

$$\mathcal{T}dS = dU + PdV, \tag{3.11}$$

which yields

$$s \equiv \frac{dS}{dV} = \frac{\rho + P}{\mathcal{T}} \tag{3.12}$$

assuming a closed (yet, not isolated) system.

The expressions (2.31), (2.34), and (3.5) of the effective density, pressure, and temperature of the ϕ-fluid then give

$$s = \frac{\mathcal{K}}{\sqrt{-\nabla^e\phi\nabla_e\phi}} \times \left[-\frac{\omega}{\phi}\nabla^e\phi\nabla_e\phi + \frac{\Box\phi}{3} - \frac{4}{3}\frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right]. \tag{3.13}$$

In a fluid in which the particle number is conserved, $\nabla_a n^a = 0$ (where $n^a = nu^a$ is the particle current density), one has [15,23,25]

$$\nabla_c s^c = \frac{P_{\text{vis}}^2}{\zeta\mathcal{T}} + \frac{q_c q^c}{\mathcal{K}\mathcal{T}^2} + \frac{\pi_{ab}\pi^{ab}}{2\eta\mathcal{T}}, \tag{3.14}$$

where the bulk viscosity term (the first term on the right-hand side) is absent for the effective ϕ-fluid. Using the fact that $q^a = -\mathcal{K}\mathcal{T}\dot{u}^a$, Eckart's entropy generation term is

$$R^a = -\mathcal{K}\dot{u}^a. \tag{3.15}$$

Equations (3.4) and (2.35) are then used to compute $\nabla_c s^c$, obtaining

$$\frac{q_c q^c}{\mathcal{K}\mathcal{T}^2} = \mathcal{K}\dot{u}_c\dot{u}^c = \frac{\mathcal{K}}{(-\nabla^e\phi\nabla_e\phi)^3}[-\nabla^e\phi\nabla_e\phi\nabla_b\phi\nabla^d\phi\nabla^b\nabla^a\phi\nabla_d\nabla_a\phi + (\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi)^2], \tag{3.16}$$

$$\pi_{ab}\pi^{ab} = 8\eta^2\sigma^2 = 2\mathcal{K}^2\mathcal{T}^2\sigma^2 \\ = \frac{(-\nabla^e\phi\nabla_e\phi)^{-2}}{32\pi^2\phi^2}\left\{\frac{1}{2}(-\nabla^e\phi\nabla_e\phi)^2\left[\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi - \frac{(\Box\phi)^2}{3}\right] + \frac{1}{3}(\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi)^2 \right. \\ \left. -(\nabla^e\phi\nabla_e\phi)\left(\nabla_a\nabla_b\phi\nabla^b\nabla_c\phi - \frac{\Box\phi}{3}\nabla_a\nabla_c\phi\right)\nabla^a\phi\nabla^c\phi\right\}, \tag{3.17}$$

and finally

$$\nabla_c s^c = \mathcal{K}\left(\dot{u}^a\dot{u}_a + \frac{\mathcal{K}\mathcal{T}\sigma^2}{\eta}\right) \\ = \mathcal{K}(\dot{u}^a\dot{u}_a - \sigma_{ab}\sigma^{ab}). \tag{3.18}$$

Since the second term in Eq. (3.18) is negative, one cannot conclude that the entropy increases. Indeed, if energy is injected into the scalar field fluid coupled to gravity, the entropy s may actually decrease, as it happens in nonisolated systems.

A special situation (if it is possible) is the one in which the ϕ-fluid is geodesic, $\dot{u}^a = 0$, which always corresponds to $q^a = 0$, $J^a = \mathcal{T}u^a$, and decreasing entropy density, consistent with the fact that the entropy generation vector (3.15) vanishes and shear contributes to decreasing s because of the negative η, as described by Eq. (3.18).

In modern constitutive theories, all constitutive relations are supposed to obey two universal principles (see, e.g., [30]): (i) the objectivity principle, i.e., independence of the observer; (ii) the entropy principle, according to which any solution of a system of constitutive equations satisfies an additional entropy balance law with a non-negative entropy production. Therefore, the result in Eq. (3.18) suggests potential violations of the latter and a consequent problem in fitting this analogy with the ϕ-fluid into the standard framework of constitutive theory. However, it is important to point out that the ϕ-fluid is hardly a real fluid—indeed, its energy density $\rho^{(\phi)}$ can be negative—and the correspondence with Eckart's theory comes as a mere comparison of kinetic and kinematic quantities characterizing the ϕ-fluid. Furthermore, as already stressed, this exotic fluid is *not* isolated since ϕ couples directly to the gravity sector, which necessarily affects the entropy balance. Hence, the analogy between properties of the ϕ-fluid and Eckart's thermodynamics holds provided that one keeps these *caveats* in mind.

D. Possible physical interpretations

Since the thermodynamic interpretation of this analogy depends heavily on the specific choice of the solution of the system (3.5) and (3.6), one can propose different formulations of the proposed approach without altering the physical scenario at hand. Here we discuss two simple possibilities.

Isolating the temperature in Eq. (3.5) and then inserting the corresponding expression into Eq. (3.6) reduces the latter to

$$h_{ab}\nabla^b \ln\left(\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{\mathcal{K}}\right) = 0. \tag{3.19}$$

One of the simplest solutions of this equation is

$$\mathcal{K} = C\sqrt{-\nabla^c\phi\nabla_c\phi}, \tag{3.20}$$

with C a positive constant. Setting, for instance, $C = 1/8\pi$ yields

$$\mathcal{T} = \frac{1}{\phi} = G_{\mathrm{eff}} \tag{3.21}$$

and

$$\mathcal{K} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi}. \tag{3.22}$$

This simple solution of Eq. (3.19) sets the stage for a curious interpretation of the thermodynamic properties of the ϕ-fluid. Specifically, the temperature measures the effective strength of the gravitational interaction while the thermal conductivity keeps track of the norm of $\nabla_a\phi$, and therefore of the variability of ϕ. Now, if one looks at the GR limit of the theory $\phi = \text{const.}$, $\mathcal{T}$ should reduce to Newton's constant whereas $\mathcal{K}$ vanishes. In other words, the GR limit of scalar-tensor gravity corresponds to the "perfect insulator" limit for the ϕ-fluid.

Alternatively, if one considers a more involved solution of Eq. (3.19) such that $\mathcal{K} \neq 0$, one finds an alternative description for the GR limit of scalar-tensor gravity: this limit corresponds to $\mathcal{T} \to 0$, i.e., GR corresponds to the absolute zero (minimum possible temperature) of the ϕ-fluid. However, finding an explicit general expression for $\mathcal{K} = \mathcal{K}(\phi, \nabla\phi)$ becomes much more involved.

IV. THE APPROACH TO THE GR EQUILIBRIUM STATE

A posteriori, the expression of $\mathcal{K}\mathcal{T}$ can be differentiated to obtain an evolution equation for this quantity. Although this may seem redundant since we already know the solution of this equation, the latter plays the role of an effective heat equation for the ϕ-fluid and is useful to understand better when, and how, the GR equilibrium state is approached.

The differentiation of Eq. (3.5) yields

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} \equiv u^c\nabla_c(\mathcal{K}\mathcal{T}) = -\frac{\sqrt{-\nabla^e\phi\nabla_e\phi}}{8\pi\phi}\frac{1}{\phi}\frac{\nabla^c\phi\nabla_c\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}} - \frac{u^c}{8\pi\phi}\frac{\nabla^e\phi\nabla_c\nabla_e\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}} = \frac{\mathcal{K}\mathcal{T}}{\phi}\sqrt{-\nabla^e\phi\nabla_e\phi} - \mathcal{K}\mathcal{T}\left(\theta - \frac{\Box\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}}\right). \tag{4.1}$$

Using again Eq. (3.5), one has

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} = 8\pi(\mathcal{K}\mathcal{T})^2 - \theta\mathcal{K}\mathcal{T} + \frac{\Box\phi}{8\pi\phi}. \tag{4.2}$$

It is not easy to interpret this equation in general since $\Box\phi$ does not have definite sign and the expansion θ depends in a rather complicated way from ϕ and its derivatives, however we can restrict to simple situations in order to gain insight into the approach to equilibrium. Consider the case of electrovacuum, $\omega = \text{const.}$, and $V(\phi) \equiv 0$; then Eq. (1.3) yields $\Box\phi = 0$. If $\theta < 0$, then $d(\mathcal{K}\mathcal{T})/d\tau > 8\pi(\mathcal{K}\mathcal{T})^2$ and $\mathcal{K}\mathcal{T}$ grows superexponentially, exploding in a finite time τ and diverging away from the GR equilibrium state. Therefore, one expects that near spacetime singularities, where worldlines of the ϕ field converge and $\theta < 0$, the deviations of scalar-tensor gravity from GR will be extreme (this idea is tested in Sec. V).

When $\theta > 0$, it is possible that the negative term $-\theta\mathcal{K}\mathcal{T}$ in the right-hand side of Eq. (4.2) dominates over the positive term $8\pi(\mathcal{K}\mathcal{T})^2$, and that the solution $\mathcal{K}\mathcal{T}$ asymptotes to zero, approaching the GR equilibrium state. However, if $\mathcal{K}\mathcal{T}$ is large, the positive term will dominate the right hand side and lead the solution away from GR, the term linear in $\mathcal{T}$ becoming negligible. Therefore, the approach to the GR equilibrium state is not granted and should not be expected all the time. In the next section, we report analytical solutions of scalar-tensor gravity where this diffusion occurs and others where it does not.

V. EXAMPLES: ANALYTICAL SOLUTIONS OF SCALAR-TENSOR GRAVITY

Here we examine certain exact solutions of scalar-tensor gravity to test the thermodynamical formalism of the effective ϕ-fluid presented in the previous sections.

A. The special case of FLRW universes

In FLRW universes, the purely spatial heat flux density q^c and the anisotropic stresses π_{ab} vanish identically as a consequence of spatial homogeneity and isotropy, and the ϕ-fluid reduces to a perfect fluid. This result is true also in Lovelock and $f(\mathcal{R},\mathcal{G})$ theories, where $\mathcal{G} \equiv \mathcal{R}^2 - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd}$ is the Gauss-Bonnet integrand, in arbitrary dimension [31], and presumably in other theories as well. Due to the absence of a spatial heat flux, we provisionally assign zero temperature $\mathcal{T}$ to FLRW spaces. This may not be the end of the story though: one can consider the possibility that, in FLRW universes, the heat flux becomes a timelike vector aligned with the four-velocity u^c of comoving observers. This situation preserves the spatial homogeneity and isotropy of FLRW space. Then, Eckart's equation (3.4) could only hold if the four-acceleration of the fluid is timelike. Indeed, this is precisely what happens in FLRW spaces sourced by a perfect fluid. The acceleration vanishes for a pressurefree dust while, for any other perfect fluid, there is a pressure gradient $\nabla_a P$ which is timelike, to preserve spatial isotropy, and points along the tangent u_a to the fluid trajectory. The latter is not geodesic because of the pressure gradient $\nabla_a P$, but it is quasigeodesic [18]: the curve, as a set of points, coincides with the geodesic but the proper time is not an affine parameter along it. A quasigeodesic coincides with a nonaffinely parametrized geodesic when the proper time of the fluid (which is, in general, the cosmic comoving time but differs from the proper time of a freely falling observer), is used as a parameter [18]. This is one of the few situations (others are listed in Ref. [18]) in which the four-acceleration $\dot{u}^a$ of a particle is not orthogonal, indeed, it is parallel, to its four-velocity u^a. Dealing with a timelike heat current density requires an extension of the formalism of [13,14] that explicitly requires the gradient $\nabla_c\phi$ to be timelike. While this extension may be possible, it goes beyond the purpose of the current manuscript and the peculiar situation of FLRW spaces with respect to Eckart's thermodynamics of scalar-tensor gravity will be discussed in detail in a separate publication.

B. A Brans-Dicke solution with a central singularity

We now search for an example of a solution of the scalar-tensor field equations with a spacetime singularity, to test whether $\mathcal{T} \to +\infty$ there and whether the deviation from the corresponding GR solution is significant. The general vacuum, static, spherically symmetric and asymptotically flat solution of the Brans-Dicke field equations with $V(\phi) = 0$ that is not a black hole is known and has a central singularity (for appropriate parameter values) [32,33], but the corresponding scalar field gradient $\nabla_a\phi$ is spacelike. We look instead for dynamical solutions with timelike $\nabla_c\phi$, and we disregard FLRW universes in which the imperfect ϕ-fluid quantities vanish identically. These criteria exclude most known analytical solutions of scalar-tensor gravity [34], but the following one, reported in Ref. [35] satisfies them. This Brans-Dicke solution is conformal to a GR solution found in [36], which generalizes an old geometry found by Wyman [37] by including a positive cosmological constant Λ. The scalar field potential in the Jordan frame is

$$V(\phi) = \frac{m^2\phi^2}{2} \tag{5.1}$$

where $m^2 = 2\Lambda/\kappa > 0$ and $\kappa = 8\pi G$. The line element and Brans-Dicke scalar read

$$ds^2 = -\kappa r^2 d\tau^2 + \left(1 - \frac{\tau}{\tau_*}\right)^2 \left(\frac{2dr^2}{1 - \frac{2\Lambda r^2}{3}} + r^2 d\Omega_{(2)}^2\right), \tag{5.2}$$

$$\phi(\tau) = \frac{\phi_*}{(1 - \frac{\tau}{\tau_*})^2}, \tag{5.3}$$

where $d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere, ω and Λ are parameters of the theory, while ϕ_* arises from an initial condition. The Ricci scalar is

$$\begin{aligned}\mathcal{R} &= \frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi + \frac{3\Box\phi}{\phi} + \frac{2V}{\phi} \\ &= \frac{1}{\kappa(1 - \frac{\tau}{\tau_*})^2}\left(2\Lambda\phi_* - \frac{4\omega}{\tau_*^2 r^2}\right).\end{aligned} \tag{5.4}$$

For any value of ω, $\mathcal{R}$ diverges as $\tau \to \tau_*^-$, corresponding to a big crunch singularity, where also ϕ diverges.

If $\omega \neq 0$, $\mathcal{R}$ diverges also when $r \to 0^+$ (see below for the case $\omega = 0$). The areal radius

$$R(\tau, r) = \left(1 - \frac{\tau}{\tau_*}\right) r, \tag{5.5}$$

vanishes as $r \to 0$ and there is a central singularity if $\omega \neq 0$.

The slices of constant time are finite with $0 \leq r \leq r_*$, where

$$r_* = \sqrt{\frac{3}{2\Lambda}}\sqrt{1 - \frac{8\pi\tilde{\phi}_0{}^2}{|2\omega + 3|\kappa}} = \sqrt{\frac{3}{2\Lambda}}\sqrt{1 - \frac{2}{\kappa\tau_*^2}} \tag{5.6}$$

(cf. Ref. [38]). We have, therefore, a naked central singularity embedded in a finite inhomogeneous universe created by Λ and ϕ, which ends at the finite future τ_*.

The case $\omega = 0$ was studied in Ref. [35]. The curvature invariant

$$\begin{aligned}\mathcal{R}_{ab}\mathcal{R}^{ab} &= \frac{1}{\phi^2}\left(\nabla_a\nabla_b\phi\nabla^a\nabla^b\phi + \frac{\Lambda^2}{\kappa^2}\right) \\ &= \frac{1}{\tau_*^4\kappa r^4(1 - \frac{\tau}{\tau_*})^4}\left(\frac{9}{\kappa\tau_*^2} - 4 + \frac{8\Lambda r^2}{3}\right) \\ &\quad + \frac{\Lambda^2}{\kappa^2\phi_*^2}\left(1 - \frac{\tau}{\tau_*}\right)^4\end{aligned} \tag{5.7}$$

diverges as $r \to 0^+$ (or as the areal radius $R \to 0^+$), therefore the naked central singularity persists for $\omega = 0$.

This Brans-Dicke solution is also a solution of purely quadratic $f(\mathcal{R})$ gravity [35]

$$f(\mathcal{R}) = \frac{\kappa\mathcal{R}^2}{4\Lambda}. \tag{5.8}$$

This theory exhibits a restricted scale-invariance and does not admit a Newtonian limit [39], however it approximates the Starobinski model $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$ of inflation [40], which fairs very well in the light of current cosmological observations [41,42]. For this solution, the gradient $\nabla_c\phi$ is timelike and Eq. (3.5) reduces to

$$\mathcal{K}\mathcal{T} = \frac{2}{(8\pi)^{3/2} r(\tau_* - \tau)}. \tag{5.9}$$

Thus, $\mathcal{K}\mathcal{T}$ diverges as $r \to 0$ at the central singularity, and also at the Big Crunch singularity $\tau \to \tau_*{}^-$. The corresponding GR solution with a positive Λ and $\phi =$ const. is de Sitter space, which has no spacetime singularities. This example illustrates the previous assertion that, where $\mathcal{K}\mathcal{T}$ diverges, the deviation of scalar-tensor gravity from GR becomes extreme.

C. Scalar-tensor black holes

Vacuum, asymptotically flat, stationary black holes coincide with those of GR, according to a host of no-hair theorems originating from an early theorem by Hawking [43]. This result was extended to more general scalar-tensor theories with varying Brans-Dicke coupling $\omega(\phi)$ and a potential $V(\phi)$ [44,45], provided that the latter has a minimum in which the scalar ϕ can lodge in stable equilibrium. The known exceptions to these no-hair theorems are maverick solutions in which the scalar ϕ diverges on the horizon, e.g., in the Bronnikov-Bocharova-Melnikov-Bekenstein extremal black hole for a conformally coupled scalar [46,47]. The proof of Hawking's theorem consists of showing that ϕ must be constant outside the horizon, and then gravity reduces to Einstein gravity in that region. Adopting the second interpretation of the thermodynamic analogy discussed in Sec. III D, one has that this occurrence corresponds to zero "theory temperature" $\mathcal{T}$ of the ϕ-fluid. The physical interpretation is that, when they form, scalar-tensor black holes freeze the extra dynamical degree of freedom ϕ outside their horizons. However, the singularity inside the horizon becomes "hot" and deviates from GR, an idea that we intend to explore in the future. Then the no-hair theorems state that, outside the horizon, GR black holes are the states of "lowest temperature" in the space of scalar-tensor black holes.

D. Thermodynamics of stealth solutions

We now examine the thermodynamics of stealth solutions of scalar-tensor gravity, i.e., of solutions in which the geometry is the same as in GR with the same matter source while the scalar field is not constant, but does not gravitate. In other words, the defining features are that the effective stress-energy tensor $T_{ab}^{(\phi)}$ vanishes in that geometry [48–57], while the scalar field still retains a nontrivial dynamics.

Example 1: $\omega = 0$, $V(\phi) = 0$, and ϕ linear in time.

Consider the action in Eq. (1.1) with $\omega = 0$ and $V(\phi) = 0$. The corresponding effective stress-energy tensor reads

$$T_{ab}^{(\phi)} = \frac{1}{\phi}(\nabla_a \nabla_b \phi - g_{ab}\Box\phi). \tag{5.10}$$

It is then easy to see that this tensor vanishes if one considers the ansatz $\phi(t) = \alpha t + \beta$, with t denoting the coordinate time of the solution of the effective Einstein field equations, and α, β are two positive constants. The linearity in time of the Brans-Dicke scalar field is actually a familiar feature in stealth solutions in Horndeski [58] and beyond Horndeski theories [58–64] (see also [34]). Furthermore, this ansatz for the scalar field forces a restriction onto the energy-momentum tensor of (ordinary) matter in the theory. Indeed, from Eq. (1.3) one finds that $T^{(m)} = g^{ab}T_{ab}^{(m)} = 0$ throughout the spacetime manifold.

Focusing on the expressions of the general imperfect fluid representation of $T_{ab}^{(\phi)}$, one finds that $\rho^{(\phi)}$, $P^{(\phi)}$, $q_a^{(\phi)}$, and $\pi_{ab}^{(\phi)}$ all vanish if the ansatz $\phi(t) = \alpha t + \beta$ is assumed. The same is true for the kinetic quantities associated with the ϕ-fluid, i.e., σ_{ab}, θ, and ω_{ab}. This means that Eckart's constitutive relations are identically satisfied by the ϕ-fluid and one can still regard Eqs. (3.5) and (3.8) as viable definitions of temperature, thermal conductivity, and shear viscosity. Furthermore, it turns out that, for this ansatz for the Brans-Dicke scalar,

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^e\phi\nabla_e\phi}}{8\pi\phi} = \frac{\sqrt{-g_{00}}\alpha}{8\pi(\alpha t + \beta)}. \tag{5.11}$$

These stealth solutions correspond to non-equilibrium states of gravity since the scalar degree of freedom is excited and propagates, even though it does not gravitate. This situation can be made sense of by considering the case of static stealth solutions. In this case one has that g_{00} is negative and corresponds to the norm of a timelike Killing vector field. Thus, the effective gravitational coupling is $G_{\text{eff}} \simeq 1/\phi$ and it evolves even though for these solutions the geometry does not, going to zero as $t \to +\infty$. Correspondingly, the scalar-tensor thermodynamic system approaches the GR state of equilibrium, or $\mathcal{K}\mathcal{T} \to 0$, at late times because gravity is switched off asymptotically.

Example 2: Static stealth solution in vacuum Brans-Dicke gravity with $\omega = -1$ and $V(\phi) = V_0\phi$.

The conditions $\omega = 0$ and $V(\phi) = 0$ are not necessary to achieve a stealth solution. As an example with $\omega \neq 0$, $V \neq 0$, consider the following solution of vacuum Brans-Dicke theory found in [65] (see also [34]), which satisfies the field equations for $\omega = -1$ and linear potential $V(\phi) = V_0\phi$ (with V_0 a positive constant).

The static, spherically symmetric geometry is [65,66]

$$ds^2 = -dt^2 + A(r)^{-\sqrt{2}}dr^2 + A(r)^{1-\sqrt{2}}r^2 d\Omega_{(2)}^2, \tag{5.12}$$

while the Brans-Dicke scalar field is

$$\phi(t,r) = \phi_0 e^{2at}A(r)^{1/\sqrt{2}} \tag{5.13}$$

and $A(r) = 1–2m/r$. Here m and $a \neq 0$ are parameters, while the constant $\phi_0 > 0$ is related to the initial conditions. This geometry is a special case of the Campanelli-Lousto static geometry [67], which is the form of the most general solution of the vacuum Brans-Dicke field equation that is static, spherically symmetric and asymptotically flat [32,33], but is expressed in a coordinate system of limited validity [33]. In general, this solution contains only naked singularities or wormhole throats, but not black holes [33,34,68]. This is not a stealth solution (it is straightforward to verify that, for example, the energy density of the effective ϕ-fluid cannot vanish if $m \neq 0$). However, the limit $m \to 0$ produces a stealth solution. For $m = 0$, the geometry reduces to the Minkowski one while $\phi(t) = \phi_0 e^{2at}$ remains dynamical and does not gravitate. We have

$$\nabla_c\phi = 2a\phi\delta_c^0, \qquad \nabla^e\phi\nabla_e\phi = -4a^2\phi^2,$$
$$\nabla_c\nabla_d\phi = 4a^2\phi\delta_c^0\delta_d^0, \tag{5.14}$$

from which one can easily infer that $\nabla_c u_d = 0$, $q_c^{(\phi)} = 0$, and $\pi_{cd}^{(\phi)} = 0$. The energy density and isotropic pressure of the ϕ-fluid are

$$\rho^{(\phi)} = -2a^2 + \frac{V_0}{2} = -P^{(\phi)}, \tag{5.15}$$

so that $T_{ab}^{(\phi)}$ vanishes identically only if

$$V_0 = 4a^2. \tag{5.16}$$

The effective gravitational coupling strength reads $G_{\text{eff}} = \phi_0^{-1}e^{-2at}$, while

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^e\phi\nabla_e\phi}}{8\pi\phi} = \frac{|a|}{4\pi} \tag{5.17}$$

remains constant in time. The stealth scalar-tensor non-equilibrium state never approaches the GR equilibrium state $\phi = \text{const.} > 0$.

Note that $\nabla^c\phi$ is parallel to the timelike Killing field $t^c = (\partial/\partial t)^c$. In order to have these two vector fields pointing in the same direction, it must be $a < 0$, which implies that $G_{\mathrm{eff}} \sim e^{2|a|t}$ diverges exponentially at late times while $\mathcal{K}\mathcal{T} = |a|/4\pi$ remains constant.

VI. CONCLUSIONS AND OUTLOOKS

We have expanded and built upon the new approach to the thermodynamics of scalar-tensor gravity presented briefly in the previous paper [7], providing details. Our approach is very different from Jacobson's thermodynamics of spacetime [1,2]: there, the temperature of spacetime is the Unruh temperature of local uniformly accelerated observers whose worldlines thread the fabric of spacetime, while here the temperature $\mathcal{T}$ arises from Eckart's first order thermodynamics for dissipative fluids, which we are led to examine following the reformulation of scalar-tensor gravity in terms of an effective ϕ-fluid in [13,14]. In [2], it was proposed that, while GR is an equilibrium state of gravity, $f(\mathcal{R})$ (which is a subclass of scalar-tensor) gravity theories and, by extension, presumably also other modified gravity theories, constitute excited non-equilibrium states. Therefore, there should be a spontaneous approach of these excited states to the GR equilibrium state. However the equations ruling this approach to equilibrium, and the order parameter(s) ruling it, have never been identified. If alternative gravity really "decays" spontaneously to GR, it should be possible to track and model this process in more than one way. Indeed, modified gravity contains extra dynamical degrees of freedom in addition to the two massless spin two modes of GR contained in the metric tensor. Therefore, a theory in which these extra degrees of freedom are excited and propagate can rightly be called an "excitation" of GR. In scalar-tensor (including metric $f(\mathcal{R})$) gravity, there is only one (scalar, massive) extra degree of freedom.

In contrast with spacetime thermodynamics, our proposal is minimalist, using less assumptions and less fundamental ones. The new approach consists of the effective fluid formulation of scalar-tensor gravity plus the application of the constitutive relations of Eckart's first order thermodynamics [15] to it. Eckart's noncausal thermodynamics is unable to describe a full relaxation process and to provide a relaxation time, however it gives us the effective temperature, the order parameter quantifying how far away a modified gravity is from GR. It provides also the information that bulk viscosity is zero, the explicit expression of the shear viscosity, a simple expression for the heat current density, as well the effective entropy density. One puzzling aspect is that the shear viscosity η is negative, but this is not too disconcerting when one realizes that the thermodynamical system (the ϕ-fluid) is not isolated but exchanges energy with its "surroundings." The most important consequence is that the entropy density is not necessarily forced to decrease, which corresponds to the fact that scalar-tensor gravity does not always approach the GR state. This fact opens the possibility that there could be other equilibrium states, i.e., other special theories of gravity, at positive values of $\mathcal{K}\mathcal{T}$. The first possibility that comes to mind is Palatini $f(\mathcal{R})$ gravity, in which the effective scalar degree of freedom is nondynamical [3–5]. However, this is not a truly new state since, in electrovacuo, it reduces to GR with a cosmological constant [3–5]. The same situation occurs in cuscuton theory [69,70] and in minimally modified gravity [71], which could also provide nontrivial equilibrium states.

In spite of the limitations of Eckart's first order thermodynamics, the effective fluid formalism is not an analogy, but a new approach to the problem of describing how gravity diffuses (or not) to the GR state of equilibrium. A seemingly self-consistent thermodynamic theory emerges in this approach: $\mathcal{K}\mathcal{T}$ is positive-definite and vanishes in the GR equilibrium state; spherical, asymptotically flat, vacuum black holes of scalar-tensor gravity (except for mavericks) correspond to the $\mathcal{K}\mathcal{T} = 0$ black holes of GR, according to the no-hair theorems; spacetime singularities correspond to formally infinite $\mathcal{K}\mathcal{T}$ and to large deviations from GR.

Several aspects of this new thermodynamics of gravity will be analyzed in future work, including: the special role of FLRW spaces; the idea that black holes have zero temperature far away from the horizon, while being "hot" and deviating from GR near the singularity; and attempts to generalize the formalism to situations in which the gradient $\nabla^c\phi$ is null or spacelike. A natural generalization of the approach proposed here has been carried out in [8] for Horndeski gravity. However, because of the much larger freedom in the choice of coupling functions and parameters in general Horndeski gravity, the interpretation of the results there is much more complicated and physical intuition relies on the results presented here.

In an alternative approach, one would trade temperature with chemical potential and assign zero temperature and entropy, but nonzero chemical potential, to the effective ϕ-fluid of modified gravity. This approach was used in Refs. [72–74] for braided kinetic gravity (note the similarity between our Eq. (3.4) and Eq. (3.35) of Ref. [72]). We will report in the future on this alternative approach for scalar-tensor gravity, as well as on both approaches for other theories of gravity alternative to GR. Eventually, the generalization from Eckart's first order, non-causal thermodynamics to realistic, causal extended thermodynamics for dissipative fluids [24,75–81] will also have to be addressed.

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