Radiative effects in two-dimensional models of quantum electrodynamics in a constant magnetic field

P. A. Eminov*
National Research University Higher School of Economics, Myasnitskaya Ulitsa 20, Moscow 101000, Russia

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The radiative energy shift of an electron in a constant magnetic field has been considered in the framework of the $(2+1)$-dimensional quantum electrodynamics with four-component fermions as well as in reduced QED_{3+1} in which photons propagate in a $(3+1)$-dimensional bulk and fermions are localized on a 2-brane. Analytical expressions are obtained for the energy of interaction of the electron spin with an external field and the radiative shift of the electron mass after averaging over the spin states of the electron. For ultrarelativistic energies of an electron and relatively weak magnetic fields, the total probability of a photon emission by a massive electron and the anomalous magnetic moment of an electron in a reduced QED_{3+1} are calculated. The dependence of the obtained values on the invariant dynamic parameter of synchrotron radiation is investigated. The total probabilities of synchrotron radiation of a charged massless fermion and the production of a pair of charged massless fermions by a photon in an external magnetic field are calculated in $(2+1)$-dimensional models of quantum electrodynamics.

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I. INTRODUCTION

The study of radiative and spin effects in low-dimensional models of quantum field theory, as well as in graphene and other planar structures, is one of the topical problems in physics [1–7]. As in the $(3+1)$-dimensional space-time, the study of quantum processes in various models of two-dimensional quantum electrodynamics, considering the influence of external conditions, such as classical fields, finite temperatures, and the density of the medium, has attracted considerable interest recently. In $(2+1)$-dimensional space-time, the algebra of Dirac matrices is described using Pauli matrices in two nonequivalent ways. For example, the work [8] uses representations

$$\gamma^0=\sigma_3,\qquad \gamma^1=i\sigma_1,\qquad \gamma^2=i\sigma_2,\tag{1.1}$$

and

$$\gamma^0=-\sigma_3,\qquad \gamma^1=-i\sigma_1,\qquad \gamma^2=-i\sigma_2,\tag{1.2}$$

where $\sigma_k(k=1,2,3)$ are the Pauli matrices.

As a result, there are two different Dirac equations for describing spinor fields

$$[\gamma^\mu(\hat{p}_\mu+eA_\mu)-m]\Psi(x^0,x^1,x^2)=0,\quad \mu=0,1,2,\tag{1.3}$$

where Ψ is a two-component spinor, $\hat{p}_x$ and $\hat{p}_y$ are the projections of the momentum operator, m is the mass of an electron with a charge $-e<0$, and $A^\mu(x)$—is the potential of the gauge field.

Along with the (1.1)–(1.2), other pairs of nonequivalent representations of Dirac gamma matrices are used also in two-dimensional quantum field theory [9–11]

$$\begin{aligned}&\gamma^0=\sigma_3,\quad \gamma^1=is\sigma_1,\quad \gamma^2=i\sigma_2,\quad s=\pm1;\\&\gamma^0=s\sigma_3,\quad \gamma^1=i\sigma_1,\quad \gamma^2=i\sigma_2,\quad s=\pm1;\end{aligned}\tag{1.4}$$

where the values $s=\pm1$ correspond to two nonequivalent representations of gamma matrices. Note also that, in contrast to the QED_{3+1}, the massive electron in the two-dimensional theory has only one spin state, that is, the electron spin in QED_{2+1} is not a pseudovector, but a pseudoscalar with respect to the Lorentz transformations [12,13].

In addition, the mass term in the Dirac equation (1.3) is odd under $P-$ and $T-$ transformations [13]. Thus, in QED_{2+1} we have two different and odd Dirac equations, each of which can be used to describe the polarization properties of an electron or positron with one spin degree of freedom. Reference [14] proposed to combine the solutions of two different Dirac equations with two-dimensional gamma matrices into one solution, which is considered as a solution describing the stationary state of a two-dimensional electron, wherein the values $s=\pm1$ correspond to two different values “of the projections of the electron spin”

*peminov@mail.ru

on the "direction of the magnetic field" (spin "up" and spin "down" states).

Such solutions of the Dirac equation in a constant magnetic field in the gauge $A_\mu = (0, 0, H)$ were found in works [5,9] in two different ways. In Ref. [9], the method of eigenfunctions in an external electromagnetic field is used, which was developed in Ref. [15].

The electron energy levels in $(2+1)$-dimensional QED and the Dirac equation normalized positive-frequency solution are described by formula [9]:

$$\Psi(x,y,z) = \frac{1}{\sqrt{2E_n}} \begin{pmatrix} \sqrt{E_n + sm} U_n(\eta) \\ -sign(eB)\sqrt{E_n + sm} U_{n-1}(\eta) \end{pmatrix} \times \exp[-iE_n t + iyp_y], \tag{1.5}$$

$$E_n = \sqrt{m^2 + 2eHn}, \tag{1.6}$$

where $n = 0, 1, 2, \ldots$ is the principle quantum number and p_y is the electron momentum projection. The Hermite function in formula (1.5) is expressed in terms of the Hermite polynomials by the formula

$$U_n(\eta) = \frac{(eH)^{\frac{1}{4}}}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{\eta^2}{2}} H_n(\eta), \quad H_n(\eta) = (-1)^n e^{\eta^2} \frac{d^n}{d\eta^n} e^{-\eta^2}, \tag{1.7}$$

and the argument of these functions

$$\eta = \sqrt{eH}\left(x + \frac{p_y}{eH}\right). \tag{1.8}$$

P-even Dirac equation for the massive electron in $(2+1)$-dimensional quantum electrodynamics was proposed in a model with a doubled fermionic representation [5,8,16–18]. In this case, there exists a fermion mass term preserving parity.

This model uses a four-dimensional (reducible) representation for Dirac gamma matrices, which contains both irreducible representations (1.1), being their direct sum:

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad \gamma^{1,2} = \begin{pmatrix} i\sigma_{1,2} & 0 \\ 0 & -i\sigma_{1,2} \end{pmatrix}. \tag{1.9}$$

The Lagrangian of the model is determined by the formula

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(\hat{p} + e\hat{A} - m)\Psi, \qquad \mu, \nu = 0, 1, 2, \tag{1.10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$—is the tensor of the electromagnetic field.

For the four-component bispinor

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{1.11}$$

the Dirac equation in a constant magnetic field in a Landau gauge

$$A^\mu(x) = (0, 0, xH), \tag{1.12}$$

shall be presented in the form

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \alpha_1 \hat{p}_x + \alpha_2(\hat{p}_y + exH) + m\gamma_0, \quad \alpha_{1,2} = \gamma^0 \gamma^{1,2}. \tag{1.13}$$

In contrast to the two-component model, the transformation of spatial inversion, in turn, transforms the two-dimensional spinors in formula (1.11) into each other:

$$P\colon \Psi_1(x^0, x^1, x^2) \to \sigma_1 \Psi_2(x^0, x^1, -x^2), \quad \Psi_2(x^0, x^1, x^2) \to \sigma_1 \Psi_1(x^0, x^1, -x^2), \tag{1.14}$$

which leads to the fermionic mass terms in the Lagrangian (1.10) and in the Dirac equation (1.13) being invariant with respect to the inversion of space.

As a result, in contrast to the two-component model, in the four-component model, the electron has two different states of spin polarization.

Explicit separation of the solutions of the Dirac equation (1.13) over spin states was carried out in the work [19]. To describe the spin of an electron in the stationary state with energy E_n in a model with the doubled fermion representation, we require that the solution of the Dirac equation to be eigenfunction of the Hamiltonian from Dirac equation, operator $\hat{p}_y = -i\frac{\partial}{\partial y}$ of the projection of the momentum to the Oy axis and a spin operator [19]

$$\hat{A} = i\varepsilon \frac{E_n}{m} \gamma^0 \gamma^1 \gamma^2,$$

which commutes with the Hamiltonian of the Dirac equation.

Positive- and negative-frequency solutions of the Dirac equation ($\varepsilon = \pm 1$) obey the additional condition

$$\hat{A}\Psi = \zeta\Psi,$$

where the quantum number $\zeta = \pm 1$ implies the projection of the electron spin on the direction of the magnetic field.

As a result, in the model with a doubled fermion representation, the wave function of the stationary state of a two-dimensional electron in a constant magnetic field is determined by the formula [19]

$$\Psi_{\varepsilon=+1} = \frac{(eH)^{\frac{1}{4}}}{\sqrt{2E_n}} \exp[-iE_n t + iyp_y] \left[D_1 \begin{pmatrix} \sqrt{E_n + m} U_{n-1}(\eta) \\ \sqrt{E_n - m} U_n(\eta) \\ 0 \\ 0 \end{pmatrix} + D_{-1} \begin{pmatrix} 0 \\ 0 \\ \sqrt{E_n - m} U_{n-1}(\eta) \\ \sqrt{E_n + m} U_n(\eta) \end{pmatrix} \right]. \tag{1.15}$$

Here the electrons energy and the argument of the Hermite functions are determined by the formulas (1.6) and (1.8), with $\zeta = +1$ it is necessary to set $D_1 = 1$, $D_{-1} = 0$ (the spin is directed along the field), and with $\zeta = -1$ vice versa, $D_1 = 0$, $D_{-1} = 1$ (the spin is directed opposite the field).

It was shown [19] that the operator $\hat{A}$ is a $(2+1)$-dimensional analogue of the projection of the operator three-dimensional spin pseudo-vector on the direction of the magnetic field in QED_{3+1}.

In a topologically massive QED_{2+1} with a doubled fermion representation, the radiative shift of the electron energy in an electron-positron plasma was investigated at finite temperature in charge-symmetric plasma [20] as well as in a degenerate electron gas [21], wherein the effect of the generation of fermionic mass at a finite temperature and a nonzero chemical potential was predicted.

The one-loop energy shift of the ground state of an electron in a constant magnetic field in QED_{2+1}, the Lagrangian of which includes the Chern-Simons term, was investigated in the work [22], and in a magnetized plasma, in the work [19].

The anomalous magnetic moment of an electron in a topologically massive QED_{2+1} at a finite temperature was first investigated based on the calculation of the vertex function in Ref. [23].

The results of this work in the region of their applicability coincide with the results of Ref. [19], where the electron AMM was obtained directly from the amplitude of the elastic scattering of an electron in a magnetized plasma in the framework of the four-component QED_{2+1} with the Chern-Simons term.

The dynamic nature of the AMM of the excited states of an electron in a constant magnetic field in the QED_{2+1} without the Chern-Simons term, as well as in topologically massive two-dimensional electrodynamics, was studied in the works [24–26]. It was shown that the magnetic field, and not the Chern-Simons term, as was previously understood [23], regulates infrared divergence in calculating the AMM of an electron. In the two-component QED_{2+1} the mass operator and the radiative correction to the fermion mass, when it lies at the lowest Landau level in a constant magnetic field, were calculated in Ref. [27].

The one-loop polarization operator and the exact expression for the amplitude of elastic photon scattering in a constant magnetic field in $(2+1)$-dimensional QED were obtained in Refs. [28,29], and the polarization operator in QED_{2+1} with a nonzero fermion density was studied in Refs. [9,10]. In graphene and in other planar nanostructures electrons, which are localized on a plane, move in $(2+1)$-dimensional space-time, while interaction between them is provided by massless gauge fields propagating in $(3+1)$-dimensional space-time. We will study radiative effects within the framework of the reduced QED model, in which photons propagate in a $(3+1)$ bulk, and fermions are localized on a 2-brane. Reduced $(3+1)$-dimensional QED, also known as the pseudo-QED model [30,31], is widely used to describe radiative effects in such structures. For example, the papers [32,33] published the results of experimental studies of the spin electron g-factor in graphene in an external magnetic field.

The theoretical analysis of the results of these experiments in the works [34,35] is conducted in the approximation linear in the external field, and the spin g-factor in Ref. [35] was determined by calculating the vertex function of the electron in the framework of the pseudo-QED model.

In the same model, in the one-loop approximation, the radiative shift of the mass of a planar electron in a constant magnetic field was calculated in Refs. [27,36] and it was demonstrated in Ref. [36] that the main term of the asymptotics of the radiative correction to the electron mass in a superstrong magnetic field is not dependent on the electron mass.

Recently, the effects of quantum electrodynamics with the participation of charged massless particles have attracted considerable interest. The elastic scattering amplitude for excited states of a charged fermion in an external constant magnetic field in a two-component pseudo-QED model was obtained in Ref. [11], where it was shown that the imaginary part of the amplitude, which determines the total probability of photon emission by a fermion, is nonzero also in the case of a massless charged fermion.

The effect of the production of a pair of massless charged fermions by a photon in a constant crossed field in the standard QED_{3+1} was investigated in Ref. [37]. The problem of radiation in massless scalar quantum electrodynamics in an external magnetic field was solved in Ref. [38].

In the present work, the radiative effects, accompanying the propagation of the fermions and photons in a constant magnetic field, are studied for the cases of QED_{2+1} with a doubled fermion representation, as well as in reduced QED_{3+1}, in which photons propagate in a $(3+1)$-dimensional bulk and fermions are localized on a 2-brane.

The calculations performed in this work are based on the results (2.25)–(2.27), which describe the radiative energy shift of an electron in the framework of the QED_{2+1} with doubled fermionic representation and were first obtained in the one-loop approximation in Sec. II.

Section III presents the new results obtained in the study of the dynamic nature of the AMM of an electron in a constant magnetic field in the framework of the reduced QED_{3+1}. The dependence of the radiative correction to the energy of the ground state of an electron on the magnetic field strength are also investigated.

In Sec. IV, we develop the quantum theory of synchrotron radiation of a massive two-dimensional electron in QED_{2+1} with a doubled fermionic representation, as well as in reduced QED_{3+1}.

It should be noted that the study of synchrotron radiation of a massive two-dimensional electron has so far been carried out only within the framework of the classical theory (see [39] and the cited literature), and the problems that arise here, as it seems to us, may be finally solved only after the creation of the quantum theory of synchrotron radiation.

In Sec. V, we presented the quantum theory of synchrotron radiation of a two-dimensional charged massless fermion in reduced QED_{3+1}. The calculation is carried out both on the basis of the exact expression (2.25) for the radiative energy shift of a massless two-dimensional charged fermion, and as a result of passing to the limit of zero electron mass in formula (4.4), obtained in Sec. IV for a massive electron.

The probability of the production of a pair of charged massless fermions by a photon in an external magnetic field in QED_{2+1} also were first calculated. In Sec. VI, the main results of this study are formulated.

II. RADIATIVE ENERGY SHIFT OF A TWO-DIMENSIONAL ELECTRON IN A MAGNETIC FIELD

In the one-loop approximation, the electron mass operator in the QED_{2+1} with the doubled fermion representation is described by the formula [15,40]

$$\Sigma(x,x') = -ie^2\gamma^\mu S_c(H,x,x')\gamma^\nu D_{\mu\nu}(x-x'), \qquad (2.1)$$

where the photon propagator in the Landau gauge

$$D_{\mu\nu}(x-x') = -ig_{\mu\nu}\int \frac{d^3p}{(2\pi)^3}\frac{\exp[-ip(x-x')]}{{p_0}^2-\vec{p}^2+i0}, \qquad (2.2)$$

and for the causal Green's function of the electron in a constant magnetic field, we use the representation [40,41]

$$S_c(H;x,x') = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\omega \exp[i\omega(t-t')] \times \sum_{s,\varepsilon=\pm1}\frac{\Psi_s^\varepsilon(\vec{x})\bar{\Psi}_s^\varepsilon(\vec{x'})}{\omega+\varepsilon E_s(1-i\delta)}. \qquad (2.3)$$

In expression (2.3), the summation is performed over all quantum numbers $s=\{n',p'_y,\zeta'\}$ of positive-frequency $(\varepsilon=+1)$ and negative-frequency $(\varepsilon=-1)$ of stationary electron states, $\Psi_{s'}^\varepsilon(\vec{x})$ is the coordinate part of the Dirac equation in a constant magnetic field in QED_{2+1} with doubled fermionic representation, which is defined by the formula (2.1).

To carry out the summation over the principal quantum number of the intermediate states of the electron in the formula (2.3), we use a method proposed in Ref. [42] for calculation of two-loop contribution to thermodynamic potential of QED_{3+1} in a constant magnetic field.

First, we expand the matrix $K^s_{\alpha\beta}=\Psi_s^\varepsilon(\vec{x})\bar{\Psi}_s^\varepsilon(\vec{x'})$ in (2.3) with respect to matrices $I,\gamma^\mu,\gamma^\mu\gamma^\nu,\gamma^0\gamma^1\gamma^2$:

$$4K = I+\gamma^\mu F_\mu + i\gamma^1\gamma^2F_{12}+i\gamma^0\gamma^1\gamma^2F_{012}, \qquad (2.4)$$

where

$$I_0 = SpK, \qquad F_\mu = Sp\gamma^\mu K,$$
$$F_{12} = Sp(i\gamma^1\gamma^2K), \qquad F_{012} = Sp(i\gamma^0\gamma^1\gamma^2K). \qquad (2.5)$$

Next, we perform summation in (2.3) over the principle quantum number n' of electron intermediate states using the formula

$$\sum_{n'=0}^{\infty} t^{n'}L^\alpha_{n'}(x) = (1-t)^{-(\alpha+1)}\exp\left[\frac{xt}{t-1}\right], \qquad (2.6)$$

where $L^\alpha_n(x)$—is a Laguerre polynomial [42].

As a result, the propagator of a two-dimensional electron in a constant magnetic field is given by the formulas:

$$S_c(H;x,x') = \exp\left[-i\frac{eH}{2}(y-y')(x+x')\right]S(H,x-x'),$$
$$S(H,x-x') = \int\frac{d^3k}{(2\pi)^3}\exp[-ik(x-x')]S_c(H,k), \qquad (2.7)$$

$$S_c(H,k) = -i\int_0^\infty ds_1\exp\left[is_1(k_0^2-m^2+i\delta)-\frac{\vec{k}^2tgeHs_1}{eHs_1}\right] \times\left[(\gamma^0k^0+m)(1+\gamma^1\gamma^2tgeHs_1)-\frac{(\vec{k},\vec{\gamma})}{\cos^2(eHs_1)}\right] \qquad (2.8)$$

The radiative shift of the electron energy is determined by the diagonal matrix element of the mass operator (2.1)

$$\Delta E_n = -ie^2 \frac{1}{T}\int d^3x d^3x' \bar{\Psi}_{n,q_y,\zeta}(x)\gamma^\mu \times S_c(H;x,x')\gamma^\nu D_{\mu\nu}(x-x')\Psi_{n,q_y,\zeta}(x'), \quad (2.9)$$

where T is the interaction time.

Calculation (2.9) is carried out using the Schwinger parameterization in the photon propagator (2.2)

$$\frac{1}{p_0^2-p^2+i0} = -i\int_0^\infty ds_2 \exp[is_2(p_0^2-p^2+i0)] \quad (2.10)$$

and the change of variables s_1 and s_2 to u and y according to the formulas

$$u = \frac{s_1}{s_1+s_2}, \qquad y = u(s_1+s_2) = s_1, \quad 0 \le u \le 1, 0 \le y < \infty. \quad (2.11)$$

Using the value of the integral [43]

$$\int_{-\infty}^{+\infty} e^{-c^2x^2} H_m(a+cx)H_n(b+cx)dx = \frac{2^n\sqrt{\pi}m!b^{n-m}}{c} L_m^{n-m}(-2ab), \quad (2.12)$$

where $H_n(x)$ and $L_n^m(x)$ Hermite and Laguerre polynomials and taking into account (2.2), (2.7)–(2.8), (2.10)–(2.12) in formula (2.9) we carry out the integration over space-time variables x^μ and x'^μ ($\mu = 0,1,2$):

$$J_{mn} = \frac{1}{LT}\int d^3xd^3x' \exp\left[-i\frac{eH}{2}(y-y')(x+x') + iE_n(t-t') - iq_y(y-y') - i\kappa^\mu(x-x')_\mu\right] U_m(\eta')U_n(\eta) = (2\pi)^2\delta(p_0+k_0-E_n)\left(\frac{2}{eH}\right)(-1)^m \exp\left[i(n-m)\left(\frac{\pi}{2}-\phi\right)\right] I_{nm}(z_0), \quad (2.13)$$

Here $p^\mu = (p^0,\vec{p})$—is 3-momentum of a virtual photon, ϕ—polar angle of vector $\vec{\kappa} = \vec{k}+\vec{p}$, T—is the interaction time, L—is the length of periodicity in direction of axis OY, δ-function of Dirac $\delta(p_0+k_0-E_n)$ expresses the energy conservation and the argument of the Laguerre functions $I_{nm}(z_0)$ [44] is determined by the formula

$$z_0 = \frac{2\vec{\kappa}^2}{eH}. \quad (2.14)$$

The integral over variable k^0 is eliminated using the Dirac δ-function, and integrals over variable p^0 are Gaussian.

After these integrations, formula (2.9) for the total shift of the radiation energy is reduced to the form:

$$\Delta E_n = (-ie^2)(-1)^n\frac{m^2}{E_n}e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{eH}}\int_0^1\frac{du}{u^{\frac{3}{2}}}\int_0^\infty\sqrt{y}dy\int d\vec{k}d\vec{p}\exp[-iE_n^2uy - iy(\kappa^2+k^2-2(\vec{k},\vec{\kappa})-i0)] \times \exp\left[i\frac{y(1-u)}{u}\left(E_n^2-m^2+i\delta-\vec{k}^2\frac{tgeHy}{eHy}\right)\right]\times\left(\zeta\frac{E_n}{m}F_1+F_2\right), \quad (2.15)$$

where

$$F_1 = I_{n,n}(z_0)\left[\frac{(2-u)e^{iz}}{\cos z}+2u\frac{e^{-iz}}{\cos z}\right] + I_{n-1,n-1}(z_0)\left[\frac{(2-u)e^{-iz}}{\cos z}+2u\frac{e^{iz}}{\cos z}\right], \quad (2.16)$$

$$F_2 = -I_{n,n}(z_0)\left[\frac{(2-u)e^{iz}}{\cos z}+2u\frac{e^{-iz}}{\cos z}\right] + I_{n-1,n-1}(z_0)\left[\frac{(2-u)e^{-iz}}{\cos z}+2u\frac{e^{iz}}{\cos z}\right] - \frac{p_\perp^2(u-1)}{m^2}\left[I_{n,n}(z_0)\frac{e^{-iz}}{\cos z} - I_{n-1,n-1}(z_0)\frac{e^{iz}}{\cos z}\right] + 2itgz\left(\frac{p_\perp}{m}\right)^2(u-1)[I_{n,n}(z_0)+I_{n-1,n-1}(z_0)] + \frac{2(\vec{k},\vec{\kappa})}{\kappa\cos^2 z}\frac{p_\perp^2}{m}I_{n,n-1}(z_0), \quad z = eHy. \quad (2.17)$$

The quantity ΔE_n in formula (2.15) is complex, since the excited states of an electron in a magnetic field are unstable with respect to the process of photon emission and transition to states with the principal quantum number $n' < n$. The real part of the total radiative correction to the electron energy is related to the electron mass shift Δm by the formula [15,40,44]

$$\Re(\Delta E_n) = \frac{m}{E_n}\Re(\Delta m), \tag{2.18}$$

and its imaginary part according to the optical theorem is proportional to the total probability of the electron radiative transition from the initial quantum state in the given external field:

$$\Im(\Delta E_n) = -\frac{1}{2}w_s. \tag{2.19}$$

In contrast to ΔE, the mass radiative correction

$$\Delta m = \frac{E_n}{m}\Delta E_n \tag{2.20}$$

is the Lorentz invariant, and similar to the QED_{3+1} it is necessary to renormalize the fermion mass by subtracting its value in the vanishing field from the nonrenormalized quantity (2.15) [15,44]. We can now pass in formulas (2.15)–(2.17) from the integration over $\vec{p}$ to integration over $\vec{\kappa} = \vec{p} + \vec{k}$, i.e.,

$$d\vec{k}d\vec{p} = d\vec{k}d\vec{\kappa} = r = kdk\kappa d\kappa d\alpha d\phi \to 2\pi kdk\kappa d\kappa d\psi, \tag{2.21}$$

where $\psi = \phi - \alpha, \psi \in [0, 2\pi]$, α and ϕ are polar angles of vectors $\vec{k}$ and $\vec{\kappa}$.

The integrals over variable ψ give Bessel function $J_\nu(b)$ of real argument

$$b = 2k\kappa y$$

of the zero and first orders, and integration over variable k is performed using the Weber formula [43]

$$\int_0^\infty \exp[-px^2]x^{\nu+1}J_\nu(cx)dx = \frac{c^\nu}{(2p)^{\nu+1}}\exp\left[-\frac{c^2}{4p}\right],$$
$$p > 0, \qquad c > 0, \qquad \Re\nu > -1. \tag{2.22}$$

Using also the relations [44]

$$I_{n,n}(\rho) = e^{-\frac{\rho}{2}}L_n(\rho), \quad I_{n,n-1}(\rho) = e^{-\frac{\rho}{2}}\sqrt{\frac{\rho}{n}}L^1_{n-1}(\rho), \tag{2.23}$$

which express the relationship between polynomials and Laguerre functions, the integral over the variable $t = \kappa^2$ is calculated by means of the formula [43]

$$\int_0^\infty t^\alpha e^{-pt}L^\alpha_n(bt) = \frac{\Gamma(\alpha+n+1)}{n!}\cdot\frac{(p-b)^n}{p^{\alpha+n+1}}, \tag{2.24}$$

where $\Gamma(x)$ is the Euler gamma-function.

As a result, the radiative correction to the electron energy in a constant magnetic field in the framework of QED_{2+1} with a doubled fermion representation is defined in the one-loop approximation by formula:

$$\Delta E_n = -\frac{m^2e^2}{16\pi^{\frac{3}{2}}E_n}e^{i\frac{\pi}{4}}\int_0^1\frac{du}{\sqrt{u}}\int_0^\infty\frac{dy}{\sqrt{y}}\frac{1}{F}$$
$$\times \exp[-ip_\perp^2 y(u-1) - 2in\ \arctan\lambda - im^2uy]$$
$$\times\left(\zeta\frac{E_n}{m}\Omega_1 + \Omega_2\right), \tag{2.25}$$

where the following notations are adopted:

$$\Omega_1 = (2 - u + 2ue^{-2iz})$$
$$- (1-\delta_{0,n})e^{2i\ \arctan\lambda}[2u + (2-u)e^{-2iz}], \tag{2.26}$$

$$\Omega_2 = -(2-u+2ue^{-2iz}) - (1-\delta_{0,n})e^{2i\ \arctan\lambda}[2u+(2-u)e^{-2iz}] + 2i\left(\frac{p_\perp}{m}\right)^2(u-1)\sin ze^{-iz}[1-e^{2i\ \arctan\lambda}]$$
$$-\left(\frac{p_\perp}{m}\right)^2(u-1)[e^{-2iz} + e^{2i\ \arctan\lambda}] + 2\left(\frac{p_\perp}{m}\right)^2\frac{u-1}{F}e^{2i\ \arctan\lambda - 2iz}, \tag{2.27}$$

$$\lambda = \frac{tgz}{1+\frac{u}{1-u}\frac{tgz}{z}}, \qquad F = 1 - u + ue^{-iz}\frac{\sin z}{z},$$
$$z = eHy, \qquad p_\perp^2 = 2eHn. \tag{2.28}$$

Note that in formula (2.25), after averaging over the electron spin, the term proportional to $\zeta\Omega_1$ vanish. This term is also equal to zero in the case of a massless electron, which also directly follows from the obtained formulas (2.25)–(2.27). The second term in (2.25), which proportional to Ω_2, was first obtained in this work and does not depend on the electron spin.

In the case of a reduced QED_{3+1}, in formula (1.2) for the photon propagator it is necessary to replace [27,30,34,35]

$$\frac{1}{p_0^2 - \vec{p}^2 + i0} \to \frac{1}{2\sqrt{p_0^2 - \vec{p}^2 + i0}}$$
$$= \left(-\frac{i}{2}\right)\sqrt{\frac{i}{\pi}}\int_0^\infty ds_2\frac{e^{is_2(p_0^2 - \vec{p}^2 + i0)}}{\sqrt{s_2}}. \tag{2.29}$$

This leads to the fact that in formula (2.25) the integrand in the double integral over the variables (u, y) is multiplied by the factor

$$\left(-\frac{i}{2}\right)\sqrt{\frac{i}{\pi}}\sqrt{\frac{u}{y(1-u)}},$$

that is, the radiative shift of the total energy of a two-dimensional electron in the reduced QED_{3+1} is described by formula (2.25), where one should replace

$$\int_0^1 \frac{du}{\sqrt{u}}\int_0^\infty \frac{dy}{\sqrt{y}} g(u,y) \to \left(-\frac{i}{2}\right)\sqrt{\frac{i}{\pi}}\int_0^1 \frac{du}{\sqrt{1-u}}\int_0^\infty \frac{dy}{y} g(u,y). \qquad (2.30)$$

In the next sections, we consider some limiting cases that follow from formulas (2.25)–(2.28), (2.30) and are of the greatest physical interest.

III. RADIATIVE CORRECTION TO THE GROUND-STATE ENERGY AND THE DYNAMIC NATURE OF THE ELECTRON AMM IN REDUCED QED

In the case of the ground state of an electron, when $n = 0$, $\zeta = -1$, we find from the formulas (2.25)–(2.28) that the renormalized correction to the mass of the ground state of an electron is determined by the formula

$$\Delta m = \frac{me^2}{8\pi^{\frac{3}{2}}} e^{i\frac{\pi}{4}} \int_0^1 \frac{du}{\sqrt{u}}\int_0^\infty \frac{dy}{\sqrt{y}} e^{-im^2 uy} \times \left[\frac{2-u+2ue^{-2iz}}{F} - (u+2)\right]. \qquad (3.1)$$

This result coincides with the results obtained earlier in works [19,22,24,25] from the amplitude of elastic scattering in the magnetic field of an electron in the ground state.

In the case of a reduced QED_{3+1}, from formula (3.1), taking into account (2.30), we find

$$(\Delta m)' = \frac{\alpha m}{2\pi}\int_0^1 \frac{du}{\sqrt{1-u}}\int_0^\infty dt e^{-t\frac{H_0}{H}} \times \left[\frac{2-u+2ue^{-\frac{2t}{u}}}{2t(1-u)+u^2(1-e^{-\frac{2t}{u}})} - \frac{(u+2)}{t}\right]. \qquad (3.2)$$

In the limiting case of a weak magnetic field, when the parameter

$$g = \frac{H}{H_0} \ll 1, \qquad (3.3)$$

where the critical field $H_0 = \frac{m^2}{e} \simeq 4,41 \times 10^{13}\, G$ has been introduced, the main contribution to the radiative mass shift of an electron (3.2) provides the domain $\frac{t}{u} \ll 1$ and we find that in an P-even reduced QED_{3+1}, as in the standard quantum electrodynamics [44,45], taking into account radiation effects in a weak magnetic field first leads to a decrease in the energy of the ground state of an electron with an increase in the magnetic field strength:

$$E_0^r \simeq m\left(1 - \frac{2\alpha}{3\pi}\cdot\frac{H}{H_0}\right), \qquad \frac{H}{H_0} \ll 1. \qquad (3.4)$$

In a superstrong magnetic field, when $H \gg H_0$, the main contribution to integral (3.2) is from the range of variables $y \gg 1$, $u \sim 1$ [45], and the correction to the electron mass is represented in the form

$$(\Delta m)' \simeq \frac{\alpha m}{2\pi}\int_0^\infty dy e^{-y\frac{H_0}{H}}\int_{u_0}^1 \frac{du}{\sqrt{1-u}[2y(1-u)+u^2]}.$$

Here, the integral over the variable u is tabular, and the leading term of its asymptotic expansion is equal to $\frac{\pi}{2}\sqrt{\frac{2}{y}}$. The remaining integral over the variable y is Gaussian, and the final expression for the leading term of the asymptotic expansion (3.2) is described by the formula

$$(\Delta m)^r \simeq \frac{\alpha m}{2\sqrt{2}}\sqrt{\frac{\pi eH}{m^2}} = \frac{\alpha}{2\sqrt{2}}\sqrt{\pi eH}, \qquad H \gg H_0. \quad (3.5)$$

An interesting feature of this result, which has no analog in the standard electrodynamics, is that in QED_{2+1} in a superstrong magnetic field, the leading term of the asymptotic radiative shift of the energy of the ground state of a two-dimensional electron does not depend on the fermion mass [36]. Result (3.5) coincides with the results (45) and (63) in [27,36], respectively.

As it follows from the formulas (3.4)–(3.5), in the reduced QED_{3+1} the ground-state electron mass shift changes sign, when passing from a weak to a superstrong magnetic field.

AMM of an electron in an external magnetic field is determined by those terms in the radiative energy shift, which are proportional to the projection of spin on the magnetic field direction [14,15]. In QED_{2+1} the electron AMM is determined by the formula [15,44,46,47]

$$\Re(\Delta E_n(\zeta)) = -\zeta H \Delta\mu. \qquad (3.6)$$

Assuming that in the ground state, when the electron spin can only be directed opposite to the magnetic field orientation, the whole amount of the energy shift is equal to the energy of interaction of the electron anomalous magnetic moment with the magnetic field, from formulas (3.4) and (3.6) we find

$$\frac{\Delta\mu}{\mu_B} \simeq -\frac{4\alpha}{3\pi}, \qquad H \ll H_0, \qquad (3.7)$$

where α is a fine-structure constant, $\mu_B = \frac{e}{2m}$ is the Bohr magneton.

The result described by formula (3.7) coincides with the vacuum value of the AMM of the electron, previously obtained in [35] on the calculation of the vertex function of the electron in the framework of pseudo-QED.

Here we studied the dynamic nature of the electron AMM in the reduced QED for ultrarelativistic electron energies, when the following conditions

$$\frac{H}{H_0} \ll 1, \qquad \frac{p_\perp}{m} \gg 1, \tag{3.8}$$

are fulfilled and the dynamic parameter of synchrotron radiation

$$\chi = \frac{e}{m^3}\sqrt{-(F_{\mu\nu}p^\nu)^2} = \frac{H}{H_0}\cdot\frac{p_\perp}{m} \tag{3.9}$$

can take any value.

Maintaining the first two expansion terms in formula (2.28) in the significant region $t \ll 1$,

$$\lambda \simeq t(1-u) + \frac{t^3 u(1-u)^2}{3}, \qquad t = eHy,$$

from formulas (2.25)–(2.28) we find the following expression for the part of the electron energy of a two-dimensional electron in the reduced QED_{3+1}, which explicitly depends on the spin quantum number:

$$\Delta E_n(\zeta) = \zeta\frac{\alpha m}{4\pi}\left(\frac{H}{H_0}\right)\int_0^\infty \frac{(\nu+2)d\nu}{(\nu+1)^{\frac{5}{2}}}\cdot z_0 f(z_0). \tag{3.10}$$

Here the Hardy-Stokes function are introduced

$$\begin{aligned} f(z_0) &= i\int_0^\infty \exp\left[-i\left(z_0 t + \frac{t^3}{3}\right)\right]dt \\ &= \Upsilon(z_0) + i\Phi(z_0), \end{aligned} \tag{3.11}$$

where the arguments of the epsilon function $\Upsilon(z_0)$ and Airy function $\Phi(z_0)$ are defined by the formula

$$z_0 = \left(\frac{u}{(1-u)\chi}\right)^{\frac{2}{3}}, \tag{3.12}$$

and introduced the spectral variable

$$\nu = \frac{u}{1-u}. \tag{3.13}$$

Thus, in the pseudo-QED model, the AMM of an electron in the quasiclassical approximation is determined by the formula

$$\frac{\Delta\mu}{\mu_B} = -\frac{\alpha}{2\pi}\int_0^\infty \frac{(\nu+2)z_0}{(\nu+1)^{\frac{5}{2}}}\Upsilon(z_0)d\nu. \tag{3.14}$$

Note that formulas (3.10) and (3.14) determine the exact values of these quantities in the case of a constant crossed field $(E = H, (\vec{E}, \vec{H}) = 0)$ [15].

To calculate the integrals in formulas (3.10) and (3.14), we use the Mellin transformation with respect to the parameter $a = \frac{1}{\chi}$.

As a result, the energy shift and the AMM of an two dimensional electron in an external magnetic field are described by the formulas

$$\begin{pmatrix} \Delta E_n(\zeta) \\ \frac{\Delta\mu}{\mu_B} \end{pmatrix} = \begin{pmatrix} \zeta\alpha m\frac{H}{4\pi H_0}G(\chi) \\ -\frac{\alpha}{2\pi}\Re G(\chi) \end{pmatrix}, \tag{3.15}$$

where the notation is adopted

$$\begin{aligned} G(\chi) &= \frac{1}{4\pi}\int_{s_0-i\infty}^{s_0+i\infty} ds\chi^{s-\frac{2}{3}}3^{-\frac{s}{2}+\frac{1}{3}}\left(\frac{4}{3}+s\right)e^{-i\frac{\pi}{4}(2s+\frac{2}{3})} \\ &\quad\times\frac{\Gamma(\frac{3s}{2})\Gamma(-\frac{s}{2}+\frac{1}{3})\Gamma(\frac{5}{3}-s)\Gamma(s-\frac{1}{6})}{\Gamma(\frac{5}{2})}, \\ &\quad \frac{1}{6} < s_0 < \frac{2}{3}. \end{aligned} \tag{3.16}$$

Here $\Gamma(z)$ is the Euler gamma-function, which is a meromorphic function with poles of the first order at the points $z = 0, -1, -2, \ldots$ and with a residue equal to $\frac{(-1)^n}{n!}$ at $z = -n$. Further, if $\chi \ll 1$ we close the integration contour in (3.7) in the right half-plane and obtain an asymptotic series in powers κ. For $\chi \gg 1$ the integration contour we must close in the left half-plane of the complex variable s and obtain a convergent series in inverse powers χ. Thus, equation (3.7) allows the following asymptotic expansions to be derived:

$$\frac{\Delta\mu}{\mu_B} = -\frac{4\alpha}{3\pi}, \quad \chi \ll 1, \tag{3.17}$$

$$\frac{\Delta\mu}{\mu_B} = -\frac{\alpha}{4\pi}\cdot\frac{3\cdot 3^{\frac{1}{4}}\Gamma^2(\frac{1}{4})}{2\sqrt{2\chi}}, \qquad \chi \gg 1. \tag{3.18}$$

Result (3.17) coincides with formula (3.7) of this work and with the result (22) in [35].

IV. RADIATIVE ENERGY SHIFT OF EXCITED STATES AND SYNCHROTRON RADIATION OF A TWO-DIMENSIONAL ELECTRON

In this section, we study the spin-independent part of the total radiative energy shift of excited states of a two-dimensional electron in a constant magnetic field. After averaging over the electron spin from formulas (2.25)–(2.27) in the quasiclassical approximation, we find the following expression for the radiative correction to the electron energy in the four-component QED_{2+1}:

$$\Delta E_n = \frac{me^2}{16\pi^{\frac{3}{2}}p_\perp}e^{i\frac{\pi}{4}}\int_0^\infty \frac{\sqrt{z_0}dv}{v(v+1)} \times \left[2\cdot\frac{3v+2}{v+1}g_1(z_0)+\frac{g'(z_0)}{z_0}\left(8+\frac{4v}{3}\right)\right], \quad (4.1)$$

where $z_0=(\frac{v}{\chi})^{\frac{2}{3}}$, $g'(z_0)=\frac{dg(z_0)}{dz_0}$ and function $g(z_0)$ specified by formula

$$g(z_0)=i\int_0^\infty \sqrt{t}\exp\left[-i\left(z_0t+\frac{t^3}{3}\right)\right]dt, \quad (4.2)$$

which is a two-dimensional analog of the Hardy-Stokes function in two-dimensional quantum electrodynamics of an intense external field [29] and the function $g_1(z_0)$ is defined by the formula

$$g_1(z_0)=\int_0^\infty \frac{dt}{\sqrt{t}}e^{-iz_0t}\left(e^{-i\frac{t^3}{3}}-1\right). \quad (4.3)$$

The result corresponding to formula (4.1) in the pseudo-QED has the form:

$$(\Delta E_n)^r = \frac{\alpha m^2}{8\pi p_\perp}\int_0^\infty \frac{dv}{(v+1)^{\frac{3}{2}}} \times \left[2\cdot\frac{3v+2}{v+1}f_1(z_0)+\frac{f'(z_0)}{z_0}\left(8+\frac{4v}{3}\right)\right]. \quad (4.4)$$

Here $f'(z_0)$ is the derivative of the Hardy-Stokes function, and

$$f_1(z_0)=\int_0^\infty \frac{dt}{t}e^{-iz_0t}\left(e^{-i\frac{t^3}{3}}-1\right)=\int_{z_0}^\infty dt\left[f(t)-\frac{1}{t}\right]. \quad (4.5)$$

According to the optical theorem (2.19), the imaginary part of the elastic scattering amplitude (4.4) determines the total probability of the photon emission by a two-dimensional massive electron in a constant magnetic field in a reduced QED_{3+1}:

$$w^{(r)} = -\frac{\alpha m^2}{2\pi p_\perp}\int_0^\infty \frac{dv}{(v+1)^{\frac{3}{2}}} \times \left[\frac{3v+2}{v+1}\int_{z_0}^\infty \Phi(x)dx+\frac{f'(z_0)}{z_0}\left(4+\frac{2v}{3}\right)\right], \quad (4.6)$$

and the integrand determines the spectral distribution of the process probability with respect to the physical variable determined by formula (3.13). Expressing Airy functions in terms of Macdonald functions [44]

$$\Phi(z_0)=\sqrt{\frac{z_0}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}z_0^{\frac{3}{2}}\right),$$
$$\Phi'(z_0)=-\frac{z_0}{\sqrt{3}}K_{\frac{2}{3}}\left(\frac{2v}{3\chi}\right), \quad (4.7)$$

formula (4.6) for the probability of synchrotron radiation of a two-dimensional electron may be represented in a more compact form:

$$w^{(r)} = \frac{\alpha m^2}{2\pi\sqrt{3}p_\perp}\int_0^\infty \frac{K_{\frac{2}{3}}(\frac{2v}{3\chi})}{(1+v)^{\frac{3}{2}}}\left[-\frac{3v+2}{(v+1)} + \frac{32}{9v}[(1+v)^{\frac{3}{2}}-1]+\frac{2v}{3}\right]dv, \quad (4.8)$$

where $v=\frac{3}{2}\chi x$.

Formulas (4.1), (4.4), and (4.8) are exact in a crossed field. From formula (4.8) we find the asymptotics for the probability of the process in the quasiquantum limit ($\chi \ll 1$) and in the ultraquantum case ($\chi \gg 1$):

$$w^{(r)} \simeq \begin{pmatrix} \frac{5}{2\sqrt{3}}\frac{\alpha m^2}{p_\perp}\chi, & \chi \ll 1, \\ C\frac{\alpha m^2}{4\pi\sqrt{3}p_\perp}\Gamma(\frac{2}{3})(3\chi)^{\frac{2}{3}}, & \chi \gg 1 \end{pmatrix}, \quad (4.9)$$

where $C \simeq 12,\ 24$.

We note that at $\chi \gg 1$, the main contribution to the integral (4.8) comes from the region $\frac{2v}{3\chi} \ll 1$, in which the function $K_\mu(y)$ has the asymptotic representation [44]:

$$K_\mu(y) \simeq 2^{\mu-1}\Gamma(\mu)y^{-\mu}, \qquad y=\frac{2v}{3\chi}.$$

The remaining integral over the variable v we have calculated numerically

$$C=\int_0^\infty \frac{dv}{v^{\frac{2}{3}}(1+v)^{\frac{3}{2}}}\left[-\frac{3v+2}{(v+1)} + \frac{2v}{3}+\frac{32}{9v}[(1+v)^{\frac{3}{2}}-1]\right] \simeq 12,24. \quad (4.10)$$

V. PHOTON EMISSION BY A CHARGED MASSLESS FERMION AND THE CREATION OF CHARGED MASSLESS FERMION PAIR BY A PHOTON IN AN EXTERNAL MAGNETIC FIELD

Study of radiation effects in an intense external field with involvement of charged massless two-dimensional fermions, such as the emission of a photon by a massless charged fermions and the creation of a pair of massless charged fermions, is of particular interest for the physics of graphene and other low-dimensional structures. In the graphene and a number of other planar structures, the dynamics of electronic excitations is described by Dirac effective two-dimensional equation for both massless and massive charged fermions. The effective electron mass arises, for example, as a result of dynamic generation due to electron-electron and other interactions in graphene without a magnetic field. This is an important stimulus for the further development of theoretical studies of the dynamic generation of the electron mass in two-dimensional systems in a magnetic field [1–4].

Within the framework of classical electrodynamics, the problem of electromagnetic radiation of a massless charged particle has been considered by many authors (see, for example, [48,49] and the cited literature). In [39], the power of synchrotron radiation of a massive charged particle was calculated in space-time of any odd dimension, including the case of $(2+1)$-dimensional classical electrodynamics.

In $(3+1)$-dimensional space-time in the eikonal approximation, the electromagnetic field created by a massless charged particle are described in [50], and the emission of a photon by an electron in a constant electric field in the Dirac graphene model is considered in [51].

The problem of synchrotron radiation of a massless charged particle in $(3+1)$ -dimensional scalar electrodynamics received its solution relatively recently within the framework of the quantum theory of synchrotron radiation in [38].

It will be noted that important electrodynamics effect—the vacuum instability in the so-called supercritical Coulomb potential—is supposed to occur in graphene with charged impurities [52–55]. The electron-hole pair production in graphene is the condensed matter analog of electron-positron pair production due to the polarization and instability of the quantum electrodynamics vacuum in the external fields.

In Ref. [38] it is shown that massless scalar quantum electrodynamics and the zero-mass limit in the initially massive theory show the same results when calculating the total probability and spectral distribution of the probability of the process of photon emission by a massless charged particle in a constant magnetic field. The rate of decay of a photon into a pair of charged massless fermions in a constant crossed field in the framework of standard QED_{3+1} was obtained in the work [37].

In the framework of reduced QED_{3+1} with two-component fermions, the total probability of photon emission by charged massless fermion in crossed electromagnetic field is calculated in [11]. The results of the works [11,37] were obtained by passing to the limit to zero fermion mass in the amplitudes of the elastic scattering of a photon in the QED_{3+1} [56] and elastic scattering of an electron in the two-component QED_{2+1} in external electromagnetic field [27].

First, we consider the process of photon emission by a two-dimensional charged massless fermion in a four-component QED_{2+1} in a constant magnetic field.

From the formulas (2.25)–(2.27), in the limiting case of massless electron, we obtain the following exact expression for the total radiative energy shift:

$$\Delta E_n(m=0)=\frac{e^2p_\perp e^{i\frac{\pi}{4}}}{16\pi^{\frac{3}{2}}}\int_0^1\frac{du(1-u)}{\sqrt{u}}\int_0^\infty\frac{dy}{yF}\times\exp[-ip_\perp^2y(u-1)-2in\arctan\lambda]\times\left[2i\sin z e^{-iz}(1-e^{2i\arctan\lambda})-(e^{-2iz}+e^{2i\arctan\lambda})+\frac{2}{F}e^{2i\arctan\lambda-2iz}\right]. \quad (5.1)$$

We will be interested in the case of high-excited Landau levels of initial and final states ($n\gg1, n'\gg1$), when the integrals can be evaluated expanding in the exponent all $z=eHy$—dependent quantities in power series of $z\ll1$. After some algebra and taking into account (2.30), we arrive at the following representation of the radiative shift of the fermion energy in the reduced QED_{3+1}:

$$\Delta E_n^{(r)}(m=0)=\frac{e^2p_\perp}{8\pi^2}\int_0^1du\sqrt{1-u}\left[2(1-u)+\frac{u}{3}\right]\times\left(\int_0^\infty tdtexp\left[-it^3\frac{2nu(1-u)^2}{3}\right]\right), \quad (5.2)$$

where $p_\perp=\sqrt{eHn}$ is the energy of massless charged fermion.

Using the formula [43]

$$\int_0^\infty x^{(\mu-1)}\begin{pmatrix}\sin\alpha x\\ \cos\alpha x\end{pmatrix}dx=\frac{\Gamma(\mu)}{a^\mu}\begin{pmatrix}\sin\frac{\pi\mu}{2}\\ \cos\frac{\pi\mu}{2}\end{pmatrix},$$

we perform in (5.2) integration over the variable t, leading to

$$\Delta E_n^{(r)}(m=0)=\frac{e^2p_\perp}{24\pi^2}e^{-i\frac{\pi}{3}}\left(\frac{3}{2n}\right)^{\frac{2}{3}}\times\Gamma\left(\frac{2}{3}\right)\int_0^1u^{-\frac{2}{3}}(1-u)^{-\frac{5}{6}}\times\left[2(1-u)+\frac{u}{3}\right]du. \quad (5.3)$$

The integral over the spectral variable u is reduced to the Euler beta function

$$B(x,y)=\int_0^1u^{\mu-1}(1-u)^{\nu-1}du=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)},\qquad \Re\mu>0,\quad \Re\nu>0.$$

Thus, for the radiative energy shift of a massless charged $2D$ fermion the following expression was obtained:

$$\Delta E_n^{(r)}(m=0) = \frac{8\alpha p_\perp}{27\sqrt{3}\pi}\left(\frac{3}{2n}\right)^{\frac{2}{3}}\Gamma\left(\frac{1}{6}\right)e^{-i\frac{\pi}{3}}, \quad (5.4)$$

where α is the fine structure constant.

In the considered regime the radiative energy shift of a massless $2D$ charged fermion can be obtained also using formula (4.4) in the limiting case $m \to 0$.

The contribution to the energy shift of a massless charged fermion in formula (4.4) is made only by the second term in the brackets, in which the passage to the limit $m \to 0$ corresponds $\chi \sim \frac{1}{m^3} \to \infty$ and $z_0 \sim m^2 \to 0$, i.e., one should put

$$f'(0) = i\Phi'(0) + \Upsilon'(0), \qquad \frac{m^2}{z_0} = \frac{(eHp_\perp)^{\frac{2}{3}}}{\nu^{\frac{2}{3}}}.$$

Considering also that [15]

$$\Upsilon'(0) = \frac{3^{-\frac{1}{3}}}{2}\Gamma\left(\frac{2}{3}\right), \qquad \Phi'(0) = -\frac{3^{\frac{1}{6}}}{2}\Gamma\left(\frac{2}{3}\right),$$

we find expression for the radiative energy shift of a massless two-dimensional charged fermion in a constant magnetic field, coinciding with the result (5.3).

Finally, the real part of the radiative energy shift of a massless $2D$ charged fermion in a constant magnetic field and the total probability of synchrotron radiation per unit time is given as follows:

$$\begin{pmatrix} w \\ \Re\Delta E_n(m=0) \end{pmatrix} = \left(\frac{2}{3}\right)^{\frac{7}{3}}\Gamma\left(\frac{1}{6}\right)\frac{\alpha\sqrt{2eH}}{\sqrt{\pi}n^{\frac{1}{6}}}\begin{pmatrix} 1 \\ \frac{1}{2\sqrt{3}} \end{pmatrix}. \quad (5.5)$$

Another interesting effect that we will consider here is the creation by a photon in a constant magnetic field of a pair of massless two-dimensional charged fermions. An analogue of this process can be the creation of an electron-hole pair in graphene, where the hole plays the role of a massless charged antiparticle [52–55].

The imaginary and real parts of the elastic scattering amplitude of a photon determine, respectively, the probability of photon decay into an electron-positron pair and the photon mass squared in a constant magnetic field [56]:

$$w = -2\Im T, \qquad \delta(m_\gamma^2) = 2\omega\Re T. \quad (5.6)$$

We consider the case of relatively weak magnetic field and high photon energies satisfying the conditions

$$H \ll H_0 = \frac{m^2}{e} \simeq 4.41\times 10^{13}G, \;\; \omega \gg m. \quad (5.7)$$

Using the Mellin transformation with respect to the parameter

$$\lambda = \frac{4\sqrt{3}}{\chi}, \qquad \chi = \sqrt{\frac{e^2(F_{\mu\nu}k^\nu)^2}{m^6}} = \frac{H}{H_0}\cdot\frac{\omega}{m}, \quad (5.8)$$

in [29] we find that the amplitude of elastic photon scattering can be represented as

$$T(\chi) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\lambda^{-s}T(s), \qquad \frac{1}{3} < s_0 < \frac{1}{2}, \quad (5.9)$$

$$\begin{aligned} T(s) = &-\frac{me^2}{(4\pi)^{\frac{3}{2}}}\frac{36\Gamma(\frac{1}{2})}{5\omega} \\ &\times \exp\left[-i\frac{\pi s}{2}\right]\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{3s}{2}-\frac{1}{2}\right) \\ &\times\left[\frac{4\Gamma(s)}{\Gamma(s+\frac{1}{2})} - \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{3}{2})}\right]. \end{aligned} \quad (5.10)$$

To determine the total probability of charged massless fermion pair creation in formulas (5.9)–(5.10) we close the integration contour in the left half-plane of the complex variable $s(\chi \sim \frac{1}{m^3})$ and in the resulting expansion of the elastic photon scattering amplitude in a series in inverse powers of the parameter $\chi \gg 1$, we retain the first term of the expansion, which does not depend on the electron mass.

As a result, for the amplitude (5.9)–(5.10) in the limiting case of zero fermionic mass, we obtain the representation

$$T(m\to 0) = -i\frac{e^2}{8\pi}\frac{24}{5}\Gamma\left(\frac{1}{3}\right)e^{-i\frac{\pi}{6}}\left(\frac{eH\omega}{4\sqrt{3}}\right)^{\frac{1}{3}}\frac{1}{\omega}, \quad (5.11)$$

and the total probability of charged massless fermion pair creation in external magnetic field is determined by the formula

$$w = \frac{e^2}{2\pi}\frac{3\sqrt{3}}{5}\Gamma\left(\frac{1}{3}\right)\left(\frac{eH\omega}{4\sqrt{3}}\right)^{\frac{1}{3}}\frac{1}{\omega}, \quad (5.12)$$

where e^2 has the dimensions of mass in QED_{2+1}.

VI. CONCLUSION

In the one-loop approximation, an exact expression is obtained for the total radiative shift of the energy of the stationary state of an electron in a constant magnetic field both in two-dimensional electrodynamics with a doubled fermionic representation and in the model of reduced QED_{3+1}. It is shown that the radiative correction to the energy of the ground state of an electron in a reduced QED changes sign when passing from a weak to a superstrong magnetic field, and in a weak field the energy of the ground state of an electron decreases. In a superstrong magnetic field, the main term of the asymptotics for the radiative shift of the energy of the ground state of a two-dimensional electron does not depend on the electron mass. This result has no analog in the standard QED_{3+1} and coincides with the previously obtained results of works [27,36].

Using the Mellin transformation with respect to the parameter $a = \frac{1}{\chi}$, where $\chi = \frac{Hp_\perp}{H_0 m}$ is the dynamic parameter of synchrotron radiation, in the quasiclassical approximation for the anomalous magnetic moment of an electron in a constant magnetic field an expression is obtained in the form of a single integral. The AMM of an electron is investigated as functions of an external magnetic field strength, as well as of the electron energy. It is shown that in the limiting case of small values of the quantum parameter, the leading term in the expansion of the AMM of a two-dimensional electron coincides with the result (22) in [35], obtained as a result of calculating the vertex function of an electron in the framework of pseudo-QED.

The spectral distribution of the probability of synchrotron radiation of a two-dimensional electron in a reduced QED is obtained. The asymptotics of the total probability of synchrotron radiation of a two-dimensional electron in a constant magnetic field are found in the quasiquantum limit ($\chi \ll 1$) and in the ultraquantum case ($\chi \gg 1$).

As expected, result (4.9) obtained by us in the reduced QED in the quasiquantum approximation coincides with the analogous result of standard quantum electrodynamics [15,44] in the absence of longitudinal motion along the magnetic field, when an electron in a constant magnetic field moves in a circle. Therefore, the power and spectral power distribution of synchrotron radiation in the classical approximation are also described by the same formulas. The power of synchrotron radiation from the relativistic electron in the same classical approximation in QED_{2+1} with a doubled fermionic representation also turns out to be a finite value proportional to the parameter $e^2 m\chi$. The same behavior for the radiation power is predicted in the classical theory of the synchrotron radiation [39].

The real part of the radiative energy shift and the total probability of synchrotron radiation of a two-dimensional charged massless fermion in a constant magnetic field is calculated. When performing the calculation, to describe the motion of an electron, the QED_{2+1} model with a doubled fermionic representation is used, and the emitted photon propagates, as in the reduced QED_{3+1}, in three-dimensional space. The square of the induced mass, calculated in the reduced QED_{3+1}, as follows from formula (5.5) is equal to

$$\delta(m^2) = 2E\delta E = \frac{2e^2\Gamma(\frac{1}{6})}{27\pi\sqrt{3}}(3eHp_\perp)^{\frac{2}{3}}. \qquad (6.1)$$

Another interesting effect considered in the work is the creation by a photon in a constant magnetic field of a pair of massless two-dimensional charged fermions (a fermion-antifermion pair), an analog of which may be the creation of an electron-hole pair in graphene.

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