$\eta(1295) \to 3\pi$ decays

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The radially excited pseudoscalar state $\eta(1295)$ is still poorly understood [as well as its $SU(3)$ partners $\pi(1300)$ and $K(1460)$] and we want to attract attention experimenters to its comprehensive study. As for the main three-body decay $\eta(1295) \to \eta\pi\pi$, here the main interest is the measurements of the shapes of the $\eta\pi$ and $\pi\pi$ mass spectra in which the contributions from the $a_0(980)$ resonance and the large $\pi\pi S$ wave (σ) should manifest themselves. To describe these mass spectra, we propose to use a simple isobar model with a single fitting parameter characterizing the relative intensity of the $a_0(980)$ and σ state production. Our main goal is to discuss the dynamics of the isospin-breaking decays $\eta(1295) \to \pi^+\pi^-\pi^0$ and $\eta(1295) \to 3\pi^0$, whose experimental studies could continue the impressive story of isospin violations in the decays of light isoscalar mesons $\omega \to 2\pi$, $\eta \to 3\pi$, $\eta' \to 3\pi$, $\eta(1405) \to 3\pi$, and $f_1(1285) \to 3\pi$. We estimate the widths of the isospin-breaking decays $\eta(1295) \to 3\pi$ produced via the mixing of the $\pi^0 - \eta$ and $a_0^0(980) - f_0(980)$ states and also due to the $\pi^0(1300) - \eta(1295)$ mixing. Processes that have the potential for detecting $\eta(1295) \to 3\pi$ decays are discussed.

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I. INTRODUCTION

Mechanisms of the isospin-breaking decays of light mesons are highly diverse. For instance, the decay of $\omega \to \pi^+\pi^-$ [1] occurs mainly due to the $\rho^0 - \omega$ mixing [2,3]. The seed mechanism of the $\eta \to 3\pi$ [4–6] and $\eta' \to 3\pi$ [7,8] decays is the $\pi^0 - \eta$ mixing, the manifestation of which is significantly enhanced due to the strong interaction of pions in the final state. This fact was gradually elucidated as a result of great efforts over more than fifty years to explain the data on the decays $\eta \to 3\pi$, see Refs. [9–24] for details. In these works, the complicated technique is presented for taking into account pair interactions in three-pion final states based on the unitarized chiral perturbation theory and solutions of dispersion equations for the $\pi\pi S$ and P waves. Decays $a_0^0(980) \to \pi^+\pi^-$ and $f_0(980) \to \eta\pi^0$ discovered by the BESIII Collaboration [25,26] are due to the $a_0^0(980) - f_0(980)$ mixing [27–30]. The significant isospin breaking in the $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ decays is explained mainly by the triangle logarithmic singularity, which is present in the transition amplitude $\eta(1405) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to 3\pi$ [31–37]. Experimental results on the search for $a_0^0(980) - f_0(980)$ mixing in the $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ decays [31] and also in the $f_1(1285) \to f_0(980)\pi^0 \to 3\pi$ decays, which were discovered by the VES [38,39] and BESIII [40] Collaborations, suggest a broader perspective on the isotopic symmetry breaking effects due to the K^+ and K^0 mass difference. It has become clear that not only the $a_0^0(980) - f_0(980)$ mixing but also any mechanism producing $K\bar{K}$ pairs with a definite isospin in an S-wave gives rise to such effects [30,41], thus suggesting a new tool for studying the nature and production mechanisms of light scalars. As for the $\eta(1295) \to 3\pi$ decays, a hint at their existence was obtained by the BESIII Collaboration in the study of the processes $e^+e^- \to J/\psi \to \gamma 3\pi$ [31]. However, it remains unclear which of the two mesons $\eta(1295)$ or $f_1(1285)$ (or both together) lead to a slight excess over the smooth background in the three-pion mass spectra around 1290 MeV [31].

In the present paper, we discuss the mechanisms that can lead to the isospin-breaking decays $\eta(1295) \to \pi^0\pi^+\pi^-$ and $\eta(1295) \to 3\pi^0$, and also the processes having the potential for detecting such decays. The paper is organized as follows. In Sec. II we construct a simple isobar model for the main decay of the $\eta(1295)$ into $\eta\pi\pi$. The model takes into account the amplitudes of subprocesses $\eta(1295) \to a_0(980)\pi \to \eta\pi\pi$ and $\eta(1295) \to \eta\sigma \to \eta(\pi\pi)_S$ the coherent sum of which defines the shapes of the $\eta\pi$ and $\pi\pi$ mass spectra or the corresponding $\eta(1295) \to \eta\pi\pi$ Dalitz plots (here σ is a symbolic notation for the virtual S-wave hadronic system with isospin $I = 0$ decaying into $\pi\pi$).

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Fitting the data on the $\eta\pi$ and $\pi\pi$ mass spectra within this model using a single parameter characterizing the relative intensity of the $a_0(980)$ and σ state production will allow easily to elucidate whether it is necessary to use for their description more complex theoretical models (similar, for example, to those discussed in Refs. [21,24] for $\eta' \to \eta\pi\pi$ decays). Note that unlike the decays $\eta \to 3\pi$, $\eta' \to 3\pi$, and $\eta' \to \eta\pi\pi$ any theoretical predictions relative to the overall normalization of the decay amplitude $\eta(1295) \to \eta\pi\pi$ is absent. That is, with a modern state of the theory, this normalization can be determined only from the experimental data on the decay width of $\eta(1295) \to \eta\pi\pi$. In Sec. III we use the above model for $\eta(1295) \to \eta\pi\pi$ to estimate the widths of the direct decays $\eta(1295) \to 3\pi$ caused by the $\pi^0 - \eta$ and $a_0^0(980) - f_0(980)$ mixing. These sources of the isospin breaking lead to very different shapes of the two-pion mass spectra. Due to the $\pi^0 - \eta$ mixing, the $\pi^0\pi^+$, $\pi^+\pi^-$, and $\pi^0\pi^0$ mass spectra turn out to be wide and rather smooth. The $a_0^0(980) - f_0(980)$ mixing amplitude is large between the K^+K^- and $K^0\bar{K}^0$ thresholds [27] and as a result leads to the narrow (about 10 MeV wide) resonance-like structures in $\pi^+\pi^-$ and $\pi^0\pi^0$ mass spectra in the 1 GeV energy region. Another source of $\eta(1295) \to 3\pi$ decays is the mixing of the $\eta(1295)$ and $\pi^0(1300)$ resonances. The presence of this mechanism, which violates isospin, fundamentally distinguishes the case under consideration from the cases of the decays $\eta \to 3\pi$, $\eta' \to 3\pi$, $\eta(1405) \to 3\pi$, and $f_1(1285) \to 3\pi$. In Sec. IV we discuss the effect of the $\pi^0(1300) - \eta(1295)$ mixing and present estimates for the $\eta(1295) \to 3\pi$ decay widths caused by the considered isospin-breaking mechanisms. Note that when estimating the widths of the direct decays $\eta(1295) \to 3\pi$ caused by $\pi^0 - \eta$ and $a_0^0(980) - f_0(980)$ mixing, we did not take into account the rescattering effects of pions in the final state. However, such approach utilizing a minimum number of free parameters presents a quite reasonable guide for the primary treatment of future data. Currently, data on the $\eta(1295) \to 3\pi$ decays is completely absent, and the applicability of the chiral perturbation theory to the analysis of $\eta(1295) \to 3\pi$ is not obvious. By the way, the transition $\eta(1295) \to \pi^0(1300) \to 3\pi$ can be considered as a peculiar kind of final state interaction. In Sec. V the processes that can be used to search for $\eta(1295) \to 3\pi$ decays are discussed. The results of our analysis are briefly summarized in Sec. VI.

II. THE $\eta(1295) \to \eta\pi^+\pi^-$ DECAY

First of all, we note that the $\eta(1295)$ meson, like its probable $SU(3)$ partners $\pi(1300)$ and $K(1460)$ [1], has not been sufficiently studied yet, despite a large number of experiments performed [1,42–52]. Progress in the investigation of the $\eta(1295)$ would be highly desirable, especially since its searches in $\gamma\gamma$ collisions [53], in central production [54,55] and in a number of experiments on the radiative J/ψ decays [56–58] did not give the expected results. Recently, the properties of excited pseudoscalar states have been discussed in Refs. [52,59–63].

The state $\eta(1295)$ was first discovered as a result of a partial-wave analysis of the $\eta\pi^+\pi^-$ system in the reaction $\pi^- p \to \eta\pi^+\pi^- n$ at 8.45 GeV [43] and then confirmed in other experiments on the reactions $\pi^- p \to \eta\pi^+\pi^- n$ [44,46,49,50], $\pi^- p \to \eta\pi^0\pi^0 n$ [48], $\pi^- p \to K^+K_S^0\pi^- n$ [45], $\pi^- p \to K^+K^-\pi^0 n$ [51], $J/\psi \to \gamma\eta\pi^+\pi^-$ [47], and $B^+ \to K^+\eta\pi\pi$ [52] (see also [1,42]). In almost all experiments, the separation of signals from $\eta(1295)$ and $f_1(1285)$ states, having common decay modes, was carried out. The results of the partial-wave analyzes indicate that the $\eta(1295)$ decays predominantly via quasi-two-body intermediate states: $\eta(1295) \to a_0(980)\pi \to \eta\pi\pi$ and $\eta(1295) \to \eta\sigma \to \eta(\pi\pi)_S$. The relation between the $\eta(1295) \to a_0(980)\pi$ and $\eta(1295) \to \eta\sigma$ modes are not well defined [1,42–52]. But there are no special indications of the dominance of any one of them. According to the Particle Data Group (PDG) [1], the mass and total width of the $\eta(1295)$ are 1294 ± 4 MeV and 55 ± 5 MeV, respectively.

To estimate the probabilities of the isospin-breaking decays $\eta(1295) \to 3\pi$, it is necessary to have a model for the main decay $\eta(1295) \to \eta\pi^+\pi^-$. The decay width of $\eta(1295) \to \eta\pi^0\pi^0$ is related to the $\eta(1295) \to \eta\pi^+\pi^-$ one by the isotopic relation $2\Gamma_{\hat{\eta}\to\eta\pi^0\pi^0} = \Gamma_{\hat{\eta}\to\eta\pi^+\pi^+}$ [here and hereinafter $\hat{\eta}$ is a short notation for $\eta(1295)$]. We use a simple isobar model (see, for example, Refs. [8,64,65] and references therein) and write the amplitude of the $\eta(1295) \to \eta\pi^+\pi^-$ decay as follows:

$$F_{\hat{\eta}\to\eta\pi^+\pi^-}(s,t,u) = T_{a_0^+}(s) + T_{a_0^-}(t) + T_\sigma^{\hat{\eta}}(u), \quad (1)$$

where s, t, and u are the $\eta\pi^+$, $\eta\pi^-$, and $\pi^+\pi^-$ invariant mass squared, respectively, and the amplitudes T have the form

$$T_{a_0^+}(s) + T_{a_0^-}(t) = \frac{g_{\hat{\eta}a_0\pi}g_{a_0\eta\pi}}{16\pi}\left(\frac{1}{D_{a_0^+}(s)} + \frac{1}{D_{a_0^-}(t)}\right),$$
$$T_\sigma^{\hat{\eta}}(u) = C_{\hat{\eta}}T_0^0(u) = C_{\hat{\eta}}\frac{\eta_0^0(u)e^{\delta_0^0(u)} - 1}{2i\rho_{\pi^+\pi^-}(u)}. \quad (2)$$

That is, the amplitudes $T_{a_0^+}(s)$ and $T_{a_0^-}(t)$ are saturated by the contributions of intermediate $a_0^+(980)$ and $a_0^-(980)$ states that manifest themselves in the $\eta(1295) \to \eta\pi^+\pi^-$ decay in the form of peaks in the $\eta\pi^+$ and $\eta\pi^-$ mass spectra, respectively; here $D_{a_0^\pm}$ is the inverse propagator of the $a_0^\pm(980)$ resonance in which the finite width corrections are taken into account (see its explicit form, for example, in Refs. [30,41,66]). For the coupling constants of the $a_0(980)$ with pairs of light pseudoscalar mesons, we use the relations valid in the four-quark model [67,68]:

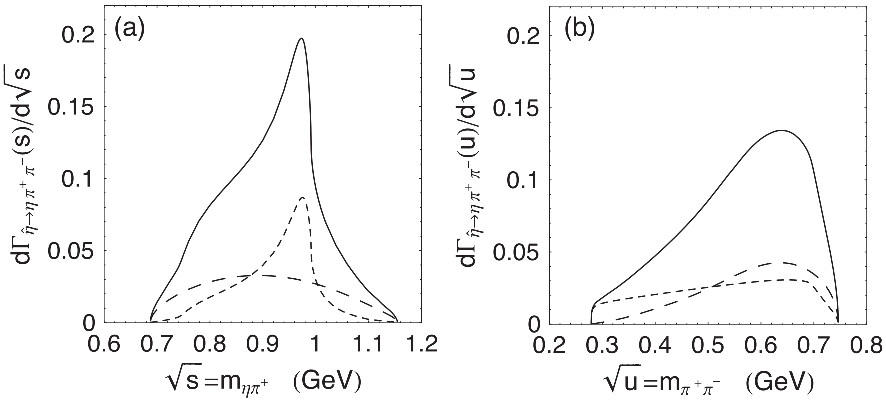

FIG. 1. The solid curves show the mass spectra (a) $\eta\pi^+$ and (b) $\pi^+\pi^-$ in the decay $\eta(1295) \to \eta\pi^+\pi^-$. Contributions from the $a_0(980)\pi$ and $\eta\sigma$ intermediate states are shown by short and long dashed curves, respectively.

$$g_{a_0\eta\pi} = \bar{g}\cos(\theta_i - \theta_p), \qquad g_{a_0\eta'\pi} = -\bar{g}\sin(\theta_i - \theta_p),$$
$$g_{a_0^+K^+\bar{K}^0} = g_{a_0^-K^0K^-} = \bar{g}, \tag{3}$$

where $\bar{g}$ is the overall coupling constant, $\theta_i = 35.3°$ is the so-called "ideal" mixing angle and $\theta_p = -11.3°$ is the mixing angle in the nonet of the light pseudoscalar mesons [1]. The corresponding decay widths have the standard form $\sqrt{s}\Gamma_{a_0\to ab}(s) = g^2_{a_0ab}\rho_{ab}(s)/(16\pi)$, where $\rho_{ab}(s) = [s^2 - 2s(m_a^2 + m_b^2) + (m_a^2 - m_b^2)^2]^{1/2}/s$. For further estimates, we set $m_{a_0} = 0.985$ GeV and $g^2_{a_0\eta\pi}/(16\pi) = 0.2\ \text{GeV}^2$ [30,41,66]. The amplitude $T_\sigma^{\hat\eta}(u)$, describing the interaction in the $\pi\pi$ channel, is taken according to Eq. (2) proportional to the S-wave $\pi\pi$ scattering amplitude with isospin $I = 0$; $\eta_0^0(u)$ and $\delta_0^0(u)$ are its inelasticity (equal to 1 for $u < 4m^2_{K^+}$) and phase, respectively. We take the amplitude $T_0^0(u)$ from Ref. [69] at the values of the parameters indicated in Table I for fitting variant I. In this work, the excellent simultaneous descriptions of the phase shifts, inelasticity, and mass distributions in the reactions $\pi\pi \to \pi\pi$, $\pi\pi \to K\bar{K}$, and $\phi \to \pi^0\pi^0\gamma$ was obtained.

The variables s, t, and u are related by the relation $\Sigma \equiv s + t + u = M^2 + m_\eta^2 + 2m^2_{\pi^+}$, where M is the invariant mass of the initial (virtual) state $\eta(1295)$. To simplify the notations, we do not indicate M among the arguments on which the amplitudes of the considered decays depend. Choosing $s \equiv m^2_{\eta\pi^+}$ and $u \equiv m^2_{\pi^+\pi^-}$ as independent variables and taking into account the adopted normalizations in Eqs. (1) and (2), we write the total decay width of $\eta(1295) \to \eta\pi^+\pi^-$ in the form

$$\Gamma_{\hat\eta\to\eta\pi^+\pi^-} = \frac{1}{\pi M^3}\int |F_{\hat\eta\to\eta\pi^+\pi^-}(s,t,u)|^2 ds du. \tag{4}$$

Integration limits for three-body decays are given in Ref. [70]. The constants $g_{\hat\eta a_0\pi}$ and $C_{\hat\eta}$ introduced in Eq. (2) can be estimated by assuming the condition of equality of the contributions from the amplitudes $T_{a_0^+}(s) + T_{a_0^-}(t)$ and $T_\sigma^{\hat\eta}(u)$ into the $\eta(1295) \to \eta\pi^+\pi^-$ decay width (see the discussion of experimental data in the second paragraph of this section) and normalizing the contribution of the module squared of their coherent sum [see Eqs. (1) and (4)] to the value $\Gamma_{\hat\eta\to\eta\pi^+\pi^-} \approx (2/3)\Gamma^{\text{tot}}_{\hat\eta} \approx 36$ MeV. For the nominal mass of the $\eta(1295)$ meson [1], we obtain $g_{\hat\eta a_0\pi} \approx 1.26$ GeV and $C_{\hat\eta} \approx 0.626$. The mass spectra of the $\eta\pi^+$ and $\pi^+\pi^-$ pairs,

$$\frac{d\Gamma_{\hat\eta\to\eta\pi^+\pi^-}(s)}{d\sqrt{s}} = \frac{2\sqrt{s}}{\pi M^3}\int |F_{\hat\eta\to\eta\pi^+\pi^-}(s,t,u)|^2 du \quad \text{and}$$
$$\frac{d\Gamma_{\hat\eta\to\eta\pi^+\pi^-}(u)}{d\sqrt{u}} = \frac{2\sqrt{u}}{\pi M^3}\int |F_{\hat\eta\to\eta\pi^+\pi^-}(s,t,u)|^2 ds, \tag{5}$$

corresponding to the above values of the parameters are shown in Fig. 1. The peak due to the $a_0^+(980)$ resonance is clearly visible in the $\eta\pi^+$ mass spectrum. The mass spectrum of $\pi^+\pi^-$ is naturally smoother. By varying the constants $g_{\hat\eta a_0\pi}$ and $C_{\hat\eta}$, one can obtain various shapes for these mass spectra and use the specified parametrization for fitting. Unfortunately, for the $\eta(1295) \to \eta\pi^+\pi^-$ decay there is still no data cleared of significant foreign admixtures. We are now in position to move on to estimating probabilities of the decays $\eta(1295) \to 3\pi$.

III. DIRECT DECAYS $\eta(1295) \to \pi^0\pi^+\pi^-$ AND $\eta(1295) \to 3\pi^0$

The diagrams responsible for the direct decays $\eta(1295) \to \pi^0\pi^+\pi^-$ are shown in Fig. 2. Diagrams (a), (b), and (c) are due to the $\pi^0 - \eta$ mixing [10], and diagram (d) is due to the $a_0^0(980) - f_0(980)$ mixing [27]. The corresponding amplitude can be written in the form

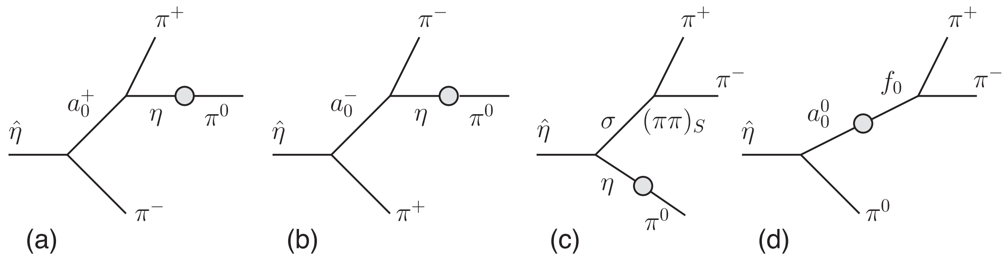

FIG. 2. Direct decays $\eta(1295) \to \pi^0\pi^+\pi^-$.

$$F^{\rm dir}_{\hat{\eta}\to\pi^0\pi^+\pi^-}(s,t,u) \equiv F^{\rm dir}_c(s,t,u) = \frac{\Pi_{\pi^0\eta}}{m^2_\eta - m^2_{\pi^0}}(T_{a_0^+}(s) + T_{a_0^-}(t) + T_\sigma(u)) + \frac{g_{\hat{\eta}a_0\pi}g_{f_0\pi^+\pi^-}}{16\pi}e^{i\delta_B(u)}G_{a_0^0f_0}(u), \tag{6}$$

where $\Pi_{\pi^0\eta}$ is the mass squared of the $\pi^0-\eta$ transition, $\delta_B(u)$ is a smooth and large phase (of about 90° for $\sqrt{u} = m_{\pi^+\pi^-} \approx 1$ GeV) of the elastic background accompanying the $f_0(980)$ resonance in the S-wave reaction $\pi\pi\to\pi\pi$ in the channel with isospin $I=0$ [27–29], and $G_{a_0^0f_0}(u)$ is the propagator of the $a_0^0(980)-f_0(980)$ transition [27,30,41];

$$G_{a_0^0f_0}(u) = \frac{\Pi_{a_0^0f_0}(u)}{D_{a_0^0}(u)D_{f_0}(u) - \Pi^2_{a_0^0f_0}(u)}, \tag{7}$$

$$\Pi_{a_0^0f_0}(u) = \frac{g_{a_0^0K^+K^-}g_{f_0K^+K^-}}{16\pi}\left[i[\rho_{K^+K^-}(u) - \rho_{K^0\bar{K}^0}(u)] - \frac{\rho_{K^+K^-}(u)}{\pi}\ln\frac{1+\rho_{K^+K^-}(u)}{1-\rho_{K^+K^-}(u)} + \frac{\rho_{K^0\bar{K}^0}(u)}{\pi}\ln\frac{1+\rho_{K^0\bar{K}^0}(u)}{1-\rho_{K^0\bar{K}^0}(u)}\right], \tag{8}$$

where $\rho_{K\bar{K}}(u) = \sqrt{1-4m_K^2/u}$ for $\sqrt{u}\geq 2m_K$; if $\sqrt{u}\leq 2m_K$, then $\rho_{K\bar{K}}(u)$ should be replaced to $i|\rho_{K\bar{K}}(u)|$. In Eq. (7) $D_{f_0}(u)$ is the inverse propagator of the $f_0(980)$ resonance coupled with $\pi\pi$, $K\bar{K}$, and $\eta\eta$ channels (see, for example, Ref. [41]). For further estimates, we set $m_{f_0} = 0.985$ GeV, $(3/2)g^2_{f_0\pi^+\pi^-}/(16\pi) = 0.098$ GeV^2, $2g^2_{f_0K^+K^-}/(16\pi) = 0.4$ GeV^2, and $g^2_{f_0\eta\eta} = g^2_{f_0K^+K^-}$ [41]. Notice that the phase of the amplitude of the $a_0^0(980)-f_0(980)$ mixing, $\Pi_{a_0^0f_0}(u)$, in the region between K^+K^- and $K^0\bar{K}^0$ thresholds changes by about 90° [28,29,41]. This fact is crucial for the observation of the $a_0^0(980)-f_0(980)$ mixing effect in polarization experiments [28,29]. According to the analysis presented in Ref. [71], we use to estimate $\Pi_{\pi^0\eta}$ the value equal to -0.004 GeV^2, see also Refs. [72,73].

The decay amplitude $\eta(1295)\to 3\pi^0$, taking into account the identity of the π^0 mesons, can be written as [20]

$$F^{\rm dir}_{\hat{\eta}\to3\pi^0}(s,t,u) \equiv F^{\rm dir}_n(s,t,u) = F^{\rm dir}_c(s,t,u) + F^{\rm dir}_c(u,s,t,) + F^{\rm dir}_c(t,u,s). \tag{9}$$

The full decay widths $\Gamma^{\rm dir}_{\eta(1295)\to\pi^0\pi^+\pi^-}$ and $\Gamma^{\rm dir}_{\eta(1295)\to3\pi^0}$ are

$$\Gamma^{\rm dir}_{\hat{\eta}\to\pi^0\pi^+\pi^-} = \frac{1}{\pi M^3}\int|F^{\rm dir}_c(s,t,u)|^2 ds du \quad \text{and} \quad \Gamma^{\rm dir}_{\hat{\eta}\to3\pi^0} = \frac{1}{6\pi M^3}\int|F^{\rm dir}_n(s,t,u)|^2 ds du. \tag{10}$$

For the above values of the parameters, we get

$$\Gamma^{\rm dir}_{\hat{\eta}\to\pi^0\pi^+\pi^-} \approx 0.027 \text{ MeV} \quad \text{and} \quad \Gamma^{\rm dir}_{\hat{\eta}\to3\pi^0} \approx 0.031 \text{ MeV}. \tag{11}$$

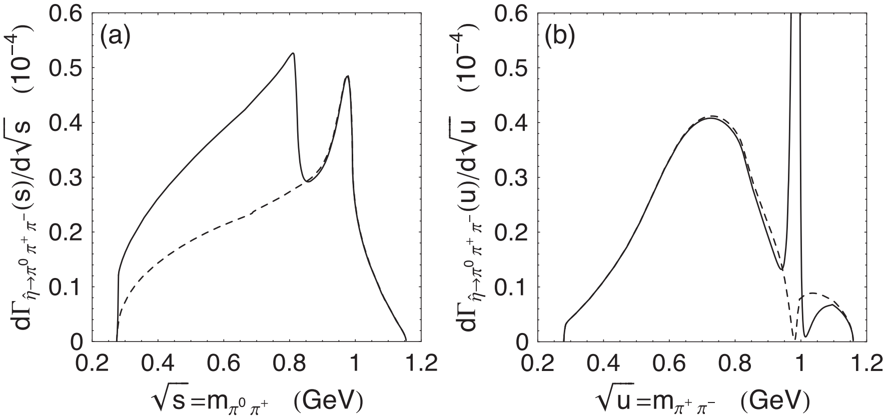

FIG. 3. The solid curves show (a) $\pi^0\pi^+$ and (b) $\pi^+\pi^-$ mass spectra in the $\eta(1295) \to \pi^0\pi^+\pi^-$ decay; due to the $a_0^0(980) - f_0(980)$ mixing mechanism the $\pi^+\pi^-$ mass spectrum reaches ≈ 11 at its maximum. The areas under the solid curves correspond to $\Gamma^{\rm dir}_{\hat\eta\to\pi^0\pi^+\pi^-} = 0.027$ MeV. The dashed curves show the contributions due to $\pi^0 - \eta$ mixing.

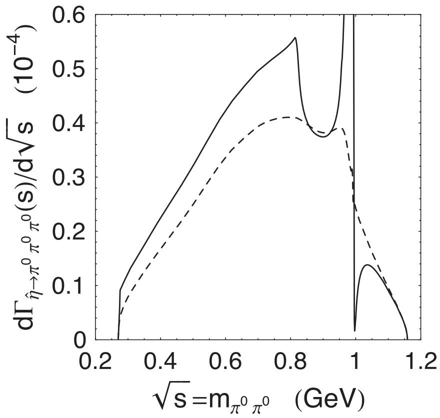

FIG. 4. The solid curve shows $\pi^0\pi^0$ mass spectrum in the $\eta(1295) \to 3\pi^0$ decay; due to the $a_0^0(980) - f_0(980)$ mixing mechanism the $\pi^0\pi^0$ mass spectrum reaches ≈ 3.4 at its maximum. The area under the solid curve corresponds to $\Gamma^{\rm dir}_{\hat\eta\to 3\pi^0} = 0.031$ MeV. The dashed curve shows the contribution due to $\pi^0 - \eta$ mixing.

As can be seen from Figs. 3 and 4, both mechanisms breaking isospin, $\pi^0 - \eta$ mixing and $a_0^0(980) - f_0(980)$ mixing, make significant contributions to $\Gamma_{\hat\eta\to 3\pi}$. The contribution from the $a_0^0(980) - f_0(980)$ mixing to the $\pi^+\pi^-$ and $\pi^0\pi^0$ mass spectra is concentrated in a narrow region near the $K\bar{K}$ thresholds, see Figs. 3(b) and 4. At the maximum, this contribution reaches ≈ 11 in the $\pi^+\pi^-$ mass spectrum [Fig. 3(b)] and ≈ 3.4 in the $\pi^0\pi^0$ one (Fig. 4). For completeness, Fig. 5 shows the Dalitz plots for distributions $|F_c^{\rm dir}(s,t,u)|^2/(\pi M^3)$ and $|F_n^{\rm dir}(s,t,u)|^2/(6\pi M^3)$. The $a_0^0(980) - f_0(980)$ mixing mechanism is responsible for the areas of the strongest blackening in these plots.

IV. $\pi^0(1300) - \eta(1295)$ MIXING

Suppose that for the states $\eta(1295)$ and $\pi^0(1300)$ that are close in mass, their mixing occurs at the same level as the $\rho^0 - \omega$ mixing [1,2,74–76]. Thus we set $\mathrm{Re}\Pi_{\hat\pi^0\hat\eta} \approx \mathrm{Re}\Pi_{\rho^0\omega} \approx \Pi_{\rho^0\omega} \approx 0.0034\ \mathrm{GeV}^2$, where $\Pi_{\hat\pi^0\hat\eta}$

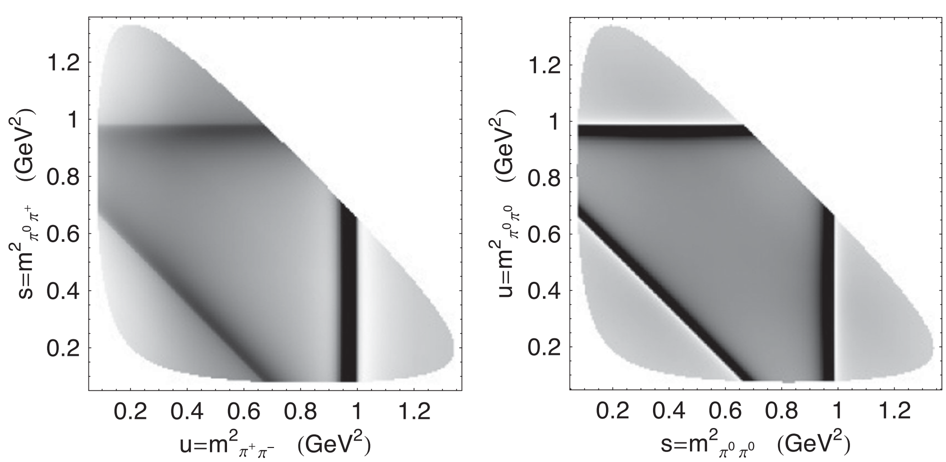

FIG. 5. The Dalitz plots for distributions (a) $|F_c^{\rm dir}(s,t,u)|^2/(\pi M^3)$ and (b) $|F_n^{\rm dir}(s,t,u)|^2/(6\pi M^3)$. The $a_0^0(980) - f_0(980)$ mixing mechanism is responsible for the areas of the strongest blackening in these plots.

and $\Pi_{\rho^0\omega}$ are the square of the messes of the $\pi^0(1300)-\eta(1295)$ and $\rho^0-\omega$ transitions, respectively [here and hereinafter $\hat\pi$ is a short notation for $\pi(1300)$]. Hence, the following rough estimate can be obtained for the width of the $\eta(1295)\to3\pi$ decay caused by the $\pi^0(1300)-\eta(1295)$ mixing:

$$\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to3\pi}\approx\left|\frac{{\rm Re}\Pi_{\hat\pi^0\hat\eta}}{-im_{\hat\pi}\Gamma^{\rm tot}_{\hat\pi}}\right|^2\Gamma_{\hat\pi\to3\pi}\approx0.023\ \text{MeV}. \qquad (12)$$

Here we put $m_{\hat\pi}\approx1300$ MeV and $\Gamma^{\rm tot}_{\hat\pi}\approx\Gamma_{\hat\pi\to3\pi}\approx300$ MeV (note that the data on the $\pi(1300)$ is very poor [1]). This value is close to the estimates of $\Gamma^{\rm dir}_{\hat\eta\to\pi^0\pi^+\pi^-}$ and $\Gamma^{\rm dir}_{\hat\eta\to3\pi^0}$ indicated in Eq. (11) for the mechanisms of the direct $\eta(1295)\to3\pi$ decays. Let us try to determine the widths of the individual decay modes that add up to $\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to3\pi}$. With the help of isotopic invariance alone, which gives

$$\begin{aligned}\Gamma_{\hat\pi^+\to3\pi}&=\Gamma_{\hat\pi^+\to\pi^+\pi^+\pi^-}+\Gamma_{\hat\pi^+\to\pi^+\pi^0\pi^0}=\Gamma_{\hat\pi^0\to3\pi}\\&=\Gamma_{\hat\pi^0\to\pi^0\pi^+\pi^-}+\Gamma_{\hat\pi^0\to3\pi^0},\\\Gamma_{\hat\pi^+\to\pi^+\pi^+\pi^-}&=\Gamma_{\hat\pi^+\to\pi^+\pi^0\pi^0}+\Gamma_{\hat\pi^0\to3\pi^0}\\&\quad(\text{or}\quad\Gamma_{\hat\pi^0\to\pi^0\pi^+\pi^-}=2\Gamma_{\hat\pi^+\to\pi^+\pi^0\pi^0}),\end{aligned}\qquad(13)$$

this cannot be done. A simple estimate of the components of $\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to3\pi}$ can be obtained if we assume that the decay $\pi(1300)\to3\pi$ occurs via $\rho\pi$ and $\sigma\pi$ intermediate states. In so doing, the possible contributions of the $\pi\pi S$-wave with isospin $I=2$ to the final three-pion states should be neglected. Experimental data on the ratio between the $\rho\pi$ and $\sigma\pi$ modes are inconsistent [1,60,77,78]. By analogy with Eq. (2), we write the decay amplitude $\pi^0(1300)\to\pi^0\sigma\to\pi^0(\pi^+\pi^-)_S$ in the form $T^{\hat\pi}_\sigma(u)=C_{\hat\pi}T^0_0(u)$. The naive quark model allows us to relate the constants $C_{\hat\pi}$ and $C_{\hat\eta}$ by the relation $C_{\hat\pi}=C_{\hat\eta}/\sin(\theta_i-\theta_p)\approx1.376C_{\hat\eta}$, if, due to the proximity of the masses of $\pi(1300)$ and $\eta(1295)$ states, we accept for the $\eta(1295)$ the quark structure of the form $(u\bar u+d\bar d)/\sqrt2$. In that case the numerical calculation similar to those done in Secs. II and III gives $\Gamma_{\hat\pi^0\to\pi^0\sigma\to\pi^0\pi^+\pi^-}\approx85$ MeV and $\Gamma_{\hat\pi^0\to\pi^0\sigma\to3\pi^0}\approx107$ MeV. Consequently, approximately 200 MeV in the $\pi(1300)\to3\pi$ decay width falls on the contribution from the $\sigma\pi$ intermediate state and the rest is due to the $\pi(1300)\to\rho\pi\to3\pi$ decay. For $\Gamma_{\hat\pi\to3\pi}\approx300$ MeV, the value of $\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to3\pi}$ indicated in Eq. (12) is thus added from three approximately equal partial widths $\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to\pi^0\sigma\to\pi^0\pi^+\pi^-}$, $\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to\pi^0\sigma\to3\pi^0}$, and $\Gamma^{\rm mix}_{\hat\eta\to\hat\pi^0\to\rho^\pm\pi^\mp\to\pi^0\pi^+\pi^-}$. Note that the decay mode $\eta(1295)\to\rho^\pm\pi^\mp\to\pi^0\pi^+\pi^-$ appears only due to the $\pi^0(1300)-\eta(1295)$ mixing.

The imaginary part of the transition amplitude $\Pi_{\hat\pi^0\hat\eta}$ is due to the contributions of real intermediate states. An example of a diagram that contributes to ${\rm Im}\Pi_{\hat\pi^0\hat\eta}$ is shown in

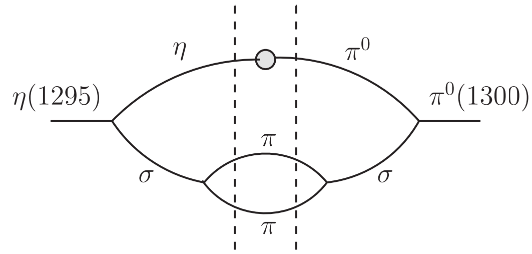

FIG. 6. An example of a diagram contributing to ${\rm Im}\Pi_{\hat\pi^0\hat\eta}$. The vertical dashed lines cutting the diagram mean that the 4-momenta of the intermediate particles, either 3π or $\eta\pi\pi$, lie on their mass shells.

Fig. 6, where the vertical dashed lines cutting the diagram mean that the 4-momenta of the intermediate particles, either 3π or $\eta\pi\pi$, lie on their mass shells. An estimate of this contribution to ${\rm Im}\Pi_{\hat\pi^0\hat\eta}$ gives

$$\begin{aligned}{\rm Im}\Pi_{\hat\pi^0\hat\eta}&=\frac{\Pi_{\pi^0\eta}}{m^2_\eta-m^2_{\pi^0}}\frac{C_{\hat\eta}C_{\hat\pi}}{\pi M^2}\frac{3}{2}\\&\quad\times\left(\int_{3\pi\,\text{p.s.}}|T^0_0(u)|^2dsdu-\int_{\eta\pi\pi\,\text{p.s.}}|T^0_0(u)|^2dsdu\right)\\&\approx(-0.0017+0.0004)\ \text{GeV}^2\approx-0.0013\ \text{GeV}^2.\end{aligned}\qquad(14)$$

Here, in the first term, integration is carried out over the three-pion phase space (3π p.s.), and in the second, over the phase space of $\eta\pi\pi$. The imaginary parts of the transition amplitudes $\eta(1295)\to a_0^\pm(980)\pi^\mp\to\eta\pi^+\pi^-\to\pi^0\pi^+\pi^-\to(\pi^0\sigma+\rho^\pm\pi^\mp)\to\pi^0(1300)$ and $\eta(1295)\to a_0^0(980)\pi^0\to(K^+K^-+K^0\bar K^0)\pi^0\to f_0(980)\pi^0\to\pi\pi\pi^0\to(\pi^0\sigma+\rho^\pm\pi^\mp)\to\pi^0(1300)$ cannot be easily estimated because of the need to take into account contributions not only from three-body, but also from five-body real intermediate states. However, they cannot greatly exceed the above estimate of the contribution to ${\rm Im}\Pi_{\hat\pi^0\hat\eta}$ from the transition $\eta(1295)\to\eta\sigma\to\pi^0\sigma\to\pi^0(1300)$. This is also confirmed by the upper estimate that roughly takes into account all contributions to $|{\rm Im}\Pi_{\hat\pi^0\hat\eta}|$:

$$\begin{aligned}&|{\rm Im}\Pi_{\hat\pi^0\hat\eta}|\\&<m_{\hat\eta}\left(\sqrt{\Gamma^{\rm dir}_{\hat\eta\to\pi^0\pi^+\pi^-}\Gamma_{\hat\pi^0\to\pi^0\pi^+\pi^-}}+\sqrt{\Gamma^{\rm dir}_{\hat\eta\to3\pi^0}\Gamma_{\hat\pi^0\to\to3\pi^0}}\right)\\&<m_{\hat\eta}\left(\sqrt{\Gamma^{\rm dir}_{\hat\eta\to\pi^0\pi^+\pi^-}}+\sqrt{\Gamma^{\rm dir}_{\hat\eta\to3\pi^0}}\right)\sqrt{\Gamma_{\hat\pi^0\to3\pi}}\approx0.0076\,\text{GeV}^2.\end{aligned}\qquad(15)$$

So the estimates in Eqs. (11), (12), (14), and (15) say that the value of $\Gamma_{\eta(1295)\to3\pi}$ may well turn out to be of the order of 0.1 MeV and, respectively, $\mathcal{B}(\eta(1295)\to3\pi)\approx0.2\%$. In so doing, one can hope that in the interference phenomena indications on the existence of the decays $\eta(1295) \to \pi^0\pi^+\pi^-$ and $\eta(1295) \to 3\pi^0$ can be detected at a level of a few percent of the main 3π signal.

For 3π production cross sections ($\pi^0\pi^+\pi^-$ or $3\pi^0$) in the pseudoscalar channel, we have

$$d\sigma(3\pi) = \left| \frac{A_{\hat{\pi}^0} F_{\hat{\pi}^0 \to 3\pi}(s,t,u)}{D_{\hat{\pi}^0}(M)} + \frac{A_{\hat{\eta}}[F^{\rm dir}_{\hat{\eta}\to 3\pi}(s,t,u) + F^{\rm mix}_{\hat{\eta}\to\hat{\pi}^0\to 3\pi}(s,t,u)]}{D_{\hat{\eta}}(M)} \right|^2 dsdu, \quad (16)$$

where $A_{\hat{\pi}^0}$, $A_{\hat{\eta}}$, and $1/D_{\hat{\pi}^0}(M)$, $1/D_{\hat{\eta}}(M)$ are the production amplitudes and the Breit-Wigner propagators of the $\pi^0(1300)$, $\eta(1295)$ resonances, respectively, $F_{\hat{\pi}^0\to 3\pi}(s,t,u)$ is the sum of the amplitudes of the $\pi^0(1300)$ decay into 3π via $\sigma\pi$ and $\rho\pi$ intermediate states, and

$$F^{\rm mix}_{\hat{\eta}\to\hat{\pi}^0\to 3\pi}(s,t,u) = \frac{\Pi_{\hat{\pi}^0\hat{\eta}} F_{\hat{\pi}^0\to 3\pi}(s,t,u)}{D_{\hat{\pi}^0}(M)}. \quad (17)$$

If the channel $3\pi^0$ is investigated, then the contribution of the $\rho\pi$ mode is absent. If the channel $\rho^\pm\pi^\mp \to \pi^0\pi^+\pi^-$ is separated, then Eq. (16) does not contain the contribution of the amplitude $F^{\rm dir}_{\hat{\eta}\to 3\pi}(s,t,u)$. Since $\Gamma^{\rm tot}_{\hat{\eta}}$ is 4–6 times smaller than $\Gamma^{\rm tot}_{\hat{\pi}}$ [1], the signal from the $\eta(1295)$ resonance has some enhancement. Of course, the interference pattern in the $\eta(1295)$ region depends fundamentally on the relative magnitude of the amplitudes $A_{\hat{\eta}}$ and $A_{\hat{\pi}^0}$.

V. WHERE TO SEARCH FOR $\eta(1295) \to 3\pi$ DECAYS?

The J/ψ radiative decays are dominated by hadron production in the states with the isospin $I = 0$. Therefore, the $\eta(1295)$ meson can manifest itself in the $J/\psi \to \gamma\eta(1295) \to \gamma 3\pi$ decays without accompaniment of the $\pi(1300)$. As already noted in the Introduction, a hint at the $J/\psi \to \gamma f_1(1285)/\eta(1295) \to \gamma\pi^0\pi^+\pi^-, \gamma\pi^0\pi^0\pi^0$ decays was obtained by the BESIII Collaboration [31] In this experiment, the invariant masses of the $\pi^+\pi^-$ and $\pi^0\pi^0$ pairs in the $\pi^0\pi^+\pi^-$ and $\pi^0\pi^0\pi^0$ mass spectra were in the $f_0(980)$ region (0.94 GeV $< m_{\pi^+\pi^-(\pi^0\pi^0)} <$ 1.04 GeV). In our model, the $\pi^+\pi^-$ and $\pi^0\pi^0$ masses spectra in this region are dominated by the transition $\eta(1295) \to a_0^0(980)\pi^0 \to f_0(980)\pi^0 \to 3\pi$ caused by $a_0^0(980) - f_0(980)$ mixing, see Figs. 3(b) and 4. Narrow peaks in the $\pi^+\pi^-$ and $\pi^0\pi^0$ mass spectra are a good indicator of the $a_0^0(980) - f_0(980)$ mixing mechanism (or in the general case of the $K\bar{K}$ loop isospin-breaking mechanism [30,41]). One can hope that searches for the signals from the $\eta(1295)$ resonance in the decays $J/\psi \to \gamma\eta(1295) \to \gamma\eta\pi\pi$ and $J/\psi \to \gamma\eta(1295) \to \gamma 3\pi$ will be successful.

Information about the $\eta(1295) \to 3\pi$ decays can also be obtained from interference experiments. For example, in the semileptonic decay $D^+(c\bar{d}) \to d\bar{d}e^+\nu_e \to 3\pi e^+\nu_e$ the $d\bar{d}$ virtual intermediate state is not has a definite isospin and can be a source of the $\eta(1295)$ and $\pi(1300)$ resonances with approximately equal production amplitudes $A_{\hat{\eta}}$ and $A_{\hat{\pi}^0}$. Then the wavelike distortion of the $\pi(1300)$ peak in the three-pion channel can make up $\approx \pm 5\%$ due to interference with the contribution from the decay $\eta(1295) \to 3\pi$. Of course, the observation of such interference phenomena implies the availability of good data on the main signal from the $\pi(1300)$ resonance. To obtain them, any reactions can be involved, for example, those in which the $\pi(1300)$ resonance was already observed earlier, i.e., peripheral reactions, nucleon-antinucleon annihilation, D meson decays, $\gamma\gamma$ collisions, etc. [1,42,60,77,78]. Methods of the partial-wave analysis are now well developed and with sufficient statistics the extraction of 3π events related to the pseudoscalar channel is although not a simple but purely technical challenge.

VI. CONCLUSION

Studies of radial excitations of light pseudoscalar mesons are of physical interest. The available data on the $\eta(1295)$, $\pi(1300)$, and $K(1460)$ states are rather poor [1]. In this paper we attract attention of experimenters to the comprehensive study of the $\eta(1295)$ state in its decay channels into $\eta\pi\pi$ and 3π. In the series of pseudoscalar isoscalar mesons η, η', $\eta(1295)$, and $\eta(1405)$ [1] the $\eta(1295)$ remained the last one that has not yet presented unexpected surprises associated with the violation of isotopic invariance in its decays into $\pi^0\pi^+\pi^-$ and $3\pi^0$. We have constructed a simple isobar model for the description of the main decay of the $\eta(1295)$ into $\eta\pi\pi$ and, based on this model, estimated the widths of the direct decays $\eta(1295) \to \pi^0\pi^+\pi^-$ and $\eta(1295) \to 3\pi^0$ caused by the mixing of the $\pi^0 - \eta$ and $a_0^0(980) - f_0(980)$ mesons. Then we discussed a possible role of the $\pi^0(1300) - \eta(1295)$ mixing and obtained rough estimates for the $\eta(1295) \to 3\pi$ decay widths with taking into account this additional mechanism of the isospin breaking. One can expect that the decay width $\eta(1295) \to 3\pi$ will be of the order of 0.1 MeV and, respectively, $\mathcal{B}(\eta(1295) \to 3\pi) \approx 0.2\%$. One can hope that in the interference phenomena indications on the existence of the decays $\eta(1295) \to \pi^0\pi^+\pi^-$ and $\eta(1295) \to 3\pi^0$ can be detected at a level of a few percent of the main 3π signal. The presented estimates are not overestimated. Finally, we have discussed the processes that can be used for experimental searches of the decays $\eta(1295) \to 3\pi$. In particular, we have noted reactions $J/\psi \to \gamma\eta(1295) \to \gamma 3\pi$ and $D^+ \to [\pi(1300) + \eta(1295)]e^+\nu_e \to 3\pi e^+\nu_e$.

Study of the decays $\eta(1295) \to \eta\pi\pi$, $\eta(1295) \to 3\pi$, and also $\pi(1300) \to 3\pi$ can be a good challenge for new high-statistics experiments.

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