

Distinguishing modified gravity with just two tensorial degrees of freedom from general relativity: Black holes, cosmology, and matter coupling

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We consider spatially covariant modified gravity in which the would-be scalar degree of freedom is made nondynamical and hence there are just two tensorial degrees of freedom, i.e., the same number of dynamical degrees of freedom as in general relativity. Focusing on a class of such modified gravity theories characterized by three functions of time, we discuss how modified gravity with two tensorial degrees of freedom can be distinguished observationally or phenomenologically from general relativity. It is checked that the theory gives the same predictions as general relativity for the parametrized post-Newtonian parameter γ and the propagation speed of gravitational waves. We also find that there is no modification to asymptotically flat black holes at rest with respect to the preferred frame. Due to a large degree of freedom to choose the time-dependent functions in the theory, the homogeneous and isotropic cosmological dynamics can be made close to or even identical to that of the Λ CDM model. We investigate the behavior of cosmological perturbations in the long- and short-wavelength limits and show that in both limits the effects of modified gravity appear only through the modification of the background evolution. Finally, it is remarked that in the presence of a Galileon field in the matter sector, the scalar degree of freedom is revived, ruining the essential feature of the theory.

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I. INTRODUCTION

According to Lovelock's theorem [1,2], the Einstein tensor plus a cosmological term is the only possible second-order Euler-Lagrange equation that is obtained from a diffeomorphism-invariant action constructed from the metric tensor alone in four spacetime dimensions. Modifying gravity amounts to relaxing the postulates of the theorem. Abandoning full spacetime diffeomorphism invariance, one can for example consider modified gravity enjoying only three-dimensional spatial diffeomorphism invariance. Basically, this way of modifying gravity is equivalent to introducing a new dynamical scalar degree of freedom (d.o.f.) that spontaneously breaks time diffeomorphism invariance. This is the underlying idea utilized in the construction of effective field theories for inflation and dark energy [3–6]. A general framework for scalar-tensor theories based on this idea has been developed in Refs. [7,8].

An interesting twist is that a scalar d.o.f. incorporated in such a framework is not necessarily dynamical and it is possible that modified gravity in which time diffeomorphism invariance is broken has only two tensorial degrees of freedom like general relativity (GR). The simplest example of such “scalarless” scalar-tensor theories is the cuscuton theory [9], in which the scalar field has an

infinite propagation speed. An attempt to identify the cuscuton-like subclass within the Horndeski [10] and beyond Horndeski [11] theories was presented in Ref. [12]. Other modified gravity theories with two d.o.f. have been constructed in Refs. [13–17], a subclass of which is in fact equivalent to the cuscuton theory [18]. On the basis of the general framework of spatially covariant theories of gravity [7,8], Ref. [19] has derived through a rigorous Hamiltonian analysis the conditions under which there are only two d.o.f. and presented a particular action satisfying the conditions. The action is composed of two derivative terms built out of three-dimensional geometrical quantities as in GR, but depends on the lapse function in a nontrivial manner. A complementary perturbative approach to spatially covariant modified gravity without a scalar d.o.f. has been proposed in Ref. [20].

The purpose of this paper is to explore phenomenological aspects of the theory of modified gravity with two tensorial d.o.f. proposed in Ref. [19] to see how the theory can be distinguished observationally from GR. We will see that it is quite difficult to find differences between the two theories in the weak gravity regime, gravitational wave propagation, black hole spacetimes, and cosmology, when restricted to a certain subset of the theories of Ref. [19]. The theory of modified gravity we study in this paper is somewhat similar to the infrared limit of Hořava gravity [21] in some respects, and we share the same motivation, e.g., as Ref. [22].

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The structure of this paper is as follows. In the next section, we review modified gravity theories with two tensorial d.o.f. [19] which we study in this paper. In Sec. III, we identify the subclass of the theories that gives the same predictions as general relativity for the parametrized post-Newtonian (PPN) parameter γ and the propagation of gravitational waves. Then, we study black hole spacetimes in Sec. IV and cosmological aspects in Sec. V. In Sec. VI, a side remark concerning a coupling to a Galileon field is given. Finally, we draw our conclusions in Sec. VII.

II. SPATIALLY COVARIANT GRAVITY WITH TWO TENSORIAL DEGREES OF FREEDOM

In this section, we introduce a class of spatially covariant modified gravity in which there are just two tensorial d.o.f. [19] and in particular there is no scalar d.o.f.

A. “Scalarless” scalar-tensor theories

Let us begin with scalar-tensor theories with one scalar d.o.f., ϕ , on top of two tensorial d.o.f. In the unitary gauge where ϕ is homogeneous on constant time hypersurfaces, ghost-free scalar-tensor theories can in general be described by the action of the form [7,8],

$$S = \int dt d^3x N \sqrt{\gamma} \mathcal{L}(N, \gamma_{ij}, K_{ij}, R_{ij}, D_i; t), \quad (1)$$

where N is the lapse function, γ_{ij} is the spatial metric, K_{ij} is the extrinsic curvature of constant time hypersurfaces,

$$K_{ij} = \frac{1}{2N} (\partial_t \gamma_{ij} - D_i N_j - D_j N_i), \quad (2)$$

with N_i being the shift vector, R_{ij} is the intrinsic curvature tensor, and D_i is the covariant derivative with respect to γ_{ij} . The Horndeski action [10,23,24] can be recast in the above form in terms of these Arnowitt-Deser-Misner (ADM) variables. One may also include the velocity of the lapse function, $\partial_t N - N^i D_i N$, and explore a larger class of healthy scalar-tensor theories with three d.o.f. [25]. Although such a generalization leads to an interesting class of theories called degenerate higher-order scalar-tensor theories [26–28], we focus for simplicity on the action of the form (1).

Thus, in general, the action (1) describes scalar-tensor theories with three d.o.f. However, in a particular subset of theories, the scalar mode obeys an elliptic equation and does not propagate. Consequently, such theories have only two tensorial d.o.f. as in GR, but the action is certainly different from that of GR. A particular example is given by [19]

$$S = \frac{1}{2} \int dt d^3x N \sqrt{\gamma} \left[\frac{\beta_0 N}{\beta_2 + N} K_{ij} K^{ij} - \frac{\beta_0}{3} \left(\frac{2N}{\beta_1 + N} + \frac{N}{\beta_2 + N} \right) K^2 + \alpha_1 + \alpha_2 R + \frac{1}{N} (\alpha_3 + \alpha_4 R) \right], \quad (3)$$

where β_A and α_A are functions of t . A rigorous Hamiltonian analysis has shown that this theory indeed has only two tensorial d.o.f. [19].¹ One might worry that the number of propagating d.o.f. may depend on the gauge choice. This concern was addressed recently in the context of higher-order scalar-tensor theories in Refs. [29,30], where it was argued that the unitary gauge analysis is sufficient. The cuscuton theory [9] is reproduced by setting $\beta_0 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = \alpha_4 = 0$, with α_3 coming from the cuscuton term $\sigma(\phi) \sqrt{-\partial_\mu \phi \partial^\mu \phi}$, while the extended cuscuton theories [12,31] correspond to the case with $\beta_1 = \beta_2$. The covariant expression for the action (3) in terms of a Stückelberg scalar field is given in Appendix B. It is instructive to rewrite the kinetic part of the Lagrangian as

$$\frac{\beta_0}{2} \left(\frac{N}{\beta_2 + N} \tilde{K}_{ij} \tilde{K}^{ij} - \frac{2}{3} \frac{N}{\beta_1 + N} K^2 \right), \quad (4)$$

where $\tilde{K}_{ij} := K_{ij} - (1/3)K\gamma_{ij}$ is the traceless part of the extrinsic curvature. This expression shows the roles of the functions β_1 and β_2 more clearly. In the following we will explore various aspects of the gravitational theory with the action (3).

Before proceeding let us comment on other examples of “scalarless” scalar-tensor theories with two d.o.f. Under the assumption that the Lagrangian is linear in the lapse function [i.e., \mathcal{L} in Eq. (1) is independent of N], the conditions for a theory to have only two tensorial d.o.f. have been addressed in Ref. [13]. References [14–16] obtained gravitational theories with two d.o.f. via canonical transformations from GR in the Hamiltonian formulation. In the context of the recent claim concerning the $D \rightarrow 4$ limit of D -dimensional Gauss-Bonnet gravity [32], a consistent theory with two d.o.f. has been proposed [17] that avoids the propagation of the scalar mode present in the previous derivations [33–37]. Symmetries that prohibit the scalar mode from propagating have been discussed in Ref. [38]. See Refs. [39–41] for further developments in “scalarless” modified gravity and Refs. [42–44] for the application to cosmology.

¹The case with $\beta_0 = 1$ was presented in Ref. [19], but it is clear from the footnote of Ref. [19] that one can include this time-dependent coefficient.

B. Quasi-Einstein frame

A field redefinition of a metric given by a disformal transformation [45] retains the number of propagating d.o.f. as long as the transformation is invertible. In terms of the ADM variables, it is expressed as

$$N \rightarrow \mathcal{N}(t, N), \quad N^i \rightarrow N^i, \quad \gamma_{ij} \rightarrow \mathcal{A}(t, N)\gamma_{ij}. \quad (5)$$

Let us consider a special case of a disformal transformation,

$$N \rightarrow b_0(t)N + b_1(t), \quad N^i \rightarrow N^i, \quad \gamma_{ij} \rightarrow \gamma_{ij}. \quad (6)$$

Under the transformation (6), the form of the action (3) is invariant, with $\beta_A(t)$ and $\alpha_A(t)$ being transformed as

$$\begin{aligned} \beta_0 &\rightarrow \beta_0/b_0, & \beta_{1,2} &\rightarrow (\beta_{1,2} + b_1)/b_0, \\ \alpha_{1,2} &\rightarrow b_0\alpha_{1,2}, & \alpha_{3,4} &\rightarrow \alpha_{3,4} + b_1\alpha_{1,2}. \end{aligned} \quad (7)$$

Therefore, via the field redefinition (6) one can always set $\beta_0 = 1$ and $\beta_2 = 0$. Since the coefficient of $K_{ij}K^{ij}$ is now standard, we dub this frame the quasi-Einstein frame. This motivates us to study the action of the form²

$$\begin{aligned} S = \frac{1}{2} \int dt d^3x N \sqrt{\gamma} &\left[K_{ij}K^{ij} - \frac{1}{3} \left(\frac{2N}{\beta + N} + 1 \right) K^2 \right. \\ &\left. + \alpha_1 + \alpha_2 R + \frac{1}{N} (\alpha_3 + \alpha_4 R) \right], \end{aligned} \quad (8)$$

where $\beta := \beta_1$, as the case of particular interest. General relativity with a cosmological constant is reproduced by setting $\beta = \alpha_3 = \alpha_4 = 0$, $\alpha_1 = \text{const}$, and $\alpha_2 = 1$. Theories with $\beta \neq 0$ are not included in the (extended) cuscuton family, and hence the impacts of β have not been investigated so far in the literature.

Note that a subtlety arises when matter is present. In this paper we assume that matter is minimally coupled to gravity in this quasi-Einstein frame.

Here we should point out the similarity and difference between the action (8) and that of the infrared limit of nonprojectable Hořava gravity [21,46,47],

$$\begin{aligned} S = \frac{1}{2} \int dt d^3x N \sqrt{\gamma} &(K_{ij}K^{ij} - \lambda K^2 \\ &+ \alpha_1 + \alpha_2 R + \eta a_i a^i), \end{aligned} \quad (9)$$

where $a_i := D_i \ln N$ and now all the coefficients are constants. The same action is also obtained in the context of Einstein-aether theory [48] if the aether field is restricted to be hypersurface orthogonal [49,50]. While being mostly similar, the essential difference can be perceived in the

²As will be clear shortly, we use units in which Newton's constant, G_N , is equal to $(8\pi\alpha_2)^{-1}$, and we will be interested mainly in the case with $\alpha_2 = 1$.

structure of the coefficient of K^2 . In the case of $\eta = 0$, nonprojectable Hořava gravity has odd dimensionality of the phase space at each spacetime point and hence is inconsistent [51,52]. By contrast, the action (8) is free of such a trouble thanks to the particular structure of the coefficient of K^2 .

III. SOLAR SYSTEM TESTS AND GRAVITATIONAL WAVES

Let us study gravitational potentials produced by non-relativistic matter and the propagation of gravitational waves in the theory described by the action (8). We assume that $\alpha_1, \alpha_3 \sim H_0^2$, where H_0 is the Hubble scale, so that they are only relevant to the large-scale cosmological dynamics. We also assume that α_2 and α_4 could vary only on cosmological time scales, because otherwise nearly static gravitational potentials would not be possible. We can then consider scalar and tensor perturbations on a Minkowski background, using the approximation $\alpha_1, \alpha_3 \ll \partial_t^2, \Delta$ ($:= \delta_{ij}\partial_i\partial_j$).

Static scalar perturbations of the ADM variables are given by

$$\begin{aligned} N &= 1 + \Phi(\vec{x}), & N_i &= \partial_i \chi(\vec{x}), \\ \gamma_{ij} &= [1 - 2\Psi(\vec{x})]\delta_{ij}. \end{aligned} \quad (10)$$

The quadratic action reads

$$\begin{aligned} S = \int dt d^3x &\left[-(\alpha_2 + \alpha_4)\Psi\Delta\Psi + 2\alpha_2\Phi\Delta\Psi \right. \\ &\left. + \frac{\beta}{3(1+\beta)}(\Delta\chi)^2 - \Phi\rho(\vec{x}) \right], \end{aligned} \quad (11)$$

where $\rho(\vec{x})$ is the matter energy density. This matter is assumed to be minimally coupled to gravity and at rest with respect to the preferred frame. Note that in deriving Eq. (11) we actually assumed that $\beta\Delta \gg H_0^2$. The equations of motion are then solved to give

$$\Delta\Phi = \left(1 + \frac{\alpha_4}{\alpha_2}\right) \frac{\rho}{2\alpha_2}, \quad \Delta\Psi = \frac{\rho}{2\alpha_2}, \quad \Delta\chi = 0. \quad (12)$$

This result implies that

$$\alpha_2 \simeq \text{const} \quad \text{and} \quad \alpha_4 \simeq 0 \quad (13)$$

are required in order to evade Solar System constraints (light bending, the Shapiro time delay, and the time variation of Newton's constant).³

³According to the current limits on the time variation of Newton's constant, $|\dot{\alpha}_2/\alpha_2|$ must be much smaller than H_0 [53].

Tensor perturbations are given by

$$N = 1, \quad N_i = 0, \quad \gamma_{ij} = \delta_{ij} + h_{ij}(t, \vec{x}), \quad (14)$$

for which the quadratic action is

$$S = \frac{1}{8} \int dt d^3x [(\partial_t h_{ij})^2 + (\alpha_2 + \alpha_4) h_{ij} \Delta h_{ij}]. \quad (15)$$

The observation of GW170817 and GRB 170817A put a tight bound on the speed of gravitational waves, $c_{\text{GW}} \simeq 1$ [54,55]. It is therefore required that

$$\alpha_2 + \alpha_4 \simeq 1. \quad (16)$$

Summarizing these results, we identify a phenomenologically interesting class characterized by

$$\alpha_2 = 1, \quad \alpha_4 = 0. \quad (17)$$

Note that in these noncosmological phenomena we do not see any effects of nonvanishing β .

A few remarks are now in order. First, here we have only considered the most basic PPN parameter, which is often denoted as $\gamma(= \Psi/\Phi)$. Clearly, this is a minimum requirement for viable modified gravity and all the other PPN parameters must eventually be evaluated. In particular, a source may move with respect to the preferred frame in which our action is defined, and then the PPN parameters that signal the preferred frame effect are expected not to vanish. Nevertheless, a study on the analogous theory with the action (9) [47] implies that the corresponding PPN parameters would vanish if only the coefficient of K^2 is modified.

The second remark is on the propagation of gravitational waves in an inhomogeneous background. The effect of small inhomogeneities produced by scalar perturbations on the gravitational wave propagation can be seen by calculating the cubic terms in the action composed of one scalar and two tensor perturbations such as $\Psi \dot{h}_{ij}^2$ and $\Phi h_{ij} \Delta h_{ij}$. It is easy to see that K^2 does not generate any such terms, and hence gravitational waves propagate in an inhomogeneous background in exactly the same way as in GR if only the coefficient of K^2 is modified.

IV. BLACK HOLE SOLUTIONS

In this section, we study black hole solutions in the theory defined by the action (8). On the basis of the discussion in the previous section, we are interested in the phenomenologically viable case with $\alpha_2 = 1$ and $\alpha_4 = 0$. Black holes under consideration are supposed to be much smaller than the size of the cosmological horizon and at rest with respect to the preferred frame. Therefore, we are mostly interested in asymptotically flat black holes obtained ignoring α_1 and α_3 . (These two functions are assumed to be characterized by

the cosmological horizon scale.) The remaining unfixed function, β , is assumed to be constant in this section, because we are looking for stationary black hole solutions. Our focus is therefore on whether the effects of β can be seen or not in black hole solutions. Nevertheless, having said that, we allow α_1 to be a nonvanishing constant and α_2 to be a constant different from 1 in the following discussion, because relaxing these assumptions does not hinder us from presenting an analytic solution below. To sum up, in this section we have three constant parameters, β , α_1 , and α_2 , though we are mostly interested in the case with $\alpha_1 = 0$ and $\alpha_2 = 1$.

A. Static and spherically symmetric solution

The ADM variables for static and spherically symmetric solutions are taken to be

$$\begin{aligned} N &= N(r), & N_i dx^i &= B(r) F(r) dr, \\ \gamma_{ij} dx^i dx^j &= F^2(r) dr^2 + r^2 d\Omega^2, \end{aligned} \quad (18)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Since the action is only invariant under spatial diffeomorphisms, at this stage we do not make a temporal coordinate transformation, $t \rightarrow T(t, r)$, to remove B . Such a coordinate transformation would result in an inhomogeneous configuration of the nondynamical scalar field.

Substituting the ADM variables (18) into the action and varying it with respect to N , B , and F , we obtain

$$\begin{aligned} \alpha_1 + \frac{2\alpha_2}{r^2} \left[\left(\frac{F^2 - 1}{F^2} \right) r \right]' + \frac{2(rB^2)'}{r^2(\beta + N)^2 F^2} \\ - \frac{2r^2}{3} \frac{\beta(\beta + 2N)}{N^2(\beta + N)^2 F^2} [(B/r)']^2 = 0, \end{aligned} \quad (19)$$

$$\frac{r^2 B}{(\beta + N)^2 F^2} [(\beta + N)F]' + \frac{\beta}{3} \left[\frac{r^4 (B/r)'}{N(\beta + N)F} \right]' = 0, \quad (20)$$

$$\begin{aligned} \alpha_1 \frac{N}{F} + \frac{2\alpha_2 N}{r^2} \frac{N}{F} \left(\frac{F^2 - 1}{F^2} - \frac{2rN'}{NF^2} \right) \\ + \frac{2(rB^2)'}{r^2 N F^3} - \frac{2\beta}{3r^4 N(\beta + N)F^3} [(r^2 B)']^2 = 0, \end{aligned} \quad (21)$$

where a dash denotes differentiation with respect to r . The system is of second order for B (provided that $\beta \neq 0$) and of first order for N and F .

Since it is difficult to obtain a general solution to Eqs. (19)–(21), we start with finding a particular solution. One can easily verify that Eqs. (19)–(21) admit the following solution:

$$N = N_0 \sqrt{f(r)}, \quad F = \frac{1}{\sqrt{f(r)}}, \quad B = \frac{N_0 b_0}{r^2}, \quad (22)$$

where

$$f(r) := 1 + \frac{\alpha_1}{6\alpha_2} r^2 - \frac{\mu_0}{r} + \frac{b_0^2}{\alpha_2 r^4}, \quad (23)$$

and N_0 , μ_0 , and b_0 are integration constants. Here, N_0 is physically less important because it may be set to 1 by rescaling the unit of t .

To see the spacetime structure of this solution more clearly, we note that the four-dimensional metric can be written as

$$ds^2 = -(N^2 - B^2) \left(dt - \frac{BF}{N^2 - B^2} dr \right)^2 + \frac{N^2 F^2}{N^2 - B^2} dr^2 + r^2 d\Omega^2, \quad (24)$$

where

$$N^2 - B^2 = N_0^2 \left(1 + \frac{\alpha_1}{6\alpha_2} r^2 - \frac{\mu_0}{r} + \frac{1 - \alpha_2 b_0^2}{\alpha_2 r^4} \right), \quad (25)$$

$$NF = N_0. \quad (26)$$

Thus, introducing a new time coordinate defined by $dT = N_0 \{ dt - [BF/(N^2 - B^2)] dr \}$, one can write the metric in a diagonal form at the expense of a homogeneous configuration of the nondynamical scalar field (see Appendix B). It is now obvious that in the phenomenologically interesting case of $\alpha_2 = 1$, the solution (22) describes Schwarzschild–(anti–)de Sitter spacetime. Importantly, we see no β dependence in this solution. Note that inside the black hole horizon there is a location at which $N = 0$. If $\alpha_1 < 0$, such a location also exists outside the cosmological horizon. This is called the universal horizon, which has been studied in Einstein-aether theory and Hořava gravity [56,57].

Aside from the cosmological constant term, essentially the same expression for a black hole solution has been obtained in Einstein-aether theory [58]. This is not surprising because the solution (22) satisfies the maximal slicing condition, $K = -(r^2 B)' / r^2 NF = 0$, while the crucial difference between the two theories is found in the coefficient of K^2 in the action. The maximal slicing condition is the very reason why there is no β dependence in the solution.⁴ Note that a similar solution was found in a different modified gravity theory with two tensorial d.o.f. [59].

⁴It should be noted here that any solution in GR satisfying the maximal slicing condition is a solution in the present theory. The maximal slicing condition is often used in numerical relativity simulations, and such realistic numerical solutions in GR are solutions in the present theory.

B. Static perturbations

We have thus obtained a particular solution (22), but at this stage it is not clear whether or not there exist other black hole solutions with the appropriate asymptotic behavior. Since B obeys the second-order differential equations, one might rather expect that there should be another integration constant in addition to N_0 , μ_0 , and b_0 . To address this point, let us consider a slightly deformed solution,

$$N = N_0 \sqrt{f + h_0(r)}, \quad F = \frac{1}{\sqrt{f + h_1(r)}},$$

$$B = N_0 \left[\frac{b_0}{r^2} + h_2(r) \right], \quad (27)$$

where h_1 , h_2 , and h_3 are to be treated as small perturbations. Equations (19) and (21) reduce, respectively, to

$$f \left[\frac{2b_0}{\alpha_2} \left(\frac{h_2}{r} \right)' - (rh_1)' \right] + \frac{3b_0^2}{\alpha_2 r^4} (h_0 - h_1) = 0, \quad (28)$$

$$f \left[\frac{2b_0}{\alpha_2} \left(\frac{h_2}{r} \right)' - (rh_0)' \right] + \left(1 + \frac{\alpha_1}{2\alpha_2} r^2 \right) (h_0 - h_1) = 0. \quad (29)$$

Eliminating h_2 from these two equations, we obtain

$$f(h_0 - h_1)' - f'(h_0 - h_1) = 0, \quad (30)$$

which can be integrated to give

$$h_0 - h_1 = C_1 f, \quad (31)$$

where C_1 is an integration constant. This integration constant can be absorbed into a redefinition of N_0 . Substituting this into Eq. (28), we have

$$h_0 = -\frac{\mu_1}{r} + C_1 \left(f - \frac{b_0^2}{\alpha_2 r^4} \right) + \frac{2b_0 h_2}{\alpha_2 r^2}, \quad (32)$$

$$h_1 = -\frac{\mu_1}{r} - \frac{C_1 b_0^2}{\alpha_2 r^4} + \frac{2b_0 h_2}{\alpha_2 r^2}, \quad (33)$$

where μ_1 is an integration constant. This integration constant simply corresponds to a constant shift of the mass parameter μ_0 . Finally, substituting all these results into Eq. (20), we arrive at

$$(\beta + N_0 \sqrt{f}) \left(h_2'' + \frac{2}{r} h_2' - \frac{2}{r^2} h_2 \right) - N_0 (\sqrt{f})' \left(h_2' + \frac{2}{r} h_2 \right) = 0, \quad (34)$$

where we assumed $\beta \neq 0$. This equation can be integrated to give

$$h_2 = \frac{b_1}{r^2} + \frac{\epsilon_1}{r^2} \int^r r^2 (\beta + N_0 \sqrt{f}) dr, \quad (35)$$

where b_1 and ϵ_1 are integration constants. The former integration constant corresponds to a constant shift of b_0 , while the latter cannot be absorbed into a shift of the integration constants of the solution (22). Let us now focus specifically on the case of a zero cosmological constant, $\alpha_1 = 0$, where the solution is asymptotically flat at zeroth order. In this case, we have

$$h_2 \sim \frac{\epsilon_1}{3} (\beta + N_0) r \quad (36)$$

for large r , which would invalidate the perturbative approximation unless $\epsilon_1 = 0$. Therefore, no asymptotically flat solution can be obtained by a small deformation of the solution (22). Our result agrees with that of the decoupling limit analysis of the khronometric theory [60].

C. Numerical solutions

We perform a numerical analysis that is complementary to and can go beyond the perturbative study in the previous subsection.

In the vicinity of the horizon, $r = r_h$, the ADM variables can be expanded as

$$N = N_h + N_1(r - r_h) + \dots, \quad (37)$$

$$F = \frac{1}{N_h} + F_1(r - r_h) + \dots, \quad (38)$$

$$B = N_h + B_1(r - r_h) + \frac{B_2}{2}(r - r_h)^2 + \dots \quad (39)$$

Here, one may set $F(r_h) = 1/N_h$ without loss of generality since doing so corresponds to fixing the scale of time. For given (r_h, N_h, B_1) , one can numerically integrate

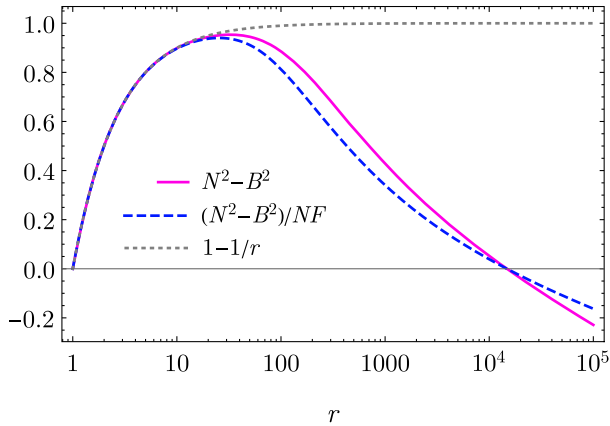


FIG. 1. Numerical solution for $r_h = 1$, $N_h = 0.1$, and $B_1 = -1.9N_h$. The theory parameters are given by $\alpha_1 = 0$, $\alpha_2 = 1$, and $\beta = 0.2$.

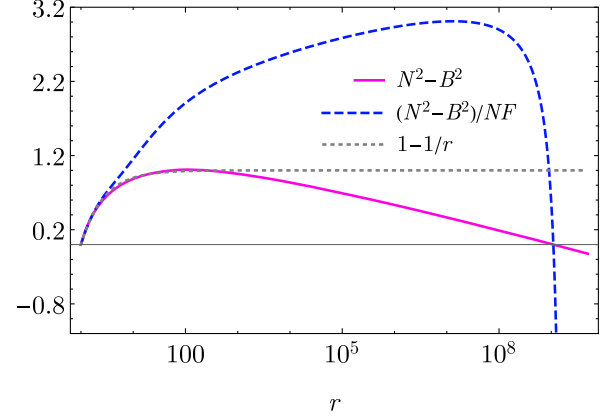


FIG. 2. Numerical solution for $r_h = 1$, $N_h = 0.1$, and $B_1 = -3N_h$. The theory parameters are given by $\alpha_1 = 0$, $\alpha_2 = 1$, and $\beta = -0.3$.

Eqs. (19)–(21) from the horizon outwards to determine the profile of N , F , and B . We thus try to find asymptotically flat solutions that are not described as a perturbation of the Schwarzschild spacetime foliated by maximal slices.

We consider again the case without the cosmological constant, $\alpha_1 = 0$. For

$$B_1 = -\frac{2N_h}{r_h}, \quad (40)$$

one has the asymptotically flat solution given by Eq. (22) (with $\alpha_1 = 0$). As far as we have investigated numerically, no asymptotically flat solutions have been found if the condition (40) is not satisfied.⁵ Typical examples obtained in the case of detuned conditions are given in Figs. 1 and 2. The large- r behavior of these numerical solutions is also different from that of asymptotically de Sitter solutions.

Our numerical analysis thus supports the conclusion that asymptotically flat, spherically symmetric black holes at rest with respect to the preferred frame in the theory defined by the action (8) with $\beta = \text{const}$, $\alpha_2 = 1$, and $\alpha_1 = \alpha_3 = \alpha_4 = 0$ are indistinguishable from those in GR.

D. Slowly rotating solution

Let us now discuss slowly rotating black hole solutions (see Refs. [61–63] for rotating black holes in Einstein-aether theory and Hořava gravity). To first order in the black hole spin, we write [64,65]

$$\delta N = 0, \quad \delta N_i dx^i = -r^2 \sin^2 \theta \omega(r, \theta) d\varphi, \quad \delta \gamma_{ij} = 0. \quad (41)$$

The $(r\varphi)$ component of the evolution equations reads

$$\partial_\theta (\sin^3 \theta \partial_\theta \omega) = 0, \quad (42)$$

⁵We have not looked for asymptotically flat solutions in the case of $\alpha_1 \neq 0$.

which, together with the regularity at the pole, yields $\omega = \omega(r)$. Then, the φ component of the momentum constraints reduces to

$$\left(\frac{r^4 \omega'}{NF}\right)' = 0. \quad (43)$$

This can be integrated to give

$$\omega(r) = -\frac{3J}{4\pi} \int^r \frac{NF}{r^4} dr + \omega_0, \quad (44)$$

where J and ω_0 are integration constants. Upon using $NF = 1$ we arrive at $\omega = J/4\pi r^3 + \omega_0$. (For simplicity, here we have set $N_0 = 1$ without loss of generality.) Noting that the time coordinate t that we are using is nonstandard, we perform a coordinate transformation

$$\varphi \rightarrow \varphi + \int^r \frac{BF}{N^2 - B^2} (\omega - \omega_0) dr + \omega_0 t. \quad (45)$$

The four-dimensional metric then becomes

$$ds^2 = -(N^2 - B^2)dT^2 + \frac{dr^2}{N^2 - B^2} + r^2 d\Omega^2 - \frac{J \sin^2 \theta}{2\pi r} dT d\varphi, \quad (46)$$

which is nothing but the Kerr metric in GR under the slow-rotation approximation. Thus, we conclude that no effect of β can be seen in slowly rotating black holes at rest with respect to the preferred frame.

V. COSMOLOGY

Our next step is the analysis of cosmology. In this section, we will mainly consider the phenomenologically most interesting case with the time-dependent functions

$$\begin{aligned} \beta_0 = 1, \quad \beta_1 = \beta(t), \quad \beta_2 = 0, \\ \alpha_1 = \alpha_1(t), \quad \alpha_2 = 1, \quad \alpha_3 = \alpha_3(t), \quad \alpha_4 = 0. \end{aligned} \quad (47)$$

However, when deriving a quadratic action for cosmological perturbations, we will work in the general action (3) where no assumptions are made on these functions, because the general form of the quadratic action will be used for another purpose in the next section.

To investigate cosmology, we add a (irrotational, barotropic) perfect fluid which is minimally coupled to gravity. Such a fluid component can be mimicked by a scalar field whose action is given by

$$S_{\text{mat}} = \int d^4x \sqrt{-g} P(Y), \quad Y := -\frac{1}{2} (\partial\varphi)^2. \quad (48)$$

The matter energy-momentum tensor reads

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \quad (49)$$

where the energy density and the four-velocity are given respectively by

$$\rho = 2YP_{,Y} - P, \quad u_\mu = -\frac{\nabla_\mu \varphi}{\sqrt{2Y}}. \quad (50)$$

The equation of motion for φ is equivalent to the conservation law, $\nabla_\mu T^\mu_\nu = 0$. Here, ∇_μ stands for the usual four-dimensional covariant derivative. The sound speed of the fluid is given by

$$c_s^2 = \frac{P_{,Y}}{P_{,Y} + 2YP_{,YY}}, \quad (51)$$

where the right-hand side is evaluated at the homogeneous background. For $P \propto Y^{(1+w)/2w}$ with $w = \text{const}$, we have $c_s^2 = P/\rho = w$.

A. Homogeneous and isotropic background

The total action is now composed of the gravity sector (8) and the action for a cosmological fluid (48).

The ADM variables for a homogeneous and isotropic universe are given by

$$N = \bar{N}(t), \quad N_i = 0, \quad \gamma_{ij} = a^2(t)\delta_{ij}. \quad (52)$$

Since time reparametrization symmetry is spontaneously broken, one cannot put $\bar{N} = 1$ in general, but, as we will show below, the lapse function is determined from the equations of motion. From the Hamiltonian constraint and the evolution equations, we obtain

$$\frac{3H^2}{(\beta/\bar{N} + 1)^2} + \frac{\alpha_1}{2} = \rho, \quad (53)$$

$$-\frac{3H^2}{\beta/\bar{N} + 1} - \frac{2}{\bar{N}} \frac{d}{dt} \left(\frac{H}{\beta/\bar{N} + 1} \right) - \frac{1}{2} \left(\alpha_1 + \frac{\alpha_3}{\bar{N}} \right) = P, \quad (54)$$

where $H := \bar{N}^{-1} d \ln a / dt$ is the Hubble parameter. (As far as the spatially flat model is concerned, the coefficients of R , i.e., α_2 and α_4 , do not appear anyway in the equations for a homogeneous universe.) The conservation law reads

$$\bar{N}^{-1} \dot{\rho} + 3H(\rho + P) = 0, \quad (55)$$

where a dot stands for differentiation with respect to t . Given the equation of state, this equation can be integrated to give ρ and P as a function of a : $\rho = \rho(a)$, $P = P(a)$. Note that Eq. (55) is not an automatic consequence of Eqs. (53) and (54). Rather, substitution of Eqs. (53) and (54) into Eq. (55) yields

$$\dot{\alpha}_1 - 3H\alpha_3 - \frac{6\beta H}{\beta/\bar{N} + 1} \left[\frac{3H^2}{\beta/\bar{N} + 1} + \frac{2}{\bar{N}} \frac{d}{dt} \left(\frac{H}{\beta/\bar{N} + 1} \right) \right] = 0. \quad (56)$$

Using Eqs. (53) and (54) again, one can write this equation as the constraint among the time-dependent functions,

$$\dot{\alpha}_1 - \sqrt{3} \left(\rho - \frac{\alpha_1}{2} \right)^{1/2} [\alpha_3 - (\alpha_1 + 2P)\beta] = 0. \quad (57)$$

In the covariant formulation presented in Appendix B, this follows from the equation of motion for a Stückelberg scalar field [see Eq. (B17)]. In the ADM formulation of usual scalar-tensor theories with two tensorial and one scalar d.o.f., the equation corresponding to Eq. (57) reduces to a first-order differential equation for the lapse function, which can be used to determine $\bar{N} = \bar{N}(t)$. In contrast, in the present case Eq. (57) does not contain \bar{N} , which is the crucial point in “scalarless” theories. Given that ρ and P are now expressed in terms of a , Eq. (57) instead allows us to write a in terms of t . Finally, substituting $a = a(t)$ into Eq. (53), one can determine the lapse function \bar{N} as a function of t .

Let us now present an example. Suppose that the universe is filled with a fluid with a constant equation of state parameter, $w = P/\rho$, and the time-dependent functions are given by

$$\beta = \text{const}, \quad \alpha_2 = 1, \quad \alpha_4 = 0, \quad (58)$$

$$\alpha_1 = 6h_0^2 \left[\frac{1}{\xi} - \frac{1}{(1+\beta)^2} \right] \coth^2 \left[\frac{3}{2}(1+w)h_0 t \right] - \frac{6h_0^2}{\xi}, \quad (59)$$

$$\alpha_3 = 6h_0^2 \left[\frac{1+w(1+\beta)}{(1+\beta)^2} - \frac{1+w}{\xi} \right] \text{csch}^2 \left[\frac{3}{2}(1+w)h_0 t \right] - \frac{6\beta h_0^2}{(1+\beta)^2}, \quad (60)$$

where ξ and h_0 are constant parameters. This example admits the following solution:

$$\bar{N} = 1, \quad a^{3(1+w)} \propto \sinh^2 \left[\frac{3}{2}(1+w)h_0 t \right]. \quad (61)$$

Noting that α_1 can be written as

$$\alpha_1 = 6 \left[\frac{1}{\xi} - \frac{1}{(1+\beta)^2} \right] H^2 - 6 \frac{h_0^2}{\xi}, \quad (62)$$

we see that Eqs. (53) and (54) read

$$3H^2 = \xi\rho + 3h_0^2, \quad (63)$$

$$-2\dot{H} = \xi(\rho + P). \quad (64)$$

This shows that the cosmological gravitational constant differs from Newton’s constant G_N by the factor ξ :

$$\frac{G_{\text{cos}}}{G_N} = \xi. \quad (65)$$

[Recall that we are working in units where $8\pi G_N = 1 (= 1/\alpha_2)$.] Aside from this modification, the background evolution obeys the standard Friedmann equations in the presence of a cosmological constant. A similar situation occurs in Einstein-aether theory and Hořava gravity, where a mild bound on the difference between G_{cos} and G_N has been obtained from the measurement of the primordial abundance of He^4 as $|\xi - 1| \lesssim 0.1$ [46,66]. In our example, ξ is a free parameter so that one may set $\xi = 1$, leading to the background evolution that is completely indistinguishable from that in GR.

In the above example, α_1 and α_3 track the cosmic evolution as $\alpha_1, \alpha_3 \sim H^2$. We present another example in which α_1 and α_3 instead remain as small as H_0^2 and simply behave as a dark energy component with a constant equation of state parameter w_{DE} . In the case where the matter component is a pressureless fluid, the model is given by

$$\beta = \text{const}, \quad \alpha_2 = 1, \quad \alpha_4 = 0, \quad (66)$$

$$\alpha_1 = -2\rho_{\text{DE}}(t), \quad (67)$$

$$\alpha_3 = 2[1 + w_{\text{DE}}(1 + \beta)]\rho_{\text{DE}}(t), \quad (68)$$

$$\rho_{\text{DE}} \propto A^{-3(1+w_{\text{DE}})}, \quad (69)$$

where A and t are related via

$$t = \frac{2\sqrt{1+r_0}}{3H_0} A^{3/2} \times {}_2F_1(1/2, -1/2w_{\text{DE}}, 1 - 1/2w_{\text{DE}}, -r_0 A^{-3w_{\text{DE}}}). \quad (70)$$

Here, ${}_2F_1$ is the hypergeometric function and $r_0 := \rho_{\text{DE}}/\rho|_{a=1}$ is the present value of the ratio between the dark energy density and the matter energy density. One can check that the solution is given by $\bar{N} = 1$ and $a = A(t)$. This example yields the background evolution subject to

$$3H^2 = (1 + \beta)^2(\rho + \rho_{\text{DE}}), \quad (71)$$

$$-2\dot{H} = (1 + \beta)^2[\rho + (1 + w_{\text{DE}})\rho_{\text{DE}}], \quad (72)$$

i.e., it mimics the cosmology with $G_{\text{cos}} = (1 + \beta)^2 G_N$ and the dark energy component.

B. Cosmological perturbations

We go on to the analysis of scalar perturbations around a cosmological background, with a particular focus on the

case of a pressureless fluid. The perturbed ADM variables are given by

$$N = \bar{N}(1 + \delta n), \quad N_i = \bar{N}\partial_i\chi, \quad \gamma_{ij} = a^2 e^{-2\psi}\delta_{ij}, \quad (73)$$

where we used the spatial gauge degrees of freedom to write γ_{ij} in the above form.

For the moment we work in the general action (3) to derive the quadratic action for the scalar perturbations, and later we will focus on the specific case of our interest. The action (3) can be expanded to quadratic order in perturbations as

$$\begin{aligned} S^{(2)} = \int dt d^3x \bar{N} a^3 & \left[-\frac{3\beta_0}{\beta_1} \left(\psi_n + \frac{H\delta n}{\beta_1} \right)^2 \right. \\ & - \frac{2\beta_0}{\beta_1} \left(\psi_n + \frac{H\delta n}{\beta_1} \right) \frac{\Delta\chi}{a^2} + \frac{\beta_0(\bar{\beta}_1 - \bar{\beta}_2)}{3\bar{\beta}_1\bar{\beta}_2} \left(\frac{\Delta\chi}{a^2} \right)^2 \\ & \left. - \left(\alpha_2 + \frac{\alpha_4}{\bar{N}} \right) \frac{\psi\Delta\psi}{a^2} + 2\alpha_2\delta n \frac{\Delta\psi}{a^2} \right], \quad (74) \end{aligned}$$

where the notations $\psi_n := \bar{N}^{-1}\partial_t\psi$ and $\bar{\beta}_{1,2} := \beta_{1,2}/\bar{N} + 1$ are used. The matter action (48) can also be expanded in terms of perturbations as

$$\begin{aligned} S_{\text{mat}}^{(2)} = \int dt d^3x \bar{N} a^3 & \left(\frac{\rho + P}{c_s^2} \right) \left[c_s^2 \frac{\delta\varphi}{\varphi_n} \frac{\Delta\chi}{a^2} + \frac{\delta n^2}{2} - \frac{\delta\varphi_n}{\varphi_n} \delta n \right. \\ & \left. - 3c_s^2 \frac{\delta\varphi_n}{\varphi_n} \psi + \frac{1}{2} \left(\frac{\delta\varphi_n}{\varphi_n} \right)^2 + \frac{c_s^2}{2} \frac{\delta\varphi\Delta\delta\varphi}{\varphi_n^2 a^2} \right], \quad (75) \end{aligned}$$

where $\delta\varphi$ is the perturbation of φ . Our total action is given by $S_{\text{tot}}^{(2)} = S^{(2)} + S_{\text{mat}}^{(2)}$.⁶ This is the complete expression for the quadratic action derived from the general action (3) in the presence of a perfect fluid (or a shift-symmetric k -essence field).

Now we focus on the case of our interest: $\beta_0 = \alpha_2 = 1$ and $\beta_2 = \alpha_4 = 0$ with the notation $\beta_1 = \beta$. For simplicity, we consider the background with $\bar{N} = 1$. We follow the analysis of density perturbations in the extended cuscuton theory [31] to derive the reduced action written solely in terms of a single variable δ defined by

$$\delta = \frac{\rho + P}{\rho c_s^2} \left(\frac{\delta\dot{\varphi}}{\dot{\varphi}} - \delta n \right) - \frac{3(\rho + P)}{\rho} \psi, \quad (76)$$

where note that the density perturbation is given by $\delta\rho = (\rho + P)/c_s^2 \cdot (\delta\dot{\varphi}/\dot{\varphi} - \delta n)$. Later at an appropriate point we will carefully take the limit $c_s^2, P \rightarrow 0$ while keeping ρ finite.

⁶We omit $\delta n\psi$ and ψ^2 from Eqs. (74) and (75) because upon combining $S^{(2)}$ and $S_{\text{mat}}^{(2)}$ they can be eliminated in the end by the use of the background equations.

We start with introducing δ as an auxiliary field and write $S_{\text{tot}}^{(2)}$ equivalently as

$$\begin{aligned} S_{\text{tot}}^{(2)} = S^{(2)} + S_{\text{mat}}^{(2)} - \int dt d^3x a^3 & \left(\frac{\rho + P}{2c_s^2} \right) \\ & \times \left[\frac{\delta\dot{\varphi}}{\dot{\varphi}} - \delta n - c_s^2 \left(\frac{\rho\delta}{\rho + P} + 3\psi \right) \right]^2. \quad (77) \end{aligned}$$

The additional third term is introduced so that it removes $\delta\dot{\varphi}^2$ in $S_{\text{tot}}^{(2)}$. Namely, one ends up with the action that depends linearly on $\delta\dot{\varphi}$. The equation of motion for δ yields Eq. (76). Substituting this into Eq. (77), the third term vanishes and the action reduces back to the original one, $S^{(2)} + S_{\text{mat}}^{(2)}$. Thus, the two representations are indeed equivalent.

Since $S_{\text{tot}}^{(2)}$ is now linear in $\delta\dot{\varphi}$, one can use the equation of motion to remove $\delta\varphi$ from the action. At this stage one may substitute $P \propto Y^{(1+w)/2w}$ and express the action in terms of ρ and w , which allows for a $w \rightarrow 0$ limit without any divergences. Then, δn and χ can also be removed from the action by the use of their equations of motion. The resulting action is of the form

$$\begin{aligned} S_{\text{tot}}^{(2)} = \int dt d^3x a^3 & [\dot{\delta}\mathcal{A}(t, \Delta)\dot{\delta} + \delta\mathcal{O}_1(t, \Delta)\delta \\ & + 2\psi\mathcal{O}_2(t, \Delta)\delta + \psi\mathcal{O}_3(t, \Delta)\psi], \quad (78) \end{aligned}$$

where

$$\mathcal{A} = \frac{\rho}{3\rho - 2\Delta/a^2}. \quad (79)$$

The explicit expressions for \mathcal{O}_1 , \mathcal{O}_2 , and \mathcal{O}_3 are messy and not illuminating. Finally, one can eliminate ψ by using its equation of motion and arrive at the reduced action expressed in terms of δ alone:

$$S_{\text{tot}}^{(2)} = \int dt d^3x a^3 [\dot{\delta}\mathcal{A}(t, \Delta)\dot{\delta} + \delta\mathcal{B}(t, \Delta)\delta]. \quad (80)$$

On the basis of this action we move to study the behavior of the perturbations in the short- and long-wavelength limits. In both limits, we will see that the effect of modification of gravity comes into play only through the underlying background model.

1. Short-wavelength limit

In the short-wavelength limit, $\Delta/a^2 \gg \rho$, we have

$$\mathcal{A} \simeq -\frac{1}{2}a^2\rho\Delta^{-1}, \quad \mathcal{B} \simeq -\frac{1}{4}a^2\rho^2\Delta^{-1}. \quad (81)$$

The time-dependent function in the action β has dropped out. Since $\rho \propto a^{-3}$, the equation of motion for δ is found to be

$$\ddot{\delta} + 2H\dot{\delta} = \frac{1}{2}\rho\delta. \quad (82)$$

Recalling that $4\pi G_N = 1/(2\alpha_2) = 1/2$, this is identical to the evolution equation for matter density perturbations in GR. Note, however, that the evolution of δ depends on the underlying background model, which could be different from that of the Λ CDM model. For example, if $G_{\text{cos}}/G_N \neq 1$ as in the example in the previous subsection, the actual evolution of the density perturbations would be modified.

The equations of motion for δn , χ , and ψ allow us to write these variables in terms of δ . In the short-wavelength limit, we obtain

$$\delta n \simeq \frac{1}{2}a^2\rho\Delta^{-1}\delta, \quad (83)$$

$$\chi \simeq -\frac{3a^4\rho}{2(1+\beta)}[H\Delta^{-2}\delta + (1+\beta)\Delta^{-2}\dot{\delta}], \quad (84)$$

$$\psi \simeq \frac{1}{2}a^2\rho\Delta^{-1}\delta. \quad (85)$$

Therefore, the metric potentials that are used in the familiar Newtonian gauge analysis,

$$\Phi := \delta n + \dot{\chi}, \quad (86)$$

$$\Psi := \psi - H\chi, \quad (87)$$

obey the standard relation,

$$\Delta\Phi = \Delta\Psi = \frac{a^2}{2}\rho\delta, \quad (88)$$

with $\delta \simeq \delta\rho/\rho$. If one ignores the cosmic expansion, then this reproduces the result obtained in Sec. III. Note that the subleading terms do depend on β , as seen from Eq. (84), and hence small nonstandard effects could in principle appear.

2. Long-wavelength limit

In the long-wavelength limit, $\Delta/a^2 \ll \rho$, we have

$$\mathcal{A} \simeq \frac{1}{3}, \quad \mathcal{B} \simeq -\frac{1}{9}\frac{\Delta}{a^2}. \quad (89)$$

Again, β dependence has dropped out. The long-wavelength solution to the equation of motion for δ is given by

$$\delta \simeq \delta_0(\vec{x}) - \frac{\Delta\delta_0(\vec{x})}{3} \int^t \frac{dt'}{a^3(t')} \int^{t'} dt'' a(t''), \quad (90)$$

where δ_0 is independent of time and the decaying mode has been discarded. From the equations of motion for the other variables in the long-wavelength limit, we obtain

$$\delta n = \mathcal{O}(\Delta\delta/a^2H^2), \quad (91)$$

$$\chi \simeq a^2\Delta^{-1}\dot{\delta}, \quad (92)$$

$$\psi \simeq -\frac{\delta}{3}, \quad (93)$$

leading to

$$\Phi = \Psi = -\frac{\delta_0(\vec{x})}{3} \left[1 - \frac{H}{a} \int^t a(t') dt' \right]. \quad (94)$$

The solution expressed in this way is the same as the long-wavelength solution for Φ and Ψ in the presence of a pressureless fluid and a cosmological constant in GR. In particular, in the matter-dominated era, $a \propto t^{2/3}$, we have the time-independent metric potentials, $\Phi = \Psi = -\delta_0/5$. Note, however, that Eq. (94) has been derived without assuming any particular form of $a(t)$ for an accelerated phase of the cosmic expansion. In our modified theory of gravity, the background evolution (of the accelerated phase) could be different from that of the Λ CDM model depending on the time-dependent functions in the action, so that the actual time dependence of Φ and Ψ could be different from that in the Λ CDM model away from the matter-dominated era.

VI. WHEN MATTER MATTERS

Finally, we discuss the problem of the coupling to generalized matter fields. Let us consider the cubic Galileon [67] whose action is given by

$$S_{\text{gal}} = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\varphi)^2 - \frac{c}{2}(\partial\varphi)^2 \square\varphi \right], \quad (95)$$

where c is a nonzero constant. In the following we will show that the extra dynamical scalar d.o.f. would reappear when one adds to the action (3) the Galileon as a matter field. This fact can be demonstrated by studying a quadratic action for scalar perturbations around a cosmological background.

Expanding the action (95) to second order in perturbations, we obtain

$$\begin{aligned}
S_{\text{gal}}^{(2)} = & \int dt d^3x \bar{N} a^3 \left[(\varphi_n - 3cH\varphi_n^2) \delta\varphi \frac{\Delta\chi}{a^2} + c\varphi_n^2 \delta\varphi_n \frac{\Delta\chi}{a^2} - c\varphi_n^3 \delta n \frac{\Delta\chi}{a^2} - c\varphi_n^2 \delta n \frac{\Delta\delta\varphi}{a^2} + \frac{\varphi_n^2}{2} (1 - 12cH\varphi_n) \delta n^2 \right. \\
& - \varphi_n (1 - 9cH\varphi_n) \delta n \delta\varphi_n - 3c\varphi_n^3 \delta n \psi_n - 3(\varphi_n - 3cH\varphi_n^2) \psi \delta\varphi_n + 3c\varphi_n^2 \psi_n \delta\varphi_n + \frac{1}{2} (1 - 6cH\varphi_n) \delta\varphi_n^2 \\
& \left. + \frac{1}{2a^2} (1 - 4cH\varphi_n - 2c\varphi_{nn}) \delta\varphi \Delta\delta\varphi \right], \tag{96}
\end{aligned}$$

where $\delta\varphi$ is the fluctuation of the Galileon field. We add this to the action (74): $S_{\text{tot}}^{(2)} = S^{(2)} + S_{\text{gal}}^{(2)}$. To highlight what causes the problem, we do not make any further simplification.

Using the equations of motion for δn and χ , one can eliminate them from the action. The resultant quadratic action is of the form

$$S_{\text{tot}}^{(2)} = \int dt d^3x \bar{N} a^3 (\mathcal{K}_{IJ} q_n^I q_n^J + \dots), \tag{97}$$

where $q^I = \{\psi, \delta\varphi\}$ and we wrote only the terms quadratic in time derivatives. A straightforward calculation yields

$$\det \mathcal{K} = - \frac{27[c\beta_0(\beta_1/\bar{N})H\varphi_n^2]^2}{\Xi}, \tag{98}$$

with

$$\begin{aligned}
\Xi := & 12\beta_0^2 H^2 - 2\beta_0 \bar{\beta}_1^2 (\bar{\beta}_1 - \bar{\beta}_2) \varphi_n^2 \\
& + 12c\beta_0 \bar{\beta}_1 [2\bar{\beta}_1 (\bar{\beta}_1 - \bar{\beta}_2) + \bar{\beta}_2] H\varphi_n^3 \\
& + 3c^2 \bar{\beta}_1^3 \bar{\beta}_2 \varphi_n^6. \tag{99}
\end{aligned}$$

This result shows that there are *two* scalar d.o.f. in the system unless $c\beta_1 = 0$; the scalar d.o.f. reappears in the presence of a Galileon field if it is minimally coupled to gravity in the frame where $\beta_1 \neq 0$.⁷ We note in passing that the revived scalar mode itself is not necessarily problematic if $\det \mathcal{K} > 0$.

What we have seen here is analogous to the issue pointed out in the context of degenerate higher-order scalar-tensor theories [68,69]. In the case of the degenerate higher-order scalar-tensor theories, the constraint associated with the degeneracy is lost if a matter field is coupled to the Christoffel symbol. In the present case, the constraint that removes the scalar d.o.f. in the gravity sector is lost due to the coupling of the cubic Galileon to the Christoffel symbol.

⁷Note that β_1 can be removed by a disformal transformation. This implies that if φ is nonminimally coupled to gravity in a particular way, then it is possible that $\beta_1 = 0$ in the frame where φ is minimally coupled to gravity.

VII. CONCLUSIONS AND OUTLOOK

In this paper, we have studied aspects of spatially covariant theories of gravity with two tensorial d.o.f., namely, modified gravity having the same number of d.o.f. as GR. We have mainly focused on a subset of the general theories developed in Ref. [19] that is characterized by three time-dependent functions. Two of the three functions are related to the cusciton terms [9] expressed in the unitary gauge, while the remaining one (denoted as β) is not included even in the extended cusciton theory [12]. More specifically, we have worked with the action (8) with $\alpha_2 = 1$ and $\alpha_4 = 0$ as a particularly interesting subclass of the general action (3), with the purpose of exploring how the theory can be distinguished observationally or phenomenologically from GR.

First, we have seen that the theory of modified gravity under consideration can evade Solar System tests as far as the parametrized post-Newtonian parameter γ is concerned. We have also seen that the speed of gravitational waves is equal to the speed of light. Therefore, there is no obvious contradiction with observations and experiments at this point.

Next, we have studied black hole solutions ignoring the cusciton terms that are supposed to be relevant only on cosmological scales. Since the only modification in the action appears in the coefficient of K^2 , where K is the trace of the extrinsic curvature of constant time hypersurfaces, the theory admits GR solutions foliated by maximal slices ($K = 0$). Therefore, we have the Schwarzschild solution. The situation here is essentially the same as that in Einstein-aether theory and the infrared limit of Hořava gravity. We have considered small deformations of the Schwarzschild solution, showing that no other asymptotically flat solutions can be obtained by a perturbative treatment. Our numerical analysis beyond the perturbative treatment supports the conclusion that the Schwarzschild solution is the only static, spherically symmetric, and asymptotically flat vacuum solution at rest with respect to the preferred frame. We have also considered a slowly rotating black hole and obtained the Kerr solution in the slow-rotation approximation.

We have then investigated the cosmological dynamics of the homogeneous and isotropic background and scalar perturbations. Since we have a large degree of freedom to choose the time-dependent functions in the action including the cusciton part, it is easy to realize the background

evolution that is very close to or even identical to that of the Λ CDM model in GR. As far as the long- and short-wavelength limits are concerned, we have found that the perturbation dynamics is modified only through the modification of the background evolution.

Combining these results, we conclude that it is quite difficult to distinguish modified gravity with the action (8) with $\alpha_2 = 1$ and $\alpha_4 = 0$ observationally or phenomenologically from GR, albeit a certain amount of modification is expected to arise on horizon scales.

Finally, we have noted, as a side remark, that in the presence of a Galileon field in the matter sector, the dynamical scalar degree of freedom eliminated from the gravitational sector could reappear in a similar way to the case analyzed in Refs. [68,69]. The result of our analysis implies that a Galileon field must be coupled nonminimally to gravity in the frame where the action takes the form of Eq. (8).

Let us comment on several issues that are left for future work.

- (1) All the parametrized post-Newtonian parameters must be evaluated to ensure that the present theory is viable. In particular, the preferred frame effect is interesting to explore. A study of a moving black hole [70] would also be necessary.
- (2) Newtonian gravity is reproduced in the weak field regime, and stationary black hole solutions in GR are obtained in the strong field regime *in vacuo*. It is then natural to move on to the study of the structure of relativistic stars.
- (3) It would be interesting to study black hole perturbations and quasinormal modes, which could help us to distinguish modified gravity with two tensorial d.o.f. from GR.
- (4) In the present paper, we have studied cosmological perturbations only in the long- and short-wavelength limits. On intermediate scales the perturbation evolution could differ from that in GR even if the background evolution coincides with that of the Λ CDM model. A more detailed investigation is necessary.
- (5) An application to inflationary cosmology would also be interesting.

These issues will be addressed in our forthcoming papers.

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APPENDIX A: EQUATIONS OF MOTION

The equations of motion in vacuum derived from the action (8) are given as follows.

- (1) The Hamiltonian constraint:

$$K_{ij}K^{ij} - \frac{1}{3} \left[2 \left(\frac{N}{\beta + N} \right)^2 + 1 \right] K^2 - \alpha_1 - \alpha_2 R = 0. \quad (\text{A1})$$

- (2) The momentum constraints:

$$D_j \pi^{ij} = 0, \quad (\text{A2})$$

where

$$\pi^{ij} := K^{ij} - \frac{1}{3} \left(\frac{2N}{\beta + N} + 1 \right) K \gamma^{ij}. \quad (\text{A3})$$

- (3) The evolution equations:

$$\begin{aligned} & \frac{1}{N\sqrt{\gamma}} \partial_t (\sqrt{\gamma} \pi^{kl}) \gamma_{ik} \gamma_{jl} - \frac{1}{2} \left(\alpha_1 + \frac{\alpha_3}{N} \right) \gamma_{ij} \\ & + 2 \left[K_{ik} K_j^k - \frac{1}{3} \left(\frac{2N}{\beta + N} + 1 \right) K K_{ij} \right] \\ & - \frac{1}{2} \left[K_{kl} K^{kl} - \frac{1}{3} \left(\frac{2N}{\beta + N} + 1 \right) K^2 \right] \gamma_{ij} \\ & + \left(\alpha_2 + \frac{\alpha_4}{N} \right) \left(R_{ij} - \frac{1}{2} R \gamma_{ij} \right) \\ & + \frac{\alpha_2}{N} (D^2 N \gamma_{ij} - D_i D_j N) + \frac{1}{N} [D^k (\pi_{ik} N_j) \\ & + D^k (\pi_{jk} N_i) - D^k (\pi_{ij} N_k)] = 0. \end{aligned} \quad (\text{A4})$$

APPENDIX B: COVARIANT FORM OF THE ACTION

In the main text, we use the ADM decomposition and consider the action of the form (3) which no longer has full diffeomorphism invariance. However, one can always restore it by introducing a Stückelberg field $\phi(x^\mu)$. In this appendix, we derive the covariant form of the action written in terms of the four-dimensional metric $g_{\mu\nu}$ and ϕ .

The Stückelberg scalar field is introduced by writing the unit normal to constant time hypersurfaces as $n_\mu = -\phi_\mu / \sqrt{2X}$, where $\phi_\mu := \nabla_\mu \phi$ and $X := -\phi^\mu \phi_\mu / 2$ with ∇_μ being the usual four-dimensional covariant derivative. The ADM variables in the action (3) are then replaced as follows:

$$N \rightarrow \frac{1}{\sqrt{2X}}, \quad (\text{B1})$$

$$\begin{aligned} K_{ij} \rightarrow \mathcal{K}_{\mu\nu} := & -\frac{\phi_{\mu\nu}}{\sqrt{2X}} + n_\mu a_\nu + n_\nu a_\mu \\ & + \frac{1}{2X} n_\mu n_\nu n^\rho \nabla_\rho X, \end{aligned} \quad (\text{B2})$$

$$R \rightarrow {}^{(4)}\mathcal{R} - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + \mathcal{K}^2 - 2\nabla_\mu(\mathcal{K}n^\mu - a^\mu), \quad (\text{B3})$$

where $\phi_{,\mu} := \nabla_\mu \nabla_\nu \phi$, $a_\mu := n^\rho \nabla_\rho n_\mu$, and ${}^{(4)}\mathcal{R}$ is the four-dimensional Ricci scalar. One thus obtains

$$S = \int d^4x \sqrt{-g} \mathcal{L}, \quad (\text{B4})$$

$$\begin{aligned} \mathcal{L} = & \left[\frac{1}{2} \tilde{B}(\phi, X) - \tilde{f}(\phi, X) \right] (\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - \mathcal{K}^2) \\ & + \tilde{C}(\phi, X)\mathcal{K}^2 + \frac{1}{2} [\tilde{\alpha}_1(\phi) + \tilde{\alpha}_3(\phi)\sqrt{2X}] \\ & + \tilde{f}(\phi, X)[{}^{(4)}\mathcal{R} - 2\nabla_\mu(\mathcal{K}n^\mu - a^\mu)], \end{aligned} \quad (\text{B5})$$

where

$$\tilde{f} = \frac{1}{2} [\tilde{\alpha}_2(\phi) + \tilde{\alpha}_4(\phi)\sqrt{2X}], \quad (\text{B6})$$

$$\tilde{B} = \frac{\tilde{\beta}_0(\phi)}{\tilde{\beta}_2(\phi)\sqrt{2X} + 1}, \quad (\text{B7})$$

$$\tilde{C} = \frac{\tilde{\beta}_0(\phi)}{3} \cdot \frac{[\tilde{\beta}_1(\phi) - \tilde{\beta}_2(\phi)]\sqrt{2X}}{[\tilde{\beta}_1(\phi)\sqrt{2X} + 1][\tilde{\beta}_2(\phi)\sqrt{2X} + 1]}. \quad (\text{B8})$$

The time-dependent functions in the action (3) are given by⁸

$$\begin{aligned} \beta_0(t) &= \tilde{\beta}_0(\phi(t)), & \beta_{1,2}(t) &= \dot{\phi}(t)\tilde{\beta}_{1,2}(\phi(t)), \\ \alpha_{1,2}(t) &= \tilde{\alpha}_{1,2}(\phi(t)), & \alpha_{3,4}(t) &= \dot{\phi}(t)\tilde{\alpha}_{3,4}(\phi(t)). \end{aligned} \quad (\text{B13})$$

With some manipulation, the Lagrangian can be written in a more suggestive form as

⁸The relation (B13) can be used to write the ϕ -dependent functions explicitly, for example, in the first concrete model presented in Sec. VA (with $w = 0$) as

$$\tilde{\alpha}_1 = 6 \left\{ \left[\frac{1}{\xi} - \frac{1}{(1+\beta)^2} \right] \phi^2 - \frac{h_0^2}{\xi} \right\}, \quad (\text{B9})$$

$$\tilde{\alpha}_3 = \frac{4}{\phi^2 - h_0^2} \left\{ \left[\frac{1}{\xi} - \frac{1}{(1+\beta)^2} \right] \phi^2 - \left(\frac{1}{\xi} - \frac{1}{1+\beta} \right) h_0^2 \right\}, \quad (\text{B10})$$

$$\tilde{\beta}_1 = -\frac{2\beta}{3(\phi^2 - h_0^2)}, \quad (\text{B11})$$

$$\tilde{\alpha}_2 = \tilde{\beta}_0 = 1, \quad \tilde{\alpha}_4 = \tilde{\beta}_2 = 0, \quad (\text{B12})$$

with the solution for the scalar field $\phi(t) = h_0 \coth[(3/2)h_0 t]$.

$$\begin{aligned} \mathcal{L} = & \frac{\tilde{\alpha}_1}{2} - \tilde{\alpha}_2''(2X - X \ln X) + \frac{\tilde{\alpha}_3}{2} \sqrt{2X} + \tilde{\alpha}_4''(2X)^{3/2} \\ & - \left(\frac{\tilde{\alpha}_2'}{2} \ln X + 2\tilde{\alpha}_4' \sqrt{2X} \right) \square\phi + \tilde{f}{}^{(4)}\mathcal{R} + \mathcal{L}_{\text{quad}}, \end{aligned} \quad (\text{B14})$$

where a prime here denotes differentiation with respect to ϕ and

$$\begin{aligned} \mathcal{L}_{\text{quad}} = & A_1 \phi_{,\mu\nu} \phi^{\mu\nu} + A_2 (\square\phi)^2 + A_3 \square\phi \phi^{\mu\nu} \phi_{,\mu\nu} \\ & + A_4 \phi^{\mu\nu} \phi_{,\mu\rho} \phi^{\rho\nu} \phi_{,\nu} + A_5 (\phi^{\mu\nu} \phi_{,\mu\nu})^2, \end{aligned} \quad (\text{B15})$$

with

$$\begin{aligned} A_1 &= \frac{\tilde{B}}{4X} - \frac{\tilde{f}}{2X}, & A_2 &= -A_1 + 4X^2 A_5, \\ A_3 &= -A_4 + 4XA_5, & A_4 &= \frac{\tilde{f}_{,X} + A_1}{X}, \\ A_5 &= \frac{\tilde{C}}{8X^3}. \end{aligned} \quad (\text{B16})$$

This is a particular case of the U-degenerate theories, i.e., higher-order scalar-tensor theories that are degenerate when restricted to the unitary gauge [29]. It is easy to check that the above Lagrangian does not satisfy the degeneracy conditions in an arbitrary gauge if $\tilde{C} \neq 0$. In general, there is one dynamical scalar d.o.f. in a U-degenerate theory in the unitary gauge. However, with the above particular form of the Lagrangian, it turns out that ϕ does not propagate.

To demonstrate that ϕ does not propagate, let us study the equation of motion for ϕ in a Friedmann-Lemaître-Robertson-Walker universe with the metric $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$. For simplicity, we consider the case with $\alpha_2 = \beta_0 = 1$ and $\alpha_4 = \beta_2 = 0$. The equation of motion for ϕ then reads

$$\tilde{\alpha}_1' \dot{\phi} - 3H\tilde{\alpha}_3 \dot{\phi} - \frac{6\tilde{\beta}_1 \dot{\phi} H}{\tilde{\beta}_1 \dot{\phi} + 1} \mathcal{P} = 0, \quad (\text{B17})$$

where

$$\mathcal{P} := 2 \frac{d}{dt} \left(\frac{H}{\tilde{\beta}_1 \dot{\phi} + 1} \right) + \frac{3H^2}{\tilde{\beta}_1 \dot{\phi} + 1}. \quad (\text{B18})$$

This equation does contain $\ddot{\phi}$ as well as \dot{H} . However, the (ij) components of the field equations have the form,

$$\mathcal{P} + \frac{1}{2} (\tilde{\alpha}_1 + \tilde{\alpha}_3 \dot{\phi}) + \dots = 0, \quad (\text{B19})$$

where the ellipsis denotes the pressure of other matter fields. The evolution equations for the metric and ϕ are thus degenerate as in extended cusciton theories [12], implying that the scalar d.o.f. is in fact not dynamical.

Let us turn to discuss black hole solutions in the case where

$$\begin{aligned} \tilde{\beta}_1 &= \text{const}, & \tilde{\beta}_2 &= 0, & \tilde{\alpha}_1 &= \alpha_1 = \text{const}, \\ \tilde{\alpha}_2 &= \alpha_2 = \text{const}, & \tilde{\alpha}_3 &= \tilde{\alpha}_4 = 0. \end{aligned} \quad (\text{B20})$$

Concerning black hole solutions, working in the covariant action results in much more involved equations. However, it is not difficult to see that the field equations admit the following configuration of the metric and ϕ :

$$ds^2 = -h(r)dT^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2, \quad (\text{B21})$$

$$\phi(T, r) = \frac{1}{N_0} \left(T + \int^r \frac{b_0/r^2}{h\sqrt{h + b_0^2/r^4}} dr \right), \quad (\text{B22})$$

where

$$h(r) := 1 + \frac{\alpha_1}{6\alpha_2} r^2 - \frac{\mu_0}{r} + \frac{1 - \alpha_2}{\alpha_2} \frac{b_0^2}{r^4}. \quad (\text{B23})$$

Note that ϕ is allowed to have the linear T dependence thanks to the shift symmetry $\phi \rightarrow \phi + \text{const}$. This is a so-called stealth solution (when $\alpha_2 = 1$), but, in contrast to the familiar examples in the literature, we have $X \neq \text{const}$ in the present case. (See Ref. [71] for a comprehensive discussion on stealth solutions in quadratic degenerate higher-order scalar-tensor theories.) Moving to the unitary gauge, $\phi(T, r) = t$, this solution reproduces the one obtained in Sec. IV A.

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