Joule-Thomson expansion of lower-dimensional black holes

Jing Liang,[1,2,*] Benrong Mu,[1,2,†] and Peng Wang[2,‡]
[1] *Physics Teaching and Research section, College of Medical Technology, Chengdu University of Traditional Chinese Medicine, Chengdu 611137, People's Republic of China*
[2] *Center for Theoretical Physics, College of Physics, Sichuan University, Chengdu 610064, People's Republic of China*

(Received 26 May 2021; accepted 24 October 2021; published 1 December 2021)

Herein, we extend Joule-Thomson expansion to the low-dimensional regime by considering the rotating Bañados-Teitelboim-Zanelli (BTZ) metric in the $(2+1)$-dimensional space-time. Specifically, the properties of three important aspects of the Joule-Thomson expansion, including the Joule-Thomson coefficient, inversion curve, and isenthalpic curve were studied. The divergence point of the Joule-Thomson coefficient and the zero point of the Hawking temperature were investigated. The inversion temperature and isenthalpic curves in the T–P plane were obtained, and the cooling-heating regions were determined. Furthermore, the minimum inversion temperature was found to be zero, and the black hole becomes an extremal black hole. The ratio of the minimum inversion and critical temperatures for BTZ black holes does not exist, since the BTZ black hole does not exhibit the critical behavior in the critical pressure P_c, critical temperature T_c, and critical volume V_c.

DOI: 10.1103/PhysRevD.104.124003

I. INTRODUCTION

Since Hawking discovered that black holes can thermodynamically emit particles, black holes have attracted significant attention. Various thermodynamic properties of black holes have been extensively investigated [1–6]. Notably, black holes, as thermodynamic systems, have many similarities with universal thermodynamic systems. These similarities are more obvious and precise for black holes in anti–de Sitter(AdS) space. Hawking and Page first presented the thermodynamics of AdS black holes, in which a phase transition was found between a Schwarzschild-AdS black hole and a thermal AdS space [7]. Subsequently, Hawking radiation and phase transition in various black holes attracted more attention [8–15].

Recently, studies on the thermodynamics of black holes in AdS space have focused on extended phase space, where the cosmological constant and its conjugate quantities are considered as thermodynamic pressure and volume, respectively:

$$P = -\frac{\Lambda}{8\pi} = \frac{(D-1)(D-2)}{16\pi l^2}, \qquad V = \left(\frac{\partial M}{\partial P}\right)_{S,Q}, \tag{1}$$

where l is the AdS space radius and the black hole mass M is considered as the enthalpy [16,17]. Many studies have explored various thermodynamics aspects of black holes in an extended phase space, such as the phase transition [18–20], critical phenomenon [21–23], compressibility [24,25], heat engine efficiency [26–28], and weak cosmic censorship conjecture [29–46].

Hawking and Page studied the phase transition between Schwarzschild-AdS black holes and thermal AdS space [7]. They found that black holes in AdS space have similar properties as general thermodynamic systems. This relationship is further enhanced in the extended phase space [47]. Joule-Thomson expansion of black holes was first reported in Ref. [48], and it was subsequently extended to other kinds of black holes, such as d-dimensional charged AdS black holes [49], Kerr-AdS black holes [50], regular (Bardeen)-AdS black holes [51], Reissner-Nordström (RN-)AdS black holes in $f(R)$ gravity [52], quintessence RN-AdS black holes [53], Bardeen-AdS black holes [54], and others [55–74]. In these studies, the inversion curves, isenthalpic curves, and heating-cooling regions in the T–P plane for different black holes were given, and the inversion curves for different black hole systems were similar.

Until now, all studies have focused on space-times with dimension $D \geq 4$, whereas the case of $D < 4$ remains to be explored. The case of $D < 4$ is well worth studying for several reasons. The growing interest in the low-dimensional gravity theory in the past decade is attributed to the confluence of evidence suggesting an effective two-dimensional Planck regime. Recently, a black hole model based on the generalized uncertainty principle was developed, and it showed reduced dimensional properties

*jingliang@stu.scu.edu.cn
†Corresponding author.
benrongmu@cdutcm.edu.cn
‡pengw@scu.edu.cn

within the sub-Planckian mass limit [75]. This suggests that the physics of quantum black holes is low dimensional (a similarly emergent two-dimensional space-time was reported in Ref. [76]). Since then, the expansion of the thermodynamic phase space of low-dimensional black holes has attracted remarkable attention.

Three-dimensional Einstein gravity is topological, implying that all geometries sourced by the same matter are locally identical. Their difference lies in the topology. To be specific, herein, we address the (rotating) Bañados-Teitelboim-Zanelli (BTZ) black hole. This study clarifies the role of topology in Joule-Thomson expansion.

Perhaps the most interesting reason is from AdS/CFT [77], where $(d+1)$-dimensional bulk (AdS) gravity corresponds to d-dimensional boundary CFT. The (rotating) BTZ geometry studied herein precisely fits the framework and should have corresponding partners in dual CFT. Since high-dimensional CFTs are usually hard to attack, most of the progress in this subject was achieved in $d=2$. Therefore, our studies on Joule-Thomson expansion of (rotating) BTZ can pave the way for further investigation of dual quantity in CFT_2.

The (rotating) BTZ black hole is a solution of the Einstein field equation in the $(2+1)$-dimensional space-time with a negative cosmological constant [78,79]. In Ref. [80], the thermodynamic quantities of BTZ black holes were discussed. The first law of thermodynamics is satisfied, and the Smarr formula for the BTZ black hole was found. On this basis, other thermodynamic properties of this black hole were investigated [81–101]. However, to date, Joule-Thomson expansion of the rotating BTZ black hole in the extended phase space has not been reported. In this study, we investigated the Joule-Thomson expansion of rotating BTZ black holes.

This paper is organized as follows: The thermodynamics of rotating BTZ black holes is reviewed in Sec. II; the Joule-Thomson expansion of a rotating BTZ black hole, including the Joule-Thomson coefficient, inversion curves, and isenthalpic curves, is discussed in Sec. III; the results are discussed in Sec. IV.

II. ROTATING BTZ BLACK HOLES

The BTZ black hole is a solution of the Einstein field equation in $(2+1)$-dimensional space-time with a negative cosmological constant $\Lambda=-l^2$, and l is the AdS radius. The corresponding action with a negative term Λ is expressed as

$$S=\frac{1}{2\pi}\int\sqrt{-g}[R+2\Lambda]. \tag{2}$$

The rotating BTZ black hole solution to Eq. (2) is given by [78,79]

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2(d\varphi+N_\phi(r)dt)^2, \tag{3}$$

where

$$f(r)=-2m+\frac{r^2}{l^2}+\frac{J^2}{4r^2}, \tag{4}$$

$$N_\phi(r)=-\frac{J}{2r^2}. \tag{5}$$

When the black hole is nonextremal, the equation $f(r)=0$ has two positive real roots, r_+ and r_-, where the largest root r_+ denotes the radius of the event horizon. When the black hole is extremal, $f(r)=0$ has only one root, r_+. The metric parameter m is related to the black hole mass M and can be expressed in terms of r_+ as

$$M=\frac{m}{4}=\frac{J^2}{32r_+^2}+\frac{r_+^2}{8l^2}. \tag{6}$$

The mass of the black hole in terms of entropy S, angular momentum J, and pressure P is given by

$$M=\frac{\pi^2J^2}{128S^2}+\frac{4PS^2}{\pi}. \tag{7}$$

The corresponding thermodynamic quantities are then [80]

$$S=\frac{\pi r_+}{2},\qquad T=\frac{r_+}{2\pi l^2}-\frac{J^2}{8\pi r_+^3}, \tag{8}$$

$$P=\frac{1}{8\pi l^2},\qquad V=\pi r^2,\qquad \Omega=\frac{J}{16r_+^2}. \tag{9}$$

A black hole, as a stable thermodynamic system, can be discussed in two phase spaces. In the normal phase space, the cosmological constant is considered as a constant, and the state parameters of the black hole satisfy the first law of thermodynamics

$$dM=TdS+\Omega dJ. \tag{10}$$

However, compared to the usual first law of thermodynamics, the VdP term is missing in Eq. (10). Based on this, the cosmological constant is considered as the pressure of the black hole, and their relationship is expressed in Eq. (1). Therefore, in the extended phase space, the first law of thermodynamics is given by [102]

$$dM=TdS+VdP+\Omega dJ, \tag{11}$$

and the Smarr formula is

$$0=TS-2PV+\Omega J. \tag{12}$$

The mass of the black hole M is defined as its enthalpy

$$M = U + PV. \tag{13}$$

Moreover, combining equations of the thermodynamic pressure, entropy, and temperature, the equation of state for the black hole is obtained as

$$P = \frac{J^2 + 8\pi r_+^3 T}{32\pi r_+^4}. \tag{14}$$

The corresponding images are shown in Fig. 1. Unlike the van der Waals fluids and charged AdS black holes, there is no inflection point in the diagram of the rotating BTZ black hole. Therefore, the rotating BTZ black hole does not have P–V critical behavior in the critical pressure P_c, critical temperature T_c, and critical volume V_c. Thus, the rotating BTZ hole is always thermodynamically stable.

III. JOULE-THOMSON EXPANSION

In Joule-Thomson expansion, gas is passed at high pressure through a porous plug or small valve in the low-pressure section of an adiabatic tube, and the enthalpy remains constant during the expansion. The expansion is characterized by a change in temperature relative to pressure. The Joule-Thomson coefficient μ, which characterizes the expansion process, is given by

$$\mu = \left(\frac{\partial T}{\partial P}\right)_H. \tag{15}$$

From Eqs. (8), (9), (11), and (12), the heat capacity at constant pressure is

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_{P,J} = \frac{1}{2}\pi r_+\left(1 - \frac{4J^2}{3J^2 + 32\pi P r_+^4}\right), \tag{16}$$

and we can obtain

$$\mu = \left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right] = \frac{2r(5J^2 - 32\pi P r_+^4)}{J^2 - 32\pi P r_+^4}. \tag{17}$$

Figure 2 shows the Joule-Thomson coefficient and Hawking temperature versus the horizon. We fix the pressure $P = 1$ and the angular momentum J as 0.5, 1, and 2. Both divergence and zero points exist for different J. Comparing these two figures, it is found that the divergence point of the Joule-Thomson coefficient is consistent with the zero point of Hawking temperature. The divergence point reveals the Hawking temperature and corresponds to extremal black holes.

Next, we focus on the ratio of the minimum inversion temperature to the T_c. The black hole equation of state is given by

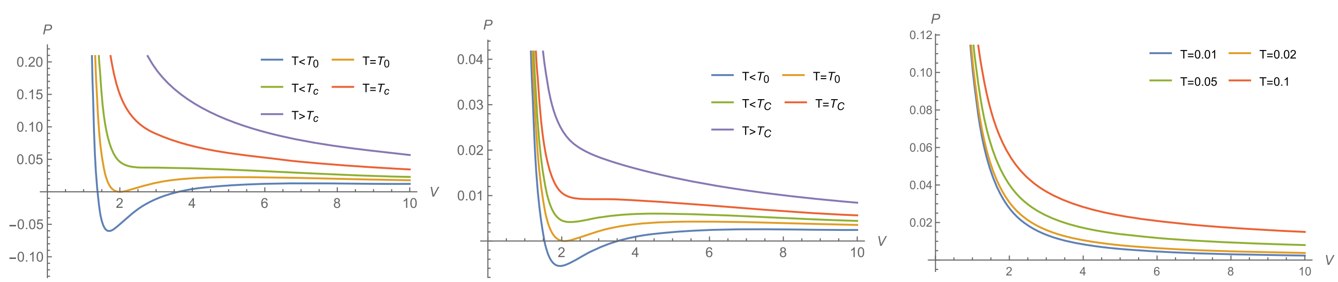
$$T = 4Pr_+ - \frac{J^2}{8\pi r_+^3}. \tag{18}$$

Then, the inversion temperature takes the form

$$T_i = V\left(\frac{\partial V}{\partial T}\right)_P = \frac{3J^2}{16\pi r_+^3} + 2Pr_+. \tag{19}$$

Subtracting Eq. (19) from Eq. (18) yields

$$\frac{5J^2}{16\pi r_+^3} - 2P_i r_+ = 0. \tag{20}$$

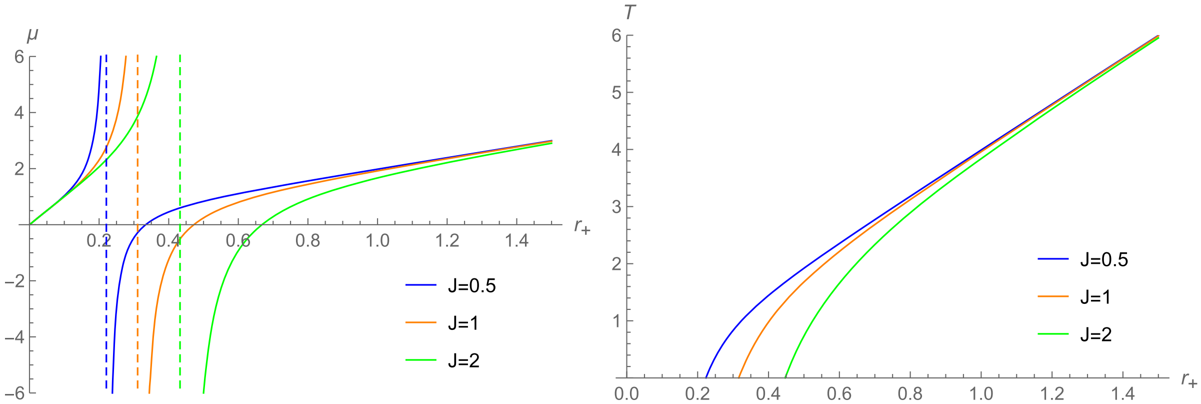


(a) $P - V$ diagram of van der Waals fluids. $T_c = \frac{8a}{27bk}$ and $T_0 = \frac{1}{4k}\frac{a}{b}$. We have set $a = b = 1$.

(b) $P - V$ diagram of charged AdS black holes. $T_c = \frac{\sqrt{6}}{18\pi Q}$ and $T_0 = \frac{\sqrt{3}}{18\pi Q}$. We have set $Q = 1$.

(c) $P - V$ diagram of rotating BTZ black holes. We have set $J = 1$.

FIG. 1. P–V diagram of van der Waals fluids, charged AdS black holes, and rotating BTZ black holes.



(a)Joule-Thomson coefficient versus event horizon.

(b)Hawking temperature T versus event horizon.

FIG. 2. Joule-Thomson coefficient and Hawking temperature T versus event horizon. Here, $P = 1$.

Solving this equation for r_+ yields four roots, of which only one is physically meaningful; others are complex or negative. The positive and real root is

$$r_+ = \left(\frac{5J^2}{32\pi P_i}\right)^{\frac{1}{4}}. \tag{21}$$

Substituting this root into Eq. (18) at $P = P_i$, the inversion temperature is given by

$$T_i = \frac{4(\frac{2}{5})^{3/4}\sqrt{J}P_i^{3/4}}{\sqrt{4\pi}}. \tag{22}$$

By setting $P_i = 0$ in the above equation, the minimum of the inversion temperature is given by

$$T_i^{\min} = 0, \tag{23}$$

which means the black hole becomes an extremal black hole. In different black holes, the expression for $T_i^{\min}$ differs. For example, in the van der Waals fluid, $T_i^{\min} = \frac{2a}{9bk}$; in the charged AdS black hole, $T_i^{\min} = \frac{1}{6\sqrt{6}\pi Q}$; and in the Kerr-AdS black hole, $T_i^{\min} = \frac{\sqrt{3}}{4(916+374\sqrt{6})^{\frac{1}{4}}\pi\sqrt{J}}$.

A rotating BTZ black hole does not have P–V critical behavior P_c, T_c, and V_c. Thus, the black hole is always thermodynamically stable, and there is no ratio between the minimum inversion temperature and T_c. This ratio differs from that of other black holes. To better investigate the Joule-Thomson expansion in a rotating BTZ black hole, the isoenthalpy and inversion curves of the black hole are depicted in Figs. 3 and 4.

The inversion temperature increases monotonically with increasing inversion pressure, but the slope of the inversion curve decreases with increasing inversion pressure (Fig. 3). In addition, in contrast to van der Waals fluids, there is only a lower inversion curve, which does not terminate at any point. This is similar to the results of charged AdS and Kerr-AdS black holes. Joule-Thomson expansion occurs in an isenthalpic process. For a black hole, the enthalpy is the mass M. The isenthalpic curve can be obtained using Eqs. (6) and (9). As shown in Fig. 4, the inversion curve divides the plane into two regions. The region above the inversion curve corresponds to the cooling region, whereas the region below the inversion curve corresponds to the heating region. The heating and cooling regions have been determined from the sign of the slope of the isoenthalpy curve. The slope is positive in the cooling region, whereas it varies in the heating region. On the other hand, cooling (heating) does not occur on the inversion curve, which acts as a boundary between the two regions.

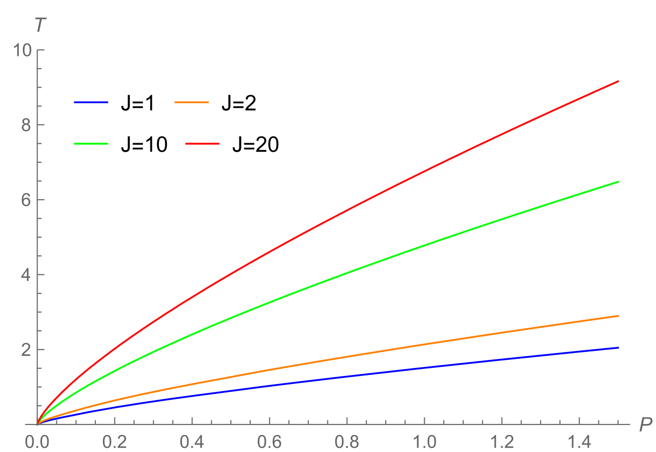
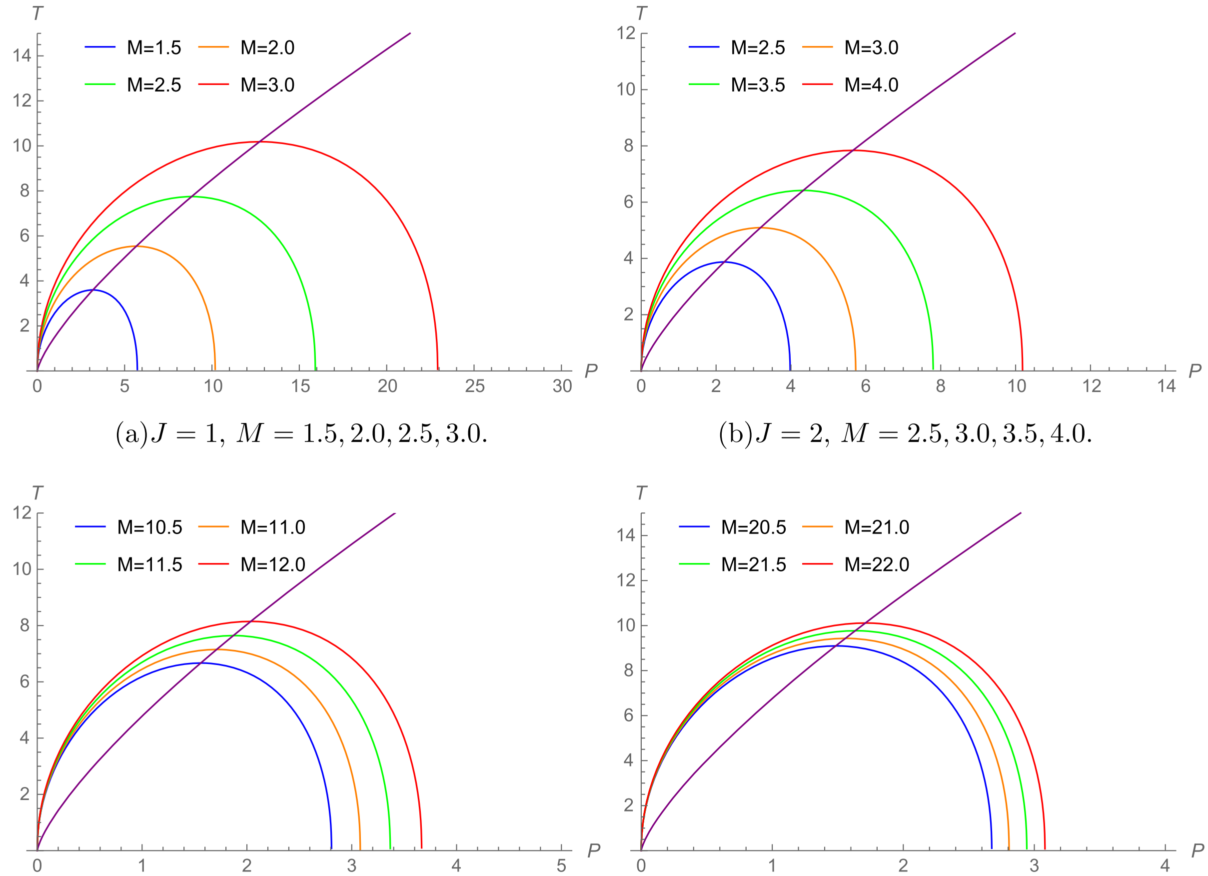


FIG. 3. Inversion curves for a BTZ black hole.



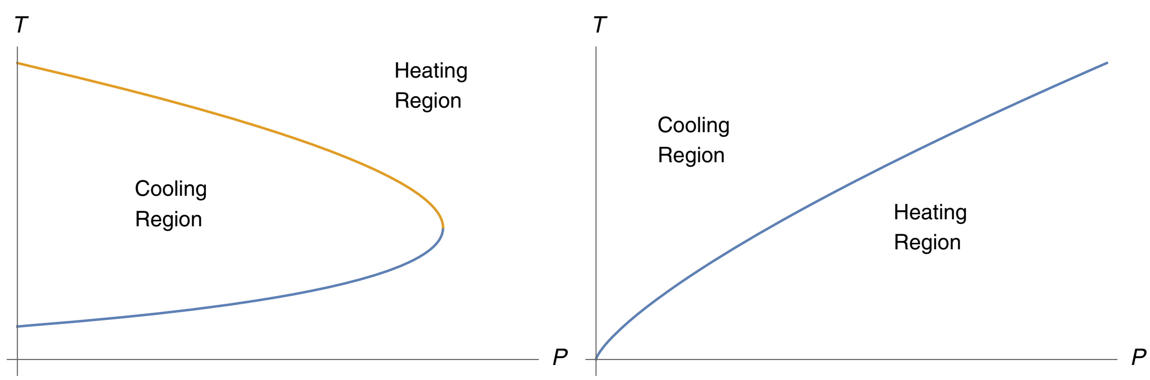
(a)$J = 1,\ M = 1.5, 2.0, 2.5, 3.0.$ (b)$J = 2,\ M = 2.5, 3.0, 3.5, 4.0.$

(c)$J = 10,\ M = 10.5, 11.0, 11.5, 12.0.$ (d)$J = 20,\ M = 20.5, 21.0, 21.5, 22.0.$

FIG. 4. Inversion and isenthalpic curves for a BTZ black hole. From bottom to top, the value of M corresponding to the isenthalpic curve increases. The purple lines are the inversion curves.

IV. CONCLUSION

The Joule-Thomson expansion of a rotating BTZ black hole in the extended phase space was investigated herein, where the cosmological constant is considered as pressure and the black hole mass is treated as enthalpy. First, we obtained the metric of the rotating BTZ black hole in three-dimensional space-time and derived the expressions for mass, entropy, Hawking temperature, pressure, volume,



(a)Inversion curves for van der Waals fluids. (b)Lower inversion curves for charged AdS, Kerr-AdS, and rotating BTZ black holes.

FIG. 5. Inversion curves for van der Waals fluids, charged AdS, Kerr-AdS, and rotating BTZ black holes.

TABLE I. Existence of critical behavior and the ratio of the minimum inversion temperature to the critical temperature in van der Waals and various black holes.

Type	Critical behavior	T_i^{min}	T_c	Ratio	Literature
Van der Waals fluid	Exist	Exist	Exist	0.75	[48]
RN-AdS BH	Exist	Exist	Exist	0.5	[53]
d-dimensional AdS BH	Exist	Exist	Exist	< 0.5	[49]
Gauss-Bonnet BH	Exist	Exist	Exist	0.4765	[58]
Toruslike BH	Not exist	Exist	Not exist	Not exist	[104]
Kerr-AdS BH	Exist	Exist	Exist	0.504622	[50]
Rotating BTZ BH	Not exist	Exist	Not exist	Not exist	

etc., of the black hole. The rotating BTZ black hole showed no P–V critical behavior P_c, T_c, and V_c, which is different from van der Waals fluids [103]. The minimum inversion temperature is zero, and the black hole becomes an extremal black hole. Moreover, there is no ratio between the minimum inversion temperature and T_c. Then, we investigated whether there is a Joule-Thomson expansion in a rotating BTZ black hole. We found that Joule-Thomson expansion is possible in rotating BTZ black holes. Joule-Thomson coefficient μ and Hawking temperature T versus the event horizon are shown in Fig. 2. The divergence point of the Joule-Thomson coefficient is consistent with the zero point of the Hawking temperature, which corresponds to an extremal black hole. Figure 3 shows the inversion curves for a black hole. Rotating BTZ black holes have only lower inversion curves. The isenthalpic and inversion curves are shown in Fig. 4. The regions above and below the inversion curve correspond to the cooling and heating regions, respectively. The inversion curve can be used to distinguish different values of the cooling and heating regions.

The BTZ black hole is very similar to the $(3+1)$-dimensional Kerr black hole in that there are two horizons, internal and external horizons, fully inscribed by the Arnowitt-Deser-Misner mass, angular momentum, and charge. However, it differs from the $(3+1)$-dimensional Kerr black hole in that it is not asymptotically flat but an asymptotically AdS black hole, and the origin is a coordinate singularity with no intrinsic singularity. In Ref. [50], the Joule-Thomson expansion was studied for Kerr-AdS black holes in the extended phase space. Kerr-AdS black holes have only lower reversal curves, and the minimum inversion temperature $T_i^{\mathrm{min}} = \frac{\sqrt{3}}{4(916+374\sqrt{6})^{\frac{1}{4}}\pi\sqrt{J}}$. Because of the presence of P–V critical behavior in Kerr-AdS black holes, the ratio of the minimum inversion temperature to T_c exists (0.504622). To compare charged AdS, Kerr-AdS, and rotating BTZ black holes with van der Waals fluids, the inversion curves of van der Waals fluids and charged AdS, Kerr AdS, and rotating BTZ black holes are shown in Fig. 5. In contrast to charged AdS, Kerr-AdS, and rotating BTZ black holes, the van der Waals fluid has upper and lower inversion curves. The cooling region is closed, and we consider both the minimum inversion temperature T_i^{min} and maximum version temperature T_i^{max} for this system. In charged AdS, Kerr-AdS, and rotating BTZ black holes, cooling always occurs above the inversion curve, whereas in van der Waals fluids, cooling occurs only in the region enclosed by the upper and lower inversion curves. Although charged AdS, Kerr-AdS, and rotating BTZ black holes have the same trend of inversion curves and the same cooling-heating regions, they do not all show P–V critical behavior. Charged AdS and Kerr-AdS black holes exhibit P–V critical behavior, whereas rotating BTZ black holes do not. Table I lists the critical behavior and ratio of the minimum inversion temperature to T_c for van der Waals fluids and various black holes. The presence or absence of Joule-Thomson expansion in a black hole is not related to the presence or absence of the ratio of the minimum inversion temperature to T_c. Furthermore, Joule-Thomson expansion does not exist in the charged BTZ black holes. This is attributed to the topology of the black holes. In future studies, we shall further discuss the deep relationship between topology and black hole thermodynamics.

Herein, we focused on the Joule-Thomson expansion of BTZ black holes in low-dimensional black holes. The results are related to many other interesting problems that deserve further investigation. For future studies, it is reasonable to study the thermodynamics of the extended phase space of low-dimensional black holes. Based on this, we shall further discuss the effect of dimensionality on the thermodynamic properties of black holes.

ACKNOWLEDGMENTS

We are grateful to Wei Hong, Haitang Yang, Jun Tao, Deyou Chen, and Xiaobo Guo for useful discussions. This work is supported in part by National Science Foundation of China (Grant No. 11747171), Natural Science Foundation of Chengdu University of TCM (Grants No. ZRYY1729 and No. ZRYY1921), Discipline Talent Promotion Program of /Xinglin Scholars (Grant No. QNXZ2018050), and the key fund project for Education Department of Sichuan (Grant No. 18ZA0173).

[1] J. D. Bekenstein, Black holes and the second law, Lett. Nuovo Cimento **4**, 737 (1972).
[2] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).
[3] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).
[4] J. D. Bekenstein, Generalized second law of thermodynamics in black hole physics, Phys. Rev. D **9**, 3292 (1974).
[5] S. W. Hawking, Black hole explosions, Nature (London) **248**, 30 (1974).
[6] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975); Erratum, Commun. Math. Phys. **46**, 206 (1976).
[7] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-De Sitter space, Commun. Math. Phys. **87**, 577 (1983).
[8] Q. Q. Jiang, S. Q. Wu, and X. Cai, Hawking radiation as tunneling from the Kerr and Kerr-Newman black holes, Phys. Rev. D **73**, 064003 (2006); Erratum, Phys. Rev. D **73**, 069902 (2006).
[9] J. Boos, V. P. Frolov, and A. Zelnikov, Ghost-free modification of the Polyakov action and Hawking radiation, Phys. Rev. D **100**, 104008 (2019).
[10] Q. Q. Jiang and S. Q. Wu, Hawking radiation of charged particles as tunneling from Reissner-Nordström-de Sitter black holes with a global monopole, Phys. Lett. B **635**, 151 (2006); Erratum, Phys. Lett. B **639**, 684 (2006).
[11] M. R. Setare, Gauge and gravitational anomalies and Hawking radiation of rotating BTZ black holes, Eur. Phys. J. C **49**, 865 (2007).
[12] Q. Q. Jiang, S. Q. Wu, and X. Cai, Hawking radiation from the $(2+1)$-dimensional BTZ black holes, Phys. Lett. B **651**, 58 (2007).
[13] Y. S. Myung, Phase transition between the BTZ black hole and AdS space, Phys. Lett. B **638**, 515 (2006).
[14] Q. Q. Jiang, S. Q. Wu, and X. Cai, Hawking radiation from the dilatonic black holes via anomalies, Phys. Rev. D **75**, 064029 (2007); Erratum, Phys. Rev. D **76**, 029904 (2007).
[15] R. Banerjee, S. K. Modak, and S. Samanta, Glassy phase transition and stability in black holes, Eur. Phys. J. C **70**, 317 (2010).
[16] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of AdS black holes, Classical Quant. Grav. **26**, 195011 (2009).
[17] D. Kubiznak, R. B. Mann, and M. Teo, Black hole chemistry: Thermodynamics with Lambda, Classical Quant. Grav. **34**, 063001 (2017).
[18] N. Altamirano, D. Kubizňák, R. B. Mann, and Z. Sherkatghanad, Kerr-AdS analogue of triple point and solid/liquid/gas phase transition, Classical Quant. Grav. **31**, 042001 (2014).
[19] S. W. Wei and Y. X. Liu, Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space, Phys. Rev. D **90**, 044057 (2014).
[20] S. H. Hendi, R. B. Mann, S. Panahiyan, and B. Eslam Panah, Van der Waals like behavior of topological AdS black holes in massive gravity, Phys. Rev. D **95**, 021501 (2017).
[21] S. W. Wei and Y. X. Liu, Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes, Phys. Rev. D **87**, 044014 (2013).
[22] R. Banerjee and D. Roychowdhury, Critical phenomena in Born-Infeld AdS black holes, Phys. Rev. D **85**, 044040 (2012).
[23] C. Niu, Y. Tian, and X. N. Wu, Critical phenomena and thermodynamic geometry of RN-AdS black holes, Phys. Rev. D **85**, 024017 (2012).
[24] B. P. Dolan, Compressibility of rotating black holes, Phys. Rev. D **84**, 127503 (2011).
[25] B. P. Dolan, The compressibility of rotating black holes in D-dimensions, Classical Quant. Grav. **31**, 035022 (2014).
[26] C. V. Johnson, Holographic heat engines, Classical Quant. Grav. **31**, 205002 (2014).
[27] S. H. Hendi, B. Eslam Panah, S. Panahiyan, H. Liu, and X. H. Meng, Black holes in massive gravity as heat engines, Phys. Lett. B **781**, 40 (2018).
[28] B. Eslam Panah, Effects of energy dependent spacetime on geometrical thermodynamics and heat engine of black holes: Gravity's rainbow, Phys. Lett. B **787**, 45 (2018).
[29] B. Gwak, Thermodynamics with pressure and volume under charged particle absorption, J. High Energy Phys. 11 (2017) 129.
[30] B. Gwak, Weak cosmic censorship conjecture in kerr-(Anti-)de Sitter black hole with scalar field, J. High Energy Phys. 09 (2018) 081.
[31] S. J. Yang, J. J. Wan, J. Chen, J. Yang, and Y. Q. Wang, Weak cosmic censorship conjecture for the novel $4D$ charged Einstein-Gauss-Bonnet black hole with test scalar field and particle, Eur. Phys. J. C **80**, 937 (2020).
[32] D. Chen, W. Yang, and X. Zeng, Thermodynamics and weak cosmic censorship conjecture in Reissner-Nordström anti-de Sitter black holes with scalar field, Nucl. Phys. **B946**, 114722 (2019).
[33] P. Wang, H. Wu, and H. Yang, Thermodynamics of nonlinear electrodynamics black holes and the validity of weak cosmic censorship at charged particle absorption, Eur. Phys. J. C **79**, 572 (2019).
[34] B. Mu and J. Tao, Minimal length effect on thermodynamics and weak cosmic censorship conjecture in anti-de Sitter black holes via charged particle absorption, Adv. High Energy Phys. **2020**, 2612946 (2020).
[35] J. Liang, X. Guo, and B. Mu, Thermodynamics with pressure and volume of black holes based on two assumptions under scalar field scattering, arXiv:2101.11414.
[36] D. Chen and S. Zeng, Overcharging problem and thermodynamics in extended phase spaces, Nucl. Phys. **B957**, 115089 (2020).
[37] B. Mu, J. Liang, and X. Guo, Thermodynamics with pressure and volume of 4D Gauss-Bonnet AdS Black Holes under the scalar field, arXiv:2011.00273.
[38] J. Liang, X. Guo, D. Chen, and B. Mu, Remarks on the weak cosmic censorship conjecture of RN-AdS black holes with cloud of strings and quintessence under the scalar field, arXiv:2008.08327.
[39] D. Chen, Thermodynamics and weak cosmic censorship conjecture in extended phase spaces of anti-de Sitter black holes with particles' absorption, Eur. Phys. J. C **79**, 353 (2019).

[40] B. Gwak, Weak cosmic censorship with pressure and volume in charged anti-de Sitter black hole under charged scalar field, J. Cosmol. Astropart. Phys. 08 (2019) 016.

[41] J. Liang, B. Mu, and J. Tao, Thermodynamics and overcharging problem in the extended phase spaces of charged AdS black holes with cloud of strings and quintessence under charged particle absorption, Chin. Phys. C **45**, 023121 (2021).

[42] X. X. Zeng and H. Q. Zhang, Thermodynamics and weak cosmic censorship conjecture in Born-Infeld-anti-de Sitter black holes, Chin. Phys. C **45**, 025112 (2021).

[43] P. Wang, H. Wu, and H. Yang, Thermodynamics and weak cosmic censorship conjecture in nonlinear electrodynamics black holes via charged particle absorption, arXiv:1904.12365.

[44] Y. W. Han, X. X. Zeng, and Y. Hong, Thermodynamics and weak cosmic censorship conjecture of the torus-like black hole, Eur. Phys. J. C **79**, 252 (2019).

[45] Y. W. Han, M. J. Lan, and X. X. Zeng, Thermodynamics and weak cosmic censorship conjecture in $(2+1)$-dimensional regular black hole with nonlinear electrodynamics sources, Eur. Phys. J. Plus **135**, 172 (2020).

[46] X. X. Zeng and H. Q. Zhang, Thermodynamics and weak cosmic censorship conjecture in the Kerr-AdS black hole, Nucl. Phys. **B959**, 115162 (2020).

[47] D. Kubiznak and R. B. Mann, *P–V* criticality of charged AdS black holes, J. High Energy Phys. 07 (2012) 033.

[48] Ö. Ökcü and E. Aydıner, Joule–Thomson expansion of the charged AdS black holes, Eur. Phys. J. C **77**, 24 (2017).

[49] J. X. Mo, G. Q. Li, S. Q. Lan, and X. B. Xu, Joule-Thomson expansion of *d*-dimensional charged AdS black holes, Phys. Rev. D **98**, 124032 (2018).

[50] Ö. Ökcü and E. Aydıner, Joule–Thomson expansion of Kerr–AdS black holes, Eur. Phys. J. C **78**, 123 (2018).

[51] J. Pu, S. Guo, Q. Q. Jiang, and X. T. Zu, Joule-Thomson expansion of the regular(Bardeen)-AdS black hole, Chin. Phys. C **44**, 035102 (2020).

[52] M. Chabab, H. El Moumni, S. Iraoui, K. Masmar, and S. Zhizeh, Joule-Thomson expansion of RN-AdS black holes in $f(R)$ gravity, Lett. High Energy Phys. **02**, 05 (2018).

[53] H. Ghaffarnejad, E. Yaraie, and M. Farsam, Quintessence Reissner Nordström Anti de Sitter black holes and Joule Thomson effect, Int. J. Theor. Phys. **57**, 1671 (2018).

[54] C. Li, P. He, P. Li, and J. B. Deng, Joule-Thomson expansion of the Bardeen-AdS black holes, Gen. Relativ. Gravit. **52**, 50 (2020).

[55] C. L. A. Rizwan, A. N. Kumara, D. Vaid, and K. M. Ajith, Joule-Thomson expansion in AdS black hole with a global monopole, Int. J. Mod. Phys. A **33**, 1850210 (2019).

[56] D. Mahdavian Yekta, A. Hadikhani, and Ö. Ökcü, Joule-Thomson expansion of charged AdS black holes in rainbow gravity, Phys. Lett. B **795**, 521 (2019).

[57] A. Cisterna, S. Q. Hu, and X. M. Kuang, Joule-Thomson expansion in AdS black holes with momentum relaxation, Phys. Lett. B **797**, 134883 (2019).

[58] S. Q. Lan, Joule-Thomson expansion of charged Gauss-Bonnet black holes in AdS space, Phys. Rev. D **98**, 084014 (2018).

[59] Y. Cao, H. Feng, W. Hong, and J. Tao, Joule-Thomson expansion of RN-AdS black hole immersed in perfect fluid dark matter, Commun. Theor. Phys. **73**, 095403 (2021).

[60] Y. Meng, J. Pu, and Q. Q. Jiang, *P–V* criticality and Joule-Thomson expansion of charged AdS black holes in the rastall gravity, Chin. Phys. C **44**, 065105 (2020).

[61] S. Bi, M. Du, J. Tao, and F. Yao, Joule-Thomson expansion of born-infeld AdS black holes, Chin. Phys. C **45**, 025109 (2021).

[62] S. Guo, Y. Han, and G. P. Li, Thermodynamic of the charged AdS black holes in rastall gravity: *P–V* critical and Joule–Thomson expansion, Mod. Phys. Lett. A **35**, 2050113 (2020).

[63] M. Rostami, J. Sadeghi, S. Miraboutalebi, A. A. Masoudi, and B. Pourhassan, Charged accelerating AdS black hole of $f(R)$ gravity and the Joule–Thomson expansion, Int. J. Geom. Methods Mod. Phys. **17**, 2050136 (2020).

[64] A. Haldar and R. Biswas, Joule-Thomson expansion of five-dimensional Einstein-Maxwell-Gauss-Bonnet-AdS black holes, Europhys. Lett. **123**, 40005 (2018).

[65] J. X. Mo and G. Q. Li, Effects of Lovelock gravity on the Joule–Thomson expansion, Classical Quant. Grav. **37**, 045009 (2020).

[66] C. H. Nam, Heat engine efficiency and Joule-Thomson expansion of non-linear charged AdS black hole in massive gravity, Gen. Relativ. Gravit. **53**, 30 (2021).

[67] H. Ghaffarnejad and E. Yaraie, Effects of a cloud of strings on the extended phase space of Einstein–Gauss–Bonnet AdS black holes, Phys. Lett. B **785**, 105 (2018).

[68] X. M. Kuang, B. Liu, and A. Övgün, Nonlinear electrodynamics AdS black hole and related phenomena in the extended thermodynamics, Eur. Phys. J. C **78**, 840 (2018).

[69] Z. W. Zhao, Y. H. Xiu, and N. Li, Throttling process of the Kerr–Newman–anti-de Sitter black holes in the extended phase space, Phys. Rev. D **98**, 124003 (2018).

[70] S. Q. Lan, Joule-Thomson expansion of neutral AdS black holes in massive gravity, Nucl. Phys. **B948**, 114787 (2019).

[71] S. Guo, Y. Han, and G. P. Li, Joule-Thomson expansion of a specific black hole in different dimensions, arXiv:1912.09590.

[72] H. Ranjbari, M. Sadeghi, M. Ghanaatian, and G. Forozani, Critical behavior of AdS Gauss–Bonnet massive black holes in the presence of external string cloud, Eur. Phys. J. C **80**, 17 (2020).

[73] J. Sadeghi and R. Toorandaz, Joule-Thomson expansion of hyperscaling violating black holes with spherical and hyperbolic horizons, Nucl. Phys. **B951**, 114902 (2020).

[74] M. Farsam, E. Yaraie, H. Ghaffarnejad, and E. Ghasami, Cooling-heating phase transition for 4D AdS Bardeen Gauss-Bonnet black hole, arXiv:2010.05697.

[75] B. J. Carr, J. Mureika, and P. Nicolini, Sub-planckian black holes and the generalized uncertainty principle, J. High Energy Phys. 07 (2015) 052.

[76] P. Nicolini and E. Spallucci, Holographic screens in ultraviolet self-complete quantum gravity, Adv. High Energy Phys. **2014**, 805684 (2014).

[77] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998).

[78] M. Banados, C. Teitelboim, and J. Zanelli, The Black Hole in Three-Dimensional Space-Time, Phys. Rev. Lett. **69**, 1849 (1992).

[79] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Geometry of the $(2+1)$ black hole, Phys. Rev. D **48**, 1506 (1993); Erratum, Phys. Rev. D **88**, 069902 (2013).

[80] A. M. Frassino, R. B. Mann, and J. R. Mureika, Lower-dimensional black hole chemistry, Phys. Rev. D **92**, 124069 (2015).

[81] J. V. Rocha and V. Cardoso, Gravitational perturbation of the BTZ black hole induced by test particles and weak cosmic censorship in AdS spacetime, Phys. Rev. D **83**, 104037 (2011).

[82] B. Wang, R. K. Su, and P. K. N. Yu, Stability of the event horizon in $(2+1)$-dimensional black holes, Phys. Rev. D **54**, 7298 (1996).

[83] S. Carlip, What we don't know about BTZ black hole entropy, Classical Quant. Grav. **15**, 3609 (1998).

[84] J. Ho, W. T. Kim, and Y. J. Park, Space-time duality of BTZ black hole, J. Korean Phys. Soc. **33**, S541 (1998).

[85] H. Saida and J. Soda, Statistical entropy of BTZ black hole in higher curvature gravity, Phys. Lett. B **471**, 358 (2000).

[86] M. I. Park, BTZ black hole with higher derivatives, the second law of thermodynamics, and statistical entropy: A new proposal, Phys. Rev. D **77**, 126012 (2008).

[87] R. A. Hennigar, D. Kubiznak, R. B. Mann, and C. Pollack, Lower-dimensional Gauss–Bonnet gravity and BTZ black holes, Phys. Lett. B **808**, 135657 (2020).

[88] R. A. Hennigar, D. Kubiznak, and R. B. Mann, Rotating Gauss-Bonnet BTZ black holes, Classical Quant. Grav. **38**, 03LT01 (2021).

[89] R. B. Mann and S. N. Solodukhin, Quantum scalar field on three-dimensional (BTZ) black hole instanton: Heat kernel, effective action and thermodynamics, Phys. Rev. D **55**, 3622 (1997).

[90] R. B. Mann, J. J. Oh, and M. I. Park, The role of angular momentum and cosmic censorship in the $(2+1)$-dimensional rotating shell collapse, Phys. Rev. D **79**, 064005 (2009).

[91] D. Chen, Weak cosmic censorship conjecture in BTZ black holes with scalar fields, Chin. Phys. C **44**, 015101 (2020).

[92] X. X. Zeng, Y. W. Han, and D. Y. Chen, Thermodynamics and weak cosmic censorship conjecture of BTZ black holes in extended phase space, Chin. Phys. C **43**, 105104 (2019).

[93] K. Düztaş, Overspinning BTZ black holes with test particles and fields, Phys. Rev. D **94**, 124031 (2016).

[94] B. Mu, J. Tao, and P. Wang, Free-fall rainbow BTZ black hole, Phys. Lett. B **800**, 135098 (2020).

[95] R. Zhao and S. L. Zhang, Canonical entropy of three-dimensional BTZ black hole, Phys. Lett. B **641**, 318 (2006).

[96] M. Akbar and A. A. Siddiqui, Charged rotating BTZ black hole and thermodynamic behavior of field equations at its horizon, Phys. Lett. B **656**, 217 (2007).

[97] B. Chen, S. x. Liu, and J. j. Zhang, Thermodynamics of black hole horizons and Kerr/CFT correspondence, J. High Energy Phys. 11 (2012) 017.

[98] R. G. Cai, Z. J. Lu, and Y. Z. Zhang, Critical behavior in $(2+1)$-dimensional black holes, Phys. Rev. D **55**, 853 (1997).

[99] R. G. Cai and J. B. Griffiths, Null particle solutions in three-dimensional (anti-)de Sitter spaces, J. Math. Phys. (N.Y.) **40**, 3465 (1999).

[100] R. G. Cai and J. H. Cho, Thermodynamic curvature of the BTZ black hole, Phys. Rev. D **60**, 067502 (1999).

[101] D. C. Zou, Y. Liu, B. Wang, and W. Xu, Thermodynamics of rotating black holes with scalar hair in three dimensions, Phys. Rev. D **90**, 104035 (2014).

[102] S. Wang, S. Q. Wu, F. Xie, and L. Dan, The first laws of thermodynamics of the $(2+1)$-dimensional BTZ black holes and Kerr-de Sitter spacetimes, Chin. Phys. Lett. **23**, 1096 (2006).

[103] J. Sadeghi and A. S. Kubeka, P–V criticality of modified BTZ black hole, Int. J. Theor. Phys. **55**, 2455 (2016).

[104] J. Liang, W. Lin, and B. Mu, Joule-Thomson expansion of the torus-like black hole, arXiv:2103.03119.