Editors' Suggestion

Dissipative superfluid relativistic magnetohydrodynamics of a multicomponent fluid: The combined effect of particle diffusion and vortices

V. A. Dommes[*] and M. E. Gusakov

Ioffe Institute, Politekhnicheskaya 26, 194021 St. Petersburg, Russia

(Received 14 September 2021; accepted 29 October 2021; published 2 December 2021)

We formulate dissipative magnetohydrodynamic equations for finite-temperature superfluid and superconducting charged relativistic mixtures, taking into account the effects of particle diffusion and possible presence of Feynman-Onsager and/or Abrikosov vortices in the system. The equations depend on a number of phenomenological transport coefficients, which describe, in particular, relative motions of different particle species and their interaction with vortices. We demonstrate how to relate these transport coefficients to the mutual friction parameters and momentum transfer rates arising in the microscopic theory. The resulting equations can be used to study, in a unified and coherent way, a very wide range of phenomena associated with dynamical processes in neutron stars, e.g., the magnetothermal evolution, stellar oscillations and damping, as well as development and suppression of various hydrodynamic instabilities in neutron stars.

DOI: 10.1103/PhysRevD.104.123008

I. INTRODUCTION

Consider a dense mixture composed of several particle species, some of which may be charged. Assume also that some components of the mixture are in a superfluid and/or superconducting state at finite temperature. In what follows, we are interested in describing the behavior of such system in the *hydrodynamic* regime, i.e., assuming that the typical particle mean-free path and collision time are much smaller than, respectively, the typical length scale and timescale of the evolution of the system.

Assume further that (i) the mixture is relativistic and can be in a strong gravitational field; (ii) the mixture is magnetized and rotating, so that there are Feynman-Onsager and Abrikosov vortices in the system (below we assume that the charged superconducting particles form a type-II superconductor); (iii) normal (nonsuperfluid and nonsuperconducting) particles of different species do *not* move with exactly the same velocities, in other words, we allow for the *diffusion* of normal particles with respect to each other. Then, the question is, what are the equations describing dynamics in such a system?

Before answering this question (which is the subject of the present study) let us explain why it is important for us to formulate such equations. The reason is that mixtures with the properties just described can be found in the inner layers (cores) of neutron stars (NSs). An NS core consists, in the simplest case, of neutrons (n), protons (p), and electrons (e) with an admixture of muons (μ). This matter is extremely compact and degenerate—its density is several times greater than the density of matter in atomic nuclei, $\rho_0 = 2.8 \times 10^{14}$ g cm^{-3}. Magnetic fields in NSs may reach enormous values $\gtrsim 10^{15}$ G [1,2], while the gravitational field is so strong that the NS radius ($\sim$10 km) is only a few times larger than the Schwarzschild radius [3]. Furthermore, according to microscopic calculations [4–7], as well as observations of cooling, glitching, and rapidly rotating NSs [8–12], baryons (in particular, neutrons and protons) in NS interiors are expected to become superfluid or superconducting at temperatures $T \lesssim 10^8 - 10^{10}$ K. This means that, if an NS is rotating and magnetized, the topological defects—neutron (Feynman-Onsager) vortices and proton (Abrikosov) flux tubes—may be present (and coexist) in the system [13,14].[1] The equations presented in this paper are designed precisely to describe various dynamical phenomena in NSs, such as NS oscillations, cooling, and magnetic field evolution.

Our paper is, of course, not the first one in a series of works that have studied the dynamics of such systems. The smooth-averaged nonrelativistic hydrodynamics describing superfluid liquid helium II with vortices was formulated by Hall and Vinen [17,18] and, independently, by Bekarevich and Khalatnikov [19]. It has been extended in subsequent studies (e.g., [20–30]) to account for charged mixtures and relativistic effects. Recently, Ref. [26] (hereafter GD16) derived the relativistic magnetohydrodynamics (MHD), which describes superfluid and superconducting mixtures

[*]vasdommes@gmail.com

[1]Here we assume that protons form a type-II superconductor, which is likely true for the outer part of the NS core but, probably, not the case for the inner part [15,16].

at finite temperatures and allows for the presence of Feynman-Onsager and Abrikosov vortices, as well as the electromagnetic field. It focuses mainly on the nondissipative equations and ignores particle diffusion, viscosity, and other dissipative effects (except for the mutual friction dissipation, which is taken into account). This work was further extended by Rau and Wasserman [29] who obtained an equivalent formulation of relativistic MHD starting from Carter's variational principle [31] and also included heat conduction and viscosity into the corresponding equations.

All these works ignore particle diffusion, i.e., relative motions of different particle species (or Bogoliubov thermal excitations, if superfluid and superconducting species are considered) with respect to each other. This is an unfortunate omission, since it is well known that diffusion plays a crucial role in the secular evolution of the magnetic field in nonsuperfluid and nonsuperconducting NSs [32–38] and, moreover, can be very efficient [39,40] in damping of NS oscillations and suppressing various instabilities in their interiors. As shown recently [41], diffusion also has a major effect on the evolution of the magnetic field in *superconducting* NSs. The reason is easy to understand. If protons form a type-II superconductor, the magnetic field in the NS core is locked to quantized proton flux tubes and its evolution is determined by the flux tube motion. To study this motion, one has to calculate the balance of forces acting on vortices, which (except for the buoyancy and tension forces [18,19,42,43]) depend on the relative velocities between vortices and different particle species that scatter on it. Because interaction (in particular, friction) of particles with vortices is very strong due to the huge amount of vortices in the system [41,44], even small mismatch in the velocities of different particle species significantly affects the force balance on vortices and, hence, the magnetic field evolution.

Up until now the MHD equations, describing relativistic charged mixtures and systematically incorporating the diffusion effects, have been studied in the very limited number of works and only neglecting the superconductivity and superfluidity effects. In particular, the most advanced MHD versions, suitable for NS modeling, were formulated in the series of papers by Andersson *et al.* [45–48] and in Ref. [49] (hereafter DGS20). In the present work we fill this gap by combining the results of GD16 and DGS20, with the aim to formulate the ready-to-use dissipative relativistic MHD for superfluid or superconducting mixtures, accounting for both vortices and diffusion effects. We follow the same approach [19,50] as in those papers. Namely, we build a first-order dissipative hydrodynamics, starting from the conservation laws and then deriving the general form of dissipative terms, which (i) are linear in thermodynamic fluxes, (ii) ensure non-negative entropy production rate, and (iii) satisfy the Onsager relations. The first-order MHD formulated in this paper is strictly valid in the hydrodynamic regime, i.e., as long as the typical length scale and timescale in the problem are much larger than the particle mean-free path and collision time, respectively. Although we did not test our MHD, it has been argued in the literature (e.g., [51,52]) that a generic first-order theory may have theoretical issues with acausality and stability due to *unphysical* high-frequency and short-wavelength modes, which lie outside the applicability domain of the hydrodynamic regime. One way to overcome these issues is to use more complicated formulations, such as the first-order theories with a specially chosen reference frame [53], second-order theories [54–56], or hydrodynamics based on Carter's variational principle [31,45,48].[2] The other (less elegant, but more pragmatic) option, which applies to those who work in the deep hydrodynamic regime, is simply to discard the unphysical modes in the solution, or filter them out, when it comes to numerical implementation. Moreover, for many practical applications, where the MHD formulated in this work can be used (e.g., modeling the NS magnetothermal evolution or oscillations and related physical instabilities), the macroscopic particle velocities appear to be nonrelativistic. Then the relativistic equations (see, e.g., Sec. V of DGS20 and the Appendix A) have a similar structure to the nonrelativistic ones; the main difference is the relativistic equation of state and, if one allows for the effects of general relativity, the metric coefficients. In this case additional degrees of freedom (which arise in the relativistic treatment and do not have Newtonian counterparts) are absent, and thus the hydrodynamics remains stable [57]. Bearing in mind the above comments, we leave detailed discussion of theoretical acausality and instability issues beyond the scope of the present work.

The paper is organized as follows. In Sec. II we formulate general hydrodynamics equations for charged superfluid or superconducting relativistic mixtures in the presence of vortices and the electromagnetic field, accounting for a number of dissipative effects: mutual friction, diffusion, viscosity, and chemical reactions. In Sec. III, we derive the entropy generation equation and in Sec. IV we use it together with the Onsager relations to derive the general form of dissipative corrections for particle currents, as well as mutual friction forces acting on vortices. In Sec. V we apply these general formulas to a number of interesting limiting cases, which are suitable for NS applications. Section VI provides a full set of hydrodynamic equations in the "MHD approximation" adopted in GD16, which is applicable for typical NS conditions and allows one to study a long-term magnetothermal evolution in superconducting NSs. Finally, we sum up in Sec. VII. The paper also contains two Appendixes. Appendix A presents a nonrelativistic limit of MHD equations from

[2]Note that in the hydrodynamic regime the higher-order corrections are typically small. This is clearly illustrated in Sec. VIII of DGS20, where it is shown that such corrections to the standard (acausal) heat equation can be safely ignored.

Sec. VI. In Appendix B we show how to express the phenomenological transport coefficients appearing in our equations through the mutual friction parameters and momentum transfer rates calculated from the microscopic theory.

Unless otherwise stated, in what follows the speed of light c and the Boltzmann constant k_B are set to unity, $c = k_B = 1$.

II. GENERAL EQUATIONS

In this section, we present dissipative equations, describing dynamics of charged finite-temperature superfluid relativistic mixtures in the presence of vortices in the *hydrodynamic* regime (see the introduction). For definiteness, and bearing in mind NS applications, we consider a mixture composed of superfluid neutrons, superconducting protons, normal electrons, and normal muons.[3] Both neutron (Feynman-Onsager) vortices and proton (Abrikosov) flux tubes can be present in the system. Generalization of these equations to more complex compositions (e.g., including hyperons) is straightforward.

The dynamical equations proposed here are very similar to those formulated in GD16 assuming type-II proton superconductivity but contain a number of extra terms: (i) the four-force G^ν in the right-hand side of Eq. (8); (ii) the particle production rate $\Delta\Gamma_i$ in the right-hand side of Eq. (1); (iii) the dissipative correction $\Delta j^\mu_{(i)}$ to the particle current density (4); (iv) the dissipative correction $\Delta\tau^{\mu\nu}$ to the energy-momentum tensor (9); and (v) the superfluid dissipative correction $\varkappa_i$ to the chemical potential μ_i in the definitions (6) and (22). Note that the first four corrections are included in the nonsuperfluid dissipative MHD of DGS20, but for superfluid or superconducting mixtures their actual form may differ.

A. Continuity equations

The four-current density $j^\mu_{(i)}$ of particle species i satisfies the continuity equation

$$\partial_\mu j^\mu_{(i)} = \Delta\Gamma_i, \tag{1}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$ is the four-gradient and $\Delta\Gamma_i$ is the corresponding production rate (source of particles i). Here and below, unless otherwise stated, Latin indices $i, k, \ldots$ refer to particle species (neutrons n, protons p, electrons e, and muons μ), whereas Greek letters $\mu, \nu \ldots = 0, 1, 2, 3$ denote the space-time indices, and summation over repeated indices is assumed.

[3] We do not assume that all neutrons and protons are necessarily in the Cooper-pair condensate. In other words, we allow for the possible presence of normal neutron and proton component in the mixture.

In the simplest case of nonsuperfluid matter in the absence of diffusion, the particle current density is $j^\mu_{(i)} = n_i u^\mu$, where u^μ is the (common for all particle species) normal four-velocity, normalized by the condition

$$u_\mu u^\mu = -1, \tag{2}$$

and n_i is the particle number density measured in the comoving frame $u^\mu = (1, 0, 0, 0)$, such that

$$u_\mu j^\mu_{(i)} = -n_i. \tag{3}$$

When accounting for superfluidity and diffusive currents, $j^\mu_{(i)}$ can generally be presented as a sum of three terms:

$$j^\mu_{(i)} = n_i u^\mu + Y_{ik} w^\mu_{(k)} + \Delta j^\mu_{(i)}, \tag{4}$$

where the four-vector $w^\mu_{(k)}$ describes the superfluid degrees of freedom [58] and satisfies the condition [25,58,59]

$$u_\mu w^\mu_{(i)} = 0. \tag{5}$$

This vector is related to the wave-function phase Φ_i of the Cooper condensate by the formula

$$w^\mu_{(i)} = \partial^\mu \phi_i - (\mu_i + \varkappa_i) u^\mu - e_i A^\mu, \tag{6}$$

where $\partial^\mu \phi_i = (\hbar/2)\partial^\mu \Phi_i$ [58], $\hbar$ is the Planck constant, μ_i is the relativistic chemical potential for particle species i, A^μ is the electromagnetic potential, and $\varkappa_i$ is the viscous dissipative correction to the chemical potential [25,59].

Further, Y_{ik} in Eq. (4) is the symmetric entrainment matrix [58,60–63], which is a relativistic analog of the nonrelativistic superfluid mass-density matrix [64–67]; and $\Delta j^\mu_{(i)}$ is the dissipative correction due to nonsuperfluid diffusive currents (see DGS20 for a similar definition of $\Delta j^\mu_{(i)}$ in normal matter).

Throughout the paper, all the thermodynamic quantities are defined (measured) in the comoving frame. This means that the relation (3) holds also in the general case (when dissipation effects are allowed for), which imposes an additional constraint on $\Delta j^\mu_{(i)}$:

$$u_\mu \Delta j^\mu_{(i)} = 0. \tag{7}$$

B. Energy-momentum conservation

The relativistic energy-momentum conservation law takes the form

$$\partial_\mu T^{\mu\nu} = G^\nu, \tag{8}$$

where G^ν is the radiation four-force density, which describes exchange of energy and momentum between matter and radiation,[4] and the energy-momentum tensor $T^{\mu\nu}$ is given by

$$\begin{aligned} T^{\mu\nu} = {}& (P+\varepsilon)u^\mu u^\nu + P g^{\mu\nu} \\ &+ Y_{ik}(w^\mu_{(i)} w^\nu_{(k)} + \mu_i w^\mu_{(k)} u^\nu + \mu_k w^\nu_{(i)} u^\mu) \\ &+ \Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} + \Delta\tau^{\mu\nu}, \end{aligned} \tag{9}$$

where P is the pressure defined by Eq. (35) below, ε is the energy density, and $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the space-time metric.[5] The energy-momentum tensor (9) is a sum of the energy-momentum tensor of a vortex-free uncharged superfluid hydrodynamics (the first three terms) plus electromagnetic and vortex contributions $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ given by Eq. (37) below and dissipative correction $\Delta\tau^{\mu\nu}$. Note that all these terms except for the last one are the same as in GD16.

In the comoving frame the energy density is given by the component T^{00} of the energy-momentum tensor, $T^{00} = \varepsilon$, which implies

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon. \tag{10}$$

This relation, in view of the expressions (5), (9), (37)–(39), (47), and (48), imposes the following constraint on the dissipative correction $\Delta\tau^{\mu\nu}$:

$$u_\mu u_\nu \Delta\tau^{\mu\nu} = 0. \tag{11}$$

Note, however, that the four-velocity u^μ itself is not uniquely defined in the system with dissipation (see, e.g., a thorough discussion of a similar issue in Ref. [50] and in DGS20). We specify u^μ by requiring the total momentum of the normal fluid component to be zero in the comoving frame. This leads to an additional condition for $\Delta\tau^{\mu\nu}$:

$$u_\nu \Delta\tau^{\mu\nu} = 0. \tag{12}$$

The condition (12) coincides with the similar condition defining the so-called Landau-Lifshitz (or transverse) frame of nonsuperfluid relativistic hydrodynamics [50].

[4]For isotropic emission $G^\nu = -Q u^\nu$, where Q is the total emissivity (e.g., it can be the neutrino emissivity due to beta processes in the NS core).

[5]In this paper, we assume that the metric is flat. Our results can easily be generalized to an arbitrary metric, provided that all relevant length scales are much smaller than the characteristic gravitational length scale. In this case, one has to replace all ordinary derivatives with their covariant counterparts and, in addition, replace the Levi-Civita tensor $\epsilon^{\mu\nu\lambda\sigma}$ with $\eta^{\mu\nu\lambda\sigma} \equiv (-\det g_{\alpha\beta})^{-1/2} \epsilon^{\mu\nu\lambda\sigma}$.

C. Maxwell equations

The electromagnetic field is described by the Maxwell equations in the medium:

$$\mathrm{div}\boldsymbol{D} = 4\pi\rho_{\mathrm{free}}, \tag{13}$$

$$\mathrm{curl}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{14}$$

$$\mathrm{div}\boldsymbol{B} = 0, \tag{15}$$

$$\mathrm{curl}\boldsymbol{H} = 4\pi \boldsymbol{J}_{\mathrm{free}} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{16}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{D}$ is the electric displacement, $\boldsymbol{H}$ is the magnetic field, ρ_{free} is the free charge density, and $\boldsymbol{J}_{\mathrm{free}}$ is the current density of free charges. Note that, generally, $\boldsymbol{D} \neq \boldsymbol{E}$ and $\boldsymbol{H} \neq \boldsymbol{B}$, since there are bound charges and bound currents in the system, associated with superfluid or superconducting vortices and their motion (for details see GD16); in the absence of vortices (and neglecting very weak magnetization and polarizability of NS matter [68]) $\boldsymbol{D} = \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}$.

The explicitly covariant form of Maxwell equations (13)–(16) is [69,70]

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{17}$$

$$\partial_\nu G^{\mu\nu} = 4\pi J^\mu_{(\mathrm{free})}, \tag{18}$$

where the antisymmetric electromagnetic tensors $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ and $G^{\mu\nu}$ are composed of components of the vectors $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{19}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}, \tag{20}$$

and $J^\mu_{(\mathrm{free})} = (\rho_{\mathrm{free}}, \boldsymbol{J}_{\mathrm{free}})$ is the four-current density of free charges,

$$J^\mu_{(\mathrm{free})} \equiv e_i j^\mu_{(i)} = e_i n_i u^\mu + e_i Y_{ik} w^\mu_{(k)} + e_i \Delta j^\mu_{(i)}, \tag{21}$$

where e_i is the electric charge for particle species i.

D. Vorticity tensor

Following GD16, we introduce the vorticity tensor

$$\mathcal{V}^{\mu\nu}_{(i)} \equiv \partial^\mu[w^\nu_{(i)} + (\mu_i + \varkappa_i)u^\nu + e_i A^\nu] - \partial^\nu[w^\mu_{(i)} + (\mu_i + \varkappa_i)u^\mu + e_i A^\mu], \qquad (22)$$

which is a relativistic generalization of the three-vector $m_i \mathrm{curl} \boldsymbol{V}_{si} + (e_i/c)\boldsymbol{B}$ (see Appendix A). In a system without topological defects (i.e., vortices), the superfluid phase Φ_i is a smooth function of coordinates satisfying the condition $\partial^\mu\partial^\nu\Phi_i - \partial^\nu\partial^\mu\Phi_i = 0$, which, in view of Eq. (6), translates into

$$\mathcal{V}^{\mu\nu}_{(i)} = 0. \qquad (23)$$

However, in the presence of vortices, the condition $\partial^\mu\partial^\nu\Phi_i - \partial^\nu\partial^\mu\Phi_i = 0$ is violated at the vortex lines. Consequently, the (smooth-averaged) vorticity tensor $\mathcal{V}^{\mu\nu}_{(i)}$ differs from zero. One can demonstrate that this tensor $\mathcal{V}^{\mu\nu}_{(i)}$ is related to the number of vortices $\mathcal{N}_{\mathrm{V}i}$ piercing the closed contour by the relation [25][6]

$$\frac{1}{2}\int df^{\mu\nu}\mathcal{V}_{(i)\mu\nu} = \pi\hbar\mathcal{N}_{\mathrm{V}i}. \qquad (24)$$

Equation (23) then should be replaced by a more general superfluid equation (59) introduced in Sec. IV below.

E. Thermodynamic relations

The dynamic equations listed above should be supplemented by the second law of thermodynamics,

$$d\varepsilon = \mu_i dn_i + T dS + \frac{Y_{ik}}{2} d(w^\alpha_{(i)} w_{(k)\alpha}) + d\varepsilon_{\mathrm{add}}, \qquad (25)$$

where T is the temperature, S is the entropy per unit volume, and the electromagnetic or vortex contribution to the energy density $d\varepsilon_{\mathrm{add}}$ reads [see Eq. (79) in GD16]

$$d\varepsilon_{\mathrm{add}} = \frac{1}{4\pi}E_\mu dD^\mu + \frac{1}{4\pi}H_\mu dB^\mu + \mathcal{V}^\mu_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^\mu_{(\mathrm{M}i)}. \qquad (26)$$

Here we introduced the auxiliary vortex-related vectors $\mathcal{W}^\mu_{(\mathrm{E}i)}$ and $\mathcal{W}^\mu_{(\mathrm{M}i)}$, in full analogy with the electromagnetic vectors D^μ and H^μ, respectively. Equation (26) should be considered as a *definition* of the vectors D^μ, H^μ, $\mathcal{W}^\mu_{(\mathrm{E}i)}$, and $\mathcal{W}^\mu_{(\mathrm{M}i)}$ [or, equivalently, the tensors $G^{\mu\nu}$ and $\mathcal{W}^{\mu\nu}_{(i)}$; see the identities (27)–(34) below]. When a microscopic model for the system energy density is specified (see, e.g., Appendix G in GD16 and Sec. VI A), one can express these vectors through the vectors E^μ, B^μ, $\mathcal{V}_{(\mathrm{E}i)\mu}$, and $\mathcal{V}_{(\mathrm{M}i)\mu}$ (or, equivalently, through the tensors $F^{\mu\nu}$ and $\mathcal{V}^{\mu\nu}_{(i)}$). The four-vectors entering Eq. (26) are related to the corresponding tensors as

$$E^\mu \equiv u_\nu F^{\mu\nu}, \qquad (27)$$

$$D^\mu \equiv u_\nu G^{\mu\nu}, \qquad (28)$$

$$B^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}, \qquad (29)$$

$$H^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu G_{\alpha\beta}, \qquad (30)$$

$$\mathcal{V}^\mu_{(\mathrm{E}i)} \equiv u_\nu \mathcal{V}^{\mu\nu}_{(i)}, \qquad (31)$$

$$\mathcal{V}^\mu_{(\mathrm{M}i)} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{V}_{(i)\alpha\beta}, \qquad (32)$$

$$\mathcal{W}^\mu_{(\mathrm{E}i)} \equiv u_\nu \mathcal{W}^{\mu\nu}_{(i)}, \qquad (33)$$

$$\mathcal{W}^\mu_{(\mathrm{M}i)} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{W}_{(i)\alpha\beta}, \qquad (34)$$

where the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$ is defined such that $\epsilon^{0123} = 1$. In the comoving frame, $u^\mu = (1, 0, 0, 0)$, the four-vectors E^μ, D^μ, B^μ and H^μ are related to, respectively, the ordinary three-vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ as $E^\mu = (0, \boldsymbol{E})$, $D^\mu = (0, \boldsymbol{D})$, $B^\mu = (0, \boldsymbol{B})$, and $H^\mu = (0, \boldsymbol{H})$.

The total pressure P is defined (see, e.g., GD16) as a partial derivative of the full system energy εV with respect to the volume V at constant total number of particles $n_i V$, total entropy SV, as well as at fixed quantities $w^\alpha_{(i)} w_{(k)\alpha}$, D^μ, B^μ, $\mathcal{W}^\mu_{(\mathrm{E}i)}$, and $\mathcal{W}^\mu_{(\mathrm{M}i)}$:

$$P \equiv -\frac{\partial(\varepsilon V)}{\partial V} = -\varepsilon + \mu_i n_i + TS, \qquad (35)$$

Using Eqs. (25), (26), and (35), one arrives at the following Gibbs-Duhem relation:

$$dP = n_i d\mu_i + S dT - \frac{Y_{ik}}{2} d(w^\alpha_{(i)} w_{(k)\alpha}) - \frac{1}{4\pi}E_\alpha dD^\alpha - \frac{1}{4\pi}H_\alpha dB^\alpha - \mathcal{V}^\mu_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} - \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^\mu_{(\mathrm{M}i)}. \qquad (36)$$

[6]This relation is satisfied for Fermi superfluids (e.g., neutrons or protons); for Bose superfluids there should be $2\pi\hbar\mathcal{N}_{\mathrm{V}i}$ in the right-hand side of the equation. Note that the factor $1/2$ was inadvertently omitted in the corresponding equation (42) in Ref. [25].

F. Electromagnetic and vortex contribution to $T^{\mu\nu}$

The electromagnetic and vortex contribution to $T^{\mu\nu}$, represented by the term $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ in Eq. (9), has been derived in GD16 and takes the form

$$\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} = \mathcal{T}^{\mu\nu}_{(\mathrm{E})} + \mathcal{T}^{\mu\nu}_{(\mathrm{M})} + \mathcal{T}^{\mu\nu}_{(\mathrm{VE})} + \mathcal{T}^{\mu\nu}_{(\mathrm{VM})}, \tag{37}$$

where the electromagnetic contributions $\mathcal{T}^{\mu\nu}_{(\mathrm{E})}$ and $\mathcal{T}^{\mu\nu}_{(\mathrm{M})}$ are given by [see Eqs. (66) and (67) in GD16]

$$\mathcal{T}^{\mu\nu}_{(\mathrm{E})} = \frac{1}{4\pi}(\perp^{\mu\nu} D^\alpha E_\alpha - D^\mu E^\nu), \tag{38}$$

$$\mathcal{T}^{\mu\nu}_{(\mathrm{M})} = \frac{1}{4\pi}({}^\perp G^{\mu\alpha\,\perp} F^\nu{}_\alpha + u^{\nu\,\perp} G^{\mu\alpha} E_\alpha + u^{\mu\,\perp} G^{\nu\alpha} E_\alpha). \tag{39}$$

Here and hereafter $\perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$, and the notation ${}^{\parallel}\mathcal{X}^{\mu\nu}$ and ${}^{\perp}\mathcal{X}^{\mu\nu}$ is introduced for arbitrary antisymmetric tensor $\mathcal{X}^{\mu\nu}$:

$$\begin{aligned}{}^{\parallel}\mathcal{X}^{\mu\nu} &= -u^\nu \mathcal{X}^\mu_{(\mathrm{E})} + u^\mu \mathcal{X}^\nu_{(\mathrm{E})} = -u^\nu u_\alpha \mathcal{X}^{\mu\alpha} + u^\mu u_\alpha \mathcal{X}^{\nu\alpha} \\ &= \begin{pmatrix} 0 & \mathcal{X}_{01} & \mathcal{X}_{02} & \mathcal{X}_{03} \\ -\mathcal{X}_{01} & 0 & 0 & 0 \\ -\mathcal{X}_{02} & 0 & 0 & 0 \\ -\mathcal{X}_{03} & 0 & 0 & 0 \end{pmatrix},\end{aligned} \tag{40}$$

$$\begin{aligned}{}^{\perp}\mathcal{X}^{\mu\nu} &= \epsilon^{\alpha\beta\mu\nu} u_\beta \mathcal{X}_{(\mathrm{M})\alpha} = \perp^{\mu\alpha}\perp^{\nu\beta}\mathcal{X}_{\alpha\beta} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{X}_{12} & \mathcal{X}_{13} \\ 0 & -\mathcal{X}_{12} & 0 & \mathcal{X}_{23} \\ 0 & -\mathcal{X}_{13} & -\mathcal{X}_{23} & 0 \end{pmatrix},\end{aligned} \tag{41}$$

where the matrix expressions are written in the comoving frame, and the "electric" and "magnetic" four-vectors $\mathcal{X}^\mu_{(\mathrm{E})}$ and $\mathcal{X}^\mu_{(\mathrm{M})}$, respectively, are defined as [cf. Eqs. (31)–(34)]

$$\mathcal{X}^\mu_{(\mathrm{E})} \equiv u_\nu \mathcal{X}^{\mu\nu}, \tag{42}$$

$$\mathcal{X}^\mu_{(\mathrm{M})} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{X}_{\alpha\beta}. \tag{43}$$

Note that the following relations are satisfied:

$$\mathcal{X}^{\mu\nu} = {}^{\parallel}\mathcal{X}^{\mu\nu} + {}^{\perp}\mathcal{X}^{\mu\nu}, \tag{44}$$

$$\perp_{\mu\nu}{}^{\parallel}\mathcal{X}^{\mu\nu} = 0, \tag{45}$$

$$u_\nu{}^{\perp}\mathcal{X}^{\mu\nu} = 0, \tag{46}$$

and ${}^{\parallel}\mathcal{X}^{\mu\nu}$ and ${}^{\perp}\mathcal{X}^{\mu\nu}$ can be expressed in terms of, respectively, electric and magnetic four-vectors $\mathcal{X}^\mu_{(\mathrm{E}i)}$ and $\mathcal{X}^\mu_{(\mathrm{M}i)}$ [see the first equalities in Eqs. (40) and (41)]. Similarly, the vortex contributions $\mathcal{T}^{\mu\nu}_{(\mathrm{VE})}$ and $\mathcal{T}^{\mu\nu}_{(\mathrm{VM})}$ to the energy-momentum tensor can be presented as [see Eqs. (88) and (89) in GD16]

$$\mathcal{T}^{\mu\nu}_{(\mathrm{VE})} = \perp^{\mu\nu}\mathcal{W}^\alpha_{(\mathrm{E}i)}\mathcal{V}_{(\mathrm{E}i)\alpha} - \mathcal{W}^\mu_{(\mathrm{E}i)}\mathcal{V}^\nu_{(\mathrm{E}i)}, \tag{47}$$

$$\begin{aligned}\mathcal{T}^{\mu\nu}_{(\mathrm{VM})} &= {}^{\perp}\mathcal{W}^{\mu\alpha\,\perp}_{(i)}\mathcal{V}^\nu_{(i)\,\alpha} + u^{\nu\,\perp}\mathcal{W}^{\mu\alpha}_{(i)}\mathcal{V}_{(\mathrm{E}i)\alpha} \\ &\quad + u^{\mu\,\perp}\mathcal{W}^{\nu\alpha}_{(i)}\mathcal{V}_{(\mathrm{E}i)\alpha}.\end{aligned} \tag{48}$$

To sum up, the dissipative equations governing dynamics of superfluid and superconducting mixture consist of the continuity equations (1) [with $j^\mu_{(i)}$ given by Eq. (4)], the energy-momentum conservation law (8) [with $T^{\mu\nu}$ given by Eqs. (9) and (37)], Maxwell equations (17) and (18), and the superfluid equation [Eq. (23) or Eq. (59) below]. These equations are supplemented by the thermodynamic relations (25), (35), and (36), as well as by the definition (12) of the comoving frame.

III. ENTROPY GENERATION EQUATION

The equations of Sec. II contain the entropy generation equation, which is crucial for determining the general form of dissipative corrections (see Sec. IV). One can derive this equation by considering the expression $u_\nu\partial_\mu T^{\mu\nu} - u_\nu G^\nu$, which vanishes in view of Eq. (8). Using Eqs. (1), (4), (5), (9), (25), and (35), as well as the identities $u_\nu\partial_\mu u^\nu = 0$ and $\partial_\mu g^{\mu\nu} = 0$, we arrive at the following entropy generation equation [cf. Eq. (33) in Ref. [59], Eq. (58) in GD16, and Eq. (25) in DGS20]:

$$\begin{aligned}\partial_\mu(S u^\mu) &= \frac{1}{T}u_\nu Y_{ik} w_{(k)\mu}[\tilde{\mathcal{V}}^{\mu\nu}_{(i)} - \partial_\mu(\varkappa_i u_\nu) + \partial_\nu(\varkappa_i u_\mu)] \\ &\quad + \frac{\mu_i}{T}\partial_\mu \Delta j^\mu_{(i)} - \frac{\mu_i}{T}\Delta\Gamma_i - \frac{u^\mu}{T}\partial_\mu \varepsilon_{\mathrm{add}} \\ &\quad + \frac{u_\nu}{T}\partial_\mu(\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} + \Delta\tau^{\mu\nu}) - \frac{Q}{T},\end{aligned} \tag{49}$$

where

$$\begin{aligned}\tilde{\mathcal{V}}^{\mu\nu}_{(i)} &\equiv \mathcal{V}^{\mu\nu}_{(i)} - e_i F^{\mu\nu} \\ &= \partial^\mu[w^\nu_{(i)} + (\mu_i + \varkappa_i)u^\nu] - \partial^\nu[w^\mu_{(i)} + (\mu_i + \varkappa_i)u^\mu]\end{aligned} \tag{50}$$

and we defined $Q \equiv u_\nu G^\nu$. Now, let us make use of Eqs. (26) and (37) and substitute expressions for $d\varepsilon_{\mathrm{add}}$ and $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$. Using Eq. (85) of GD16, we present the term $-u^\mu\partial_\mu\varepsilon_{\mathrm{add}}$ as

$$\begin{aligned}-u^\mu\partial_\mu\varepsilon_{\mathrm{add}} &= u^\nu F_{\mu\nu}\partial_\alpha\left(\frac{1}{4\pi}G^{\mu\alpha}\right) + u^\nu\mathcal{V}_{(i)\mu\nu}\partial_\alpha\mathcal{W}^{\mu\alpha}_{(i)} \\ &\quad - u_\nu\partial_\mu\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}.\end{aligned} \tag{51}$$

Then, employing Maxwell equations (18) together with the relation $u_\mu \partial_\nu w^\mu_{(i)} = -w^\mu_{(i)}\partial_\nu u_\mu$ [which follows from Eq. (5)] and substituting Eqs. (21), (37), (50), and (51) into Eq. (49), we obtain

$$\partial_\mu S^\mu = \frac{n_i}{T} u^\nu \mathcal{V}_{(i)\mu\nu} W^\mu_{(i)} - \Delta j^\mu_{(i)} d_{(i)\mu} - \varkappa_i {}^\perp\nabla_\mu \left(\frac{Y_{ik} w^\mu_{(k)}}{T}\right) - \Delta\tau^{\mu\nu}\partial_\mu\left(\frac{u_\nu}{T}\right) - \frac{\mu_i}{T}\Delta\Gamma_i - \frac{Q}{T}, \quad (52)$$

where we introduced the entropy four-current

$$S^\mu = S u^\mu - \frac{\mu_i}{T}\Delta j^\mu_{(i)} - \frac{\varkappa_i}{T} Y_{ik} w^\mu_{(k)} - \frac{u_\nu}{T}\Delta\tau^{\mu\nu}, \quad (53)$$

the four-vector $W^\mu_{(i)}$,

$$W^\mu_{(i)} \equiv \frac{1}{n_i}[Y_{ik} w^\mu_{(k)} + \perp^{\mu\nu}\partial^\alpha \mathcal{W}_{\nu\alpha(i)}], \quad (54)$$

the displacement vector (see DGS20)

$$d_{(i)\mu} \equiv {}^\perp\nabla_\mu\left(\frac{\mu_i}{T}\right) - \frac{e_i E_\mu}{T}, \quad (55)$$

and the orthogonal part of the four-gradient

$${}^\perp\nabla_\mu \equiv \perp_{\mu\nu}\partial^\nu. \quad (56)$$

Note that $d_{(i)\mu}$ and $W^\mu_{(i)}$ can be defined up to an arbitrary term proportional to u^μ, which does not affect the entropy generation equation (52) due to the condition (7) and antisymmetry of $\mathcal{V}^{\mu\nu}_{(i)}$, respectively. For further convenience, we define these vectors in a way that ensures that they are both orthogonal to u^μ.[7]

If u^μ is specified by the condition (12), Eqs. (52) and (53) reduce to[8]

$$\partial_\mu S^\mu = \frac{\mu_i n_i^2}{T} f_{(i)\mu} W^\mu_{(i)} - \Delta j^\mu_{(i)} d_{(i)\mu} - \varkappa_i {}^\perp\nabla_\mu\left(\frac{Y_{ik} w^\mu_{(k)}}{T}\right) - \Delta\tau^{\mu\nu}\frac{{}^\perp\nabla_\mu u_\nu}{T} - \frac{\mu_i}{T}\Delta\Gamma_i - \frac{Q}{T}, \quad (57)$$

$$S^\mu = S u^\mu - \frac{\mu_i}{T}\Delta j^\mu_{(i)} - \frac{\varkappa_i}{T} Y_{ik} w^\mu_{(k)}. \quad (58)$$

Here we introduced the four-vector $f^\mu_{(i)}$ as

$$u_\nu \mathcal{V}^{\mu\nu}_{(i)} = \mu_i n_i f^\mu_{(i)}, \quad (59)$$

where no summation over repeated index i is assumed. Note that $f^\mu_{(i)}$ is orthogonal to u^μ, since the vorticity tensor $\mathcal{V}^{\mu\nu}_{(i)}$ is antisymmetric:

$$u_\mu f^\mu_{(i)} = 0. \quad (60)$$

Equation (59) can be regarded as a superfluid equation [25,26], which replaces the potentiality condition $\mathcal{V}^{\mu\nu}_{(i)} = 0$ of a vortex-free system.

The right-hand side of Eq. (57) describes entropy generation and must be non-negative (except for the arbitrary last term) for all possible fluid configurations. It includes vortex-mediated mutual friction between normal and superfluid components (first term) [14], diffusion (second term), viscosity (third and fourth terms), chemical reactions (such as Urca processes; fifth term) and radiation (sixth term).

Note, in passing, that different formulations of the first-order hydrodynamics (i.e., different forms of dissipative corrections) are possible even if u^μ is specified unambiguously [53]. This is due to the fact that various derivatives entering the dissipative corrections are not all independent but can be expressed (up to higher-order terms) through one another using the zero-order (nondissipative) hydrodynamic equations. For example, one can relate $u^\nu\partial_\nu u^\mu$ to ${}^\perp\nabla^\mu P$ via the momentum conservation law ${}^\perp\nabla_\nu T^{\mu\nu} = 0$. We follow here the approach of Ref. [50], so that in our formulation the right-hand side of Eq. (57) (and, consequently, the dissipative corrections) in the comoving frame contains only spatial derivatives and does not contain the terms like $u^\nu\partial_\nu u^\mu$ or $u^\nu\partial_\nu T$.

IV. DIFFUSIVE CURRENTS AND MUTUAL FRICTION FORCES

The entropy generation equation (57) allows one to find the general form of the unknown dissipative corrections,[9] namely, $f^\mu_{(i)}$, $\Delta j^\mu_{(i)}$, $\varkappa_i$, $\Delta\tau^{\mu\nu}$, and $\Delta\Gamma_i$ (here and below we ignore the last term, $-Q/T$, which can be arbitrary). Following Landau and Lifshitz [50] and DGS20, we express the dissipative corrections as linear combinations of thermodynamic forces $W^\mu_{(i)}$, $d_{(i)\mu}$, ${}^\perp\nabla_\mu(Y_{ik}w^\mu_{(k)}/T)$, ${}^\perp\nabla_\mu u_\nu$, and μ_i[10] and require that the right-hand side of

[7]GD16 uses a slightly different definition for $W^\mu_{(i)}$: $W^\mu_{(i)} \equiv (1/n_i)[Y_{ik}w^\mu_{(k)} + \partial_\alpha \mathcal{W}^{\mu\alpha}_{(i)}]$. If one prefers to use that definition, then one should replace $W^\mu_{(i)}$ with $\perp^{\mu\nu} W_{\nu(i)}$ [which is equivalent to Eq. (54) due to the condition (5)] everywhere in the paper.

[8]As in DGS20, we make use of the condition (12) and replace $\Delta\tau^{\mu\nu}\partial_\mu(u_\nu/T)$ with $\Delta\tau^{\mu\nu}({}^\perp\nabla_\mu u_\nu)/T$ in the right-hand side of Eq. (57).

[9]Note that some of these corrections may, in fact, contain nondissipative terms, but, for brevity, we call them "dissipative."

[10]Actually, μ_i should enter these expressions only in particular combinations that represent chemical potential imbalances for a given reaction (e.g., $\mu_n - \mu_p - \mu_e$ for the direct or modified Urca processes [71]); see DGS20 for more details.

Eq. (57) would be a positively defined quadratic form, so that the entropy would not decrease for all possible fluid configurations. The coefficients arising in these linear combinations can be scalars, vectors, or tensors, that can only depend on the system properties in the absence of dissipation; they are collectively called *transport coefficients*. We require, in addition, that these coefficients must satisfy the Onsager relations.

In the completely isotropic (in the comoving frame) matter, the transport coefficients depend only on the equilibrium scalar thermodynamic quantities, as well as on u^μ and $g^{\mu\nu}$ (or $\perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$). In the presence of preferred directions (e.g., vortex lines or magnetic field), the coefficients, generally, depend also on the corresponding vectors and the angles between them. These vectors include superfluid vectors $w^\mu_{(i)}$, electromagnetic vectors E^μ, D^μ, B^μ, and H^μ, and vortex-related vectors $\mathcal{V}^\mu_{(\mathrm{E}i)}$, $\mathcal{V}^\mu_{(\mathrm{M}i)}$, $\mathcal{W}^\mu_{(\mathrm{E}i)}$, and $\mathcal{W}^\mu_{(\mathrm{M}i)}$. However, the situation is considerably simplified in the MHD approximation described in Sec. VI A (see also GD16). This approximation is mainly based on the fact that the magnetic induction $\boldsymbol{B}$ is much larger than the fields $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ in the comoving frame and is locked to superconducting proton flux tubes. In this limit the only preferred directions[11] are defined by the neutron vortex lines $\mathcal{V}^\mu_{(\mathrm{M}n)}$, proton vortex lines $\mathcal{V}^\mu_{(\mathrm{M}p)}$ [or, equivalently, the magnetic induction B^μ; see Eq. (104)], and the superfluid neutron current $Y_{nk} w^\mu_{(k)}$.[12] Below, following Refs. [50,72,73], we neglect small terms that explicitly depend on $w^\mu_{(k)}$ (or, equivalently, on $Y_{nk} w^\mu_{(k)}$) *in the expressions for the transport coefficients*. These terms are usually ignored in the literature [50,72,73] when deriving the dissipative hydrodynamic equations for superfluid helium 4. In the context of neutron stars, the same approximation has been adopted and discussed in Ref. [59]. As a result, we are left with only two preferred directions, specified by the neutron vortices $\mathcal{V}^\mu_{(\mathrm{M}n)}$ and magnetic field or proton flux tubes B^μ [or $\mathcal{V}^\mu_{(\mathrm{M}p)}$], which determine anisotropy of transport coefficients.

[11]That these preferred directions are the only ones that should be taken into account in the MHD approximation is independently justified by the results of Appendix B, where it is shown that the more microscopic approach leads to exactly the same dissipative corrections as those obtained in this section. Generally, any additional preferred direction can be ignored as long as one can neglect the corresponding force in the force balance equations for particles or vortices. For example, in the non-superfluid MHD in the limit $\boldsymbol{B} \to 0$ an anisotropic correction to the diffusion coefficients $\mathcal{D}^{\mu\nu}_{e\mu}$ is of the order $\sim (e_p n_i B)/(cJ_{e\mu}) \sim$ (Lorentz force)/($e\mu$ friction force); see DGS20. Correspondingly, the magnetic field does not provide a preferred direction in this limit.

[12]In the thermodynamic equilibrium, the superconducting proton current $Y_{pk} w^\mu_{(k)}$ vanishes in the MHD approximation due to the screening condition [see Eq. (129) with $\Delta j^\mu_{(i)} = 0$].

Under the above assumptions, the vectors $\Delta j^\mu_{(i)}$ and $f^\mu_{(i)}$ can only depend on the thermodynamic forces $W^\mu_{(i)}$ and $d^\mu_{(i)}$ [and are independent of the forces ${}^\perp\nabla_\mu (Y_{ik} w^\mu_{(k)}/T)$, ${}^\perp\nabla_\mu u_\nu$, and μ_i][13]:

$$-\frac{\mu_i n_i^2}{T} f^\mu_{(i)} = -\mathcal{A}^{\mu\nu}_{ik} W_{(k)\nu} - \mathcal{B}^{\mu\nu}_{ik} d_{(k)\nu}, \tag{61}$$

$$\Delta j^\mu_{(i)} = -\mathcal{C}^{\mu\nu}_{ik} W_{(k)\nu} - \mathcal{D}^{\mu\nu}_{ik} d_{(k)\nu}, \tag{62}$$

where *no* summation over i in the left-hand side of Eq. (61) is implied. The transport coefficient $\mathcal{A}^{\mu\nu}_{ik}$ describes the mutual friction effects [14], as well as (possible) interaction between neutron vortices and proton flux tubes.[14] The coefficient $\mathcal{D}^{\mu\nu}_{ik}$ is responsible for the diffusion, thermodiffusion and thermal conductivity effects (see DGS20). Finally, the cross-coefficients $\mathcal{B}^{\mu\nu}_{ik}$ and $\mathcal{C}^{\mu\nu}_{ik}$ describe the impact of diffusive currents on the mutual friction forces on vortices, and vice versa.

In the present work, we are mainly interested in studying the joint effects of diffusion and vortices (represented by the vectors $\Delta j^\mu_{(i)}$ and $f^\mu_{(i)}$) on the structure of superfluid MHD. To study these effects, it is sufficient to consider only the first two terms in Eq. (57), since they do not interfere with the other terms in this equation and constitute a positively defined quadratic form themselves [see Eqs. (61) and (62)]. Thus, in what follows, we shall ignore viscosity ($\varkappa_i = \Delta\tau^{\mu\nu} = 0$) and chemical reactions ($\Delta\Gamma_i = 0$): the related dissipative corrections can be studied separately and, in fact, have already been analyzed in the past (see, e.g., Refs. [29,48,59], and DGS20). With this simplification, the entropy generation equation (57) becomes

$$\partial_\mu \left(S u^\mu - \frac{\mu_i}{T} \Delta j^\mu_{(i)} \right) = \frac{\mu_i n_i^2}{T} f_{(i)\mu} W^\mu_{(i)} - \Delta j^\mu_{(i)} d_{(i)\mu}. \tag{63}$$

[13]See Appendix B of DGS20, where it is demonstrated, for a similar problem, that $\Delta j^\mu_{(i)}$ cannot depend on the tensor ${}^\perp\nabla_\mu u_\nu$. The same consideration also applies to $f^\mu_{(i)}$ and can be readily generalized to an arbitrary number of preferred axial vectors (such as $\mathcal{V}^\mu_{(\mathrm{M}n)}$ and/or $\mathcal{V}^\mu_{(\mathrm{M}p)}$) in the system. In turn, it is also easy to verify that $\Delta j^\mu_{(i)}$ and $f^\mu_{(i)}$ cannot depend on the scalar thermodynamic forces, such as ${}^\perp\nabla_\nu (Y_{ik} w^\nu_{(k)}/T)$. This dependence may only lead to additional terms $\propto u^\mu {}^\perp\nabla_\nu (Y_{ik} w^\nu_{(k)}/T)$ in Eqs. (61) and (62), but these terms must vanish to satisfy the conditions (7) and (60).

[14]Note that the vortex–flux tube interaction should be accounted for in the expressions for $\mathcal{W}^\mu_{(\mathrm{M}i)}$ [which enter the definition (54) for $W^\mu_{(i)}$]. In Sec. VI A we employ a simple model which ignores this effect [see Eq. (105)]; however, such a simplification does not affect the general expression (72) for the coefficient $\mathcal{A}^{\mu\nu}_{ik}$.

The coefficients $\mathcal{A}^{\mu\nu}_{ik}$, $\mathcal{B}^{\mu\nu}_{ik}$, $\mathcal{C}^{\mu\nu}_{ik}$, and $\mathcal{D}^{\mu\nu}_{ik}$ in Eqs. (61) and (62) depend on the vectors $\mathcal{V}^{\mu}_{(\mathrm{M}n)}$ and B^μ, as well as on the scalar thermodynamic quantities and on u^μ and $\perp^{\mu\nu}$. Below we provide expressions for these coefficients for the system with two preferred directions and demonstrate how these expressions can be simplified in the case of only one preferred direction.

A. General case: Two preferred directions

Let us introduce the following quantities:

$$b^\mu \equiv \frac{B^\mu}{\sqrt{B_\alpha B^\alpha}}, \tag{64}$$

$$b^{\mu\nu} \equiv \frac{{}^{\perp}F^{\mu\nu}}{\sqrt{B_\alpha B^\alpha}}, \tag{65}$$

$$\omega^\mu \equiv \frac{\mathcal{V}^{\mu}_{(\mathrm{M}n)}}{\sqrt{\mathcal{V}_{(\mathrm{M}n)\alpha}\mathcal{V}^{\alpha}_{(\mathrm{M}n)}}}, \tag{66}$$

$$\omega^{\mu\nu} \equiv \frac{{}^{\perp}\mathcal{V}^{\mu\nu}_{(\mathrm{M}n)}}{\sqrt{\mathcal{V}_{(\mathrm{M}n)\alpha}\mathcal{V}^{\alpha}_{(\mathrm{M}n)}}}. \tag{67}$$

In the comoving frame $b^\mu = (0, \boldsymbol{b})$, $\omega^\mu = (0, \boldsymbol{\omega})$, where $\boldsymbol{b}$ and $\boldsymbol{\omega}$ are the unit vectors in the direction of the magnetic field and neutron vortices, respectively.

The Onsager principle leads to conditions[15]

$$\mathcal{A}^{\mu\nu}_{ik}(\boldsymbol{b}, \boldsymbol{\omega}) = \mathcal{A}^{\nu\mu}_{ki}(-\boldsymbol{b}, -\boldsymbol{\omega}), \tag{68}$$

$$\mathcal{D}^{\mu\nu}_{ik}(\boldsymbol{b}, \boldsymbol{\omega}) = \mathcal{D}^{\nu\mu}_{ki}(-\boldsymbol{b}, -\boldsymbol{\omega}), \tag{69}$$

$$\mathcal{C}^{\mu\nu}_{ik}(\boldsymbol{b}, \boldsymbol{\omega}) = -\mathcal{B}^{\nu\mu}_{ki}(-\boldsymbol{b}, -\boldsymbol{\omega}). \tag{70}$$

From the constraints $u_\mu f^\mu_{(i)} = 0$ [Eq. (60)] and $u_\mu \Delta j^\mu_{(i)} = 0$ [Eq. (7)] it also follows that

$$u_\mu \mathcal{A}^{\mu\nu}_{ik} = u_\mu \mathcal{B}^{\mu\nu}_{ik} = u_\mu \mathcal{C}^{\mu\nu}_{ik} = u_\mu \mathcal{D}^{\mu\nu}_{ik} = 0. \tag{71}$$

Relations (68)–(71) imply that all transport coefficients are purely spatial in the comoving frame and may depend on u^μ only through the tensor $\perp^{\mu\nu}$.

Let us start with the transport coefficient $\mathcal{A}^{\mu\nu}_{ik}$. Generally, it can be presented as a sum of nine linearly independent tensors,[16] which we choose in the following form that allows us to separate symmetric (the first six terms) and antisymmetric (the last three terms) parts of the tensor:

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{ik} &= \mathcal{A}^{\perp}_{ik}\perp^{\mu\nu} + \mathcal{A}^{\omega\omega}_{ik}\omega^\mu\omega^\nu + \mathcal{A}^{bb}_{ik}b^\mu b^\nu \\ &\quad + \mathcal{A}^{\omega b}_{ik}(\omega^\mu b^\nu + \omega^\nu b^\mu) + \mathcal{A}^{\omega\omega b}_{ik}(\omega^\mu\omega_\alpha b^{\nu\alpha} + \omega^\nu\omega_\alpha b^{\mu\alpha}) \\ &\quad + \mathcal{A}^{b\omega b}_{ik}(b^\mu\omega_\alpha b^{\nu\alpha} + b^\nu\omega_\alpha b^{\mu\alpha}) \\ &\quad + \mathcal{A}^{\omega-b}_{ik}(\omega^\mu b^\nu - \omega^\nu b^\mu) + \mathcal{A}^{\omega}_{ik}\omega^{\mu\nu} + \mathcal{A}^{b}_{ik}b^{\mu\nu},\end{aligned} \tag{72}$$

where the scalar coefficients $\mathcal{A}^{\perp}_{ik}$, $\mathcal{A}^{\omega\omega}_{ik}$, etc., may depend on the equilibrium quantities and the angle between $\boldsymbol{b}$ and $\boldsymbol{\omega}$. To clarify the meaning of different terms in Eq. (72), it is instructive to write out the expression for the vector $\mathcal{A}^{\mu\nu}_{ik}W_{(k)\nu}$ in the comoving frame. The zeroth component of this four-vector vanishes, while its spatial part reads

$$\begin{aligned}&\mathcal{A}^{\perp}_{ik}\boldsymbol{W}_k + \mathcal{A}^{\omega\omega}_{ik}\boldsymbol{\omega}(\boldsymbol{\omega}\boldsymbol{W}_k) + \mathcal{A}^{bb}_{ik}\boldsymbol{b}(\boldsymbol{b}\boldsymbol{W}_k) \\ &+ \mathcal{A}^{\omega b}_{ik}[\boldsymbol{\omega}(\boldsymbol{b}\boldsymbol{W}_k) + \boldsymbol{b}(\boldsymbol{\omega}\boldsymbol{W}_k)] \\ &+ \mathcal{A}^{\omega\omega b}_{ik}\{\boldsymbol{\omega}([\boldsymbol{\omega}\times\boldsymbol{b}]\boldsymbol{W}_k) + [\boldsymbol{\omega}\times\boldsymbol{b}](\boldsymbol{\omega}\boldsymbol{W}_k)\} \\ &+ \mathcal{A}^{b\omega b}_{ik}\{\boldsymbol{b}([\boldsymbol{\omega}\times\boldsymbol{b}]\boldsymbol{W}_k) + [\boldsymbol{\omega}\times\boldsymbol{b}](\boldsymbol{b}\boldsymbol{W}_k)\} \\ &+ \mathcal{A}^{\omega-b}_{ik}[\boldsymbol{\omega}(\boldsymbol{b}\boldsymbol{W}_k) - \boldsymbol{b}(\boldsymbol{\omega}\boldsymbol{W}_k)] + \mathcal{A}^{\omega}_{ik}[\boldsymbol{W}_k\times\boldsymbol{\omega}] + \mathcal{A}^{b}_{ik}[\boldsymbol{W}_k\times\boldsymbol{b}],\end{aligned} \tag{73}$$

where $\boldsymbol{W}_k$ is the spatial part of the four-vector $W^\mu_{(k)}$: $W^\mu_{(k)} = (0, \boldsymbol{W}_k)$.

Plugging Eq. (72) into the Onsager relation (68), we get

$$\begin{aligned}&\mathcal{A}^{\perp}_{ik} = \mathcal{A}^{\perp}_{ki}, \quad \mathcal{A}^{\omega\omega}_{ik} = \mathcal{A}^{\omega\omega}_{ki}, \quad \mathcal{A}^{bb}_{ik} = \mathcal{A}^{bb}_{ki}, \\ &\mathcal{A}^{\omega b}_{ik} = \mathcal{A}^{\omega b}_{ki}, \quad \mathcal{A}^{\omega\omega b}_{ik} = -\mathcal{A}^{\omega\omega b}_{ki}, \quad \mathcal{A}^{b\omega b}_{ik} = -\mathcal{A}^{b\omega b}_{ki}, \\ &\mathcal{A}^{\omega-b}_{ik} = -\mathcal{A}^{\omega-b}_{ki}, \quad \mathcal{A}^{\omega}_{ik} = \mathcal{A}^{\omega}_{ki}, \quad \mathcal{A}^{b}_{ik} = \mathcal{A}^{b}_{ki}.\end{aligned} \tag{74}$$

As one can check by substituting Eqs. (61), (72), and (74) into the entropy generation equation (63), the coefficients $\mathcal{A}^{\omega\omega b}_{ik}$, $\mathcal{A}^{b\omega b}_{ik}$, $\mathcal{A}^{\omega}_{ik}$, and $\mathcal{A}^{b}_{ik}$ are nondissipative and do not contribute to the entropy generation.

---

[15]The minus sign in Eq. (70) appears because $d_{(k)\nu}$ and $W_{(k)\nu}$ have different parity under time reversal $t \to -t$ [74].

[16]To make this point clearer, let us work in the comoving frame, choosing x axis along the direction $\boldsymbol{\omega}$ and z axis along $[\boldsymbol{\omega}\times\boldsymbol{b}]$. Then, introducing unit vectors $y^\mu \equiv \frac{b^\mu - b^\alpha\omega_\alpha\omega^\mu}{\| b^\nu - b^\alpha\omega_\alpha\omega^\nu \|} = (0,0,1,0)$ and $z^\mu \equiv -y_\alpha\omega^{\mu\alpha} = (0,0,0,1)$, one can generally decompose $\mathcal{A}^{\mu\nu}_{ik}$ into the sum of nine linearly independent tensors:

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{ik} &= \mathcal{A}^{11}_{ik}\omega^\mu\omega^\nu + \mathcal{A}^{12}_{ik}\omega^\mu y^\nu + \mathcal{A}^{13}_{ik}\omega^\mu z^\nu + \mathcal{A}^{21}_{ik}y^\mu\omega^\nu + \mathcal{A}^{22}_{ik}y^\mu y^\nu \\ &\quad + \mathcal{A}^{23}_{ik}y^\mu z^\nu + \mathcal{A}^{31}_{ik}z^\mu\omega^\nu + \mathcal{A}^{32}_{ik}z^\mu y^\nu + \mathcal{A}^{33}_{ik}z^\mu z^\nu,\end{aligned}$$

where the scalar coefficients $\mathcal{A}^{11}_{ik}, \mathcal{A}^{12}_{ik}\ldots$ may depend on the angle between $\boldsymbol{\omega}$ and $\boldsymbol{b}$. One can directly check that the nine tensors entering Eq. (72) are indeed linearly independent and they can be expressed as linear combinations of $\omega^\mu\omega^\nu$, $\omega^\mu y^\nu$, $\omega^\mu z^\nu$, etc.

The same consideration also applies to the transport coefficients $\mathcal{B}_{ik}^{\mu\nu}$, $\mathcal{C}_{ik}^{\mu\nu}$, and $\mathcal{D}_{ik}^{\mu\nu}$. The result is

$$\begin{aligned}\mathcal{B}_{ik}^{\mu\nu} &= \mathcal{B}_{ik}^{\perp}\perp^{\mu\nu} + \mathcal{B}_{ik}^{\omega\omega}\omega^\mu\omega^\nu + \mathcal{B}_{ik}^{bb}b^\mu b^\nu \\ &+ \mathcal{B}_{ik}^{\omega b}(\omega^\mu b^\nu + \omega^\nu b^\mu) + \mathcal{B}_{ik}^{\omega\omega b}(\omega^\mu\omega_\alpha b^{\nu\alpha} + \omega^\nu\omega_\alpha b^{\mu\alpha}) \\ &+ \mathcal{B}_{ik}^{b\omega b}(b^\mu\omega_\alpha b^{\nu\alpha} + b^\nu\omega_\alpha b^{\mu\alpha}) \\ &+ \mathcal{B}_{ik}^{\omega - b}(\omega^\mu b^\nu - \omega^\nu b^\mu) + \mathcal{B}_{ik}^{\omega}\omega^{\mu\nu} + \mathcal{B}_{ik}^{b}b^{\mu\nu}, \end{aligned} \tag{75}$$

$$\begin{aligned}\mathcal{C}_{ik}^{\mu\nu} &= \mathcal{C}_{ik}^{\perp}\perp^{\mu\nu} + \mathcal{C}_{ik}^{\omega\omega}\omega^\mu\omega^\nu + \mathcal{C}_{ik}^{bb}b^\mu b^\nu \\ &+ \mathcal{C}_{ik}^{\omega b}(\omega^\mu b^\nu + \omega^\nu b^\mu) + \mathcal{C}_{ik}^{\omega\omega b}(\omega^\mu\omega_\alpha b^{\nu\alpha} + \omega^\nu\omega_\alpha b^{\mu\alpha}) \\ &+ \mathcal{C}_{ik}^{b\omega b}(b^\mu\omega_\alpha b^{\nu\alpha} + b^\nu\omega_\alpha b^{\mu\alpha}) \\ &+ \mathcal{C}_{ik}^{\omega - b}(\omega^\mu b^\nu - \omega^\nu b^\mu) + \mathcal{C}_{ik}^{\omega}\omega^{\mu\nu} + \mathcal{C}_{ik}^{b}b^{\mu\nu}, \end{aligned} \tag{76}$$

$$\begin{aligned}\mathcal{D}_{ik}^{\mu\nu} &= \mathcal{D}_{ik}^{\perp}\perp^{\mu\nu} + \mathcal{D}_{ik}^{\omega\omega}\omega^\mu\omega^\nu + \mathcal{D}_{ik}^{bb}b^\mu b^\nu \\ &+ \mathcal{D}_{ik}^{\omega b}(\omega^\mu b^\nu + \omega^\nu b^\mu) + \mathcal{D}_{ik}^{\omega\omega b}(\omega^\mu\omega_\alpha b^{\nu\alpha} + \omega^\nu\omega_\alpha b^{\mu\alpha}) \\ &+ \mathcal{D}_{ik}^{b\omega b}(b^\mu\omega_\alpha b^{\nu\alpha} + b^\nu\omega_\alpha b^{\mu\alpha}) \\ &+ \mathcal{D}_{ik}^{\omega - b}(\omega^\mu b^\nu - \omega^\nu b^\mu) + \mathcal{D}_{ik}^{\omega}\omega^{\mu\nu} + \mathcal{D}_{ik}^{b}b^{\mu\nu}. \end{aligned} \tag{77}$$

The Onsager principle for $\mathcal{B}_{ik}^{\mu\nu}$ and $\mathcal{C}_{ik}^{\mu\nu}$ (70) implies

$$\begin{aligned} &\mathcal{C}_{ik}^{\perp} = -\mathcal{B}_{ki}^{\perp}, \qquad \mathcal{C}_{ik}^{\omega\omega} = -\mathcal{B}_{ki}^{\omega\omega}, \qquad \mathcal{C}_{ik}^{bb} = -\mathcal{B}_{ki}^{bb}, \\ &\mathcal{C}_{ik}^{\omega b} = -\mathcal{B}_{ki}^{\omega b}, \qquad \mathcal{C}_{ik}^{\omega\omega b} = \mathcal{B}_{ki}^{\omega\omega b}, \qquad \mathcal{C}_{ik}^{b\omega b} = \mathcal{B}_{ki}^{b\omega b}, \\ &\mathcal{C}_{ik}^{\omega-b} = \mathcal{B}_{ki}^{\omega-b}, \qquad \mathcal{C}_{ik}^{\omega} = -\mathcal{B}_{ki}^{\omega}, \qquad \mathcal{C}_{ik}^{b} = -\mathcal{B}_{ki}^{b}. \end{aligned} \tag{78}$$

Note that the coefficients $\mathcal{B}_{ik}^{\perp}$, $\mathcal{B}_{ik}^{\omega\omega}$, $\mathcal{B}_{ik}^{bb}$, $\mathcal{B}_{ik}^{\omega b}$, and $\mathcal{B}_{ik}^{\omega-b}$ are nondissipative, in contrast to the analogous coefficients $\mathcal{A}_{ik}^{\perp}$, $\mathcal{A}_{ik}^{\omega\omega}$, $\mathcal{A}_{ik}^{bb}$, $\mathcal{A}_{ik}^{\omega b}$, and $\mathcal{A}_{ik}^{\omega-b}$.

The Onsager principle for $\mathcal{D}_{ik}^{\mu\nu}$ (69) leads to

$$\begin{aligned} &\mathcal{D}_{ik}^{\perp} = \mathcal{D}_{ki}^{\perp}, \qquad \mathcal{D}_{ik}^{\omega\omega} = \mathcal{D}_{ki}^{\omega\omega}, \qquad \mathcal{D}_{ik}^{bb} = \mathcal{D}_{ki}^{bb}, \\ &\mathcal{D}_{ik}^{\omega b} = \mathcal{D}_{ki}^{\omega b}, \qquad \mathcal{D}_{ik}^{\omega\omega b} = -\mathcal{D}_{ki}^{\omega\omega b}, \qquad \mathcal{D}_{ik}^{b\omega b} = -\mathcal{D}_{ki}^{b\omega b}, \\ &\mathcal{D}_{ik}^{\omega-b} = -\mathcal{D}_{ki}^{\omega-b}, \qquad \mathcal{D}_{ik}^{\omega} = \mathcal{D}_{ki}^{\omega}, \qquad \mathcal{D}_{ik}^{b} = \mathcal{D}_{ki}^{b}. \end{aligned} \tag{79}$$

The coefficients $\mathcal{D}_{ik}^{\omega\omega b}$, $\mathcal{D}_{ik}^{b\omega b}$, $\mathcal{D}_{ik}^{\omega}$, and $\mathcal{D}_{ik}^{b}$ are nondissipative, similarly to $\mathcal{A}_{ik}^{\omega\omega b}$, $\mathcal{A}_{ik}^{b\omega b}$, $\mathcal{A}_{ik}^{\omega}$, and $\mathcal{A}_{ik}^{b}$.

In this section we have derived the general expressions for the transport coefficients $\mathcal{A}_{ik}^{\mu\nu}$ (72), $\mathcal{B}_{ik}^{\mu\nu}$ (75), $\mathcal{C}_{ik}^{\mu\nu}$ (76), and $\mathcal{D}_{ik}^{\mu\nu}$ (77), which describe mutual friction (61) and diffusion (62) effects, for the system with two preferred directions. These coefficients have similar tensor structure and can be presented as a sum of six symmetric and three antisymmetric tensor terms, which are purely spatial in the comoving frame, and describe anisotropy of mutual friction and diffusion effects in such a system. The Onsager principle (68)–(70) reduces the number of independent coefficients, imposing additional constraints on $\mathcal{A}_{ik}^{\mu\nu}$ and $\mathcal{D}_{ik}^{\mu\nu}$ and allowing one to express the coefficients $\mathcal{C}_{ik}^{\mu\nu}$ through $\mathcal{B}_{ik}^{\mu\nu}$. Note also that the transport coefficients (and, consequently, the quantities $f_{(i)}^{\mu}$ and $\Delta j_{(i)}^{\mu}$) have both dissipative and nondissipative contributions; i.e., not all the terms in the expressions for $f_{(i)}^{\mu}$ and $\Delta j_{(i)}^{\mu}$ lead to entropy generation in Eq. (63).

B. One preferred direction

Now let us assume that there is only one preferred direction in the system, $b^\mu = \omega^\mu$; i.e., either proton and neutron vortices are aligned with each other, or there is only one sort of vortices in the system. In this case, the expressions (72) and (75)–(77) acquire the same form as the diffusion coefficients from DGS20:

$$\mathcal{A}_{ik}^{\mu\nu} = \mathcal{A}_{ik}^{\parallel}\omega^\mu\omega^\nu + \mathcal{A}_{ik}^{\perp}(\perp^{\mu\nu} - \omega^\mu\omega^\nu) + \mathcal{A}_{ik}^{H}\omega^{\mu\nu}, \tag{80}$$

$$\mathcal{B}_{ik}^{\mu\nu} = \mathcal{B}_{ik}^{\parallel}\omega^\mu\omega^\nu + \mathcal{B}_{ik}^{\perp}(\perp^{\mu\nu} - \omega^\mu\omega^\nu) + \mathcal{B}_{ik}^{H}\omega^{\mu\nu}, \tag{81}$$

$$\mathcal{C}_{ik}^{\mu\nu} = \mathcal{C}_{ik}^{\parallel}\omega^\mu\omega^\nu + \mathcal{C}_{ik}^{\perp}(\perp^{\mu\nu} - \omega^\mu\omega^\nu) + \mathcal{C}_{ik}^{H}\omega^{\mu\nu}, \tag{82}$$

$$\mathcal{D}_{ik}^{\mu\nu} = \mathcal{D}_{ik}^{\parallel}\omega^\mu\omega^\nu + \mathcal{D}_{ik}^{\perp}(\perp^{\mu\nu} - \omega^\mu\omega^\nu) + \mathcal{D}_{ik}^{H}\omega^{\mu\nu}, \tag{83}$$

where $\mathcal{A}_{ik}^{\parallel} \equiv \mathcal{A}_{ik}^{\perp} + \mathcal{A}_{ik}^{\omega\omega} + \mathcal{A}_{ik}^{bb} + 2\mathcal{A}_{ik}^{\omega b}$, $\mathcal{A}_{ik}^{H} \equiv \mathcal{A}_{ik}^{\omega} + \mathcal{A}_{ik}^{b}$, and analogous definitions apply to $\mathcal{B}_{ik}^{\parallel}$, $\mathcal{B}_{ik}^{H}$, $\mathcal{C}_{ik}^{\parallel}$, $\mathcal{C}_{ik}^{H}$, $\mathcal{D}_{ik}^{\parallel}$, and $\mathcal{D}_{ik}^{H}$. The Onsager relations (74), (78), and (79) then imply

$$\mathcal{A}_{ik}^{\parallel} = \mathcal{A}_{ki}^{\parallel}, \qquad \mathcal{A}_{ik}^{\perp} = \mathcal{A}_{ki}^{\perp}, \qquad \mathcal{A}_{ik}^{H} = \mathcal{A}_{ki}^{H}, \tag{84}$$

$$\mathcal{C}_{ik}^{\parallel} = -\mathcal{B}_{ki}^{\parallel}, \qquad \mathcal{C}_{ik}^{\perp} = -\mathcal{B}_{ki}^{\perp}, \qquad \mathcal{C}_{ik}^{H} = -\mathcal{B}_{ki}^{H}, \tag{85}$$

$$\mathcal{D}_{ik}^{\parallel} = \mathcal{D}_{ki}^{\parallel}, \qquad \mathcal{D}_{ik}^{\perp} = \mathcal{D}_{ki}^{\perp}, \qquad \mathcal{D}_{ik}^{H} = \mathcal{D}_{ki}^{H}. \tag{86}$$

The coefficients $\mathcal{A}_{ki}^{H}$, $\mathcal{D}_{ki}^{H}$, $\mathcal{B}_{ik}^{\parallel}$, $\mathcal{B}_{ik}^{\perp}$, $\mathcal{C}_{ik}^{\parallel}$, and $\mathcal{C}_{ik}^{\perp}$ are nondissipative.

C. Summary

To sum up, in this section we found a general form of the four-vectors $f_{(i)}^{\mu}$ (61), which encode all the information about the forces acting on neutron and proton vortices, and the diffusive currents $\Delta j_{(i)}^{\mu}$ (62), which describe diffusion, thermodiffusion and thermal conductivity effects. These vectors are expressed as linear combinations of the vectors $W_{(k)\nu}$ and $d_{(k)\nu}$. The transport coefficients $\mathcal{A}_{ik}^{\mu\nu}$, $\mathcal{B}_{ik}^{\mu\nu}$, $\mathcal{C}_{ik}^{\mu\nu}$, and $\mathcal{D}_{ik}^{\mu\nu}$ in these relations depend on the directions of neutron vortices and the magnetic field; they are given by Eqs. (72) and (75)–(77), which reduce to Eqs. (80)–(83) in the case of single preferred direction. The transport coefficients satisfy the Onsager relations (68)–(70), which imply Eqs. (74), (78), and (79) for a system with two preferred directions, and Eqs. (84)–(86) for a system with a single preferred direction.

We emphasize the presence of cross-coefficients $\mathcal{B}_{ik}^{\mu\nu}$ and $\mathcal{C}_{ik}^{\mu\nu}$, describing the interplay of diffusion and mutual friction effects: the diffusive forces $d_{(k)\nu}$ affect particle velocities (or currents $\Delta j^\mu_{(i)}$), which, in turn, influence the vortex motion via the mutual friction mechanism (and vice versa).

V. DIFFUSION AND MUTUAL FRICTION IN NS MATTER: SPECIAL CASES

Let us apply the general formulas from the previous section to a number of interesting limiting cases, in which these formulas can be substantially simplified.

A. Isotropic matter: Neutrons are superfluid, protons are superconducting, no vortices

In the absence of vortices and any preferred direction the four-vectors $f^\mu_{(i)}$ vanish in view of Eqs. (23) and (59). Therefore, due to Eqs. (61) and (70), $\mathcal{A}_{ik}^{\mu\nu}=\mathcal{B}_{ik}^{\mu\nu}=\mathcal{C}_{ik}^{\mu\nu}=0$. As in normal (nonsuperfluid and nonsuperconducting) MHD (see DGS20), the generalized diffusion coefficient $\mathcal{D}_{ik}^{\mu\nu}$ in the isotropic matter is then simply given by

$$\mathcal{D}_{ik}^{\mu\nu}=\mathcal{D}_{ki}^{\mu\nu}=\mathcal{D}_{ik}\perp^{\mu\nu}, \tag{87}$$

and the entropy generation equation (63) reduces to

$$\partial_\mu\left(Su^\mu-\frac{\mu_i}{T}\Delta j^\mu_{(i)}\right)=\mathcal{D}_{ik}d_{(i)\mu}d_{(k)\mu}. \tag{88}$$

The generalized diffusion coefficients $\mathcal{D}_{ik}$ in superfluid matter can be expressed through the momentum transfer rates of microscopic theory similarly to how it is done in DGS20 for normal matter [75].

B. Magnetized $npe\mu$ matter with superfluid neutrons and normal protons, no vortices

Now let us consider magnetized $npe\mu$ matter with superfluid neutrons in the absence of vortices. Then the only preferred direction is that of the magnetic field, b^μ. The four-vector $f^\mu_{(i)}$ vanishes in view of Eqs. (23) and (59), but $W^\mu_{(i)}$, generally, differs from zero. Therefore, due to Eqs. (61) and (85), $\mathcal{A}_{ik}^{\mu\nu}=\mathcal{B}_{ik}^{\mu\nu}=\mathcal{C}_{ik}^{\mu\nu}=0$. As a result, $\Delta j^\mu_{(i)}$ has exactly the same form as in the nonsuperfluid magnetized matter (cf. DGS20):

$$\Delta j^\mu_{(i)}=-\mathcal{D}^{\parallel}_{ik}b^\mu b^\nu d_{(k)\nu}-\mathcal{D}^{\perp}_{ik}(\perp^{\mu\nu}-b^\mu b^\nu)d_{(k)\nu} - \mathcal{D}^{H}_{ik}b^{\mu\nu}d_{(k)\nu}, \tag{89}$$

where $i,k=n,\ p,\ e,\ \mu$. The entropy generation equation (63) reduces to

$$\partial_\mu\left(Su^\mu-\frac{\mu_i}{T}\Delta j^\mu_{(i)}\right)=\mathcal{D}^{\parallel}_{ik}b^\mu b^\nu d_{(i)\mu}d_{(k)\nu} + \mathcal{D}^{\perp}_{ik}(\perp^{\mu\nu}-b^\mu b^\nu)d_{(i)\mu}d_{(k)\nu}. \tag{90}$$

C. Unmagnetized $npe\mu$ matter with superfluid neutron vortices

In this example, we discuss the unmagnetized $npe\mu$ matter, allowing for the presence of superfluid neutron vortices and diffusion. Protons can be either normal or superconducting. The dynamic equations for such a system allow us to simultaneously study the combined effect of particle diffusion [40] and mutual friction dissipation [76] on damping of NS oscillations and development of various instabilities in NSs.

Since in real NSs the typical areal density of neutron vortices is small [24] (the intervortex spacing is much larger than the particle mean-free path), they have a negligible effect on the diffusion coefficients $\mathcal{D}_{ik}^{\mu\nu}$, which remain approximately isotropic. Because of the same reason, the difference between the velocities of normal particle species (e.g., electrons and muons or electrons and neutron Bogoliubov thermal excitations) is small in comparison to the difference between any of these velocities and the neutron vortex velocity $\boldsymbol{V}_{\mathrm{L}n}$. Consequently, when calculating the force acting on neutron vortices from a particle species i [see Eq. (B2), where a similar force on proton vortices is presented], one can replace $\boldsymbol{V}_i-\boldsymbol{V}_{\mathrm{L}n}$ with $\boldsymbol{V}_{\mathrm{norm}}-\boldsymbol{V}_{\mathrm{L}n}$, where $\boldsymbol{V}_{\mathrm{norm}}$ is the average velocity of normal (nonsuperfluid) component (A3). This approximation allows one to neglect the cross-coefficients $\mathcal{B}_{ik}^{\mu\nu}$ and $\mathcal{C}_{ik}^{\mu\nu}$,[17] that is, to decouple the diffusion and mutual friction mechanisms. As a result, with the help of Eqs. (80) and (87), Eqs. (61) and (62) reduce to

$$-\frac{\mu_n n_n^2}{T}f^\mu_{(n)}=-\mathcal{A}^{\parallel}_{nn}\omega^\mu\omega^\nu W_{(n)\nu}-\mathcal{A}^{\perp}_{nn}(\perp^{\mu\nu}-\omega^\mu\omega^\nu)W_{(n)\nu} - \mathcal{A}^{H}_{nn}\omega^{\mu\nu}W_{(n)\nu}, \tag{91}$$

$$f^\mu_{(p)}=0, \tag{92}$$

$$\Delta j^\mu_{(i)}=-\mathcal{D}_{ik}d^\mu_{(k)}. \tag{93}$$

Here the coefficients $\mathcal{A}^{\perp}_{nn}$, $\mathcal{A}^{\parallel}_{nn}$, and $\mathcal{A}^{H}_{nn}$ describe the mutual friction effect. In order to relate them to the

[17] In principle, these coefficients can be calculated in exactly the same way as it is done for superfluid and superconducting $npe\mu$ matter with proton flux tubes in Appendix B (see also Sec. V D). Note, however, that the typical areal density of proton flux tubes in NSs is comparable to particle mean-free path [44]; hence, the cross-coefficients $\mathcal{B}_{ik}^{\mu\nu}$ and $\mathcal{C}_{ik}^{\mu\nu}$ for this problem are not small and should be accounted for.

commonly used mutual friction parameters α_n, β_n, and γ_n [25,26,72], one has to compare Eq. (91) with the analogous equation (98) in GD16, which reads, in our notation,

$$f^{\mu}_{(n)} = \alpha_n \mathcal{V}_{(\mathrm{M}n)} \omega^{\mu\nu} W_{(n)\nu} + (\beta_n - \gamma_n)\mathcal{V}_{(\mathrm{M}n)} \omega^{\mu\alpha}\omega^{\nu}{}_{\alpha} W_{(n)\nu} + \gamma_n \mathcal{V}_{(\mathrm{M}n)} \perp^{\mu\nu} W_{(n)\nu}, \tag{94}$$

where $\mathcal{V}_{(\mathrm{M}n)}$ is defined by Eq. (32). Using the identity $\omega^{\mu\alpha}\omega^{\nu}{}_{\alpha} \equiv \perp^{\mu\nu} - \omega^{\mu}\omega^{\nu}$, we find

$$\mathcal{A}^{H}_{nn} = \frac{\mu_n n_n^2}{c^3 T}\mathcal{V}_{(\mathrm{M}n)}\alpha_n, \qquad \mathcal{A}^{\perp}_{nn} = \frac{\mu_n n_n^2}{c^3 T}\mathcal{V}_{(\mathrm{M}n)}\beta_n,$$
$$\mathcal{A}^{\parallel}_{nn} = \frac{\mu_n n_n^2}{c^3 T}\mathcal{V}_{(\mathrm{M}n)}\gamma_n, \tag{95}$$

where we, for practical convenience, restored the speed of light c. We should stress that, generally, diffusion affects the coefficients $\mathcal{A}^{\mu\nu}_{ik}$ (see Sec. V D and Appendix B), and they cannot be always expressed only through the mutual friction parameters α_i, β_i, and γ_i of nondiffusive superfluid hydrodynamics.

It is also worth noting that, if we allow for the presence of the magnetic field (assuming that protons are nonsuperconducting and thus $f^{\mu}_{(p)} = 0$) but neglect its effect on the neutron vortices, then expression (91) for $f^{\mu}_{(n)}$ will remain the same, while the expression for $\Delta j^{\mu}_{(i)}$ should be replaced with Eq. (89) to account for anisotropy of diffusion in the magnetic field.

D. Magnetized *npeμ* matter with superfluid neutrons (no vortices) and type-II proton superconductivity

This limit is interesting if we want to study magnetothermal evolution in slowly rotating superconducting neutron stars with type-II proton superconductivity. It is expected that in this problem neutron vortices do not play a major role [77] and can be neglected in the first approximation. At the same time, the combined effect of diffusion (i.e., relative motions of different particle species) and mutual friction dissipation related to the presence of proton vortices (flux tubes) appears to be crucial for this problem [41] and should be accounted for. Note that, for instance, electron–flux tube interaction is comparable to (and even stronger than) the electron-muon interaction (see, e.g., Ref. [41] and Appendix B). Thus, in contrast to the previous case, here we cannot decouple diffusion and mutual friction effects.

Since we ignore neutron vortices, we are left with only one preferred direction, $\boldsymbol{b}$. The full system of dynamic equations in this situation is provided in Sec. VI, and here we only present the expressions for $f^{\mu}_{(i)}$ and $\Delta j^{\mu}_{(i)}$. In the absence of neutron vortices $f^{\mu}_{(n)}$ vanishes, as do the coefficients $\mathcal{A}^{\mu\nu}_{nk} = \mathcal{B}^{\mu\nu}_{nk} = \mathcal{C}^{\mu\nu}_{kn} = 0$. Thus, the general form of the vectors $f^{\mu}_{(i)}$ and $\Delta j^{\mu}_{(i)}$ is ($i, k = n,\ p,\ e,\ \mu$)

$$f^{\mu}_{(n)} = 0, \tag{96}$$

$$-\frac{\mu_p n_p^2}{T} f^{\mu}_{(p)} = -\mathcal{A}^{\mu\nu}_{pp} W_{(p)\nu} - \mathcal{B}^{\mu\nu}_{pk} d_{(k)\nu}, \tag{97}$$

$$\Delta j^{\mu}_{(i)} = -\mathcal{C}^{\mu\nu}_{ip} W_{(p)\nu} - \mathcal{D}^{\mu\nu}_{ik} d_{(k)\nu}, \tag{98}$$

or, using Eqs. (80), (81)–(83), and (85) (with ω^{μ} replaced by b^{μ} and with $\omega^{\mu\nu}$ replaced by $b^{\mu\nu}$),

$$-\frac{\mu_p n_p^2}{T} f^{\mu}_{(p)} = -\mathcal{A}^{\parallel}_{pp} b^{\mu}b^{\nu} W_{(p)\nu} - \mathcal{A}^{\perp}_{pp}(\perp^{\mu\nu} - b^{\mu}b^{\nu}) W_{(p)\nu} - \mathcal{A}^{H}_{pp} b^{\mu\nu} W_{(p)\nu} - \mathcal{B}^{\parallel}_{pk} b^{\mu}b^{\nu} d_{(k)\nu} - \mathcal{B}^{\perp}_{pk}(\perp^{\mu\nu} - b^{\mu}b^{\nu}) d_{(k)\nu} - \mathcal{B}^{H}_{pk} b^{\mu\nu} d_{(k)\nu}, \tag{99}$$

$$\Delta j^{\mu}_{(i)} = -\mathcal{C}^{\parallel}_{ip} b^{\mu}b^{\nu} W_{(p)\nu} - \mathcal{C}^{\perp}_{ip}(\perp^{\mu\nu} - b^{\mu}b^{\nu}) W_{(p)\nu} - \mathcal{C}^{H}_{ip} b^{\mu\nu} W_{(p)\nu} - \mathcal{D}^{\parallel}_{ik} b^{\mu}b^{\nu} d_{(k)\nu} - \mathcal{D}^{\perp}_{ik}(\perp^{\mu\nu} - b^{\mu}b^{\nu}) d_{(k)\nu} - \mathcal{D}^{H}_{ik} b^{\mu\nu} d_{(k)\nu}. \tag{100}$$

The phenomenological coefficients in Eqs. (99) and (100) can be expressed through microscopic quantities (mutual friction parameters and momentum transfer rates), as shown in Appendix B in the simple case of vanishing entrainment and $T = 0$. The cross terms in Eq. (100), containing the coefficients $\mathcal{B}^{\parallel}_{pi} = -\mathcal{C}^{\parallel}_{ip}$, $\mathcal{B}^{\perp}_{pi} = -\mathcal{C}^{\perp}_{ip}$, and $\mathcal{B}^{H}_{pi} = -\mathcal{C}^{H}_{ip}$, lead to interference between the diffusion and mutual friction effects.

Note, in passing, that if the neutron vortices are present but do not affect the diffusive currents (see Sec. V C) and do not interact with proton vortices, then the expressions for $f^{\mu}_{(p)}$ (99) and $\Delta j^{\mu}_{(i)}$ (100) will remain the same, whereas $f^{\mu}_{(n)}$ will be given by Eq. (91).

VI. FULL SYSTEM OF EQUATIONS IN THE MHD APPROXIMATION FOR *npeμ* MIXTURE WITH PROTON VORTICES

In this section we formulate the full system of MHD equations for magnetized $npe\mu$ matter, accounting for neutron superfluidity as well as type-II proton superconductivity and adopting the "MHD approximation" from GD16. The resulting set of equations, presented in Sec. VI B, is suitable for, e.g., studying the combined quasistationary evolution of the magnetic field and temperature in slowly rotating superconducting NSs. For practical convenience, below in this section we do not set $c = 1$.

A. "Magnetohydrodynamic" approximation

First, let us briefly summarize the main consequences of the "MHD approximation" formulated in Sec. VIII of GD16, which allows us to substantially simplify the general equations of Sec. II. This approximation is mainly based on the fact that, under typical NS conditions (and assuming type-II proton superconductivity), the magnetic induction $\boldsymbol{B}$ is much larger than the fields $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ defined in the comoving frame. For actual calculations, one also has to specify a microscopic model that allows one to express the four-vectors D^μ, H^μ, $\mathcal{W}^\mu_{(\mathrm{E}i)}$ and $\mathcal{W}^\mu_{(\mathrm{M}i)}$ through E^μ, B^μ, $\mathcal{V}^\mu_{(\mathrm{E}i)}$ and $\mathcal{V}^\mu_{(\mathrm{M}i)}$. For definiteness, below we use the simple model of noninteracting vortices from Appendix G2 of GD16; note, however, that the MHD approximation can be formulated for other microscopic models in a similar way.

As discussed in Ref. [24] and GD16, the magnetic field $\boldsymbol{H}$ is related to the magnetic induction $\boldsymbol{B}$ as[18]

$$\boldsymbol{H} = \boldsymbol{B} - \boldsymbol{B}_{\mathrm{V}n} - \boldsymbol{B}_{\mathrm{V}p}, \tag{101}$$

where $\boldsymbol{B}_{\mathrm{V}i}$ is the magnetic induction associated with neutron ($i = n$) or proton ($i = p$) vortices. In other words, $\boldsymbol{H}$ coincides with the London field generated by NS rotation, $|\boldsymbol{H}| \sim 2 \times 10^{-2}[\Omega/(100\ \mathrm{s}^{-1})]\ \mathrm{G} \ll |\boldsymbol{B}| \sim 10^{12}$ G, where Ω is the NS spin frequency. This field, as well as $\boldsymbol{B}_{\mathrm{V}n}$,[19] is neglected in comparison to $\boldsymbol{B}_{\mathrm{V}p}$ in the MHD approximation: all the magnetic induction is assumed to be locked to proton vortices, $\boldsymbol{B} \approx \boldsymbol{B}_{\mathrm{V}p}$.

Similarly, the fields $\boldsymbol{D}$ and $\boldsymbol{E}$ are related as

$$\boldsymbol{D} = \boldsymbol{E} - \boldsymbol{E}_{\mathrm{V}n} - \boldsymbol{E}_{\mathrm{V}p}. \tag{102}$$

Here the electric field $\boldsymbol{E}_{\mathrm{V}i}$ is generated by vortex motion, $\boldsymbol{E}_{\mathrm{V}i} = -(1/c)\boldsymbol{V}_{\mathrm{L}i} \times \boldsymbol{B}_{\mathrm{V}i}$, where $\boldsymbol{V}_{\mathrm{L}i}$ is the vortex velocity, which is assumed to be nonrelativistic; the electric induction $\boldsymbol{D}$ is of the order of small gradients of thermodynamic functions, $|\boldsymbol{D}| \sim |\boldsymbol{\nabla}\mu_i|/e_p$. Both $\boldsymbol{E}$ and $\boldsymbol{D}$ are much smaller than $\boldsymbol{B}$.

Since the vectors $\boldsymbol{D}$ and $\boldsymbol{H}$ are small, it follows from the second pair of Maxwell equations (18) that the total free electric current density $J^\mu_{(\mathrm{free})}$ should also be exceptionally small, much smaller than the individual contributions to $J^\mu_{(\mathrm{free})}$ from each particle species. This observation enables us to make further simplification by discarding Maxwell equations (18) but instead requiring that the free electric current density $J^\mu_{(\mathrm{free})}$ should vanish [this approximation is well known in the literature and is further discussed by us around Eqs. (128) and (129)]:

$$J^\mu_{(\mathrm{free})} = e_i n_i u^\mu + e_i Y_{ik} w^\mu_{(k)} + e_i \Delta j^\mu_{(i)} = 0. \tag{103}$$

Now let us turn to the vortex-related vectors $\mathcal{V}^\mu_{(\mathrm{E}i)}$, $\mathcal{V}^\mu_{(\mathrm{M}i)}$, $\mathcal{W}^\mu_{(\mathrm{E}i)}$, and $\mathcal{W}^\mu_{(\mathrm{M}i)}$ [or, equivalently, to the corresponding tensors ${}^{\parallel}\mathcal{V}^{\mu\nu}_{(i)}$, ${}^{\perp}\mathcal{V}^{\mu\nu}_{(i)}$, ${}^{\parallel}\mathcal{W}^{\mu\nu}_{(i)}$, and ${}^{\perp}\mathcal{W}^{\mu\nu}_{(i)}$; see Eqs. (40) and (41)]. The number of proton vortices is typically larger by more than 10 orders of magnitude than the number of neutron vortices (see, e.g., Ref. [24]). Consequently, the four-vector $\mathcal{V}^\mu_{(\mathrm{M}n)}$ can be neglected in comparison to $\mathcal{V}^\mu_{(\mathrm{M}p)}$ in the expressions for $d\varepsilon_{\mathrm{add}}$ (26) and $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ (37), since the lengths of these vectors are proportional to the number of vortices, as follows from Eq. (24). Note also that in the comoving frame $|\mathcal{V}_{(\mathrm{E}i)}| \sim (V_{\mathrm{L}i}/c)|\mathcal{V}_{(\mathrm{M}i)}|$; thus, $\mathcal{V}^\mu_{(\mathrm{E}i)}$ can be neglected in comparison to $\mathcal{V}^\mu_{(\mathrm{M}i)}$, and, similarly, $\mathcal{W}^\mu_{(\mathrm{E}i)}$ can be neglected in comparison to $\mathcal{W}^\mu_{(\mathrm{M}i)}$.

Under the above assumptions, the four-vector $\tilde{\mathcal{V}}^\mu_{(\mathrm{M}p)} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu \tilde{\mathcal{V}}_{(i)\alpha\beta}$, which reduces to $(0, m_p \mathrm{curl}\boldsymbol{V}_{sp})$ in the nonrelativistic limit, can be neglected in comparison to $(e_p/c)B^\mu$. Thus, the four-vector $\mathcal{V}^\mu_{(\mathrm{M}p)} = \tilde{\mathcal{V}}^\mu_{(\mathrm{M}p)} + (e_p/c)B^\mu$ [see Eqs. (43) and (50)] reduces to

$$\mathcal{V}^\mu_{(\mathrm{M}p)} = \frac{e_p}{c} B^\mu, \tag{104}$$

which physically means that the magnetic induction is produced by proton vortices.

For a simple microscopic model of noninteracting vortices, the four-vectors $\mathcal{W}^\mu_{(\mathrm{M}i)}$ are related to $\mathcal{V}^\mu_{(\mathrm{M}i)}$ as [see Eqs. (124) and (G9)–(G11) in GD16]

$$\mathcal{W}^\mu_{(\mathrm{M}i)} = \frac{\hat{E}_{\mathrm{V}i}}{\pi\hbar} \frac{\mathcal{V}^\mu_{(\mathrm{M}i)}}{\mathcal{V}_{(\mathrm{M}i)}}, \tag{105}$$

where $\mathcal{V}_{(\mathrm{M}i)} \equiv \sqrt{\mathcal{V}_{(\mathrm{M}i)\alpha}\mathcal{V}^\alpha_{(\mathrm{M}i)}}$, $\hat{E}_{\mathrm{V}i}$ is the vortex energy per unit length specified below, and no summation over i is assumed. $\mathcal{W}^\mu_{(\mathrm{M}p)}$ can also be rewritten in terms of the critical magnetic field $H_{\mathrm{c}1}$ [80]:

$$\mathcal{W}^\mu_{(\mathrm{M}p)} = \frac{c}{4\pi e_p} H_{\mathrm{c}1} \frac{B^\mu}{B}. \tag{106}$$

In this formula $B \equiv (B_\mu B^\mu)^{1/2}$, and $H_{\mathrm{c}1}$ is expressed through $\hat{E}_{\mathrm{V}p}$ as

$$H_{\mathrm{c}1} = \frac{4\pi \hat{E}_{\mathrm{V}p}}{\hat{\phi}_{p0}}, \tag{107}$$

where $\hat{\phi}_{p0} = (\pi\hbar c/e_p)$ is the magnetic flux associated with the proton vortex. The energy $\hat{E}_{\mathrm{V}i}$ per unit length for neutron and proton vortices is given by [see Eqs. (E17) and (E18) in GD16]

[18]Some authors (e.g., [29,78,79]) use a different definition for $\boldsymbol{H}$, identifying it with the critical field $H_{\mathrm{c}1}$; we find that definition less convenient since $\boldsymbol{H}$ defined that way does not satisfy the Maxwell equation (16). Note, however, that both approaches are, in principle, possible and the resulting equations are completely equivalent [29].

[19]$\boldsymbol{B}_{\mathrm{V}i}$ is proportional to the number of vortices per unit area $N_{\mathrm{V}i}$; for typical NS conditions $N_{\mathrm{V}n}$ is less than $N_{\mathrm{V}p}$ by more than 10 orders of magnitude and thus $|\boldsymbol{B}_{\mathrm{V}n}| \ll |\boldsymbol{B}_{\mathrm{V}p}|$.

$$\hat{E}_{Vn} \approx \frac{\pi}{4}\hbar^2 c^2 \frac{Y_{nn}Y_{pp} - Y_{np}^2}{Y_{pp}} \ln\left(\frac{b_n}{\xi_n}\right), \tag{108}$$

$$\hat{E}_{Vp} \approx \frac{\pi}{4}\hbar^2 c^2 Y_{pp} \ln\left(\frac{\delta_p}{\xi_p}\right). \tag{109}$$

In Eqs. (108) and (109) ξ_i is the coherence length for particle species i, δ_p is the London penetration depth for protons, and b_n is some "external" radius of the order of the typical intervortex spacing [25,72]. Note that Eq. (109) (see also Ref. [81] for a nonrelativistic expression) is only applicable to a strong type-II superconductor, i.e., in the limit $\delta_p \gg \xi_p$.

We remind the reader that the expressions (105) for $\mathcal{W}^{\mu}_{(\mathrm{M}i)}$ are valid only for a simple model of noninteracting vortices. If one accounts, e.g., for vortex–flux tube interaction, then both these vectors will depend on $\mathcal{V}^{\mu}_{(\mathrm{M}p)}$ and $\mathcal{V}^{\mu}_{(\mathrm{M}n)}$ simultaneously.

Using the approximations discussed above, one can also simplify the thermodynamic relations. First, all the thermodynamic quantities (e.g., the energy density ε) can be expressed as functions of the variables n_i, S, $w^{\mu}_{(i)}w_{(k)\mu}$, and B:

$$\varepsilon = \varepsilon(n_i, S, w^{\mu}_{(i)}w_{(k)\mu}, B). \tag{110}$$

Second, only the term $\mathcal{W}_{(\mathrm{M}p)\mu}d\mathcal{V}^{\mu}_{(\mathrm{M}p)}$ can be retained in the expression (26) for $d\varepsilon_{\mathrm{add}}$. Thus, in view of the relations (104) and (106), the second law of thermodynamics (25) becomes

$$d\varepsilon = \mu_i dn_i + TdS + \frac{Y_{ik}}{2}d(w^{\alpha}_{(i)}w_{(k)\alpha}) + \frac{1}{4\pi}H_{c1}dB, \tag{111}$$

and the Gibbs-Duhem relation (36), consequently, takes the form

$$dP = n_i d\mu_i + SdT - \frac{Y_{ik}}{2}d(w^{\alpha}_{(i)}w_{(k)\alpha}) - \frac{1}{4\pi}H_{c1}dB. \tag{112}$$

Similarly, only the last term (and only for proton vortices, $i = p$) survives in the expression for $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ (37):

$$\begin{aligned}\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} &= \mathcal{T}^{\mu\nu}_{(\mathrm{VM})} \\ &= {}^{\perp}\mathcal{W}^{\mu\alpha}_{(p)}\,{}^{\perp}\mathcal{V}^{\nu}_{(p)\alpha} + u^{\nu}\,{}^{\perp}\mathcal{W}^{\mu\alpha}_{(p)}\mathcal{V}_{(\mathrm{E}p)\alpha} \\ &\quad + u^{\mu}\,{}^{\perp}\mathcal{W}^{\nu\alpha}_{(p)}\mathcal{V}_{(\mathrm{E}p)\alpha}.\end{aligned} \tag{113}$$

Noting that $\mathcal{V}^{\mu}_{(\mathrm{E}p)} = \mu_p n_p f^{\mu}_{(p)}/c^3$ [see Eq. (59)], and also using the relations (104) and (106), one can transform Eq. (113) to

$$\begin{aligned}\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} =& \frac{H_{c1}B}{4\pi}b^{\mu\alpha}b^{\nu}{}_{\alpha} \\ &+ \frac{\mu_p n_p H_{c1}}{4\pi e_p c^2}(u^{\mu}b^{\nu\alpha}f_{(p)\alpha} + u^{\nu}b^{\mu\alpha}f_{(p)\alpha})\end{aligned} \tag{114}$$

or, equivalently, to

$$\begin{aligned}&\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} \\ &= \frac{H_{c1}B}{4\pi}(\perp^{\mu\nu} - b^{\mu}b^{\nu}) \\ &\quad + \frac{\mu_p n_p H_{c1}}{4\pi e_p c^2}(u^{\mu}\epsilon^{\nu\alpha\beta\gamma}u_{\alpha}f_{(p)\beta}b_{\gamma} + u^{\nu}\epsilon^{\mu\alpha\beta\gamma}u_{\alpha}f_{(p)\beta}b_{\gamma}).\end{aligned} \tag{115}$$

Repeating the derivation of the entropy generation equation (52) with $d\varepsilon$ given by Eq. (111) and $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ given by Eq. (113), one can find that the four-vectors $W^{\mu}_{(n)}$ and $W^{\mu}_{(p)}$ [see Eq. (54)] in the MHD limit should be defined as

$$W^{\mu}_{(n)} \equiv \frac{1}{n_n}cY_{nk}w^{\mu}_{(k)}, \tag{116}$$

$$W^{\mu}_{(p)} = \frac{1}{n_p}\left[cY_{pk}w^{\mu}_{(k)} + \frac{c}{4\pi e_p}\perp^{\mu\nu}\partial^{\alpha}(H_{c1}b_{\nu\alpha})\right]. \tag{117}$$

B. MHD equations

Now, working in the MHD approximation described above, let us formulate the dynamic equations for superconducting NSs with $npe\mu$ cores. We assume that protons form a type-II superconductor, and neutrons are superfluid. However, we ignore the effects of NS rotation and hence assume that there are no neutron vortices in the system, $\mathcal{V}^{\mu\nu}_{(n)} = 0$. Note that neutron vortices can be included separately (see Remark 3). As for the dissipative effects, we consider only diffusion and mutual friction, thus ignoring chemical reactions as well as viscosity (i.e., we set $Q = \Delta\Gamma_i = \Delta\tau^{\mu\nu} = \varkappa_i = 0$). The latter effects can easily be incorporated separately if needed.

The full set of equations allows one to find seven unknown functions B^{μ}, u^{μ}, $w^{\mu}_{(n)}$, n_n, n_e, n_{μ}, and S (all other unknown quantities can be expressed algebraically through these functions) and includes the following.

(1) Continuity equations for neutrons, electrons, and muons describing evolution of n_n, n_e, and n_{μ}, respectively:

$$\partial_{\alpha}j^{\alpha}_{(n)} = \partial_{\alpha}(n_n u^{\alpha} + Y_{nk}w^{\alpha}_{(k)} + \Delta j^{\alpha}_{(n)}) = 0, \tag{118}$$

$$\partial_{\alpha}j^{\alpha}_{(e)} = \partial_{\alpha}(n_e u^{\alpha} + \Delta j^{\alpha}_{(e)}) = 0, \tag{119}$$

$$\partial_{\alpha}j^{\alpha}_{(\mu)} = \partial_{\alpha}(n_{\mu} u^{\alpha} + \Delta j^{\alpha}_{(\mu)}) = 0. \tag{120}$$

(2) Total energy ($\mu = 0$) and momentum ($\mu = 1, 2, 3$) conservation laws (8) describing evolution of the energy density ε and four-velocity u^μ:

$$\partial_\nu T^{\mu\nu} = 0, \tag{121}$$

where

$$T^{\mu\nu} = (P+\varepsilon)u^\mu u^\nu + P g^{\mu\nu} + Y_{ik}(w^\mu_{(i)} w^\nu_{(k)} + \mu_i w^\mu_{(k)} u^\nu + \mu_k w^\nu_{(i)} u^\mu) + \Delta T^{\mu\nu}_{(\mathrm{EM+vortex})} \tag{122}$$

and $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ is specified by Eq. (114). Instead of the energy conservation law, it is convenient to use the entropy generation equation (63):

$$\partial_\mu S^\mu = \partial_\mu\left(S u^\mu - \frac{\mu_i}{T}\Delta j^\mu_{(i)}\right) = \frac{\mu_p n_p^2}{c^3 T} f_{(p)\mu} W^\mu_{(p)} - \Delta j^\mu_{(i)} d_{(i)\mu}. \tag{123}$$

(3) The four-vector $w^\mu_{(n)}$ satisfies the superfluid equation for neutrons, which, in the absence of vortices, reads

$$\mathcal{V}^{\mu\nu}_{(n)} \equiv \frac{1}{c}[\partial^\mu(w^\nu_{(n)} + \mu_n u^\nu) - \partial^\nu(w^\mu_{(n)} + \mu_n u^\mu)] = 0. \tag{124}$$

(4) Magnetic induction evolves according to Maxwell equation (17),

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{125}$$

which, in terms of the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, reads

$$\mathrm{curl}\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{126}$$

$$\mathrm{div}\boldsymbol{B} = 0. \tag{127}$$

The set of equations (118)–(127) contains also unknown quantities n_p, $w^\mu_{(p)}$, $\boldsymbol{E}$, $f^\mu_{(p)}$, and $\Delta j^\mu_{(p)}$, which are expressed algebraically through the seven functions defined above.

First, the quantities n_p and $w^\mu_{(p)}$ can be found from the condition $J^\mu_{(\mathrm{free})} = 0$ (103), which, in view of the constraints (5) and (7), leads to the well-known (and often employed in the literature) quasineutrality (128) and screening (129) conditions [24,26,82]:

$$n_p = n_e + n_\mu, \tag{128}$$

$$Y_{pk} w^\mu_{(k)} + \Delta j^\mu_{(p)} - \Delta j^\mu_{(e)} - \Delta j^\mu_{(\mu)} = 0. \tag{129}$$

Next, the quantities $f^\mu_{(p)}$ (99) and $\Delta j^\mu_{(i)}$ (100) have the following form [note that we restored the factor c^3 in the left-hand side of Eq. (130)]:

$$-\frac{\mu_p n_p^2}{c^3 T} f^\mu_{(p)} = -\mathcal{A}^{\parallel}_{pp} b^\mu b^\nu W_{(p)\nu} - \mathcal{A}^{\perp}_{pp}(\perp^{\mu\nu} - b^\mu b^\nu) W_{(p)\nu} - \mathcal{A}^H_{pp} b^{\mu\nu} W_{(p)\nu} - \mathcal{B}^{\parallel}_{pk} b^\mu b^\nu d_{(k)\nu} - \mathcal{B}^{\perp}_{pk}(\perp^{\mu\nu} - b^\mu b^\nu) d_{(k)\nu} - \mathcal{B}^H_{pk} b^{\mu\nu} d_{(k)\nu}, \tag{130}$$

$$\Delta j^\mu_{(i)} = -\mathcal{C}^{\parallel}_{ip} b^\mu b^\nu W_{(p)\nu} - \mathcal{C}^{\perp}_{ip}(\perp^{\mu\nu} - b^\mu b^\nu) W_{(p)\nu} - \mathcal{C}^H_{ip} b^{\mu\nu} W_{(p)\nu} - \mathcal{D}^{\parallel}_{ik} b^\mu b^\nu d_{(k)\nu} - \mathcal{D}^{\perp}_{ik}(\perp^{\mu\nu} - b^\mu b^\nu) d_{(k)\nu} - \mathcal{D}^H_{ik} b^{\mu\nu} d_{(k)\nu}, \tag{131}$$

where $d_{(i)\mu}$ and $W_{(p)\nu}$ are given by Eqs. (55) and (117), respectively. The transport coefficients $\mathcal{A}^{\parallel}_{pp}$, $\mathcal{A}^{\perp}_{pp}$, $\mathcal{A}^H_{pp}$, $\mathcal{B}^{\parallel}_{pk}$, $\mathcal{B}^{\perp}_{pk}$, $\mathcal{B}^H_{pk}$, $\mathcal{C}^{\parallel}_{ip}$, $\mathcal{C}^{\perp}_{ip}$, $\mathcal{C}^H_{ip}$, $\mathcal{D}^{\parallel}_{ik}$, $\mathcal{D}^{\perp}_{ik}$, and $\mathcal{D}^H_{ik}$ should be expressed through microscopic mutual friction parameters and momentum transfer rates. We discuss these relations in Appendix B.

Finally, the electric field E^μ can be expressed algebraically from the superfluid proton equation (59):

$$u_\nu \mathcal{V}^{\mu\nu}_{(p)} \equiv \frac{1}{c} u_\nu\{\partial^\mu[w^\nu_{(p)} + \mu_p u^\nu] - \partial^\nu[w^\mu_{(p)} + \mu_p u^\mu]\} + \frac{e_p}{c} E^\mu = \frac{\mu_p n_p}{c^3} f^\mu_{(p)}. \tag{132}$$

Note that the right-hand sides of Eqs. (130) and (131) implicitly contain $\Delta j^\mu_{(i)}$ and E^μ[20]; therefore one has to solve Eqs. (130)–(132) simultaneously in order to obtain closed-form expressions for $f^\mu_{(p)}$, $\Delta j^\mu_{(i)}$, and E^μ.

The nonrelativistic version of MHD equations from this section is provided in Appendix A.

Remark 1.—If $\mathcal{A}^{\parallel}_{pp} = \mathcal{B}^{\parallel}_{pi} = 0$, one can define the vortex velocity $v^\mu_{(\mathrm{L}p)}$, satisfying the vorticity transfer equation [25]

$$v_{(\mathrm{L}p)\nu} \mathcal{V}^{\mu\nu}_{(p)} = 0. \tag{133}$$

In analogy with GD16 [see Eq. (101) there], one can find that, up to arbitrary terms parallel to b^μ,

[20] $W^\mu_{(p)}$ depends on the quantity $Y_{pk} w^\mu_{(k)}$ [see the definition (54)], which is expressed through $\Delta j^\mu_{(i)}$ with the help of the screening condition (129). In addition, $d^\mu_{(k)}$ depends on E^μ [see the definition (55)].

$$v^{\mu}_{(\mathrm{L}p)} = u^{\mu} - \frac{cT}{n_p e_p B}(\mathcal{A}^{H}_{pp} W_{(p)\nu} + \mathcal{B}^{H}_{pk} d_{(k)\nu})\perp^{\mu\nu} + \frac{cT}{n_p e_p B}(\mathcal{A}^{\perp}_{pp} W_{(p)\nu} + \mathcal{B}^{\perp}_{pk} d_{(k)\nu}) b^{\mu\nu}. \tag{134}$$

Remark 2.—The MHD equations presented in this section are very similar to those of Sec. VIII in GD16. For the reader's convenience, let us list their main differences from GD16.

(1) Particle currents include the dissipative corrections $\Delta j^{\mu}_{(i)}$.
(2) We use a slightly different definition of $W^{\mu}_{(p)}$ (see footnote 7).
(3) The term $Y_{pk} w^{\mu}_{(k)}$ in the expression (117) for $W^{\mu}_{(p)}$ does not vanish due to the presence of diffusive currents.
(4) $f^{\mu}_{(p)}$ (and thus $v^{\mu}_{(\mathrm{L}p)}$) includes additional terms proportional to $d^{\mu}_{(k)}$ (if transport coefficients $\mathcal{B}^{\mu\nu}_{ik} \neq 0$).
(5) Neutron vortices are absent: $\mathcal{V}^{\mu\nu}_{(n)} = 0$.

Remark 3.—One can easily account for the presence of neutron vortices, provided that we neglect their effect on diffusion and ignore vortex–flux tube interaction (see Sec. V C). Under these assumptions, all equations of this section remain the same, except for Eq. (124), which should be replaced with

$$u_{\nu}\mathcal{V}^{\mu\nu}_{(n)} \equiv \frac{1}{c} u_{\nu}\{\partial^{\mu}[w^{\nu}_{(n)} + \mu_n u^{\nu}] - \partial^{\nu}[w^{\mu}_{(n)} + \mu_n u^{\mu}]\} = \frac{\mu_n n_n}{c^3} f^{\mu}_{(n)}, \tag{135}$$

and Eq. (123), which should be replaced with

$$\partial_{\mu} S^{\mu} = \partial_{\mu}\left(S u^{\mu} - \frac{\mu_i}{T}\Delta j^{\mu}_{(i)}\right) = \frac{\mu_p n_p^2}{c^3 T} f_{(p)\mu} W^{\mu}_{(p)} + \frac{\mu_n n_n^2}{c^3 T} f_{(n)\mu} W^{\mu}_{(n)} - \Delta j^{\mu}_{(i)} d_{(i)\mu}, \tag{136}$$

where $f^{\mu}_{(n)}$ is [see Eq. (91)]

$$-\frac{\mu_n n_n^2}{c^3 T} f^{\mu}_{(n)} = -\mathcal{A}^{\parallel}_{nn} \omega^{\mu}\omega^{\nu} W_{(n)\nu} - \mathcal{A}^{\perp}_{nn}(\perp^{\mu\nu} - \omega^{\mu}\omega^{\nu}) W_{(n)\nu} - \mathcal{A}^{H}_{nn}\omega^{\mu\nu} W_{(n)\nu} \tag{137}$$

and $W^{\mu}_{(n)}$ is given by Eq. (116).

VII. SUMMARY

In the present study we have formulated equations of dissipative relativistic finite-temperature MHD describing superfluid and superconducting charged mixtures in the presence of vortices and electromagnetic field. For the first time, the corresponding MHD equations systematically and simultaneously take into account the combined effects of particle diffusion and mutual friction forces acting on superfluid or superconducting vortices. It is important to stress that these two effects interfere with one another: diffusion affects particle velocities which, in turn, influence the vortex motion via the mutual friction mechanism (and vice versa); as a result, the cross-coefficients $\mathcal{B}^{\mu\nu}_{ik}$ and $\mathcal{C}^{\mu\nu}_{ik}$ in Eqs. (61) and (62) differ from zero.

We have obtained the general MHD equations and derived the entropy generation equation, following the same phenomenological approach [19,50] as in our previous papers [25,26,49] (see Secs. II and III). These equations extend the results of GD16 (which neglects all the dissipative processes except for the mutual friction dissipation) by accounting for the diffusion, viscosity, chemical reactions, and radiation. Then, starting from the Onsager principle and the condition of non-negative entropy production rate, we have derived in Sec. IV the general expressions for the mutual friction forces and diffusive currents adopting the MHD approximation from GD16 (see Sec. VI A), that utilizes the fact that in typical NS conditions the magnetic induction $\boldsymbol{B}$ is much larger than the fields $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{H}$. Note that, in this approximation, mutual friction and diffusion (which are the main focus of our study) appear to be completely decoupled from other dissipative mechanisms, which can be studied separately.

We have applied the formulated MHD to a number of special cases, where it can be considerably simplified (some of these cases are interesting because of their application to NSs). In particular, simplifications arising for unmagnetized NSs are discussed in Sec. V C. The resulting equations allow one to easily study the effect of diffusion and mutual friction dissipation on damping of stellar oscillations and various dynamical instabilities in NSs [40,76]. In turn, Sec. V D provides equations suitable for studying the quasistationary magnetic field evolution in superconducting NS cores [41]. The full system of equations in this limit is presented in Sec. VI and describes $npe\mu$ matter with type-II proton superconductivity, accounting for an interplay of mutual friction and particle diffusion dissipation.

The MHD equations discussed above contain a number of phenomenological transport coefficients, that have to be determined from microphysics. We have shown (see Appendix B) how to establish a connection between our formalism and the microscopic approach, by expressing the phenomenological coefficients arising in our theory through the microscopic mutual friction parameters D_i and momentum transfer rates J_{ik} in the low-temperature limit. We emphasize that *all* these phenomenological coefficients, generally, depend on both D_i and J_{ik} due to interference between the diffusion and mutual friction mechanisms.

We see two main immediate practical applications of our results. First, the dissipative MHD equations, presented in this work, allow one to realistically model long-term magnetothermal evolution in superconducting NSs, accounting for the macroscopic particle flows, diffusive currents, mutual friction, and finite temperatures, as well as special and general relativistic effects. Second, with the help of these equations, one can study the combined effect of diffusion and mutual friction on oscillations and hydrodynamic instabilities in NSs: these effects are extremely efficient dissipative agents in superfluid and superconducting NS cores [40,76].

The presented magnetohydrodynamics can be generalized in a number of ways. First, one can easily consider a more complex particle composition (e.g., including hyperons) within the presented framework. Another straightforward step is to consider viscosity and chemical reactions in the presence of two preferred directions in the system (specified by the two types of vortices) and to derive general form of the corresponding dissipative corrections following the same procedure as in Sec. IV. Further, an important task would be to describe pinning of neutron vortices to proton flux tubes and the vortex creep. In principle, our general equations should account for these effects, but for practical applications one also has to find a relation between the phenomenological quantities (such as the vector $\mathcal{W}^{\mu}_{(\mathrm{M}i)}$ or the transport coefficient $\mathcal{A}^{\mu\nu}_{ik}$) and the microscopic parameters describing vortex–flux tube interaction [77,83–87]. We expect that all these improvements will enable further progress toward realistic modeling of the various dynamical processes in NSs.

ACKNOWLEDGMENTS

We are very grateful to E. M. Kantor for reading the draft version of the paper and valuable comments. This study is partially supported by the Foundation for the Advancement of Theoretical Physics and Mathematics BASIS [Grant No. 17-12-204-1] and by Russian Foundation for Basic Research [Grant No. 19-52-12013].

APPENDIX A: NONRELATIVISTIC LIMIT OF EQUATIONS OF SEC. VI

In this Appendix we present three-dimensional version of MHD equations of Sec. VI (see analogous equations in Appendix I of GD16), assuming that all macroscopic velocities are nonrelativistic (the "low-velocity" limit). At the same time, we employ a relativistic equation of state and discuss transition to the fully nonrelativistic limit separately. To proceed to the latter limit, one has to assume that not only macroscopic velocities, but also the equation of state is nonrelativistic. Then one has to replace the chemical potential μ_i for particle species i with the particle rest energy, $m_i c^2$ [note, however, that in the superfluid equations for neutrons (A23) and protons (A41), as well as in Eq. (A36), one should retain the small quantity $\breve{\mu}_i \equiv (\mu_i - m_i c^2)/m_i$] and express the entrainment matrix Y_{ik} through the nonrelativistic matrix ρ_{ik} by the formula [58]

$$\rho_{ik} = m_i m_k c^2 Y_{ik}, \tag{A1}$$

where no summation over repeated indices is assumed. In the absence of entrainment $\rho_{ik} = \rho_{si}\delta_{ik}$—i.e., the off-diagonal elements of the matrix vanish—and diagonal elements contain superfluid mass densities ρ_{si} for particle species i. In the fully nonrelativistic limit, the pressure P can be neglected in comparison to the energy density ε, which equals the rest energy density:

$$P \ll \varepsilon \approx \rho c^2, \tag{A2}$$

where $\rho \equiv m_i n_i$ is the total mass density. The components of $\Delta T^{\mu\nu}_{(\mathrm{EM+vortex})}$ are also much smaller than ρc^2.

Below, all the three-vectors (shown in boldface) are defined in the laboratory frame. Note that all scalar thermodynamic quantities (e.g., particle number density n_i) in this paper are measured in the comoving frame; however, in the laboratory frame they have the same values in the low-velocity limit.

1. Nonrelativistic three-velocities

For convenience, let us first introduce some nonrelativistic quantities. The four-velocity u^μ is expressed through the normal (nonsuperfluid) velocity $\boldsymbol{V}_{\mathrm{norm}}$ of nonrelativistic hydrodynamics by the formula

$$u^\mu \equiv (u^0, \boldsymbol{u}) = \left(\frac{1}{\sqrt{1-\frac{V^2_{\mathrm{norm}}}{c^2}}}, \frac{\boldsymbol{V}_{\mathrm{norm}}}{c\sqrt{1-\frac{V^2_{\mathrm{norm}}}{c^2}}},\right) \approx \left(1, \frac{\boldsymbol{V}_{\mathrm{norm}}}{c}\right). \tag{A3}$$

In what follows, we retain only leading-order terms in $\boldsymbol{V}_{\mathrm{norm}}/c$ and $\boldsymbol{V}_{\mathrm{s}i}/c$ in all equations.

The four-vector $w^\mu_{(i)}$ is related to the nonrelativistic superfluid velocity $\boldsymbol{V}_{\mathrm{s}i}$ by [25,58]

$$w^\mu_{(i)} = m_i c V^\mu_{(\mathrm{s}i)} - \mu_i u^\mu, \tag{A4}$$

where $V^\mu_{(\mathrm{s}i)} \equiv (V^0_{(\mathrm{s}i)}, \boldsymbol{V}_{\mathrm{s}i})$ and $V^0_{(\mathrm{s}i)}$ can be found from Eqs. (5) and (A4):

$$V^0_{(\mathrm{s}i)} = \frac{\mu_i}{m_i c u^0} + \frac{\boldsymbol{u}\boldsymbol{V}_{\mathrm{s}i}}{u^0}. \tag{A5}$$

In the low-velocity limit

$$w^\mu_{(i)} = (w^0_{(i)}, \boldsymbol{w}_{(i)}) \approx \left(0, m_i c \boldsymbol{V}_{\mathrm{s}i} - \mu_i \frac{\boldsymbol{V}_{\mathrm{norm}}}{c}\right). \tag{A6}$$

For nonrelativistic particles $\mu_i \approx m_i c^2$, and, in the fully nonrelativistic limit, the vector $\boldsymbol{w}_{(i)}$ reduces to

$$\boldsymbol{w}_{(i)} = m_i c(\boldsymbol{V}_{\mathrm{s}i} - \boldsymbol{V}_{\mathrm{norm}}). \tag{A7}$$

Being expressed in terms of $V^{\mu}_{(\mathrm{s}i)}$, the vorticity tensor $\mathcal{V}^{\mu\nu}_{(i)}$ (22) reads (recall that, starting from Sec. IV, we ignore viscosity and set $\varkappa_i = 0$)

$$\mathcal{V}^{\mu\nu}_{(i)} = m_i[\partial^{\mu} V^{\nu}_{(\mathrm{s}i)} - \partial^{\nu} V^{\mu}_{(\mathrm{s}i)}] + \frac{e_i}{c} F^{\mu\nu}. \tag{A8}$$

In the fully nonrelativistic limit it is also convenient to introduce the nonsuperfluid particle velocities $\boldsymbol{V}_i$, in order to express the spatial part of the particle current $j^{\mu}_{(i)} \equiv (j^0_{(i)}, \boldsymbol{j}_i)$, as a sum of nonsuperfluid and superfluid currents (with velocities $\boldsymbol{V}_i$ and $\boldsymbol{V}_{\mathrm{s}i}$, respectively):

$$\boldsymbol{j}_i = \left(n_i - \frac{1}{m_i}\sum_k \rho_{ik}\right)\frac{\boldsymbol{V}_i}{c} + \frac{1}{m_i}\sum_k \frac{\rho_{ik}\boldsymbol{V}_{\mathrm{s}k}}{c}. \tag{A9}$$

Note that no summation over index i is assumed in Eqs. (A9)–(A11), and only linear terms in velocities are taken into account. Comparing Eq. (A9) with definitions (4), (A3), and (A6), one can express $\Delta\boldsymbol{j}_i$ through $\boldsymbol{V}_i$ as

$$\Delta\boldsymbol{j}_i = \left(n_i - \frac{1}{m_i}\sum_k \rho_{ik}\right)\frac{\boldsymbol{V}_i - \boldsymbol{V}_{\mathrm{norm}}}{c}. \tag{A10}$$

For nonsuperfluid particles Eq. (A10) reduces to

$$\Delta\boldsymbol{j}_i = n_i \frac{(\boldsymbol{V}_i - \boldsymbol{V}_{\mathrm{norm}})}{c}. \tag{A11}$$

Using the above definitions, below we present the low-velocity version of equations of Sec. VI and also discuss how they will be modified in the fully nonrelativistic limit. The full set of equations contains dynamic equations for seven unknown functions $\boldsymbol{B}$, $\boldsymbol{V}_{\mathrm{norm}}$, $\boldsymbol{w}_n$, n_n, n_e, n_μ, and S, supplemented by algebraic relations allowing one to find all other quantities.

2. Dynamic equations

(1) In the low-velocity limit the continuity equations for neutrons (118), electrons (119) and muons (120) read, respectively,

$$\frac{\partial n_n}{\partial t} + \boldsymbol{\nabla}[n_n \boldsymbol{V}_{\mathrm{norm}} + cY_{nk}\boldsymbol{w}_k + c\Delta\boldsymbol{j}_n] = 0, \tag{A12}$$

$$\frac{\partial n_e}{\partial t} + \boldsymbol{\nabla}[n_e \boldsymbol{V}_{\mathrm{norm}} + c\Delta\boldsymbol{j}_e] = 0, \tag{A13}$$

$$\frac{\partial n_\mu}{\partial t} + \boldsymbol{\nabla}[n_\mu \boldsymbol{V}_{\mathrm{norm}} + c\Delta\boldsymbol{j}_\mu] = 0. \tag{A14}$$

In the fully nonrelativistic limit these equations can be presented, in terms of the velocities $\boldsymbol{V}_i$ and $\boldsymbol{V}_{\mathrm{s}i}$ [64], as

$$\frac{\partial \rho_n}{\partial t} + \boldsymbol{\nabla}[(\rho_n - \rho_{nn} - \rho_{np})\boldsymbol{V}_n + \rho_{nk}\boldsymbol{V}_{\mathrm{s}k}] = 0, \tag{A15}$$

$$\frac{\partial \rho_e}{\partial t} + \boldsymbol{\nabla}(\rho_e \boldsymbol{V}_e) = 0, \tag{A16}$$

$$\frac{\partial \rho_\mu}{\partial t} + \boldsymbol{\nabla}(\rho_\mu \boldsymbol{V}_\mu) = 0, \tag{A17}$$

where $\rho_i \equiv m_i n_i$ and no summation over i is assumed.

(2) The entropy generation equation (123), which is convenient to use instead of the energy conservation law, reduces to

$$\frac{1}{c}\frac{\partial S}{\partial t} + \boldsymbol{\nabla}\left(S\frac{\boldsymbol{V}_{\mathrm{norm}}}{c} - \frac{\mu_i}{T}\Delta\boldsymbol{j}_i\right) = \frac{\mu_p n_p^2}{c^3 T}\boldsymbol{f}_p \boldsymbol{W}_p - \Delta\boldsymbol{j}_i \boldsymbol{d}_{(i)}, \tag{A18}$$

and the total momentum conservation equation reads

$$\frac{1}{c}\frac{\partial T^{0l}}{\partial t} + \nabla_m T^{lm} = 0, \tag{A19}$$

where the spatial indices l and m run over $l, m = 1, 2, 3$, and the energy-momentum tensor $T^{\mu\nu}$ is specified by Eq. (122). In the fully nonrelativistic limit the momentum density T^{0l}/c reduces simply to $T^{0l}/c = \rho V^l_{\mathrm{norm}} + \sum_{ik}\rho_{ik}(V^l_{\mathrm{s}k} - V^l_{\mathrm{norm}})$, while T^{lm} is given by Eq. (A33) below. Then Eq. (A19), with the help of the Gibbs-Duhem relation (A28), can be represented as

$$\frac{\partial}{\partial t}\left[\rho V^l_{\mathrm{norm}} + \sum_{ik}\rho_{ik}(V^l_{\mathrm{s}k} - V^l_{\mathrm{norm}})\right] + \nabla^m\left[\rho V^l_{\mathrm{norm}}V^m_{\mathrm{norm}} + \sum_{ik}\rho_{ik}(V^l_{\mathrm{s}i}V^m_{\mathrm{s}k} - V^l_{\mathrm{norm}}V^m_{\mathrm{norm}})\right]$$
$$= -n_i\nabla^l\mu_i - S\nabla^l T + \rho_{ik}\nabla^l\left[\frac{(\boldsymbol{V}_{\mathrm{s}i} - \boldsymbol{V}_{\mathrm{norm}})(\boldsymbol{V}_{\mathrm{s}k} - \boldsymbol{V}_{\mathrm{norm}})}{2}\right] - \frac{1}{4\pi}[\boldsymbol{B}\times \mathrm{curl}(H_{\mathrm{c}1}\boldsymbol{b})]^l. \tag{A20}$$

Here the last term in the right-hand side describes buoyancy and tension forces acting on proton flux tubes. This term replaces the Lorentz term $\boldsymbol{J}_{\rm free}\times\boldsymbol{B}$ of the ordinary MHD, which vanishes due to the screening of electric current inside the superconductor (see, e.g., Refs. [24,78][21]).

(3) Superfluid equation (124), written for neutrons in the absence of vortices, in the three-dimensional form reduces to the two equations

$$\frac{1}{c}\frac{\partial \boldsymbol{V}_{\rm sn}}{\partial t}+\boldsymbol{\nabla}V^0_{\rm sn}=0, \tag{A21}$$

$$\mathrm{curl}\boldsymbol{V}_{\rm sn}=0, \tag{A22}$$

where V^0_{sn} is given by Eq. (A5). One can also obtain a nonrelativistic version of Eq. (A21), assuming that velocities are small and neutrons are nonrelativistic (see Ref. [25], Appendix C):

$$\frac{\partial \boldsymbol{V}_{\rm sn}}{\partial t}+(\boldsymbol{V}_{\rm sn}\boldsymbol{\nabla})\boldsymbol{V}_{sn}+\boldsymbol{\nabla}\left[\breve{\mu}_n-\frac{1}{2}|\boldsymbol{V}_{\rm sn}-\boldsymbol{V}_{\rm norm}|^2\right]=0, \tag{A23}$$

where $\breve{\mu}_n\equiv(\mu_n-m_nc^2)/m_n$.

(4) The "magnetic evolution" equation [the same as Eq. (I23) in GD16] is obtained from Maxwell equation

$$\mathrm{curl}\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \tag{A24}$$

by substituting $\boldsymbol{E}$ from Eq. (A40) (see below) and neglecting the terms depending on curl$\boldsymbol{V}_{sp}$ in comparison to the similar terms depending on $e_p/(m_pc)\boldsymbol{B}$:

$$\frac{\partial\boldsymbol{B}}{\partial t}+\mathrm{curl}\left(\frac{\mu_pn_p}{e_pc}\boldsymbol{f}_p+\boldsymbol{B}\times\boldsymbol{V}_{\rm norm}\right)=0. \tag{A25}$$

The above equations describe time evolution of magnetic field $\boldsymbol{B}$, velocities $\boldsymbol{V}_{\rm norm}$ and $\boldsymbol{V}_{sn}$ (or, equivalently, $\boldsymbol{w}_n$), as well as scalar thermodynamic quantities (n_i and S). Note that the superfluid velocity for protons, $\boldsymbol{V}_{sp}$ (or $\boldsymbol{w}_p$), is expressed from the screening condition (A31) and, thus, does not provide an additional dynamic degree of freedom; the diffusive currents $\Delta\boldsymbol{j}_i$ (or velocities $\boldsymbol{V}_i$ of nonsuperfluid components) are also expressed algebraically via Eq. (A35).

3. Algebraic relations

(1) In the low-velocity limit the small quantity $w^\mu_{(i)}w_{(k)\mu}$ that enters the thermodynamic relations (110)–(112) reduces to $\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)}$ [see Eq. (A6)]. As a result, any thermodynamic quantity (e.g., the energy density ε) should be expressed as functions of the variables n_i, S, $\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)}$, and B:

$$\varepsilon=\varepsilon(n_i,S,\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)},B), \tag{A26}$$

whereas the second law of thermodynamics and the Gibbs-Duhem relation read, respectively,

$$d\varepsilon=\mu_idn_i+TdS+\frac{Y_{ik}}{2}d(\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)})+\frac{1}{4\pi}H_{c1}dB, \tag{A27}$$

$$dP=n_id\mu_i+SdT-\frac{Y_{ik}}{2}d(\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)})-\frac{1}{4\pi}H_{c1}dB. \tag{A28}$$

In the fully nonrelativistic limit the term $\frac{Y_{ik}}{2}d(\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)})$ reduces, in view of Eqs. (A1) and (A7), to

$$\frac{Y_{ik}}{2}d(\boldsymbol{w}_{(i)}\boldsymbol{w}_{(k)})=\rho_{ik}d\frac{(\boldsymbol{V}_{\rm si}-\boldsymbol{V}_{\rm norm})(\boldsymbol{V}_{\rm sk}-\boldsymbol{V}_{\rm norm})}{2}. \tag{A29}$$

(2) Proton number density n_p and superfluid proton velocity $\boldsymbol{V}_{sp}$ can be found from the quasineutrality (128) and screening (129) conditions:

$$n_p=n_e+n_\mu, \tag{A30}$$

$$\boldsymbol{j}_p-\boldsymbol{j}_e-\boldsymbol{j}_\mu=Y_{pk}\boldsymbol{w}_k+(\Delta\boldsymbol{j}_p-\Delta\boldsymbol{j}_e-\Delta\boldsymbol{j}_\mu)=0. \tag{A31}$$

For nonrelativistic matter the screening condition (A31), written in terms of $\boldsymbol{V}_i$ and $\boldsymbol{V}_{sp}$, takes the form

$$\frac{\rho_{pk}}{m_p}(\boldsymbol{V}_{\rm sk}-\boldsymbol{V}_p)+n_p\boldsymbol{V}_p-n_e\boldsymbol{V}_e-n_\mu\boldsymbol{V}_\mu=0. \tag{A32}$$

(3) The energy-momentum tensor $T^{\mu\nu}$, employed in Eq. (A19), is specified by Eqs. (122) and (115). In the fully nonrelativistic limit its spatial part T^{lm} ($l,m=1,2,3$), with the help of relations (A1)–(A3) and (A7), reduces to [cf. Ref. [64] and Eq. (I22) in GD16]

$$T^{lm}=\left(\rho-\sum_{ik}\rho_{ik}\right)V^l_{\rm norm}V^m_{\rm norm}+\sum_{ik}\rho_{ik}V^l_{\rm si}V^m_{\rm sk}+P\delta^{lm}+\frac{H_{c1}}{4\pi}\left(B\delta^{lm}-\frac{B^lB^m}{B}\right). \tag{A33}$$

[21]Note that the force $F^i_{\rm mag}$ in Eq. (95) of Ref. [24] contains an additional term, $-(\rho_p)/(4\pi)\nabla^i(B\partial H_c/\partial\rho)$; in our formulation this term is included in $\boldsymbol{\nabla}\mu_i$ due to renormalization of the chemical potential—see Eq. (G25) in GD16.

(4) $\Delta\boldsymbol{j}_i$ and $\boldsymbol{f}_p$ are expressed through $\boldsymbol{d}_k$ and $\boldsymbol{W}_p$ [see Eqs. (130) and (131)]:

$$-\frac{\mu_p n_p^2}{c^3 T}\boldsymbol{f}_p = -\mathcal{A}^{\parallel}_{pp}\boldsymbol{W}_{p\parallel} - \mathcal{A}^{\perp}_{pp}\boldsymbol{W}_{p\perp} - \mathcal{A}^{H}_{pp}[\boldsymbol{W}_{p\perp}\times\boldsymbol{b}] - \mathcal{B}^{\parallel}_{pk}\boldsymbol{d}_{k\parallel} - \mathcal{B}^{\perp}_{pk}\boldsymbol{d}_{k\perp} - \mathcal{B}^{H}_{pk}[\boldsymbol{d}_{k\perp}\times\boldsymbol{b}], \quad (A34)$$

$$\Delta\boldsymbol{j}_i = -\mathcal{C}^{\parallel}_{ip}\boldsymbol{W}_{p\parallel} - \mathcal{C}^{\perp}_{ip}\boldsymbol{W}_{p\perp} - \mathcal{C}^{H}_{ip}[\boldsymbol{W}_{p\perp}\times\boldsymbol{b}] - \mathcal{D}^{\parallel}_{ik}\boldsymbol{d}_{k\parallel} - \mathcal{D}^{\perp}_{ik}\boldsymbol{d}_{k\perp} - \mathcal{D}^{H}_{ik}[\boldsymbol{d}_{k\perp}\times\boldsymbol{b}], \quad (A35)$$

where

$$\boldsymbol{d}_{k\parallel} \equiv (\boldsymbol{d}_k\boldsymbol{b})\boldsymbol{b}, \quad \boldsymbol{d}_{k\perp} \equiv \boldsymbol{d}_k - (\boldsymbol{d}_k\boldsymbol{b})\boldsymbol{b},$$
$$\boldsymbol{W}_{p\parallel} \equiv (\boldsymbol{W}_p\boldsymbol{b})\boldsymbol{b}, \quad \boldsymbol{W}_{p\perp} \equiv \boldsymbol{W}_p - (\boldsymbol{W}_p\boldsymbol{b})\boldsymbol{b}, \quad (A36)$$

$$\boldsymbol{b} \equiv \frac{\boldsymbol{B}}{B}, \quad (A37)$$

$$\boldsymbol{d}_k = \boldsymbol{\nabla}\left(\frac{\mu_k}{T}\right) - \frac{e_k}{T}\left[\boldsymbol{E} + \frac{\boldsymbol{V}_{\rm norm}}{c}\times\boldsymbol{B}\right], \quad (A38)$$

$$\boldsymbol{W}_p = \frac{cY_{pk}}{n_p}\boldsymbol{w}_{(k)} + \frac{c}{4\pi e_p n_p}\mathrm{curl}(H_{c1}\boldsymbol{b}). \quad (A39)$$

(5) The electric field $\boldsymbol{E}$ is expressed from the superfluid equation (132) for protons:

$$\frac{\partial \boldsymbol{V}_{sp}}{\partial t} + c\boldsymbol{\nabla}V^0_{sp} + \mathrm{curl}\boldsymbol{V}_{sp}\times\boldsymbol{V}_{\rm norm} = -\frac{\mu_p n_p}{m_p c^2}\boldsymbol{f}_p + \frac{e_p}{m_p}\left(\boldsymbol{E} + \frac{\boldsymbol{V}_{\rm norm}}{c}\times\boldsymbol{B}\right), \quad (A40)$$

which, in the nonrelativistic limit, takes the form [cf. GD16, Eq. (I7)]

$$\frac{\partial \boldsymbol{V}_{sp}}{\partial t} + (\boldsymbol{V}_{sp}\boldsymbol{\nabla})\boldsymbol{V}_{sp} + \boldsymbol{\nabla}\left[\breve{\mu}_p - \frac{1}{2}|\boldsymbol{V}_{sp} - \boldsymbol{V}_{\rm norm}|^2\right] = -\mathrm{curl}\boldsymbol{V}_{sp}\times(\boldsymbol{V}_{\rm norm} - \boldsymbol{V}_{sp}) - n_p\boldsymbol{f}_p + \frac{e_p}{m_p}\left(\boldsymbol{E} + \frac{\boldsymbol{V}_{\rm norm}}{c}\times\boldsymbol{B}\right), \quad (A41)$$

where $\breve{\mu}_p \equiv (\mu_p - m_p c^2)/m_p$.

Note that the right-hand sides of Eqs. (A34) and (A35) implicitly contain $\Delta\boldsymbol{j}_i$ and $\boldsymbol{E}$ (see footnote 20); therefore, one has to solve Eqs. (A34), (A35), and (A41) *simultaneously* in order to obtain closed-form expressions for $\boldsymbol{f}_p$, $\Delta\boldsymbol{j}_i$, and $\boldsymbol{E}$.

Remark 1.—If neutrons and protons are completely superfluid, then $\Delta\boldsymbol{j}_n$ and $\Delta\boldsymbol{j}_p$ (which describe dissipative corrections to the nonsuperfluid currents) vanish together with the corresponding transport coefficients.

Remark 2.—The magnetic evolution equation (A25) can be further simplified if transport coefficients $\mathcal{A}^{\parallel}_{pp}$ and $\mathcal{B}^{\parallel}_{pi}$ in Eq. (A34) are small. Then $\boldsymbol{f}_p$ can be presented as

$$\boldsymbol{f}_p = \frac{e_p c}{\mu_p n_p}[\boldsymbol{B}\times(\boldsymbol{V}_{{\rm L}p} - \boldsymbol{V}_{\rm norm})], \quad (A42)$$

where

$$\boldsymbol{V}_{{\rm L}p} = \boldsymbol{V}_{\rm norm} - \frac{c^2 T}{e_p n_p B}(\mathcal{A}^H_{pp}\boldsymbol{W}_p + \mathcal{B}^H_{pk}\boldsymbol{d}_k) + \frac{c^2 T}{e_p n_p B}(\mathcal{A}^{\perp}_{pp}\boldsymbol{W}_p + \mathcal{B}^{\perp}_{pk}\boldsymbol{d}_k)\times\boldsymbol{b} \quad (A43)$$

is the nonrelativistic velocity of proton vortices [spatial part of the four-vector $v^{\mu}_{({\rm L}p)}$ multiplied by c; see Eq. (134)]. Equation (A25) can then be rewritten in the form [cf. GD16, Eq. (I24)]

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathrm{curl}(\boldsymbol{B}\times\boldsymbol{V}_{{\rm L}p}) = 0, \quad (A44)$$

which simply states that the magnetic field is transferred by the vortices.

Remark 3.—One can easily account for the presence of neutron vortices, provided that we neglected their effect on diffusion and ignore vortex–flux tube interaction (see Sec. V C). Under these assumptions, all equations of this section remain the same, except for Eqs. (A21)–(A23), which should be replaced with

$$\frac{\partial \boldsymbol{V}_{sn}}{\partial t} + c\boldsymbol{\nabla}V^0_{sn} + \mathrm{curl}\boldsymbol{V}_{sn}\times\boldsymbol{V}_{\rm norm} = -\frac{\mu_n n_n}{m_n c^2}\boldsymbol{f}_n, \quad (A45)$$

and Eq. (A18), which should be replaced with

$$\frac{1}{c}\frac{\partial S}{\partial t} + \boldsymbol{\nabla}\left(S\frac{\boldsymbol{V}_{\rm norm}}{c} - \frac{\mu_i}{T}\Delta\boldsymbol{j}_i\right) = \frac{\mu_p n_p^2}{c^3 T}\boldsymbol{f}_p\boldsymbol{W}_p + \frac{\mu_n n_n^2}{c^3 T}\boldsymbol{f}_n\boldsymbol{W}_n - \Delta\boldsymbol{j}_i\boldsymbol{d}_{(i)}, \quad (A46)$$

where $\boldsymbol{W}_n$ is given by [see Eq. (116)]

$$\boldsymbol{W}_n \equiv \frac{1}{n_n}cY_{nk}\boldsymbol{w}_{(k)} \quad (A47)$$

and $\boldsymbol{f}_n$ is [see Eq. (91)]

$$-\frac{\mu_n n_n^2}{c^3 T}\boldsymbol{f}_n = -\mathcal{A}^{\parallel}_{nn}\boldsymbol{W}_{n\parallel} - \mathcal{A}^{\perp}_{nn}\boldsymbol{W}_{n\perp} - \mathcal{A}^{H}_{nn}[\boldsymbol{W}_{n\perp}\times\boldsymbol{\omega}], \quad (A48)$$

$$\boldsymbol{W}_{n\parallel} \equiv (\boldsymbol{W}_n\boldsymbol{\omega})\boldsymbol{\omega}, \quad \boldsymbol{W}_{n\perp} \equiv \boldsymbol{W}_n - (\boldsymbol{W}_n\boldsymbol{\omega})\boldsymbol{\omega}, \quad \boldsymbol{\omega} \equiv \frac{\boldsymbol{\mathcal{V}}_{({\rm M}n)}}{\mathcal{V}_{({\rm M}n)}}. \quad (A49)$$

In the nonrelativistic limit Eq. (A45) reduces to [cf. GD16, Eq. (I7)]

$$\frac{\partial \boldsymbol{V}_{\rm sn}}{\partial t} + (\boldsymbol{V}_{\rm sn}\boldsymbol{\nabla})\boldsymbol{V}_{\rm sn} + \boldsymbol{\nabla}\left[\breve{\mu}_n - \frac{1}{2}|\boldsymbol{V}_{\rm sn} - \boldsymbol{V}_{\rm norm}|^2\right] = -{\rm curl}\boldsymbol{V}_{\rm sn} \times (\boldsymbol{V}_{\rm norm} - \boldsymbol{V}_{\rm sn}) - n_n \boldsymbol{f}_n. \quad \text{(A50)}$$

APPENDIX B: PHENOMENOLOGICAL TRANSPORT COEFFICIENTS IN THE LOW-TEMPERATURE LIMIT

Here we establish a connection between our transport coefficients and the mutual friction parameters and momentum transfer rates of microscopic theory. To this aim, we analyze the equation of motion for individual proton vortices, as well as the Euler-like equations for nonsuperfluid particles in the $npe\mu$ matter.[22] We present an algorithm that allows us to find microscopic expressions for $\Delta \boldsymbol{j}_i$ and $\boldsymbol{V}_{{\rm L}p}$, compare them with the phenomenological equations (A35) and (A43), and, finally, obtain the expressions for the phenomenological transport coefficients $\mathcal{A}^{\mu\nu}_{ik}$, $\mathcal{B}^{\mu\nu}_{ik}$, $\mathcal{C}^{\mu\nu}_{ik}$, and $\mathcal{D}^{\mu\nu}_{ik}$.

As in Appendix A, we work in the MHD limit, ignore neutron vortices, and assume that all macroscopic velocities are nonrelativistic. For the sake of simplicity, we further make some additional assumptions. Namely, we adopt the low-temperature limit ($T \to 0$), ignore all the terms depending on $\boldsymbol{\nabla} T$, and assume that protons and neutrons are completely superfluid (no Bogoliubov thermal excitations), so that only electrons and muons can scatter off the vortex cores. In addition, we also neglect entrainment between superfluid neutrons and protons, i.e., set $Y_{np} = 0$.

The proton vortex velocity $\boldsymbol{V}_{{\rm L}p}$ enters the equation describing the balance of forces acting on a proton vortex. Neglecting small vortex mass, the latter equation takes the form [44]

$$\sum_{i=e,\mu} \boldsymbol{F}_{i\to V} + \boldsymbol{F}_{\rm ext} = 0, \quad \text{(B1)}$$

where

$$\boldsymbol{F}_{i\to V} = -D_i[\boldsymbol{b} \times [\boldsymbol{b} \times (\boldsymbol{V}_i - \boldsymbol{V}_{{\rm L}p})]] + D_i'[\boldsymbol{b} \times (\boldsymbol{V}_i - \boldsymbol{V}_{{\rm L}p})] \quad \text{(B2)}$$

is the velocity-dependent force per unit length acting on a vortex from particle species i, $\boldsymbol{V}_i \equiv c\boldsymbol{j}_i/n_i$ is the velocity of particle species i, and coefficients D_i and D_i' are calculated from microphysics (see Ref. [44] and references therein). In the absence of diffusion the phenomenological mutual friction parameters α_p, β_p, and γ_p employed in GD16 can be expressed through D_i and D_i' as, respectively,

$$\frac{\mu_p n_p}{c^2}\alpha_p = \frac{\pi\hbar n_p (D_e' + D_\mu')}{(D_e + D_\mu)^2 + (D_e' + D_\mu')^2}, \quad \text{(B3)}$$

$$\frac{\mu_p n_p}{c^2}\beta_p = \frac{\pi\hbar n_p (D_e + D_\mu)}{(D_e + D_\mu)^2 + (D_e' + D_\mu')^2}, \quad \text{(B4)}$$

$$\gamma_p = 0. \quad \text{(B5)}$$

To obtain these relations, one has to solve Eq. (B1) with $\boldsymbol{V}_e = \boldsymbol{V}_\mu = \boldsymbol{V}_{\rm norm}$ and compare the result with Eqs. (101) and (I25) of GD16.

The first and the second term in Eq. (B2) describe the (dissipative) drag force and the (nondissipative) transverse force, respectively. $\boldsymbol{F}_{\rm ext}$ is the velocity-independent force per unit length; it is the sum of buoyancy and tension forces [43]:

$$\boldsymbol{F}_{\rm ext} = -\frac{\hbar c}{4 e_p}[\boldsymbol{b} \times {\rm curl}(H_{c1}\boldsymbol{b})]. \quad \text{(B6)}$$

Using Eqs. (A31) and (A39) and noting that $\Delta \boldsymbol{j}_p = 0$ (since all protons are superconducting), one can present $\boldsymbol{F}_{\rm ext}$ as

$$\boldsymbol{F}_{\rm ext} = -\pi\hbar n_p \left[\boldsymbol{b} \times \left(\boldsymbol{W}_p - \frac{c}{n_p}\Delta \boldsymbol{j}_e - \frac{c}{n_p}\Delta \boldsymbol{j}_\mu\right)\right]. \quad \text{(B7)}$$

The velocities $\boldsymbol{V}_e$ and $\boldsymbol{V}_\mu$ can be found from the Euler equations [41] ($i = e$, μ and no summation over i is assumed)

$$n_i\left[\frac{\partial}{\partial t} + (\boldsymbol{V}_i \boldsymbol{\nabla})\right]\left(\frac{\mu_i}{c^2}\boldsymbol{V}_i\right) = -n_i \boldsymbol{\nabla}\mu_i - \frac{\mu_i n_i}{c^2}\boldsymbol{\nabla}\phi - \sum_{k\neq i} J_{ik}(\boldsymbol{V}_i - \boldsymbol{V}_k) - N_{{\rm V}p}\boldsymbol{F}_{i\to V}, \quad \text{(B8)}$$

where ϕ is the gravitational potential, $J_{ik} = J_{ki}$ is the momentum transfer rate per unit volume between particle species i and k, and $N_{{\rm V}p} = B/\hat{\phi}_{p0} = e_p B/(\pi\hbar c)$ is the number of proton vortices per unit area. The Lorentz force is contained in the last term in the right-hand side of Eq. (B8), since we assume that all the electromagnetic field is generated by proton vortices. Note that, e.g., in the similar equations of Ref. [41] the vector $\boldsymbol{\mathcal{F}}_{pi}$ from this reference includes only the drag force [the second term in Eq. (B2)], whereas the Lorentz force [the first term in Eq. (B2)] is written out separately.

Since in the hydrodynamic regime the velocities $\boldsymbol{V}_i$ are close to one another, one can simplify the left-hand side of Eq. (B8) by replacing $\boldsymbol{V}_i$ with the average mass velocity of nonsuperfluid particles $\boldsymbol{U} \equiv (\mu_e n_e \boldsymbol{V}_e + \mu_\mu n_\mu \boldsymbol{V}_\mu)/(\mu_e n_e + \mu_\mu n_\mu)$ [88,89], which, in the low-temperature limit,

[22]These Euler-like equations follow from the transport equations written for each particle species; see, e.g., Refs. [32,40,49,88].

coincides with $\boldsymbol{V}_{\rm norm}$ introduced in Eq. (A3). Below we work in the comoving frame, specified by the condition $\boldsymbol{V}_{\rm norm}=0$ or, in terms of $\boldsymbol{V}_e$ and $\boldsymbol{V}_\mu$,

$$\mu_e n_e \boldsymbol{V}_e + \mu_\mu n_\mu \boldsymbol{V}_\mu = 0. \tag{B9}$$

The left-hand side of Eq. (B8) in this frame reduces to $(\mu_i n_i/c^2)\partial \boldsymbol{U}/\partial t$. Then, subtracting Euler equations (B8) (divided by $\mu_i n_i$) for electrons and muons, we obtain

$$-\frac{\boldsymbol{\nabla}\mu_e}{\mu_e}+\frac{\boldsymbol{\nabla}\mu_\mu}{\mu_\mu}-\left(\frac{1}{\mu_e n_e}+\frac{1}{\mu_\mu n_\mu}\right)J_{e\mu}(\boldsymbol{V}_e-\boldsymbol{V}_\mu) - \frac{1}{\mu_e n_e}N_{{\rm V}p}\boldsymbol{F}_{e\to V}+\frac{1}{\mu_\mu n_\mu}N_{{\rm V}p}\boldsymbol{F}_{\mu\to V}=0. \tag{B10}$$

The set of linear algebraic equations (B1), (B9), and (B10) allows one to find the quantities $\boldsymbol{V}_{{\rm L}p}$, $\Delta\boldsymbol{j}_e$, and $\Delta\boldsymbol{j}_\mu$. To express them through $\boldsymbol{W}_p$, $\boldsymbol{d}_e$, and $\boldsymbol{d}_\mu$, one has to make the following substitutions in these equations:

(1) substitute $\boldsymbol{F}_{i\to V}$ and $\boldsymbol{F}_{\rm ext}$ from Eqs. (B2) and (B7);
(2) replace $\boldsymbol{\nabla}\mu_i$ with $T\boldsymbol{d}_i+e_i\boldsymbol{E}$ [see Eq. (A36); recall that we ignore the terms depending on $\boldsymbol{\nabla}T$];
(3) replace $\boldsymbol{E}$ with $(-1/c)\boldsymbol{V}_{{\rm L}p}\times\boldsymbol{B}$ [this condition follows from the assumption that the electric field is generated only by the vortex motion; see Eq. (G15) in GD16];
(4) replace $\boldsymbol{V}_i$ with $c\Delta\boldsymbol{j}_i/n_i$ (note that we work in the comoving frame, $\boldsymbol{V}_{\rm norm}=0$).

Then, solving the system of equations (B1), (B9), and (B10) and comparing the results with Eqs. (A35) and (A43), one can determine the coefficients $\mathcal{A}^{\mu\nu}_{pp}$, $\mathcal{B}^{\mu\nu}_{pk}$, $\mathcal{C}^{\mu\nu}_{ip}$, and $\mathcal{D}^{\mu\nu}_{ik}$ and directly check that the Onsager relations (74), (78), and (79) are satisfied.

Since the resulting expressions are very lengthy, we do not provide them for the most general case. Instead, we write them out in the limit $J_{e\mu}\ll N_{{\rm V}p}D_i\ll|N_{{\rm V}p}D'_i|$, which is realistic for typical NS conditions (see, e.g., Fig. 1 in Ref. [41]). We also set $D'_i=-\pi\hbar n_i$, as argued in Refs. [24,44]. Then the transport coefficients have, up to the first order in $J_{e\mu}/|N_{{\rm V}p}D'_i|=(cJ_{e\mu})/(e_p n_i B)$ and $D_i/|D'_i|=D_i/(\pi\hbar n_i)$, the following form[23]:

$$\mathcal{A}^{\parallel}_{pp}=0, \tag{B11}$$

$$\mathcal{A}^{\perp}_{pp}=\frac{e_pB(D_e+D_\mu)}{\pi\hbar c^2T}, \tag{B12}$$

$$\mathcal{A}^{H}_{pp}=-\frac{e_pn_pB}{c^2T}, \tag{B13}$$

$$\mathcal{B}^{\parallel}_{pk}=0, \tag{B14}$$

$$\mathcal{B}^{\perp}_{pe}=\frac{\mu_\mu(\mu_\mu n_\mu D_e-\mu_e n_e D_\mu)[n_eD_\mu(\mu_e^2n_e^2+2\mu_e^2n_en_\mu+\mu_\mu^2n_\mu^2)+n_\mu D_e(\mu_e^2n_e^2+2\mu_\mu^2n_en_\mu+\mu_\mu^2n_\mu^2)]}{\pi^2\hbar^2cn_en_\mu(n_e+n_\mu)(\mu_e^2n_e+\mu_\mu^2n_\mu)^2} + \frac{\mu_\mu J_{e\mu}(\mu_en_e+\mu_\mu n_\mu)^2(\mu_\mu n_\mu D_e-\mu_en_eD_\mu)}{\pi\hbar e_pBn_en_\mu(\mu_e^2n_e+\mu_\mu^2n_\mu)^2}\approx 0, \tag{B15}$$

$$\mathcal{B}^{\perp}_{p\mu}=-\frac{\mu_e}{\mu_\mu}\mathcal{B}^{\perp}_{pe}, \tag{B16}$$

$$\mathcal{B}^{H}_{pe}=\frac{\mu_\mu(\mu_en_eD_\mu-\mu_\mu n_\mu D_e)}{\pi\hbar c(\mu_e^2n_e+\mu_\mu^2n_\mu)},\qquad \mathcal{B}^{H}_{p\mu}=-\frac{\mu_e}{\mu_\mu}\mathcal{B}^{H}_{pe}, \tag{B17}$$

$$\mathcal{C}^{\parallel}_{ip}=0,\qquad \mathcal{C}^{\perp}_{ip}=-\mathcal{B}^{\perp}_{pi},\qquad \mathcal{C}^{H}_{ip}=-\mathcal{B}^{H}_{pi}, \tag{B18}$$

$$\mathcal{D}^{\parallel}_{e\mu}=\mathcal{D}^{\parallel}_{\mu e}=-\frac{\mu_e\mu_\mu n_e^2n_\mu^2T}{cJ_{e\mu}(\mu_en_e+\mu_\mu n_\mu)^2}, \tag{B19}$$

$$\mathcal{D}^{\perp}_{e\mu}=\mathcal{D}^{\perp}_{\mu e}=-\frac{\mu_e\mu_\mu T(\mu_e^2n_e^2D_\mu+\mu_\mu^2n_\mu^2D_e)}{\pi\hbar e_pB(\mu_e^2n_e+\mu_\mu^2n_\mu)^2} - \frac{c\mu_e\mu_\mu TJ_{e\mu}(\mu_en_e+\mu_\mu n_\mu)^2}{e_p^2B^2(\mu_e^2n_e+\mu_\mu^2n_\mu)^2} \approx -\frac{\mu_e\mu_\mu T(\mu_e^2n_e^2D_\mu+\mu_\mu^2n_\mu^2D_e)}{\pi\hbar e_pB(\mu_e^2n_e+\mu_\mu^2n_\mu)^2}, \tag{B20}$$

$$\mathcal{D}^{H}_{e\mu}=\mathcal{D}^{H}_{\mu e}=\frac{\mu_e\mu_\mu n_en_\mu T}{e_pB(\mu_e^2n_e+\mu_\mu^2n_\mu)}, \tag{B21}$$

$$\mathcal{D}^{\parallel,\perp,H}_{ee}=-\frac{\mu_\mu}{\mu_e}\mathcal{D}^{\parallel,\perp,H}_{e\mu},\qquad \mathcal{D}^{\parallel,\perp,H}_{\mu\mu}=-\frac{\mu_e}{\mu_\mu}\mathcal{D}^{\parallel,\perp,H}_{e\mu}. \tag{B22}$$

A number of comments regarding these equations are listed below.

(1) The coefficients $\mathcal{A}^{\parallel}_{pp}$ and $\mathcal{B}^{\parallel}_{pi}$ vanish since there is no force acting along the vortex line in Eq. (B1).

[23]Note that the expression (B11) for $\mathcal{B}^{\perp}_{pe}$ is of the second order in the small parameter (D_i/D'_i); we write it down to emphasize that, generally, it does not vanish. We also point out that we retain the (small) second term $\propto J_{e\mu}$ in the intermediate equality in (B20), because only this term survives in the expression for $\mathcal{D}^{\perp}_{e\mu}$ in the nonsuperfluid MHD of DGS20.

(2) $\mathcal{A}^H_{pp}$ and $\mathcal{A}^\perp_{pp}$ do not depend, in the leading order, on the electron-muon momentum transfer rate $J_{e\mu}$; these coefficients are proportional to, respectively, the mutual friction parameters α_p and β_p of non-diffusive hydrodynamics [26] [cf. Eq. (95)]. Note, however, that generally *all* coefficients, except for $\mathcal{D}^\parallel_{ik}$, depend on both $J_{e\mu}$ and D_i.

(3) The cross-coefficient $\mathcal{B}^H_{pi}$, which describes force acting on a vortex due to gradients of chemical potentials $\boldsymbol{\nabla}\mu_i$, differs from zero. This interference of diffusion and mutual friction has the following physical meaning: diffusion affects particle velocities $\boldsymbol{V}_i$ which, in turn, affect the vortex motion via the mutual friction mechanism (and vice versa).

(4) The dissipative cross-coefficient $\mathcal{B}^\perp_{pi}$, generally, differs from zero but vanishes in the first order in $J_{e\mu}/|N_{\mathrm{V}p}D'_i| = (cJ_{e\mu})/(e_p n_i B)$ and $D_i/|D'_i| = D_i/(\pi\hbar n_i)$ and, thus, can be neglected.

(5) The expression for $\mathcal{D}^\parallel_{e\mu}$, which describes diffusion of electrons and muons along the vortex lines, has exactly the same form as in the nonsuperfluid matter (see DGS20), since the only force acting along the vortex line is the electron-muon friction.

(6) In contrast, the dominant first term in $\mathcal{D}^\perp_{e\mu}$ depends on the mutual friction parameters D_e and D_μ. This means that, for electrons and muons moving across the vortex array, the momentum exchange between particles is mediated mainly by vortices [via the friction force; see the first term in Eq. (B2)], instead of direct electron-muon interaction [the term $J_{e\mu}(\boldsymbol{V}_e - \boldsymbol{V}_\mu)$ in the Euler equation (B10)].

(7) The (nondissipative) coefficient $\mathcal{D}^H_{e\mu}$ has, in the leading order, the same form as for nonsuperfluid matter. This is not surprising, since this coefficient describes the Lorentz force acting on electrons and muons.

Remark 1.—If we consider another limit and neglect the friction force between flux tubes and electrons or muons, i.e., set $D_e = D_\mu = 0$ (without assuming that $J_{e\mu}$ is small), then diffusion and mutual friction are completely decoupled, $B^{\mu\nu}_{pi} = C^{\mu\nu}_{pi} = 0$. In addition, $\mathcal{A}^\parallel_{pp}$ and $\mathcal{A}^\perp_{pp}$ also vanish, $\mathcal{A}^\parallel_{pp} = \mathcal{A}^\perp_{pp} = 0$, so that the force on a vortex is described only by nondissipative coefficient $\mathcal{A}^H_{pp} = -e_p n_p B/(c^2 T)$. In turn, the generalized diffusion coefficients $D^\parallel_{ik}$, $D^\perp_{ik}$, and D^H_{ik} in this approximation take exactly the same form as in the nonsuperfluid matter (see DGS20).

In conclusion, we note that the presented scheme for calculating the phenomenological transport coefficients can readily be generalized to arbitrary temperatures and particle compositions.

[1] V. M. Kaspi and A. M. Beloborodov, Magnetars, Annu. Rev. Astron. Astrophys. **55**, 261 (2017).

[2] A. M. Beloborodov and X. Li, Magnetar heating, Astrophys. J. **833**, 261 (2016).

[3] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev, *Neutron Stars 1: Equation of State and Structure*, Astrophysics and Space Science Library Vol. 326 (Springer, New York, 2007).

[4] A. Sedrakian and J. W. Clark, Superfluidity in nuclear systems and neutron stars, Eur. Phys. J. A **55**, 167 (2019).

[5] D. Ding, A. Rios, H. Dussan, W. H. Dickhoff, S. J. Witte, A. Carbone, and A. Polls, Pairing in high-density neutron matter including short- and long-range correlations, Phys. Rev. C **94**, 025802 (2016).

[6] A. Gezerlis, C. J. Pethick, and A. Schwenk, Pairing and superfluidity of nucleons in neutron stars, arXiv:1406.6109.

[7] U. Lombardo and H.-J. Schulze, in *Physics of Neutron Star Interiors*, Lecture Notes in Physics Vol. 578, edited by D. Blaschke, N. K. Glendenning, and A. Sedrakian (Springer-Verlag, Berlin, 2001), p. 30.

[8] D. G. Yakovlev and C. J. Pethick, Neutron star cooling, Annu. Rev. Astron. Astrophys. **42**, 169 (2004).

[9] D. Page, J. M. Lattimer, M. Prakash, and A. W. Steiner, Stellar superfluids, arXiv:1302.6626.

[10] E. M. Kantor, M. E. Gusakov, and V. A. Dommes, Constraining Neutron Superfluidity with R-Mode Physics, Phys. Rev. Lett. **125**, 151101 (2020).

[11] P. S. Shternin, D. D. Ofengeim, W. C. G. Ho, C. O. Heinke, M. J. P. Wijngaarden, and D. J. Patnaude, Model-independent constraints on superfluidity from the cooling neutron star in Cassiopeia A, Mon. Not. R. Astron. Soc. **506**, 709 (2021).

[12] B. Haskell and A. Melatos, Models of pulsar glitches, Int. J. Mod. Phys. D **24**, 1530008 (2015).

[13] J. Sauls, Superfluidity in the interiors of neutron stars, in *Timing Neutron Stars*, NATO Advanced Study Institute (ASI) Series C Vol. 262, edited by H. Ögelman and E. P. J. van den Heuvel (Springer, Berlin, 1989), p. 457.

[14] B. Haskell and A. Sedrakian, Superfluidity and superconductivity in neutron stars, in *Astrophysics and Space Science Library* edited by L. Rezzolla, P. Pizzochero, D. I. Jones, N. Rea, and I. Vidaña (Springer, Cham, 2018), Vol. 457, p. 401, https://doi.org/10.1007/978-3-319-97616-7.

[15] A. Sedrakian, Type-I superconductivity and neutron star precession, Phys. Rev. D **71**, 083003 (2005).

[16] T. S. Wood, V. Graber, and W. G. Newton, Superconducting phases in a two-component microscale model of neutron star cores, arXiv:2011.02873.

[17] H. E. Hall and W. F. Vinen, The rotation of liquid helium II. II. The theory of mutual friction in uniformly rotating helium II, Proc. R. Soc. A **238**, 215 (1956).
[18] H. E. Hall, The rotation of liquid helium II, Adv. Phys. **9**, 89 (1960).
[19] I. Bekarevich and I. Khalatnikov, Phenomenological derivation of the equations of vortex motion in He II, Sov. J. Exp. Theor. Phys. **13**, 643 (1961).
[20] G. A. Vardanyan and D. M. Sedrakyan, Magnetohydrodynamics of superfluid solutions, Sov. J. Exp. Theor. Phys. **54**, 919 (1981).
[21] G. Mendell and L. Lindblom, The coupling of charged superfluid mixtures to the electromagnetic field, Ann. Phys. (N.Y.) **205**, 110 (1991).
[22] A. D. Sedrakian and D. M. Sedrakian, Superfluid core rotation in pulsars. I. Vortex cluster dynamics, Astrophys. J. **447**, 305 (1995).
[23] B. Carter and D. Langlois, Relativistic models for superconducting-superfluid mixtures, Nucl. Phys. **531B**, 478 (1998).
[24] K. Glampedakis, N. Andersson, and L. Samuelsson, Magnetohydrodynamics of superfluid and superconducting neutron star cores, Mon. Not. R. Astron. Soc. **410**, 805 (2011).
[25] M. E. Gusakov, Relativistic formulation of the Hall-Vinen-Bekarevich-Khalatnikov superfluid hydrodynamics, Phys. Rev. D **93**, 064033 (2016).
[26] M. E. Gusakov and V. A. Dommes, Relativistic dynamics of superfluid-superconducting mixtures in the presence of topological defects and an electromagnetic field with application to neutron stars, Phys. Rev. D **94**, 083006 (2016).
[27] N. Andersson, S. Wells, and J. A. Vickers, Quantised vortices and mutual friction in relativistic superfluids, Classical Quantum Gravity **33**, 245010 (2016).
[28] L. Gavassino and M. Antonelli, Thermodynamics of uncharged relativistic multifluids, Classical Quantum Gravity **37**, 025014 (2020).
[29] P. B. Rau and I. Wasserman, Relativistic, finite temperature multifluid hydrodynamics in a neutron star from a variational principle, Phys. Rev. D **102**, 063011 (2020).
[30] N. Andersson and G. L. Comer, Relativistic fluid dynamics: Physics for many different scales, Living Rev. Relativity **24**, 3 (2021).
[31] B. Carter, Convective variational approach to relativistic thermodynamics of dissipative fluids, Proc. R. Soc. A **433**, 45 (1991).
[32] P. Goldreich and A. Reisenegger, Magnetic field decay in isolated neutron stars, Astrophys. J. **395**, 250 (1992).
[33] D. A. Shalybkov and V. A. Urpin, Ambipolar diffusion and anisotropy of resistivity in neutron star cores, Mon. Not. R. Astron. Soc. **273**, 643 (1995).
[34] F. Castillo, A. Reisenegger, and J. A. Valdivia, Magnetic field evolution and equilibrium configurations in neutron star cores: the effect of ambipolar diffusion, Mon. Not. R. Astron. Soc. **471**, 507 (2017).
[35] A. Passamonti, T. Akgün, J. A. Pons, and J. A. Miralles, The relevance of ambipolar diffusion for neutron star evolution, Mon. Not. R. Astron. Soc. **465**, 3416 (2017).
[36] M. E. Gusakov, E. M. Kantor, and D. D. Ofengeim, Evolution of the magnetic field in neutron stars, Phys. Rev. D **96**, 103012 (2017).
[37] D. D. Ofengeim and M. E. Gusakov, Fast magnetic field evolution in neutron stars: The key role of magnetically induced fluid motions in the core, Phys. Rev. D **98**, 043007 (2018).
[38] F. Castillo, A. Reisenegger, and J. A. Valdivia, Two-fluid simulations of the magnetic field evolution in neutron star cores in the weak-coupling regime, Mon. Not. R. Astron. Soc. **498**, 3000 (2020).
[39] K. Kraav and M. Gusakov, Diffusion as a damping mechanism for neutron-star oscillations, J. Phys. Conf. Ser. **1697**, 012023 (2020).
[40] K. Y. Kraav, M. E. Gusakov, and E. M. Kantor, Diffusion as a leading dissipative mechanism in superconducting neutron stars, Mon. Not. R. Astron. Soc. **506**, L74 (2021).
[41] M. E. Gusakov, E. M. Kantor, and D. D. Ofengeim, Magnetic field evolution time-scales in superconducting neutron stars, Mon. Not. R. Astron. Soc. **499**, 4561 (2020).
[42] A. G. Muslimov and A. I. Tsygan, Vortex lines in neutron star superfluids and decay of pulsar magnetic fields, Astrophys. Space Sci. **115**, 43 (1985).
[43] V. A. Dommes and M. E. Gusakov, Vortex buoyancy in superfluid and superconducting neutron stars, Mon. Not. R. Astron. Soc. **467**, L115 (2017).
[44] M. E. Gusakov, Force on proton vortices in superfluid neutron stars, Mon. Not. R. Astron. Soc. **485**, 4936 (2019).
[45] N. Andersson, Resistive relativistic magnetohydrodynamics from a charged multifluids perspective, Phys. Rev. D **86**, 043002 (2012).
[46] N. Andersson, G. L. Comer, and I. Hawke, A variational approach to resistive relativistic plasmas, Classical Quantum Gravity **34**, 125001 (2017).
[47] N. Andersson, I. Hawke, K. Dionysopoulou, and G. L. Comer, Beyond ideal magnetohydrodynamics: From fibration to $3+1$ foliation, Classical Quantum Gravity **34**, 125003 (2017).
[48] N. Andersson, K. Dionysopoulou, I. Hawke, and G. L. Comer, Beyond ideal magnetohydrodynamics: Resistive, reactive and relativistic plasmas, Classical Quantum Gravity **34**, 125002 (2017).
[49] V. A. Dommes, M. E. Gusakov, and P. S. Shternin, Dissipative relativistic magnetohydrodynamics of a multicomponent mixture and its application to neutron stars, Phys. Rev. D **101**, 103020 (2020).
[50] L. Landau and E. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).
[51] W. A. Hiscock and L. Lindblom, Stability and causality in dissipative relativistic fluids, Ann. Phys. (N.Y.) **151**, 466 (1983).
[52] W. A. Hiscock and L. Lindblom, Generic instabilities in first-order dissipative relativistic fluid theories, Phys. Rev. D **31**, 725 (1985).
[53] P. Kovtun, First-order relativistic hydrodynamics is stable, J. High Energy Phys. 10 (**2019**) 034.
[54] W. Israel, Nonstationary irreversible thermodynamics: A causal relativistic theory, Ann. Phys. (N.Y.) **100**, 310 (1976).

[55] W. Israel and J. M. Stewart, Transient relativistic thermodynamics and kinetic theory, Ann. Phys. (N.Y.) **118**, 341 (1979).

[56] I. S. Liu, I. Müller, and T. Ruggeri, Relativistic thermodynamics of gases, Ann. Phys. (N.Y.) **169**, 191 (1986).

[57] L. Gavassino, M. Antonelli, and B. Haskell, When the entropy has no maximum: A new perspective on the instability of the first-order theories of dissipation, Phys. Rev. D **102**, 043018 (2020).

[58] M. E. Gusakov and N. Andersson, Temperature-dependent pulsations of superfluid neutron stars, Mon. Not. R. Astron. Soc. **372**, 1776 (2006).

[59] M. E. Gusakov, Bulk viscosity of superfluid neutron stars, Phys. Rev. D **76**, 083001 (2007).

[60] M. E. Gusakov, E. M. Kantor, and P. Haensel, Relativistic entrainment matrix of a superfluid nucleon-hyperon mixture: The zero temperature limit, Phys. Rev. C **79**, 055806 (2009).

[61] M. E. Gusakov, E. M. Kantor, and P. Haensel, Relativistic entrainment matrix of a superfluid nucleon-hyperon mixture. II. Effect of finite temperatures, Phys. Rev. C **80**, 015803 (2009).

[62] M. E. Gusakov, Transport equations and linear response of superfluid Fermi mixtures in neutron stars, Phys. Rev. C **81**, 025804 (2010).

[63] M. E. Gusakov, P. Haensel, and E. M. Kantor, Physics input for modelling superfluid neutron stars with hyperon cores, Mon. Not. R. Astron. Soc. **439**, 318 (2014).

[64] A. F. Andreev and E. P. Bashkin, Three-velocity hydrodynamics of superfluid solutions, Sov. J. Exp. Theor. Phys. **42**, 164 (1976).

[65] M. E. Gusakov and P. Haensel, The entrainment matrix of a superfluid neutron proton mixture at a finite temperature, Nucl. Phys. **A761**, 333 (2005).

[66] L. B. Leinson, The entrainment matrix of a superfluid nucleon mixture at finite temperatures, Mon. Not. R. Astron. Soc. **479**, 3778 (2018).

[67] V. Allard and N. Chamel, Entrainment effects in neutron-proton mixtures within the nuclear energy-density functional theory. II. Finite temperatures and arbitrary currents, Phys. Rev. C **103**, 025804 (2021).

[68] A. Broderick, M. Prakash, and J. M. Lattimer, The equation of state of neutron star matter in strong magnetic fields, Astrophys. J. **537**, 351 (2000).

[69] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Oxford, 1960).

[70] I. Toptygin, *Electromagnetic Phenomena in Matter: Statistical and Quantum Approaches* (Wiley-VCH, New York, 2015).

[71] D. G. Yakovlev, A. D. Kaminker, O. Y. Gnedin, and P. Haensel, Neutrino emission from neutron stars, Phys. Rep. **354**, 1 (2001).

[72] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Westview, New York, 2000).

[73] S. Putterman, *Superfluid Hydrodynamics*, North-Holland Series in Low Temperature Physics (North-Holland, Amsterdam, 1974).

[74] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics - Pergamon International Library of Science, Technology, Engineering and Social Studies, Oxford: Pergamon Press, and Reading: Addison-Wesley, —c1969, 2nd rev. - enlarg.ed.* (1969).

[75] O. A. Goglichidze and M. E. Gusakov (to be published).

[76] B. Haskell, R-modes in neutron stars: Theory and observations, Int. J. Mod. Phys. E **24**, 1541007 (2015).

[77] E. Gügercinoğlu and M. A. Alpar, Microscopic vortex velocity in the inner crust and outer core of neutron stars, Mon. Not. R. Astron. Soc. **462**, 1453 (2016).

[78] I. Easson and C. J. Pethick, Stress tensor of cosmic and laboratory type-II superconductors, Phys. Rev. D **16**, 275 (1977).

[79] T. Akgün and I. Wasserman, Toroidal magnetic fields in type II superconducting neutron stars, Mon. Not. R. Astron. Soc. **383**, 1551 (2008).

[80] L. D. Landau and E. M. Lifshitz, *Statistical Physics. Pt.2* (Pergamon, Oxford, 1980).

[81] G. Mendell, Superfluid hydrodynamics in rotating neutron stars. I - Nondissipative equations. II - Dissipative effects, Astrophys. J. **380**, 515 (1991).

[82] P. B. Jones, Neutron superfluid spin-down and magnetic field decay in pulsars., Mon. Not. R. Astron. Soc. **253**, 279 (1991).

[83] M. A. Alpar, S. A. Langer, and J. A. Sauls, Rapid postglitch spin-up of the superfluid core in pulsars, Astrophys. J. **282**, 533 (1984).

[84] T. Sidery and M. A. Alpar, The effect of quantized magnetic flux lines on the dynamics of superfluid neutron star cores, Mon. Not. R. Astron. Soc. **400**, 1859 (2009).

[85] B. Link, Thermally activated post-glitch response of the neutron star inner crust and core. I. Theory, Astrophys. J. **789**, 141 (2014).

[86] M. A. Alpar, Flux-vortex pinning and neutron star evolution, J. Astrophys. Astron. **38**, 44 (2017).

[87] A. Sourie and N. Chamel, Force on a neutron quantized vortex pinned to proton fluxoids in the superfluid core of cold neutron stars, Mon. Not. R. Astron. Soc. **493**, 382 (2020).

[88] D. G. Iakovlev and D. A. Shalybkov, Electrical conductivity of neutron star cores in the presence of a magnetic field - I. General solution for a multicomponent Fermi liquid - II. - A free particle model of npeSigma/-/-matter, Astrophys. Space Sci. **176**, 171 (1991).

[89] S. I. Braginskii, Transport processes in a plasma, Rev. Plasma Phys. **1**, 205 (1965).