Electroweak and left-right phase transitions in $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification

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The electroweak phase transition in grand unified theory inspired $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification is shown to be weakly first order and occurs at $T = T_c^{\rm EW} \sim 163$ GeV, which is very similar to the behavior in the standard model in perturbation theory. A new phase appears at higher temperatures. $SU(2)_L \times U(1)_Y$ ($\theta_H = 0$) and $SU(2)_R \times U(1)_{Y'}$ ($\theta_H = \pi$) phases become almost degenerate above $T \sim m_{\rm KK}$ where $m_{\rm KK}$ is the Kaluza-Klein mass scale (typically around 13 TeV) and θ_H is the Aharonov-Bohm phase along the fifth dimension. The two phases become degenerate at $T = T_c^{\rm LR} \sim m_{\rm KK}$. As the temperature drops in the evolution of the early Universe the $SU(2)_R \times U(1)_{Y'}$ phase becomes unstable. The tunneling rate from the $SU(2)_R \times U(1)_{Y'}$ phase to the $SU(2)_L \times U(1)_Y$ phase becomes sizable and a first-order phase transition takes place at $T = 2.5$–2.6 TeV. The amount of gravitational waves produced in this left-right phase transition is small, far below the reach of the sensitivity of LISA. A detailed analysis of the $SU(2)_R \times U(1)_{Y'}$ phase is also given. It is shown that the W boson, Z boson and photon, with θ_H varying from 0 to π, are transformed to gauge bosons in the $SU(2)_R \times U(1)_{Y'}$ phase. Gauge couplings and wave functions of quarks, leptons, and dark fermions in the $SU(2)_R \times U(1)_{Y'}$ phase are determined.

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I. INTRODUCTION

The standard model (SM), $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory, has been firmly established at low energies. Implications of the model in the history of the Universe have also been discussed intensively. There remain a few important mysteries such as dark matter and baryon number generation in the Universe. Something beyond the SM is necessary.

The Higgs boson with a mass of 125 GeV was discovered in 2012, whose properties (observed so far) have been consistent with the SM within experimental errors. Yet it is not clear whether the observed boson is precisely what the SM assumes to exist. We need further measurements of the Higgs boson's couplings to itself and other particles to make a judgment. With the possibility of the appearance of cracks of the SM in mind, many alternative models with an extension of the scalar sector have been proposed.

Furthermore the Higgs sector in the SM has the severe gauge hierarchy problem when implemented in a larger theory, such as grand unification. One possible answer to this problem is gauge-Higgs unification (GHU) in which the Higgs boson is identified with a zero mode of the fifth dimensional component of the gauge potential [1–9]. It appears as a fluctuation mode of an Aharonov-Bohm (AB) phase θ_H in the fifth dimension. The mass of the Higgs boson is generated at the one-loop level, the finiteness of which is guaranteed by gauge invariance. Among various GHU models $SO(5) \times U(1)_X \times SU(3)_C$ GHU model in the Randall-Sundrum (RS) warped space reproduces nearly the same phenomenology at low energies as the SM for $\theta_H \lesssim 0.1$ [10–14]. Gauge couplings of quarks and leptons are almost the same as in the SM. The Cabibbo-Kobayashi-Maskawa (CKM) mass mixing is incorporated with natural suppression of flavor-changing neutral currents (FCNCs). Yukawa couplings of quarks and leptons are suppressed by a factor of $\cos\theta_H$ or $\cos^2 \frac{1}{2}\theta_H$ compared to those in the SM. The model predicts Z' bosons as Kaluza-Klein (KK) excited modes of γ, Z and Z_R where Z_R is the gauge boson of $SU(2)_R$ in $SO(4) \simeq SU(2)_L \times SU(2)_R \subset SO(5)$. It has been shown that effects of Z' bosons can be observed in fermion pair production at electron-positron (e^-e^+) collider experiments. Significant interference effects should be seen even in the early stage of the planned International Linear Collider (ILC) experiments at energies of 250 GeV by measuring the dependence on polarization of electron and positron beams [15–22].

Natural questions arise about the behavior of $SO(5) \times U(1)_X \times SU(3)_C$ GHU at finite temperature. At which temperature is the electroweak (EW) symmetry, $SU(2)_L \times U(1)_Y$, restored? Is the transition first order or second order? Is there any difference from the SM [23–26]? These are main themes analyzed in this paper. Previously phase transitions in GHU have been analyzed in various models [27–31]. Adachi and Maru analyzed their $SU(3) \times U(1)$ GHU model to show that the EW phase transition is strongly first-order [$v_c/T_c = O(1)$] though v_c and T_c turn out about 1 GeV, being too small. In $SO(5) \times U(1)_X \times SU(3)_C$ GHU the $SU(2)_L \times U(1)_Y$ symmetric phase corresponds to $\theta_H = 0$, which dynamically breaks down to $U(1)_{\rm EM}$ with nonvanishing $\theta_H \sim 0.1$ at zero temperature. It will be shown below that the $SU(2)_L \times U(1)_Y$ symmetry is restored around $T = T_c^{\rm EW} \sim 163$ GeV. The transition is shown to be weakly first order, just as in the SM in perturbation theory. A new phase emerges at higher temperatures. In $SO(5) \times U(1)_X \times SU(3)_C$ GHU, $SO(5)$ symmetry breaks down to $SO(4) \simeq SU(2)_L \times SU(2)_R$ by orbifold boundary conditions. Although a brane scalar at the UV brane spontaneously breaks $SU(2)_R \times U(1)_X$ to $U(1)_Y$ at $T = 0$, a new minimum of the effective potential at finite temperature appears at $\theta_H = \pi$. It will be seen that above $T = T_{c1}^{\rm LR} \sim m_{\rm KK}$ the $\theta_H = 0$ phase and the $\theta_H = \pi$ phase become almost degenerate. The two phases are separated by a barrier so that a domain structure will be formed in the Universe as the Universe expands and the temperature drops to around $T_{c1}^{\rm LR}$. As the Universe cools down further, the $\theta_H = \pi$ phase becomes absolutely unstable at $T = T_{c2}^{\rm LR} \sim 2.3$ TeV. The transition from the $\theta_H = \pi$ phase to the $\theta_H = 0$ phase takes place by tunneling at $T = T_{\rm decay}^{\rm LR} \sim 2.5$ TeV. In the $\theta_H = 0$ phase there is $SU(2)_L \times U(1)_Y$ gauge invariance, while in the $\theta_H = \pi$ phase $SU(2)_R \times U(1)_{Y'}$ gauge symmetry emerges. The transition from the $\theta_H = \pi$ phase to the $\theta_H = 0$ phase is called as the left-right (LR) transition.

In this paper we consider GHU models defined in the RS warped space with fixed curvature and size [32]. The stabilization of the RS warped space can be achieved by the Goldberger-Wise mechanism [33]. It has been discussed in the literature that in this case a decompactification phase transition from the RS space to the black-brane phase may take place below the KK scale [34,35]. The discussion of the LR transition in this paper remains valid as long as the RS warped space is stable around $T = T_{\rm decay}^{\rm LR}$.

The paper is organized as follows. In Sec. II the $SO(5) \times U(1)_X \times SU(3)_C$ GHU model is introduced, and its effective potential $V_{\rm eff}(\theta_H; T)$ at finite temperature is given in Sec. III. The EW phase transition is examined in Sec. IV, and the LR transition is studied in Sec. V. The LR transition is the transition from the $\theta_H = \pi$ phase to the $\theta_H = 0$ phase. We give a detailed analysis of the $\theta_H = \pi$ phase in Sec. VI. It will be shown that the $\theta_H = \pi$ phase corresponds to $SU(2)_R \times U(1)_{Y'}$ gauge symmetry, which becomes manifest in the twisted gauge. Gauge couplings of quarks, leptons, and dark fermions at $\theta_H = 0$ and $\theta_H = \pi$ are clarified. Wave functions of those fields are summarized in Appendixes. Section VII is devoted to a summary and discussion.

II. MODEL

We analyze $SO(5) \times U(1)_X \times SU(3)_C$ GHU models defined in the RS warped space. We focus on the grand unified theory (GUT) inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU model specified in Refs. [15–17]. The metric of the RS space is given by

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dz^2}{k^2}\right) \quad \text{for } 1 \le z \le z_L. \qquad (2.1)$$

The bulk region $1 < z < z_L$ is anti–de Sitter spacetime with a cosmological constant $\Lambda = -6k^2$, which is sandwiched by the UV brane at $z = 1$ and the IR brane at $z = z_L$. The KK mass scale is $m_{\rm KK} = \pi k/(z_L - 1) \simeq \pi k z_L^{-1}$ for $z_L \gg 1$. In addition to the gauge fields $A_M^{SU(3)_C}$, $A_M^{SO(5)}$, and $A_M^{U(1)_X}$ of $SU(3)_C$, $SO(5)$, and $U(1)_X$, matter fields are introduced in the bulk and on the UV brane. They are summarized in Table I.

The action of the model is given in Refs. [15,17]. It has been shown that the model reproduces the SM phenomenology at low energies. The bulk part of the action for the fermion multiplets are given, with $\bar\Psi = i\Psi^\dagger\gamma^0$, by

$$\begin{aligned}
S_{\rm bulk}^{\rm fermion} &= \int d^5x\sqrt{-\det G}\Big\{\sum_J \bar\Psi^J \mathcal{D}(c_J)\Psi^J \\
&\quad - \sum_\alpha \big(m_{D_\alpha}\bar\Psi^{+\alpha}_{(\mathbf{3},\mathbf{1})}\Psi^{-\alpha}_{(\mathbf{3},\mathbf{1})} + {\rm H.c.}\big) \\
&\quad - \sum_\gamma \big(m_{V_\gamma}\bar\Psi^{+\gamma}_{(\mathbf{1},\mathbf{5})}\Psi^{-\gamma}_{(\mathbf{1},\mathbf{5})} + {\rm H.c.}\big)\Big\}, \\
\mathcal{D}(c) &= \gamma^A e_A{}^M\left(D_M + \frac{1}{8}\omega_{MBC}[\gamma^B,\gamma^C]\right) - ck, \\
D_M &= \partial_M - ig_S A_M^{SU(3)} - ig_A A_M^{SO(5)} - ig_B Q_X A_M^{U(1)},
\end{aligned} \qquad (2.2)$$

TABLE I. Matter fields in the bulk and on the UV brane. Content of each field in $\mathcal{G} = SU(3)_C \times SO(5) \times U(1)_X$ and $G_{22} = SU(2)_L \times SU(2)_R (\subset SO(5))$ is shown. Parity assignment (P_0, P_1) of left- and right-handed quarks, leptons, and dark fermion multiplets in the bulk is shown.

Field		$\mathcal{G}$	G_{22}	Left	Right	Name
Quark	$\Psi^\alpha_{(\mathbf{3},\mathbf{4})}$	$(\mathbf{3},\mathbf{4})_{\frac{1}{6}}$	$[\mathbf{2},\mathbf{1}]$	$(+,+)$	$(-,-)$	$u\ c\ t$ $d\ s\ b$
			$[\mathbf{1},\mathbf{2}]$	$(-,-)$	$(+,+)$	$u'\ c'\ t'$ $d'\ s'\ b'$
	$\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$	$(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}$	$[\mathbf{1},\mathbf{1}]$	$(\pm,\pm)$	$(\mp,\mp)$	$D^\pm_d\ D^\pm_s\ D^\pm_b$
Lepton	$\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$	$(\mathbf{1},\mathbf{4})_{-\frac{1}{2}}$	$[\mathbf{2},\mathbf{1}]$	$(+,+)$	$(-,-)$	$\nu_e\ \nu_\mu\ \nu_\tau$ $e\ \mu\ \tau$
			$[\mathbf{1},\mathbf{2}]$	$(-,-)$	$(+,+)$	$\nu'_e\ \nu'_\mu\ \nu'_\tau$ $e'\ \mu'\ \tau'$
darkF	Ψ^β_F	$(\mathbf{3},\mathbf{4})_{\frac{1}{6}}$	$[\mathbf{2},\mathbf{1}]$	$(-,+)$	$(+,-)$	F
			$[\mathbf{1},\mathbf{2}]$	$(+,-)$	$(-,+)$	F'
darkF$_\ell$	$\Psi^\beta_{F_\ell}$	$(\mathbf{1},\mathbf{4})_{-\frac{1}{2}}$	$[\mathbf{2},\mathbf{1}]$	$(-,+)$	$(+,-)$	F_ℓ
			$[\mathbf{1},\mathbf{2}]$	$(+,-)$	$(-,+)$	F'_ℓ
darkV	$\Psi^{\pm\gamma}_V$	$(\mathbf{1},\mathbf{5})_0$	$[\mathbf{2},\mathbf{2}]$	$(\pm,\pm)$	$(\mp,\mp)$	$N^\pm\ E^\pm$ $E'^\pm\ N'^\pm$
			$[\mathbf{1},\mathbf{1}]$	$(\mp,\mp)$	$(\pm,\pm)$	$S^\pm$
Brane fermion	χ^α	$(\mathbf{1},\mathbf{1})_0$	$[\mathbf{1},\mathbf{1}]$	$\cdots$	$\cdots$	χ
Brane scalar	Φ_S	$(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$	$[\mathbf{2},\mathbf{1}]$	$\cdots$	$\cdots$	$\Phi_{[2,1]}$
			$[\mathbf{1},\mathbf{2}]$	$\cdots$	$\cdots$	$\Phi_{[1,2]}$

where the sum $\sum_J$ extends over $\Psi^J = \Psi^\alpha_{(\mathbf{3},\mathbf{4})}, \Psi^\alpha_{(\mathbf{1},\mathbf{4})}, \Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}, \Psi^\beta_F, \Psi^\beta_{F_\ell}$, and $\Psi^{\pm\gamma}_V$. The bulk mass parameter c_J of each fermion multiplet is important to specify a mass and wave function of the lowest (zero) mode. In the GUT inspired model bulk mass parameters of $\Psi^\alpha_{(\mathbf{3},\mathbf{4})}$ and $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ are taken to be negative. $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$ and $\Psi^{\pm\gamma}_V$ have additional Dirac-type masses m_{D_α} and m_{V_γ}, respectively.

The original RS metric is given by

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{2.3}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$, $\sigma(y) = \sigma(y+2L) = \sigma(-y)$, and $\sigma(y) = ky$ for $0 \le y \le L$. The coordinates y and z are related by $z = e^{ky}$ for $0 \le y \le L$. The fifth dimension in the RS space has topology of S^1/Z_2. In the y coordinate the orbifold boundary conditions are given by $(A_\mu, A_y)(x, y_j - y) = P_j(A_\mu, -A_y)(x, y_j + y)P_j$ ($j = 0$, 1), where $(y_0, y_1) = (0, L)$ and $(P_j)^2 = 1$. It follows that $A_y(x, y+2L) = P_1 P_0 A_y(x,y) P_0 P_1$. We take the orbifold boundary conditions $P_0 = P_1 = \mathrm{diag}(1,1,1,1,-1)$, which breaks $SO(5)$ to $SO(4) \simeq SU(2)_L \times SU(2)_R$. Further, the brane scalar field Φ_S located at $z = 1$, develops a nonvanishing expectation value to spontaneously break $SU(2)_R \times U(1)_X$ to $U(1)_Y$.

The $4D$ Higgs boson $\Phi_H(x)$ is the zero mode of the $SO(5)/SO(4)$ part of $A_z^{SO(5)}$,

$$A_z^{(j5)}(x,z) = \frac{1}{\sqrt{k}}\phi_j(x)u_H(z) + \cdots \quad (j = 1\text{–}4),$$
$$u_H(z) = \sqrt{\frac{2}{z_L^2 - 1}}\,z,$$
$$\Phi_H(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_2 + i\phi_1 \\ \phi_4 - i\phi_3 \end{pmatrix}. \tag{2.4}$$

At the quantum level Φ_H develops a nonvanishing expectation value. Without loss of generality we assume $\langle\phi_1\rangle, \langle\phi_2\rangle, \langle\phi_3\rangle = 0$ and $\langle\phi_4\rangle \neq 0$, which is related to the Aharonov-Bohm phase θ_H in the fifth dimension. Eigenvalues of

$$\hat{W} = P\,\exp\left\{ig_A\int_{-L}^{L} dy A_y\right\}$$
$$= P\,\exp\left\{2ig_A\int_{1}^{z_L} dz A_z\right\} \tag{2.5}$$

are gauge invariant. For $A_z = (2k)^{-1/2}\phi_4(x)u_H(z)T^{(45)}$

$$\hat{W} = \exp\{if_H^{-1}\phi_4(x)\cdot 2T^{(45)}\},$$
$$f_H = \frac{2}{g_A}\sqrt{\frac{k}{z_L^2 - 1}} = \frac{2}{g_w}\sqrt{\frac{k}{L(z_L^2 - 1)}},$$
$$\theta_H = \frac{\langle\phi_4\rangle}{f_H}, \tag{2.6}$$

where $g_w = g_A/\sqrt{L}$ is the $4D$ $SU(2)_L$ gauge coupling.

At zero temperature the effective potential $V_{\mathrm{eff}}(\theta_H)$ has a global minimum at $\theta_H \neq 0$ which breaks $SU(2)_L \times U(1)_Y$ to $U(1)_{\mathrm{EM}}$. W bosons, Z bosons, quarks, and leptons acquire masses with $\theta_H \neq 0$.

The RS metric has two parameters, k and L. With one of them (or the KK mass scale $m_{\mathrm{KK}} = \pi k(z_L - 1)^{-1} \sim \pi k z_L^{-1}$) given, the other parameter is fixed by the Z boson mass m_Z once the resultant values of θ_H and the weak mixing angle $\sin^2\theta_W$ are specified. The bulk mass parameters c_J of quark multiplets $\Psi^\alpha_{(\mathbf{3},\mathbf{4})}$ and lepton multiplets $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ are determined from masses of up-type quarks and charged leptons. Masses of down-type quarks are reproduced through bulk actions for $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}, \Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$, and brane interactions among $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$, $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$, and Φ_S. It has been shown that the CKM mixing matrix can be generated with natural suppression of FCNCs in the quark sector [16]. The brane fermions χ^α are Majorana fermions. Brane interactions among χ^α, $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$, and Φ_S induce the gauge-Higgs seesaw mechanism [36] similar to the inverse seesaw mechanism in grand unified theories [37]. Tiny neutrino masses are naturally explained.

Dark fermions are relevant to have dynamical-electroweak symmetry breaking by the Hosotani mechanism. There are five parameters $(n_F, c_F, n_V, c_V, m_V)$ to be specified in the dark fermion sector where n_F (n_V) and c_F (c_V)

are the number and bulk mass parameter of $\Psi_F(\Psi_V^\pm)$ multiplets, and m_V is a Dirac-type mass in (2.2). Rigorously speaking, there are additional parameters (n_{F_ℓ}, c_{F_ℓ}) associated with Ψ_{F_ℓ}. In the evaluation of the effective potential $V_{\rm eff}(\theta_H)$, contributions coming from Ψ_{F_ℓ} are summarized by the replacement $n_F \to n_F + \frac{1}{3} n_{F_\ell}$ for $c_{F_\ell} = c_F$. For $|c_{F_\ell}| > \frac{1}{2}$ its contributions are negligible. As seen below, such physical quantities as transition temperature do not depend on n_F so much, and therefore we suppress the reference to (n_{F_ℓ}, c_{F_ℓ}) in discussing $V_{\rm eff}(\theta_H)$ below. The parameters $(n_F, c_F, n_V, c_V, m_V)$ are chosen such that $V_{\rm eff}(\theta_H)$ has a global minimum at a desired value of θ_H and the resultant Higgs boson mass $m_H = f_H^{-1}\{d^2 V_{\rm eff}(\theta)/d\theta^2|_{\theta=\theta_H}\}^{1/2}$ is 125.1 GeV. This procedure leaves three parameters unfixed. Surprisingly there appears the θ_H universality in physics at low energies [17]. Gauge couplings of quarks, leptons, and W and Z bosons are almost independent of θ_H and other parameters. Yukawa couplings of quarks and leptons, Higgs couplings of W and Z are suppressed, compared to those in the SM, by a factor $\cos^2 \frac{1}{2}\theta_H$ or $\cos\theta_H$, but do not depend on details of the parameters in the dark fermion sector. Similarly cubic and quartic self-couplings of the Higgs boson become smaller than those in the SM, depending solely on θ_H, but not on the choice of the parameters in the dark fermion sector. The resultant phenomenology at low energies is nearly the same as that in the SM.

Distinct signals of GHU appear in physics of KK excited modes of gauge fields and fermions. For $\theta_H \sim 0.1$ the KK mass scale $m_{\rm KK}$ turns out in the range 10 TeV to 15 TeV. Masses of the first KK bosons of γ, Z, and Z_R, which play the role of Z' bosons, are around $0.8\, m_{\rm KK}$. Couplings of quarks and leptons to those Z' bosons exhibit large parity violation, and couplings of either left-handed or right-handed quarks and leptons become rather large. It has been shown that a large deviation from the SM can be observed in the $e^- e^+ \to f\bar{f}$ processes, where f is a lepton or quark, at energies well below $m_{Z'}$. Significant deviation can be observed even in the early stage of the planned International Linear Collider (ILC) at $\sqrt{s} = 250$ GeV. The interference effect between the two amplitudes for $e^- e^+ \to \gamma,\ Z \to f\bar{f}$ and $e^- e^+ \to Z' \to f\bar{f}$ becomes very large, and cross sections reveal a distinct dependence on the polarization of incident e^- and e^+ beams [18–22].

In this paper we explore the behavior of the model at finite temperature, particularly in the context of cosmological evolution of the Universe. In the SM the electroweak $SU(2)_L \times U(1)_Y$ symmetry is restored at high temperature. In perturbation theory the transition is weakly first order with T_c around 160 GeV. We will show that the behavior of the EW phase transition in GHU is very similar to that in the SM, though the mechanism of EW symmetry breaking at zero temperature is quite different. It will be shown further that a new phase transition, called the LR phase transition, emerges around 2.5 TeV in the GUT inspired GHU.

III. EFFECTIVE POTENTIAL AT FINITE TEMPERATURE

At zero temperature the effective potential $V_{\rm eff}(\theta_H, T = 0)$ at the one-loop level is evaluated from the mass spectra of all fields which depend on θ_H. It is given by

$$V_{\rm eff}(\theta_H, T=0) = \sum_a \frac{(-1)^{\eta_a}}{2} \int \frac{d^4 p_E}{(2\pi)^4} \sum_n \ln\{p_E^2 + m_n^a(\theta_H)^2\}, \tag{3.1}$$

where $\sum_a$ extends over all field multiplets and $\eta_a = 0$ or 1 for bosons or fermions, respectively. When the KK spectrum $\{m_n^a(\theta_H)\}$ is determined by the zeros of a function $\rho_a(z;\theta_H)$; namely, by $\rho_a(m_n^a;\theta_H) = 0$, $(n = 1, 2, 3, \ldots)$, then $V_{\rm eff}$ is given [38] by

$$V_{\rm eff}(\theta_H, T=0) = \sum_a \frac{(-1)^{\eta_a}}{(4\pi)^2} \int_0^\infty dy y^3 \ln \rho_a(iy;\theta_H). \tag{3.2}$$

The θ_H-dependent part of $V_{\rm eff}^{\rm 1loop}(\theta_H)$ is finite and independent of the cutoff and regularization method employed. As explained in the previous section the parameters of the model are determined such that $V_{\rm eff}(\theta_H, T = 0)$ has a global minimum at a desired value of θ_H and the resultant Higgs boson mass is $m_H = 125.1$ GeV.

At finite temperature $T \neq 0$, the effective potential becomes

$$V_{\rm eff}(\theta_H, T) = \sum_a \frac{(-1)^{\eta_a}}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\beta} \sum_{\ell=-\infty}^{\infty} \sum_n \ln\{\omega_\ell^2 + \vec{p}^{\,2} + m_n^a(\theta_H)^2\},$$
$$\beta = \frac{1}{T}, \qquad \omega_\ell = \frac{2\pi}{\beta}\left(\ell + \frac{\eta_a}{2}\right). \tag{3.3}$$

There appears summation over Matsubara frequencies and KK modes. There are two ways to evaluate it. One way is to first sum over Matsubara frequencies. Employing the identity [23]

$$\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\beta}\sum_{\ell}\ln\{\omega_\ell^2+\vec{p}^2+m^2\}=\int\frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2}\omega(p)+\frac{1}{\beta}\ln(1-(-1)^\eta e^{-\beta\omega(p)})\right\}+m\text{-independent terms,} \tag{3.4}$$

where $\omega(p)=\sqrt{\vec{p}^2+m^2}$, one finds

$$V_{\rm eff}(\theta_H,T)=V_{\rm eff}(\theta_H,0)+\Delta V_{\rm eff}(\theta_H,T),$$
$$\Delta V_{\rm eff}(\theta_H,T)=\sum_a\sum_n\frac{(-1)^{\eta_a}}{2\pi^2\beta^4}\int_0^\infty dx\,x^2\,\ln(1-(-1)^{\eta_a}e^{-\sqrt{x^2+(\beta m_n^a)^2}}), \tag{3.5}$$

where $m_n^a=m_n^a(\theta_H)$. $\Delta V_{\rm eff}(\theta_H,T)$ is finite. The sum over KK modes converges. Contributions from modes with $m_n\gg\beta^{-1}=T$ are negligible. In the following sections we numerically evaluate $V_{\rm eff}(\theta_H,0)$ by (3.2) and $\Delta V_{\rm eff}(\theta_H,T)$ by (3.5).

Alternatively one can evaluate $V_{\rm eff}$ by first summing over contributions from the KK modes. The key observation is that $\sqrt{\omega_\ell^2+m_n^2}$ in the expression in (3.3) can be viewed as a mass of the ℓth Matsubara mode in $(3+1)$ dimensions. When the spectrum $\{m_n(\theta_H)\}$ is determined by $\rho(m_n;\theta_H)=0$, $(n=1,2,3,\ldots)$, then the spectrum $\{z_n=\sqrt{\omega_\ell^2+m_n^2}\}$ is determined by $\bar\rho(z_n;\theta_H)=\rho(\sqrt{z_n^2-\omega_\ell^2};\theta_H)=0$. Hence one finds [39]

$$V_{\rm eff}(\theta_H,T)=\sum_a\sum_\ell\frac{(-1)^{\eta_a}}{4\pi^2\beta}\int_0^\infty dy y^2\ln\rho_a\left(i\sqrt{y^2+\omega_\ell^2};\theta_H\right). \tag{3.6}$$

In RS space, spectrum-determining functions $\rho(z;\theta_H)$ involve Bessel functions so that the y-integral in (3.6) for each Matsubara mode demands some time to find the accurate θ_H dependence of $V_{\rm eff}(\theta_H)$. For this reason we employ the first method using (3.2) and (3.5) to evaluate $V_{\rm eff}(\theta_H,T)$ below. $V_{\rm eff}(\theta_H,0)$ has been already obtained in Ref. [17].

IV. ELECTROWEAK PHASE TRANSITION

At zero temperature the EW symmetry is dynamically broken by the Hosotani mechanism in GHU. Dominant contributions to the θ_H-dependent part of $V_{\rm eff}(\theta_H,0)$ come from gauge-field multiplets, top-quark multiplets, and dark fermion multiplets at the one-loop level. In phenomenologically interesting cases $V_{\rm eff}(\theta_H,0)$ has a global minimum around $\theta_H\sim 0.1$ and the KK mass scale $m_{\rm KK}$ turns out to be around 10 TeV to 15 TeV. In this section we address the question of when and how the EW symmetry is restored at finite temperature.

In the SM the EW symmetry is spontaneously broken at the tree level, and is restored at finite temperature. In perturbation theory the transition occurs at $T_c^{\rm EW}\sim 160\,$GeV, and is weakly first order [23,24]. In the lattice simulation the transition is observed to be smoother [26]. Although the EW symmetry breaking mechanism at $T=0$ in GHU is quite different from that in the SM, the behavior of the EW symmetry restoration at the weak scale is expected to be similar. In GHU, $T_c^{\rm EW}\ll m_{\rm KK}$, so that only SM particles are expected to give relevant contributions to $\Delta V_{\rm eff}(\theta_H,T)$ in (3.5) at $T\lesssim T_c^{\rm EW}$.

Recall that only KK towers with θ_H-dependent $m_n(\theta_H)$ are relevant to $\Delta V_{\rm eff}(\theta_H,T)$ in (3.5). They are W towers, Z towers, A_z (Higgs) towers, top-quark towers, bottom-quark towers, dark-fermion Ψ_F (darkF) towers, and dark-fermion Ψ_V (darkV) towers. The spectrum-determining $\rho(z;\theta_H)$ functions are tabulated in Appendix A of Ref. [17]. Other quark and lepton multiplets have θ_H-dependent spectra $m_n(\theta_H)$, but the magnitude of their θ_H-dependence is small and almost irrelevant to $\Delta V_{\rm eff}(\theta_H,T)$. The spectra $\{m_n^a(\theta)\}$ for W, top, darkF, and darkV towers are displayed in Fig. 1.

The mass spectrum of the top quark has the largest θ_H dependence. The spectrum of the darkF has the second largest θ_H dependence. As opposed to the top-quark case, the darkF is massive at $\theta_H=0$ while it becomes massless at $\theta_H=\pi$. The spectrum of the darkV tower has much weaker

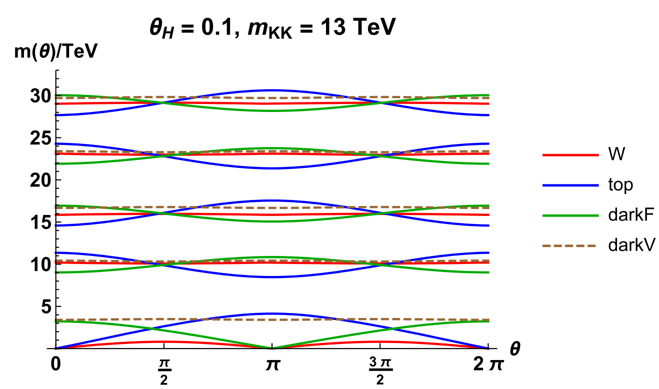

FIG. 1. Mass spectra for W boson ($W^{(n)}, \hat{W}^{(n)}$), top ($t^{(n)}$ and $t'^{(n)}$), darkF, and darkV towers when $V_{\rm eff}(\theta_H,0)$ has a global minimum at $\theta_H^{\rm min}=0.1$, and other parameters are given by $m_{\rm KK}=13$ TeV and $(n_F,n_V,c_V)=(2,4,0.2)$. For the W series levels from the bottom are $W^{(0)},\hat{W}^{(1)},W^{(1)},\hat{W}^{(2)},W^{(2)},\cdots$, while for the top series they are $t^{(0)},t'^{(1)},t^{(1)},t'^{(2)},t^{(2)},\cdots$.

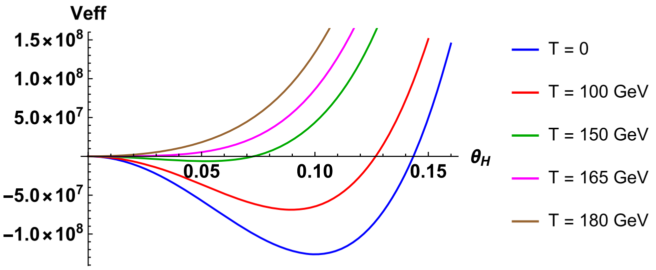

FIG. 2. Behavior of $V_{\rm eff}(\theta_H, T)$ in units of GeV^4 for $T = 0$–180 GeV. At $T = 0$ $V_{\rm eff}$ has a minimum at $\theta_H^{\rm min} = 0.1$.

θ_H dependence. Although it is important at zero temperature, it gives little effect for the behavior of $V_{\rm eff}(\theta_H, T)$ at finite temperature.

We insert those mass spectra $\{m_n^a(\theta)\}$ into (3.5) to find $V_{\rm eff}(\theta_H, T)$. Its behavior for $0\ \mathrm{GeV} \le T \le 180\ \mathrm{GeV}$ and $0 \le \theta_H \le 0.15$ is depicted in Fig. 2. Here and below, $V_{\rm eff}(\theta_H, T) - V_{\rm eff}(0, T)$ has been plotted in the figures. For $T < 300$ GeV, only contributions from the SM fields, namely W, Z, Higgs, and top-quark fields, are relevant to $\Delta V_{\rm eff}(\theta_H, T)$. The EW symmetry is restored around 163 GeV. Near the critical temperature one needs careful evaluation. As in the SM a small bump develops for small θ_H as a result of $T|\phi|^3$-type contributions from bosons. When $V_{\rm eff}(\theta_H, 0)$ has a global minimum at $\theta_H = \theta_H^{\rm min} = 0.1$ and other parameters are given by $m_{\rm KK} = 13$ TeV, $n_F = 2$, $n_V = 4$ and $c_V = 0.2$, the critical temperature is found to be $T_c^{\rm EW} = 163.2$ GeV. $V_{\rm eff}(\theta_H, T_c^{\rm EW})$ for $0 \le \theta_H < 0.013$ is evaluated numerically and is depicted in Fig. 3. The degenerate minimum is located at $\theta_H^c = 0.0104$ and $v_c = \theta_H^c f_H = 25.54$ GeV. The ratio $v_c / T_c^{\rm EW}$ is 0.156. The transition is weakly first order. It is known in the SM that $T_c^{\rm EW,SM} \simeq 163.4$ GeV, $v_c^{\rm SM} \simeq 24.3$ GeV, and $v_c^{\rm SM}/T_c^{\rm EW,SM} \simeq 0.15$ in the one-loop approximation [24]. Although the EW symmetry breaking mechanism in GHU is quite different from that in the SM, the behavior at finite temperature ($T < 300$ GeV) in GHU is almost the same as in the SM.

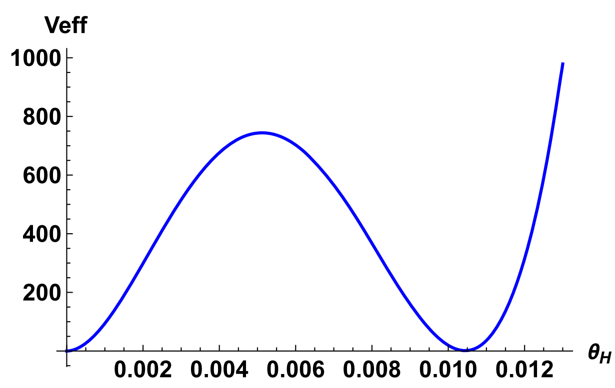

FIG. 3. $V_{\rm eff}(\theta_H, T_c^{\rm EW})$ in units of GeV^4 is plotted at $T = T_c^{\rm EW} = 163.2$ GeV for $(\theta_H^{\rm min}, n_F, n_V, c_V) = (0.1, 2, 4, 0.2)$. Degenerate minima are located at $\theta_H = 0$ and $\theta_H^c = 0.0104$. $v_c = \theta_H^c f_H = 25.54$ GeV so that $v_c / T_c^{\rm EW} = 0.156$.

We remark that this behavior does not depend on detailed values of various parameters in the model. As mentioned above, there are a few parameters in the dark fermion sector which can be taken differently. One finds that $T_c^{\rm EW} = 163.2$ GeV for $(\theta_H^{\rm min}, n_F, n_V, c_V) = (0.1, 2, 4, 0.2)$, and $T_c^{\rm EW} = 163.3$ GeV for (0.1,5,2,0.2). The θ_H universality remains valid for the quantity $T_c^{\rm EW}$; $T_c^{\rm EW}$ depends on $\theta_H^{\rm min}$ very little, too. One finds, for instance, that $T_c^{\rm EW} = 163.3$ GeV for $(\theta_H^{\rm min}, n_F, n_V, c_V) = (0.11, 2, 4, 0.2)$. All evaluations of the critical temperature in this section have been carried out at the one-loop level. Higher-order corrections may affect the values just as in the SM.

V. LEFT-RIGHT PHASE TRANSITION

As the temperature is raised further, a new feature emerges in the global behavior of $V_{\rm eff}(\theta_H, T)$. $\theta_H = \pi$ becomes a local minimum of $V_{\rm eff}(\theta_H, T)$ at $T = T_{c2}^{\rm LR} \sim 2.3$ TeV, and becomes a global minimum at $T = T_{c1}^{\rm LR} \sim m_{\rm KK}$. Its behavior is plotted in Fig. 4.

In expression (3.5) of $V_{\rm eff}(\theta_H, T)$ the contributions from W, Z, Higgs, and darkV towers are periodic in θ_H with a period π, giving the same amount of contributions at $\theta_H = 0$ and π. On the other hand, contributions from top-quark and darkF towers have periodicity with a period 2π, giving rise to a difference between $\theta_H = 0$ and π. Furthermore, the top quark is massless and the darkF is massive at $\theta_H = 0$, whereas the top quark is massive and darkF is massless at $\theta_H = \pi$. In effect, the role of top quark and darkF is interchanged.

As T is increased further above $m_{\rm KK}$, it is expected from (3.6) that contributions from boson fields dominate over those from fermion fields. For fermions, the Matsubara frequency $|\omega_\ell|$ is equal to or larger than πT, whereas for bosons there exist zero frequency modes, $\omega_0 = 0$. Fermion contributions are suppressed compared to boson contributions, which in turn implies, in the current case, that $\theta_H = 0$ and $\theta_H = \pi$ phases become almost degenerate at sufficiently high temperature.

This leads to an important consequence in the history of the evolution of the early Universe. As the Universe

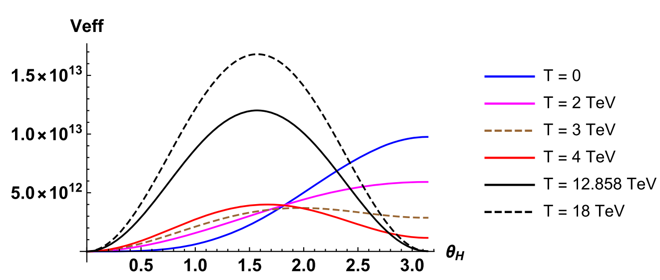

FIG. 4. Behavior of $V_{\rm eff}(\theta_H, T)$ in the unit of GeV^4 for $T = 0$–18 TeV. $\theta_H^{\rm min} = 0.1$, $m_{\rm KK} = 13$ TeV, and $(n_F, n_V, c_V) = (2, 4, 0.2)$.

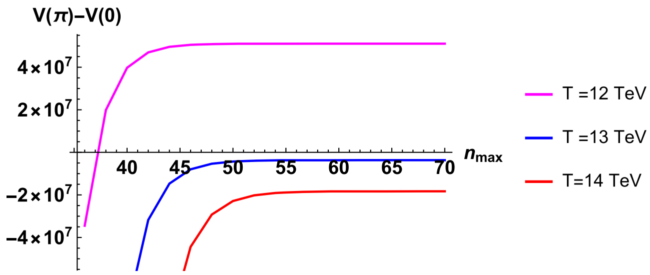

FIG. 5. $V_{\rm eff}^{\rm LR}(T)$ in (5.1) in units of GeV^4 is plotted with varying an even integer $n_{\rm max}$. $m_{\rm KK} = 13$ TeV, $\theta_H^{\rm min} = 0.1$, and $(n_F, n_V, c_V) = (2, 4, 0.2)$.

expands and the temperature drops to $T \sim m_{\rm KK}$, the $\theta_H = 0$ and $\theta_H = \pi$ states are almost degenerate so that the Universe would settle in the domain structure. As is seen below, the $\theta_H = \pi$ state remains stable until T drops further to $T_{\rm decay}^{\rm LR} \sim 2.6$ TeV at which time tunneling from $\theta_H = \pi$ to $\theta_H = 0$ rapidly takes place.

We shall see in Sec. VI that the role of $SU(2)_L$ and $SU(2)_R$ is interchanged in the $\theta_H = 0$ and $\theta_H = \pi$ states. For this reason the $\theta_H = 0 \leftrightarrow \pi$ transition is called the left-right transition.

A. Critical temperatures $T_{c1}^{\rm LR}$ and $T_{c2}^{\rm LR}$

There is a critical temperature $T_{c1}^{\rm LR}$ at which $V_{\rm eff}(0, T_{c1}^{\rm LR}) = V_{\rm eff}(\pi, T_{c1}^{\rm LR})$. For $T > T_{c1}^{\rm LR}$, $V_{\rm eff}(0,T) > V_{\rm eff}(\pi, T)$. It should be remembered that $V_{\rm eff}(0,T) - V_{\rm eff}(\pi, T) \ll V_{\rm eff}(\frac{1}{2}\pi, T) - V_{\rm eff}(\pi, T)$ for $T > T_{c1}^{\rm LR}$. There is another critical temperature $T_{c2}^{\rm LR}$. While the $\theta_H = \pi$ state remains as a local minimum for $T_{c2}^{\rm LR} < T < T_{c1}^{\rm LR}$, the $\theta_H = \pi$ state becomes a maximum of $V_{\rm eff}(\theta_H, T)$ for $T < T_{c2}^{\rm LR}$, hence becoming absolutely unstable.

To find the values of $T_{c1}^{\rm LR}$ and $T_{c2}^{\rm LR}$ one need to sum over the contributions from a large number of KK modes in (3.5). Since only top quark and darkF towers are relevant for this quantity, one can write, for $V_{\rm eff}^{\rm LR}(T) \equiv V_{\rm eff}(\pi, T) - V_{\rm eff}(0, T)$,

$$V_{\rm eff}^{\rm LR}(T) = V_{\rm eff}^{\rm LR}(0) + \delta V_{\rm eff}^{\rm LR}(T),$$
$$\delta V_{\rm eff}^{\rm LR}(T) \simeq {\sum_a}' \sum_{n=1}^{n_{\rm max}} \frac{-1}{2\pi^2 \beta^4} \int_0^\infty dx\, x^2 \ln \frac{1 + e^{-\sqrt{x^2 + [\beta m_n^a(\pi)]^2}}}{1 + e^{-\sqrt{x^2 + [\beta m_n^a(0)]^2}}}, \tag{5.1}$$

where the sum $\sum_a'$ extends over top quark and darkF towers.

For $T > m_{\rm KK}$ a large number of KK modes contribute. As seen from the spectrum depicted in Fig. 1, the behavior of the θ_H dependence of $m_n^a(\theta_H)$ alternates as n. There results partial cancellation between the nth mode and $(n+1)$th mode in Eq. (5.1). $V_{\rm eff}^{\rm LR}(T)$ in (5.1) is plotted in Fig. 5 with an even integer $n_{\rm max}$ varied. One can see that $n_{\rm max} \geq 50$ is necessary near the critical temperature to reach an asymptotic value.

The critical temperature $T_{c1}^{\rm LR}$ turns out to be very close to $m_{\rm KK}$, and has little dependence on $\theta_H^{\rm min}$. The other critical temperature $T_{c2}^{\rm LR}$ turns out to be around 2.3 TeV. It is tabulated in Table II with various choices of the parameters.

B. Bounce solutions and $T_{\rm decay}^{\rm LR}$

Although $V_{\rm eff}(0,T) < V_{\rm eff}(\pi, T)$ for $T < T_{c1}^{\rm LR}$, the transition from the $\theta_H = \pi$ phase to the $\theta_H = 0$ phase does not proceed immediately. The temperature must drop further before a rapid transition takes place. We need to evaluate bounce solutions at the finite temperature to estimate the tunneling rate.

As shown in Fig. 4 in the case of $\theta_H^{\rm min} = 0.1$, $m_{\rm KK} = 13$ TeV, and $(n_F, n_V, c_V) = (2, 4, 0.2)$; the $\theta_H = \pi$ state is at a local minimum of $V_{\rm eff}(\theta_H, T)$ at $T = 4$ TeV. The tunneling rate per unit time per unit volume is given in the form $A(T) e^{-S_3/T}$ where S_3 is the three-dimensional action of a bounce solution [23]

$$S_3 = \int d^3x \left\{ \frac{1}{2} (\nabla\phi)^2 + V_{\rm eff}\left(\frac{\phi}{f_H}, T\right) - V_{\rm eff}(\pi, T) \right\},$$
$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} - \frac{1}{f_H} V_{\rm eff}^{(1)}\left(\frac{\phi}{f_H}, T\right) = 0,$$
$$\lim_{r\to\infty} \phi(r) = \pi f_H,$$
$$\left.\frac{d\phi}{dr}\right|_{r=0} = 0. \tag{5.2}$$

In terms of dimensionless quantities $\theta = \phi/f_H$, $t = f_H r$, and $U(\theta, T) = -f_H^{-4} V_{\rm eff}(\theta, T)$, we have

TABLE II. $T_{c1}^{\rm LR}$ and $T_{c2}^{\rm LR}$. $(\theta_H^{\rm min}, m_{\rm KK}, n_F, n_V, c_V)$ are input parameters.

$\theta_H^{\rm min}$	$m_{\rm KK}$	n_F	n_V	c_V	c_F	m_V/k	$T_{c1}^{\rm LR}$	$T_{c2}^{\rm LR}$
0.1	13 TeV	2	4	0.2	0.358042	0.086414	12.86 TeV	2.348 TeV
		5	2	0.2	0.456079	0.071245	12.85 TeV	2.238 TeV
	11 TeV	2	4	0.2	0.236826	0.106592	10.88 TeV	2.277 TeV
0.11	13 TeV	2	4	0.2	0.392398	0.075104	12.86 TeV	2.215 TeV

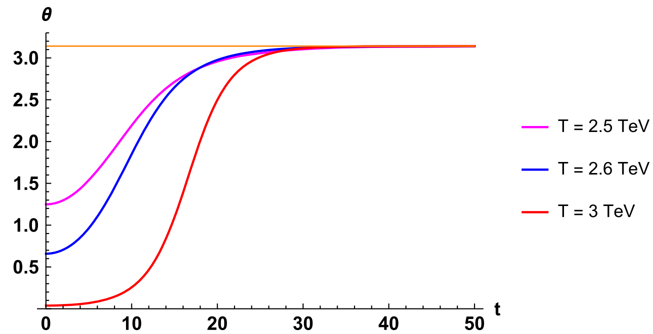

FIG. 6. Bounce solutions $\theta(t)$, where $t = f_H r$, for $T = 2.5$ TeV, 2.6 TeV, and 3 TeV in the case of $\theta_H^{\rm min} = 0.1$, $m_{\rm KK} = 13$ TeV, and $(n_F, n_V, c_V) = (2, 4, 0.2)$.

$$S_3 = 4\pi f_H \int_0^\infty dt t^2 \left\{ \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - U(\theta, T) + U(\pi, T) \right\},$$
$$\frac{d^2\theta}{dt^2} + \frac{2}{t}\frac{d\theta}{dt} + \frac{dU}{d\theta} = 0, \tag{5.3}$$

with conditions $d\theta/dt|_{t=0} = 0$ and $\theta|_{t=\infty} = \pi$. The problem is reduced to determining the motion of a particle in a potential U.

Bounce solutions can be easily found. Solutions for $T = 2.5$ TeV, $T = 2.6$ TeV, and 3 TeV are displayed in Fig. 6. For higher temperatures, say, $T = 4$ TeV, $\theta(0)$ must be very close to 0, in which case the thin-wall approximation becomes legitimate.

For the tunneling rate the most relevant quantity is S_3/T. The result is summarized in Table III. Bubble nucleation rate becomes sufficiently large so that the LR transition from the $\theta_H = \pi$ state to the $\theta_H = 0$ state rapidly proceeds at $T = T_{\rm decay}^{\rm LR}$ when S_3/T becomes $O(130\text{–}140)$ [40–43]. It is seen that in the case of $m_{\rm KK} = 13$ TeV, $T_{\rm decay}^{\rm LR} \sim 2.6$ TeV for $\theta_H^{\rm min} = 0.1$ and $T_{\rm decay}^{\rm LR} \sim 2.45$ TeV for $\theta_H^{\rm min} = 0.11$, respectively.

The $\theta_H = \pi$ state corresponds to $v_c = \pi f_H = 7.71$ TeV for $\theta_H^{\rm min} = 0.1$ and $m_{\rm KK} = 13$ TeV, which gives $v_c/T_{\rm decay}^{\rm LR} \sim 2.97$. One might wonder whether or not gravitational waves (GWs) generated in the LR phase transition can be detected in future GW observations. There are two relevant quantities denoted as α and β in the literature for describing dynamics of a first-order phase transition in association with generations of GWs [44–46]. α is the ratio between the false vacuum energy (latent heat) density and the thermal energy density at $T_{\rm decay}^{\rm LR}$, which gives a measure of the transition strength. β is the rate of time variation of the nucleation rate at the transition. The number g_* of relativistic degrees of freedom at $T_{\rm decay}^{\rm LR}$ is $96.25 + 42 n_F$ (180.25 for $n_F = 2$). We have found that $\alpha \sim 0.004$ and $\beta/H_* \sim 2100$ in the case $\theta_H^{\rm min} = 0.1$, $m_{\rm KK} = 13$ TeV, and $(n_F, n_V, c_V) = (2, 4, 0.2)$, where H_* is the Hubble parameter at the transition. The amount of energy released in the LR transition is small, giving a tiny value of α. A GW signal from the LR transition is far below the reach of the sensitivity of, say, LISA.

Before closing this section we summarize the cosmological history of the Universe in Table IV after the temperature drops around $T = m_{\rm KK}$. As remarked in the introduction, the scenario is valid as long as the RS warped space is stable around $T_{\rm decay}^{\rm LR}$. If the Universe is in a decompactified phase discussed in Refs. [34,35] at $T < m_{\rm KK}$, the scenario may need to be modified accordingly.

VI. GAUGE SYMMETRY AND COUPLINGS AT $\theta_H = 0$ AND π

In the previous section we have seen that the Universe forms domain structure above $T = T_{\rm decay}^{\rm LR}$ in which the

TABLE III. S_3/T for bounce solutions. $(n_F, n_V, c_V) = (2, 4, 0.2)$. Initial values $\theta(0)$ are tuned to yield bounce solutions.

$\theta_H^{\rm min}$	$m_{\rm KK}$	T	$\theta(0)$	S_3/T
0.1	13 TeV	3 TeV	0.0379	616
		2.65 TeV	0.4761	190
		2.6 TeV	0.6585	150
		2.55 TeV	0.9094	112
		2.5 TeV	1.2490	77
	11 TeV	3 TeV	0.0791	1107
		2.5 TeV	0.7000	160
		2.45 TeV	1.0050	113
		2.4 TeV	1.4246	71
0.11	13 TeV	3 TeV	0.0864	840
		2.55 TeV	0.3490	212
		2.5 TeV	0.4875	171
		2.45 TeV	0.6802	133
		2.4 TeV	0.9488	98

TABLE IV. Schematic view of the history of the Universe is shown.

Temperature	Phase of the Universe
$m_{\rm KK} \sim 13$ TeV	: Domain structure of $\theta_H = 0$ and $\theta_H = \pi$ phases is formed.
↓	domains consisting of $\begin{cases} \theta_H = 0: SU(2)_L \times U(1)_Y \text{ phase} \\ \theta_H = \pi: SU(2)_R \times U(1)_{Y'} \text{ phase} \end{cases}$
$T_{\rm decay}^{\rm LR} \sim 2.5$ TeV	: LR first-order transition from $\theta_H = \pi$ phase to $\theta_H = 0$ phase
↓	$\theta_H = 0: SU(2)_L \times U(1)_Y$ phase
$T_c^{\rm EW} \sim 163$ GeV	: EW weakly first-order transition to $\theta_H \neq 0$
↓	$\theta_H \neq 0: SU(2)_L \times U(1)_Y$ is broken to $U(1)_{\rm EM}$
$T \sim 0$	: The present universe

$\theta_H = 0$ and $\theta_H = \pi$ states coexist. The $\theta_H = 0$ state is the $SU(2)_L \times U(1)_Y$ symmetric phase. It is important to understand what the $\theta_H = \pi$ state is.

In this section it is shown that the $\theta_H = \pi$ state corresponds to a state with $SU(2)_R \times U(1)_{Y'}$ symmetry, which becomes manifest in the twisted gauge. In $SO(5) \times U(1)_X$ GHU the $SU(2)_L \times U(1)_Y$ and $SU(2)_R \times U(1)_{Y'}$ phases are connected smoothly by θ_H. Mass spectrum and gauge couplings of quarks and leptons continuously change as θ_H varies from 0 to π. To find those gauge couplings, wave functions of gauge fields and fermion fields in the fifth dimension must be first determined. Details of wave functions are given in the Appendixes.

A. Twisted gauge

When $V_{\rm eff}(\theta_H, T=0)$ is minimized at $\theta_H \neq 0$, the EW symmetry is spontaneously broken to $U(1)_{\rm EM}$ in general. It has been known that gauge couplings and other physical quantities in the vacuum with $\theta_H \neq 0$ can be most conveniently evaluated in the twisted gauge. This remains valid at $T \neq 0$.

In GHU one can make a large gauge transformation such that the AB phase θ_H defined in (2.6) becomes zero in a new gauge [38,47]. To be more explicit, consider an $SO(5)$ gauge transformation

$$\tilde{A}_M = \tilde{\Omega} A_M \tilde{\Omega}^{-1} + \frac{i}{g_A} \tilde{\Omega} \partial_M \tilde{\Omega}^{-1},$$
$$\tilde{\Omega} = e^{i\theta(z) T^{45}}, \qquad \theta(z) = \theta_H \frac{z_L^2 - z^2}{z_L^2 - 1}. \tag{6.1}$$

As $\langle \tilde{A}_z \rangle = \langle A_z \rangle + g_A^{-1} \theta'(z) T^{45}$, the AB phase in the new gauge becomes $\tilde{\theta}_H = 0$. This gauge is called the twisted gauge. Quantities in the twisted gauge are denoted with a tilde in this section and in the Appendixes. Although the AB phase vanishes, orbifold boundary conditions are changed to

$$\tilde{P}_0^{\rm vec} = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& \cos 2\theta_H & -\sin 2\theta_H \\ &&& -\sin 2\theta_H & -\cos 2\theta_H \end{pmatrix},$$
$$\tilde{P}_1^{\rm vec} = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& -1 \end{pmatrix},$$
$$\tilde{P}_0^{\rm sp} = \sigma^0 \otimes (\cos\theta_H \sigma^3 + \sin\theta_H \sigma^2), \quad \tilde{P}_1^{\rm sp} = \sigma^0 \otimes \sigma^3. \tag{6.2}$$

Here $\tilde{P}_j^{\rm vec}$ and $\tilde{P}_j^{\rm sp}$ represent orbifold boundary condition matrices in the vectorial and spinorial representations, respectively. In the twisted gauge, boundary conditions at $z = 1$ becomes nontrivial and θ_H dependent, but equations and wave functions of various fields in the bulk $(1 < z \leq z_L)$ become simple as the background field $\tilde{\theta}_H$ vanishes. Physics does not depend on the gauge.

$SO(5)$ gauge fields are decomposed as

$$A_M = \frac{1}{\sqrt{2}} \sum_{1 \leq j < k \leq 5} A_M^{(jk)} T^{jk},$$
$$[T^{jk}, T^{\ell m}] = i(\delta^{j\ell} T^{km} - \delta^{jm} T^{k\ell} - \delta^{k\ell} T^{jm} + \delta^{km} T^{j\ell}). \tag{6.3}$$

Four-dimensional components in the twisted gauge are given by

$$\tilde{A}_\mu = \frac{1}{\sqrt{2}} \sum_{1 \leq j < k \leq 5} \tilde{A}_\mu^{(jk)} T^{jk} = \tilde{\Omega} A_\mu \tilde{\Omega}^{-1}. \tag{6.4}$$

Generators of $SU(2)_L$ and $SU(2)_R$ are

$$\begin{pmatrix} T^{a_L} \\ T^{a_R} \end{pmatrix} = \frac{1}{2} \left(\frac{1}{2} \epsilon^{abc} T^{bc} \pm T^{a4} \right), \quad a, b, c = 1\text{–}3. \tag{6.5}$$

To investigate the relation between the original and twisted gauges, let us define $\tilde{T}^{jk}(z) = \tilde{\Omega} T^{jk} \tilde{\Omega}^{-1}$. It follows that

$$\begin{bmatrix} \tilde{T}^{a4}(z) \\ \tilde{T}^{a5}(z) \end{bmatrix} = \begin{bmatrix} \cos\theta(z) & -\sin\theta(z) \\ \sin\theta(z) & \cos\theta(z) \end{bmatrix} \begin{bmatrix} T^{a4} \\ T^{a5} \end{bmatrix}, \quad (a = 1, 2, 3) \tag{6.6}$$

and other components remain unchanged. Recall that $\theta(1) = \theta_H$ and $\theta(z_L) = 0$. In particular for $\theta_H = \pi$, $\tilde{T}^{a4}(1) = -T^{a4}$, and $\tilde{T}^{a4}(z_L) = +T^{a4}$. In the basis of $\{\tilde{T}^{jk}(z)\}$ the role of T^{a_L} and T^{a_R} is interchanged as z varies from $z = 1$ to z_L. Indeed this property becomes crucial in discussing gauge symmetry in the $\theta_H = \pi$ state.

B. Gauge symmetry and couplings

Wave functions of KK towers of gauge fields in the twisted gauge are given in Appendix B. With $A_\mu^{a_{L/R}} = 2^{-1/2}(\frac{1}{2}\epsilon^{abc} A_\mu^{(bc)} \pm A_\mu^{(a4)})$ and $A_\mu^{\hat{a}} = A_\mu^{a4}$, a set $(A_\mu^{b_L}, A_\mu^{b_R}, A_\mu^{\hat{b}})$ $(b = 1,\ 2)$ forms charged gauge-field towers, which are decomposed into W, $\hat{W}$, and W_R towers. Mass eigenvalues are given by $\{k\lambda_n\}$ in each KK tower. They are determined by

$$W \text{ tower: } 2SC'(1; \lambda_{W^{(n)}}) + \lambda_{W^{(n)}} \sin^2\theta_H = 0,$$
$$\hat{W} \text{ tower: } 2SC'(1; \lambda_{\hat{W}^{(n)}}) + \lambda_{\hat{W}^{(n)}} \sin^2\theta_H = 0,$$
$$W_R \text{ tower: } C(1; \lambda_{W_R^{(n)}}) = 0, \tag{6.7}$$

where functions $C(z;\lambda), S(z;\lambda)$, etc. are defined in (A1). The spectra of the W and $\hat{W}$ towers for $\sin\theta_H = 0$ reduce to those determined by $C'(1;\lambda_{W^{(n)}})=0$ and $S(1;\lambda_{\hat{W}^{(n)}})=0$, respectively. The W tower has a zero mode $\lambda_{W^{(0)}}$ for $\sin\theta_H=0$, whereas the $\hat{W}$ tower does not. As depicted in Fig. 1 the spectra of the W and $\hat{W}$ towers alternate. The lowest level $[W^{(0)}]$ corresponds to W boson.

For $\theta_H=0$ and π, the W boson becomes massless, but the property of the W boson at $\theta_H=\pi$ is quite different from that at $\theta_H=0$. The W boson at $\theta_H=0$ is a purely $SU(2)_L$ gauge boson. As is manifest from the expression in (B1), $W_\mu^{(n)}(x)$ fields are purely $SU(2)_R$ at $\theta_H=\pi$ in the twisted gauge.

In the sector of neutral gauge bosons, $(A_\mu^{3_L}, A_\mu^{3_R}, A_\mu^{\hat{3}}, B_\mu)$, there are Z, $\hat{Z}$, Z_R, and γ towers. (A neutral tower from $A_\mu^{\hat{4}}$ does not couple with quarks and leptons.) Mass eigenvalues in each KK tower are determined by

$$
\begin{aligned}
&Z \text{ tower: } 2SC'(1;\lambda_{Z^{(n)}})+(1+s_\phi^2)\lambda_{Z^{(n)}}\sin^2\theta_H=0,\\
&\hat{Z} \text{ tower: } 2SC'(1;\lambda_{\hat{Z}^{(n)}})+(1+s_\phi^2)\lambda_{\hat{Z}^{(n)}}\sin^2\theta_H=0,\\
&Z_R \text{ tower: } C(1;\lambda_{Z_R^{(n)}})=0,\\
&\gamma \text{ tower: } C'(1;\lambda_{\gamma^{(n)}})=0, \qquad (6.8)
\end{aligned}
$$

where

$$
c_\phi=\frac{g_A}{\sqrt{g_A^2+g_B^2}}, \qquad s_\phi=\frac{g_B}{\sqrt{g_A^2+g_B^2}}. \qquad (6.9)
$$

The spectra of the Z and $\hat{Z}$ towers for $\sin\theta_H=0$ reduce to those determined by $C'(1;\lambda_{Z^{(n)}})=0$ and $S(1;\lambda_{\hat{Z}^{(n)}})=0$, respectively. The Z tower has zero mode $\lambda_{Z^{(0)}}$ for $\sin\theta_H=0$, whereas the $\hat{Z}$ tower does not. As in the case of W and $\hat{W}$ towers, the spectra of the Z and $\hat{Z}$ towers alternate. The lowest level $[Z^{(0)}]$ corresponds to Z boson.

A photon is always massless; $\lambda_{\gamma^{(0)}}=0$.

The bare weak mixing angle θ_W^0 is defined by

$$
\sin\theta_W^0=\frac{s_\phi}{\sqrt{1+s_\phi^2}}, \qquad \cos\theta_W^0=\frac{1}{\sqrt{1+s_\phi^2}}. \qquad (6.10)
$$

It has been shown that in the case of $\theta_H^{\rm min}=0.1$ and $m_{\rm KK}=13$ TeV, for instance, $\sin^2\theta_W^0=0.2305$ yields nearly the same phenomenology at low energies as that of the SM with $\sin^2\theta_W=0.2312$. In particular, it gives the forward-backward asymmetry $A_{FB}(e^-e^+\to\mu^-\mu^+)=0.01693$ at $\sqrt{s}=m_Z$ [16,18].

The Z boson becomes massless in the $\theta_H=0$ and π state. As is seen from (B3), the wave function of Z boson in the twisted gauge is nonvanishing in the T^{3_R} and $U(1)_{\rm EM}$ components for $\theta_H=\pi$. Note that $\tilde{A}_\mu=\sum_{a=1}^3\{\tilde{A}_\mu^{a_L}T^{a_L}+\tilde{A}_\mu^{a_R}T^{a_R}+\tilde{A}_\mu^{\hat{a}}T^{\hat{a}}\}+\tilde{A}_\mu^{\hat{4}}T^{\hat{4}}$ and $g_A s_\phi=g_B c_\phi$. Inserting (B1) and (B3) into $g_A\tilde{A}_\mu+g_B Q_X B_\mu$, one finds that the gauge couplings of $W_\mu^{(0)}$, $Z_\mu^{(0)}$, and $A_\mu^{\gamma(0)}$ are given by

$$
\begin{aligned}
&\underline{\text{for } \theta_H=0;}\\
&\frac{g_w}{\sqrt{2}}\{W_\mu^{(0)}(T^{1_L}+iT^{2_L})+W_\mu^{(0)\dagger}(T^{1_L}-iT^{2_L})\}+\frac{g_w}{\cos\theta_W^0}Z_\mu^{(0)}(T^{3_L}-\sin^2\theta_W^0 Q_{\rm EM})+eQ_{\rm EM}A_\mu^{\gamma(0)}\\
&\quad=g_w\{W_\mu^{1(0)}T^{1_L}+W_\mu^{2(0)}T^{2_L}+W_\mu^{3(0)}T^{3_L}\}+g_w\tan\theta_W^0 B_\mu(T^{3_R}+Q_X),\\
&\underline{\text{for } \theta_H=\pi;}\\
&\frac{g_w}{\sqrt{2}}\{W_\mu^{(0)}(T^{1_R}+iT^{2_R})+W_\mu^{(0)\dagger}(T^{1_R}-iT^{2_R})\}+\frac{g_w}{\cos\theta_W^0}Z_\mu^{(0)}(T^{3_R}-\sin^2\theta_W^0 Q_{\rm EM})+eQ_{\rm EM}A_\mu^{\gamma(0)}\\
&\quad=g_w\{W_\mu^{1(0)}T^{1_R}+W_\mu^{2(0)}T^{2_R}+W_\mu^{3(0)}T^{3_R}\}+g_w\tan\theta_W^0 B_\mu(T^{3_L}+Q_X), \qquad (6.11)
\end{aligned}
$$

where

$$
\begin{aligned}
&g_w=\frac{g_A}{\sqrt{L}}, \qquad e=g_w\sin\theta_W^0, \qquad Q_{\rm EM}=T^{3_L}+T^{3_R}+Q_X,\\
&W_\mu^{(0)}=\frac{1}{\sqrt{2}}(W_\mu^{1(0)}-iW_\mu^{2(0)}), \qquad \begin{bmatrix} Z_\mu^{(0)}\\ A_\mu^{\gamma(0)}\end{bmatrix}=\begin{bmatrix}\cos\theta_W^0 & -\sin\theta_W^0\\ \sin\theta_W^0 & \cos\theta_W^0\end{bmatrix}\begin{bmatrix}W_\mu^{3(0)}\\ B_\mu\end{bmatrix}. \qquad (6.12)
\end{aligned}
$$

In the $\theta_H=\pi$ state, there exists $SU(2)_R\times U(1)_{Y'}$ gauge symmetry in the twisted gauge where the $U(1)_{Y'}$ charge is given by $Y'=T^{3_L}+Q_X$.

Gauge couplings of quarks, leptons, and dark fermions in the $\theta_H=\pi$ state are quite different from those in the $\theta_H=0$ state. For a fermion field $\Psi(x,z)$ it is most convenient to express its KK expansion as $\check{\Psi}(x,z)=z^{-2}\Psi(x,z)$.

For up-type quarks in $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}$ in Table I the KK expansion is given, for the first generation pair (u, u') for instance, by

$$\begin{bmatrix} \check{\tilde{u}} \\ \check{\tilde{u}}' \end{bmatrix} = \sqrt{k}\left\{ u^{(0)} + \sum_{n=1}^{\infty} u^{(n)} + \sum_{n=1}^{\infty} u'^{(n)} \right\}. \tag{6.13}$$

Wave functions are given in Appendix Sec. C 1. The spectrum is determined by

$$\begin{aligned} &u \text{ tower}:\ S_L S_R(1; \lambda_{u^{(n)}}, c_u) + \sin^2\frac{\theta_H}{2} = 0, \\ &u' \text{ tower}:\ S_L S_R(1; \lambda_{u'^{(n)}}, c_u) + \sin^2\frac{\theta_H}{2} = 0, \end{aligned} \tag{6.14}$$

where functions $S_{L/R}$, and $C_{L/R}$ are given in (A2). We note $S_L S_R(1;\lambda,c) + \sin^2\frac{1}{2}\theta_H = C_L C_R(1;\lambda,c) - \cos^2\frac{1}{2}\theta_H$. The lowest zero modes, $\hat{u}_L^{(0)}(x)$ and $\hat{u}_R^{(0)}(x)$ have chiral structure. They are massless ($\lambda_{u^{(0)}} = 0$) for $\theta_H = 0$. Their wave functions behave differently from those of the $n \geq 1$ modes. The spectrum-determining equation for the u tower for $c_u < 0$ reduces to $S_R(1;\lambda_{u^{(n)}}, c_u) = 0$ at $\theta_H = 0$ and $C_L(1;\lambda_{u^{(n)}}, c_u) = 0$ at $\theta_H = \pi$, while for the u' tower it reduces to $S_L(1;\lambda_{u'^{(n)}}, c_u) = 0$ at $\theta_H = 0$ and $C_R(1;\lambda_{u'^{(n)}}, c_u) = 0$ at $\theta_H = \pi$. We note that the spectrum $\{\lambda_{u^{(n)}}, \lambda_{u'^{(n)}}\}$ at $\theta_H = \pi$ is different from that at $\theta_H = 0$. In particular $\lambda_{u^{(0)}}|_{\theta_H=0} = 0$ but $\lambda_{u^{(0)}}|_{\theta_H=\pi} > 0$.

Mass eigenstates of down-type quarks are more involved, the details of which have been given in Refs. [15,16]. Down-type quarks in $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}$ and $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$ fields in Table I mix with each other by brane interactions, which in the most general case induce the CKM mass-mixing matrix as well. For the sake of simplicity we consider the case in which brane interactions are diagonal in the generation space. The spectrum for the first generation (d, d' and $D^{\pm}$ towers) is determined by

$$\begin{aligned} &\left(S_L^Q S_R^Q + \sin^2\frac{\theta_H}{2}\right)(\mathcal{S}_{L1}^D \mathcal{S}_{R1}^D - \mathcal{S}_{L2}^D \mathcal{S}_{R2}^D) \\ &\quad + |\mu_1|^2 C_R^Q S_R^Q (\mathcal{S}_{L1}^D \mathcal{C}_{L1}^D - \mathcal{S}_{L2}^D \mathcal{C}_{L2}^D) = 0, \end{aligned} \tag{6.15}$$

where $S_{L/R}^Q = S_{L/R}(1;\lambda, c_u)$, $\mathcal{S}_{Lj}^D = \mathcal{S}_{Lj}(1;\lambda, c_{D_d}, \tilde{m}_{D_d})$, etc. Functions $\mathcal{S}_{L/Rj}$, and $\mathcal{C}_{L/Rj}$ are given in (A3). c_{D_α} is the bulk mass parameter of the $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$ field, and $\tilde{m}_{D_\alpha} = m_{D_\alpha}/k$. μ_1 parametrizes the strength of a brane interaction among $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}$, $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$, and Φ_S, which is relevant to reproduce a mass of each down-type quark. There are four KK towers, $\mathbf{d} = (d, d', D^+, D^-)$. Their KK expansion can be written as

$$\begin{bmatrix} \check{\tilde{d}} \\ \check{\tilde{d}}' \\ \check{\tilde{D}}^+ \\ \check{\tilde{D}}^- \end{bmatrix} = \sqrt{k}\left\{ \sum_{n=0}^{\infty} d^{(n)} + \sum_{n=1}^{\infty} d'^{(n)} + \sum_{n=1}^{\infty} D^{+(n)} + \sum_{n=1}^{\infty} D^{-(n)} \right\}. \tag{6.16}$$

Details of wave functions of all KK modes are given in Appendix Sec. C 1. For $\theta_H = 0$ there appears a massless mode $d^{(0)}$ with chiral wave functions, which is identified with a down quark. For $\theta_H = \pi$ there is no zero mode.

W and Z couplings of quarks are easily found with the use of (6.11). We note that $\lambda_{u^{(n)}} = \lambda_{d^{(n)}}$ at $\theta_H = 0$ and $\lambda_{u'^{(n)}} = \lambda_{d'^{(n)}}$ at $\theta_H = \pi$. Couplings with $W_\mu^{(0)}$, $Z_\mu^{(0)}$, and $A_\mu^{\gamma(0)}$ are given by

for $\theta_H = 0$;

$$\begin{aligned} &\frac{g_w}{\sqrt{2}}\left\{ W_\mu^{(0)}\left(\bar{\hat{u}}_L^{(0)}\gamma^\mu \hat{d}_L^{(0)} + \sum_{n=1}^{\infty} \bar{\hat{u}}^{(n)}\gamma^\mu \hat{d}^{(n)} \right) + W_\mu^{(0)\dagger}\left(\bar{\hat{d}}_L^{(0)}\gamma^\mu \hat{u}_L^{(0)} + \sum_{n=1}^{\infty} \bar{\hat{d}}^{(n)}\gamma^\mu \hat{u}^{(n)} \right) \right\} \\ &\quad + \frac{g_w}{2\cos\theta_W^0} Z_\mu^{(0)} \left\{ (\bar{\hat{u}}_L^{(0)}\gamma^\mu \hat{u}_L^{(0)} - \bar{\hat{d}}_L^{(0)}\gamma^\mu \hat{d}_L^{(0)}) + \sum_{n=1}^{\infty} (\bar{\hat{u}}^{(n)}\gamma^\mu \hat{u}^{(n)} - \bar{\hat{d}}^{(n)}\gamma^\mu \hat{d}^{(n)}) \right\} + \left\{ -g_w \frac{\sin^2\theta_W^0}{\cos\theta_W^0} Z_\mu^{(0)} + eA_\mu^{\gamma(0)} \right\} J_{\rm EM}^\mu, \end{aligned}$$

for $\theta_H = \pi$;

$$\begin{aligned} &\frac{g_w}{\sqrt{2}}\left\{ W_\mu^{(0)} \sum_{n=1}^{\infty} \bar{\hat{u}}'^{(n)}\gamma^\mu \hat{d}'^{(n)} + W_\mu^{(0)\dagger} \sum_{n=1}^{\infty} \bar{\hat{d}}'^{(n)}\gamma^\mu \hat{u}'^{(n)} \right\} + \frac{g_w}{2\cos\theta_W^0} Z_\mu^{(0)} \sum_{n=1}^{\infty} (\bar{\hat{u}}'^{(n)}\gamma^\mu \hat{u}'^{(n)} - \bar{\hat{d}}'^{(n)}\gamma^\mu \hat{d}'^{(n)}) \\ &\quad + \left\{ -g_w \frac{\sin^2\theta_W^0}{\cos\theta_W^0} Z_\mu^{(0)} + eA_\mu^{\gamma(0)} \right\} J_{\rm EM}^\mu, \end{aligned} \tag{6.17}$$

where

$$J^{\mu}_{\rm EM} = \sum_{n=0}^{\infty}\bigg(\frac{2}{3}\bar{\hat{u}}^{(n)}\gamma^{\mu}\hat{u}^{(n)} - \frac{1}{3}\bar{\hat{d}}^{(n)}\gamma^{\mu}\hat{d}^{(n)}\bigg) + \sum_{n=1}^{\infty}\bigg(\frac{2}{3}\bar{\hat{u}}^{\prime(n)}\gamma^{\mu}\hat{u}^{\prime(n)} - \frac{1}{3}\bar{\hat{d}}^{\prime(n)}\gamma^{\mu}\hat{d}^{\prime(n)} - \frac{1}{3}\bar{\hat{D}}^{+(n)}\gamma^{\mu}\hat{D}^{+(n)} - \frac{1}{3}\bar{\hat{D}}^{-(n)}\gamma^{\mu}\hat{D}^{-(n)}\bigg). \quad (6.18)$$

At $\theta_H = 0$ couplings of massless modes are chiral, but those of all massive modes are vectorlike. At $\theta_H = \pi$, $W^{(0)}_{\mu}$ couplings to $\bar{\hat{u}}^{(0)}_L\gamma^{\mu}\hat{d}^{(0)}_L$ and $\bar{\hat{u}}^{(0)}_R\gamma^{\mu}\hat{d}^{(0)}_R$ vanish, and for $Z^{(0)}_{\mu}$ couplings of $\hat{u}^{(0)}$ and $\hat{d}^{(0)}$, the T^{3_R} part vanishes with only the $U(1)_{\rm EM}$ part surviving. All modes are massive and their gauge couplings are vectorlike. We also note that a top quark becomes very heavy at $\theta_H = \pi$, but all other quarks and leptons remain light. For $\theta^{\rm min}_H = 0.1$, $m_{\rm KK} = 13$ TeV, for instance, $m_t = 4.14$ TeV and $m_c = 12.4$ GeV at $\theta_H = \pi$.

The gauge couplings in (6.17) can be clearly and neatly understood from quantum numbers in the $\theta_H = 0$ and π phases as summarized in Table V.

Gauge couplings in the lepton sector are found in the same manner. Details are given in Appendix Sec. C 2. We note that the gauge couplings in the lepton sector, given by (C23), can be summarized as in Table V for the quark sector. One needs to replace (u, d, u', d') by (ν_e, e, ν'_e, e'), and $(\frac{1}{6}, \frac{2}{3}, -\frac{1}{3})$ in $U(1)_Y$ and $U(1)_{Y'}$ charges by $(-\frac{1}{2}, 0, -1)$ there.

Dark fermions in the spinor representation Ψ^{β}_F (darkF fermions) are denoted, in the twisted gauge, as

$$\check{\tilde{\Psi}}_F(x,z) = \begin{bmatrix} \check{\tilde{F}}_1 \\ \check{\tilde{F}}_2 \\ \check{\tilde{F}}'_1 \\ \check{\tilde{F}}'_2 \end{bmatrix}. \quad (6.19)$$

The spectrum of F and F' towers, $\{\lambda_{F^{(n)}}, \lambda_{F'^{(n)}}\}$, is determined by

TABLE V. Charge assignment of u, u', d, and d' towers under $SU(2)_L \times U(1)_Y$ in the $\theta_H = 0$ phase and $SU(2)_R \times U(1)_{Y'}$ in the $\theta_H = \pi$ phase. The index n runs as $n = 1, 2, 3, \ldots$. Only the zero modes $u^{(0)}$ and $d^{(0)}$ in the $\theta_H = 0$ phase are chiral.

	$\theta_H = 0$ phase		$\theta_H = \pi$ phase	
	$SU(2)_L$	$U(1)_Y$	$SU(2)_R$	$U(1)_{Y'}$
$u^{(0)}_L, d^{(0)}_L$	**2**	$\frac{1}{6}$	**1**	$\frac{2}{3}, -\frac{1}{3}$
$u^{(0)}_R, d^{(0)}_R$	**1**	$\frac{2}{3}, -\frac{1}{3}$	**1**	$\frac{2}{3}, -\frac{1}{3}$
$u^{(n)}, d^{(n)}$	**2**	$\frac{1}{6}$	**1**	$\frac{2}{3}, -\frac{1}{3}$
$u'^{(n)}, d'^{(n)}$	**1**	$\frac{2}{3}, -\frac{1}{3}$	**2**	$\frac{1}{6}$

$$S_L S_R(1;\lambda_n, c_F) + \cos^2\frac{\theta_H}{2} = C_L C_R(1;\lambda_n, c_F) - \sin^2\frac{\theta_H}{2} = 0. \quad (6.20)$$

Massless modes appear at $\theta_H = \pi$. The spectrum-determining equation for the F tower reduces to $C_L(1;\lambda_{F^{(n)}}, c_F) = 0$ at $\theta_H = 0$ and $S_R(1;\lambda_{F^{(n)}}, c_F) = 0$ at $\theta_H = \pi$, while for the F' tower it reduces to $C_R(1;\lambda_{F'^{(n)}}, c_F) = 0$ at $\theta_H = 0$ and $S_L(1;\lambda_{F'^{(n)}}, c_F) = 0$ at $\theta_H = \pi$. For $c_F > 0$ zero modes appear in the F' tower, and the KK expansion is given by

$$\begin{bmatrix} \check{\tilde{F}}_j \\ \check{\tilde{F}}'_j \end{bmatrix} = \sqrt{k}\bigg\{\sum_{n=1}^{\infty} F^{(n)}_j + \sum_{n=0}^{\infty} F'^{(n)}_j\bigg\}. \quad (6.21)$$

Wave functions of each mode are given in Appendix Sec. C 3. For $c_F < 0$ zero modes appear in the F tower, and the KK expansion is given by

$$\begin{bmatrix} \check{\tilde{F}}_j \\ \check{\tilde{F}}'_j \end{bmatrix} = \sqrt{k}\bigg\{\sum_{n=0}^{\infty} F^{(n)}_j + \sum_{n=1}^{\infty} F'^{(n)}_j\bigg\}. \quad (6.22)$$

We stress that massless modes appear at $\theta_H = \pi$ in the darkF sector. Gauge couplings at $\theta_H = \pi$ are given, for $c_F > 0$, by

$$\frac{g_w}{\sqrt{2}}\bigg\{\bigg[W^{(0)}_{\mu}\bigg(\bar{\hat{F}}'^{(0)}_{1R}\gamma^{\mu}\hat{F}'^{(0)}_{2R} + \sum_{n=1}^{\infty}\bar{\hat{F}}'^{(n)}_{1}\gamma^{\mu}\hat{F}'^{(n)}_{2}\bigg) + {\rm H.c.}\bigg]\bigg\}$$
$$+ \frac{g_w}{2\cos\theta^0_W}Z^{(0)}_{\mu}\bigg\{\bar{\hat{F}}'^{(0)}_{1R}\gamma^{\mu}\hat{F}'^{(0)}_{1R} - \bar{\hat{F}}'^{(0)}_{2R}\gamma^{\mu}\hat{F}'^{(0)}_{2R} + \sum_{n=1}^{\infty}(\bar{\hat{F}}'^{(n)}_{1}\gamma^{\mu}\hat{F}'^{(n)}_{1} - \bar{\hat{F}}'^{(n)}_{2}\gamma^{\mu}\hat{F}'^{(n)}_{2})\bigg\}$$
$$+ \bigg\{-g_w\frac{\sin^2\theta^0_W}{\cos\theta^0_W}Z^{(0)}_{\mu} + eA^{\gamma(0)}_{\mu}\bigg\}\bigg\{\sum_{n=1}^{\infty}(\bar{\hat{F}}^{(n)}_{1}, \bar{\hat{F}}^{(n)}_{2})\gamma^{\mu}Q_{\rm EM}\begin{pmatrix}\hat{F}^{(n)}_1 \\ \hat{F}^{(n)}_2\end{pmatrix} + \sum_{n=0}^{\infty}(\bar{\hat{F}}'^{(n)}_{1}, \bar{\hat{F}}'^{(n)}_{2})\gamma^{\mu}Q_{\rm EM}\begin{pmatrix}\hat{F}'^{(n)}_1 \\ \hat{F}'^{(n)}_2\end{pmatrix}\bigg\}. \quad (6.23)$$

TABLE VI. Charge assignment of F_1, F_2, F'_1, and F'_2 towers under $SU(2)_L \times U(1)_Y$ in the $\theta_H = 0$ phase and $SU(2)_R \times U(1)_{Y'}$ in the $\theta_H = \pi$ phase for $c_F > 0$. The index n runs as $n = 1, 2, 3, \ldots$. Only the zero modes $F_1^{\prime(0)}$ and $F_2^{\prime(0)}$ in the $\theta_H = \pi$ phase are chiral.

	$\theta_H = 0$ phase		$\theta_H = \pi$ phase	
	$SU(2)_L$	$U(1)_Y$	$SU(2)_R$	$U(1)_{Y'}$
$F_1^{(n)}, F_2^{(n)}$	**2**	$\frac{1}{6}$	**1**	$\frac{2}{3}, -\frac{1}{3}$
$F_{1L}^{\prime(0)}, F_{2L}^{\prime(0)}$	**1**	$\frac{2}{3}, -\frac{1}{3}$	**1**	$\frac{2}{3}, -\frac{1}{3}$
$F_{1R}^{\prime(0)}, F_{2R}^{\prime(0)}$	**1**	$\frac{2}{3}, -\frac{1}{3}$	**2**	$\frac{1}{6}$
$F_1^{\prime(n)}, F_2^{\prime(n)}$	**1**	$\frac{2}{3}, -\frac{1}{3}$	**2**	$\frac{1}{6}$

The gauge couplings in (6.23) at $\theta_H = \pi$, as well as those at $\theta_H = 0$, can be understood from quantum numbers in each phase (summarized in Table VI).

Formulas for $c_F < 0$ are obtained by replacing $\hat{F}_j^{\prime(0)}$ by $\hat{F}_j^{(0)}$. One can flip the orbifold boundary conditions for Ψ_F^β, too. By reversing the parity assignment for Ψ_F^β in Table I, the role of the left-handed and the right-handed components are interchanged.

Formulas for $SU(3)_C$-singlet darkF_ℓ ($\Psi_{F_\ell}^\beta$) fields take the same form as for $SU(3)_C$-triplet darkF fields. The KK expansions are the same. The only change is in their $U(1)$ charges. As in the lepton case, $(\frac{1}{6}, \frac{2}{3}, -\frac{1}{3})$ in $U(1)_Y$ and $U(1)_{Y'}$ charges should be replaced by $(-\frac{1}{2}, 0, -1)$.

It is observed that all modes of the darkF tower are massive and their gauge couplings are vectorlike at $\theta_H = 0$, but there appear chiral massless modes at $\theta_H = \pi$. In other words a theory with Ψ_F^β but no $\Psi_{F_\ell}^\beta$ would become anomalous at $\theta_H = \pi$. The anomaly cancellation is achieved with a set $(\Psi_F^\beta, \Psi_{F_\ell}^\beta)$ just as in the cancellation in the quark-lepton sector at $\theta_H = 0$. We would like to stress that the appearance of a set $(\Psi_F^\beta, \Psi_{F_\ell}^\beta)$ is a natural consequence from the viewpoint of grand unification. One set is contained in the **32** representation in $SO(11)$ gauge-Higgs grand unification [48,49].

Dark fermions in the vector representation $\Psi_V^{\pm\gamma}$ (darkV fermions) are, as depicted in Fig. 1, always massive and very heavy, and therefore they do not affect the behavior of the model at the finite temperature $T \lesssim T_{\mathrm{decay}}^{\mathrm{LR}}$ very much. Their gauge couplings are summarized in Appendix Sec. C 4. It is shown there that all couplings are vectorlike.

C. Chiral fermions, anomaly, and nontopological solitons

In the above we have shown how the $\theta_H = 0$ phase is smoothly connected to the $\theta_H = \pi$ phase in GHU, and have clarified gauge couplings of fermions in each phase. It is remarkable and intriguing that chiral fermions (quark and lepton multiplets) in the $\theta_H = 0$ phase are continuously transformed to vectorlike fermions in the $\theta_H = \pi$ phase. The situation is reversed for dark fermions in the spinor representation (darkF fermions). They are massive and vectorlike in the $\theta_H = 0$ phase, becoming chiral in the $\theta_H = \pi$ phase. This fact immediately leads to an important question about anomalies. What is the fate of anomalies in the $\theta_H = 0$ phase when θ_H is continuously changed to $\theta_H = \pi$? More generally we need to understand what kinds of anomalies arise in general θ_H states in GHU.

Another point of interest is the possibility of having $\theta_H = \pi$ solitons—nontopologoical solitons similar to Fermi balls [50]. The lowest modes of darkF fermions are very heavy both in the $\theta_H = 0$ phase and in the $\theta_H = \theta_H^{\mathrm{min}}$ state (the current Universe) at $T = 0$, whereas they become massless in the $\theta_H = \pi$ phase. There can be a nontopological soliton such that in its inside $\theta_H = \pi$ and massless darkF fermions are filled, but its outside is in the $\theta_H = 0$ or θ_H^{min} state. Although the energy density of the $\theta_H = \pi$ state is larger than that of the θ_H^{min} state, darkF fermions inside the ball cannot freely go outside as their masses in the θ_H^{min} state are large. The phase θ_H cannot changes from π to θ_H^{min} either for the same reason. Pair annihilation processes are involved as well. As shown in the preceding subsections, some of dark fermion pairs can annihilate to virtual gauge bosons, which subsequently annihilate to quark-lepton pairs. Gauge bosons need to tunnel out from the inside to the outside of the soliton as quarks/leptons are heavy inside the soliton. It would make more difficult for an object to decay, if the total darkF number is nonvanishing. As a whole such an object can become stable; its size can be large. There may be important cosmological consequences of such solitons.

We would like to leave these intriguing questions for future investigation. The existence of the $\theta_H = \pi$ state in GHU may have profound implications.

VII. SUMMARY AND DISCUSSIONS

In the present paper we have investigated the behavior of the GUT inspired $SO(5) \times U(1) \times SU(3)$ GHU model at finite temperature. At zero temperature the EW symmetry $SU(2)_L \times U(1)_Y$ is dynamically broken to $U(1)_{\mathrm{EM}}$ by the Hosotani mechanism. As the temperature is raised, the EW symmetry is restored around $T = 163$ GeV. We have shown that the transition is weakly first order just as in the SM in perturbation theory. Although the EW symmetry breaking mechanism at $T = 0$ is quite different from that in the SM, the behavior at finite temperature $T \lesssim 1$ TeV is almost the same as in the SM. This is due to the fact that the particle spectrum at low energies is the same as in the SM.

As the temperature is increased further, a new feature emerges in GHU. Above $T_{c1}^{\rm LR} \sim m_{\rm KK}$ the $\theta_H = 0$ and $\theta_H = \pi$ states become almost degenerate. In the effective potential $V_{\rm eff}(\theta_H; T)$ these two states are separated by a barrier so that domain structure will be formed as the Universe expands and the temperature drops to $\sim T_{c1}^{\rm LR}$. Eventually the $\theta_H = \pi$ state becomes totally unstable for $T < T_{c2}^{\rm LR} \sim 2.3$ TeV. We have shown that the transition from the $\theta_H = \pi$ state to the $\theta_H = 0$ state takes place rapidly around $T = T_{\rm decay}^{\rm LR} \sim 2.6$ TeV.

The $\theta_H = 0$ and $\theta_H = \pi$ states are characterized as the $SU(2)_L \times U(1)_Y$ and $SU(2)_R \times U(1)_{Y'}$ phases, respectively. W bosons, Z bosons, and photons become gauge bosons of $SU(2)_R \times U(1)_{Y'}$ symmetry in the $\theta_H = \pi$ state. The transition from the $\theta_H = \pi$ state to the $\theta_H = 0$ state is called the left-right transition. Gauge couplings of quarks, leptons, and dark fermions in the $SU(2)_R \times U(1)_{Y'}$ phase differ from those in the $SU(2)_L \times U(1)_Y$ phase. We showed, for instance, that quarks do not couple to W bosons, and their couplings to Z bosons come solely from the $U(1)_{\rm EM}$ part in the $\theta_H = \pi$ state.

In the history of the early Universe the LR transition is a first-order phase transition, taking place by tunneling through bubble nucleation. GWs are produced in this transition, the amount of which, however, turns out small. A GW signal from the LR transition is far below the reach of the sensitivity of LISA, etc.

The $4D$ Higgs boson corresponds to the $4D$ fluctuation mode of the AB phase θ_H in the fifth dimension in GHU. There is only one Higgs boson in the GHU model under investigation. This single boson connects the $U(1)_{\rm EM}$ phase at $T = 0$, the $SU(2)_L \times U(1)_Y$ phase and the $SU(2)_R \times U(1)_{Y'}$ phase. It would be of great interest to know the physical consequences of the existence of the $SU(2)_R \times U(1)_{Y'}$ phase in the context of the history of the evolution of the early Universe.

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APPENDIX A: BASIS FUNCTIONS

Wave functions of gauge fields and fermions are expressed in terms of the following basis functions. For gauge fields we introduce

$$\begin{aligned} C(z;\lambda) &= \frac{\pi}{2}\lambda z z_L F_{1,0}(\lambda z, \lambda z_L), \\ S(z;\lambda) &= -\frac{\pi}{2}\lambda z F_{1,1}(\lambda z, \lambda z_L), \\ C'(z;\lambda) &= \frac{\pi}{2}\lambda^2 z z_L F_{0,0}(\lambda z, \lambda z_L), \\ S'(z;\lambda) &= -\frac{\pi}{2}\lambda^2 z F_{0,1}(\lambda z, \lambda z_L), \\ F_{\alpha,\beta}(u,v) &\equiv J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v), \end{aligned} \tag{A1}$$

where $J_\alpha(u)$ and $Y_\alpha(u)$ are Bessel functions of the first and second kind. A relation $CS' - SC' = \lambda z$ holds. For fermion fields with a bulk mass parameter c, we define

$$\begin{aligned} \begin{pmatrix} C_L \\ S_L \end{pmatrix}(z;\lambda,c) &= \pm\frac{\pi}{2}\lambda\sqrt{z z_L} F_{c+\frac{1}{2},c\mp\frac{1}{2}}(\lambda z, \lambda z_L), \\ \begin{pmatrix} C_R \\ S_R \end{pmatrix}(z;\lambda,c) &= \mp\frac{\pi}{2}\lambda\sqrt{z z_L} F_{c-\frac{1}{2},c\pm\frac{1}{2}}(\lambda z, \lambda z_L). \end{aligned} \tag{A2}$$

These functions satisfy $C_L C_R - S_L S_R = 1$, $C_L(z;\lambda,-c) = C_R(z;\lambda,c)$, and $S_L(z;\lambda,-c) = -S_R(z;\lambda,c)$. Also note that $S_{L/R}(z;0,c) = 0$ and $C_{L/R}(z;0,c) \neq 0$. To treat down-type quarks and dark fermions we also use

$$\begin{aligned} \mathcal{C}_{L1}(z;\lambda,c,\tilde m) &= C_L(z;\lambda,c+\tilde m) + C_L(z;\lambda,c-\tilde m), \\ \mathcal{C}_{L2}(z;\lambda,c,\tilde m) &= S_L(z;\lambda,c+\tilde m) - S_L(z;\lambda,c-\tilde m), \\ \mathcal{S}_{L1}(z;\lambda,c,\tilde m) &= S_L(z;\lambda,c+\tilde m) + S_L(z;\lambda,c-\tilde m), \\ \mathcal{S}_{L2}(z;\lambda,c,\tilde m) &= C_L(z;\lambda,c+\tilde m) - C_L(z;\lambda,c-\tilde m), \\ \mathcal{C}_{R1}(z;\lambda,c,\tilde m) &= C_R(z;\lambda,c+\tilde m) + C_R(z;\lambda,c-\tilde m), \\ \mathcal{C}_{R2}(z;\lambda,c,\tilde m) &= S_R(z;\lambda,c+\tilde m) - S_R(z;\lambda,c-\tilde m), \\ \mathcal{S}_{R1}(z;\lambda,c,\tilde m) &= S_R(z;\lambda,c+\tilde m) + S_R(z;\lambda,c-\tilde m), \\ \mathcal{S}_{R2}(z;\lambda,c,\tilde m) &= C_R(z;\lambda,c+\tilde m) - C_R(z;\lambda,c-\tilde m). \end{aligned} \tag{A3}$$

APPENDIX B: GAUGE FIELDS

Wave functions of KK towers of gauge fields in the twisted gauge can be summarized in simple forms. With $A_\mu^{a_{L/R}} = 2^{-1/2}(\frac{1}{2}\epsilon^{abc}A_\mu^{(bc)} \pm A_\mu^{(a4)})$ and $A_\mu^{\hat a} = A_\mu^{a4}$, a set $(A_\mu^{b_L}, A_\mu^{b_R}, A_\mu^{\hat b})$ $(b = 1, 2)$ forms charged gauge field towers, W, $\hat W$, and W_R towers,

$$
\begin{bmatrix} \tilde{A}_\mu^{1_L} - i\tilde{A}_\mu^{2_L} \\ \tilde{A}_\mu^{1_R} - i\tilde{A}_\mu^{2_R} \\ \tilde{A}_\mu^{\hat{1}} - i\tilde{A}_\mu^{\hat{2}} \end{bmatrix} = \begin{bmatrix} (1+c_H)\overset{\circ}{W}_\mu \\ (1-c_H)\overset{\circ}{W}_\mu \\ -\sqrt{2}s_H \overset{\circ}{W}{}_\mu^S \end{bmatrix} + \begin{bmatrix} s_H(1+c_H)\overset{\circ}{\hat{W}}_\mu \\ s_H(1-c_H)\overset{\circ}{\hat{W}}_\mu \\ \sqrt{2}\overset{\circ}{\hat{W}}{}_\mu^S \end{bmatrix} + \begin{bmatrix} (1-c_H)\overset{\circ}{W}_{R\mu} \\ -(1+c_H)\overset{\circ}{W}_{R\mu} \\ 0 \end{bmatrix},
$$

$$
\begin{bmatrix} \overset{\circ}{W}_\mu(x,z) \\ \overset{\circ}{W}{}_\mu^S(x,z) \end{bmatrix} = \sqrt{k}\sum_{n=0}^{\infty} W_\mu^{(n)}(x)\frac{1}{\sqrt{r_{W^{(n)}}}}\begin{bmatrix} C(z;\lambda_{W^{(n)}}) \\ \hat{S}(z;\lambda_{W^{(n)}}) \end{bmatrix},
$$

$$
\begin{bmatrix} \overset{\circ}{\hat{W}}_\mu(x,z) \\ \overset{\circ}{\hat{W}}{}_\mu^S(x,z) \end{bmatrix} = \sqrt{k}\sum_{n=1}^{\infty} \hat{W}_\mu^{(n)}(x)\frac{1}{\sqrt{r_{\hat{W}^{(n)}}}}\begin{bmatrix} C(z;\lambda_{\hat{W}^{(n)}}) \\ \check{S}(z;\lambda_{\hat{W}^{(n)}}) \end{bmatrix},
$$

$$
\overset{\circ}{W}_{R\mu}(x,z) = \sqrt{k}\sum_{n=1}^{\infty} W_{R\mu}^{(n)}(x)\frac{1}{\sqrt{r_{W_R^{(n)}}}}C(z;\lambda_{W_R^{(n)}}),
$$

$$
r_{W^{(n)}} = \int_1^{z_L}\frac{dz}{z}\{(1+c_H^2)C(z;\lambda_{W^{(n)}})^2 + s_H^2\hat{S}(z;\lambda_{W^{(n)}})^2\},
$$

$$
r_{\hat{W}^{(n)}} = \int_1^{z_L}\frac{dz}{z}\{s_H^2(1+c_H^2)C(z;\lambda_{\hat{W}^{(n)}})^2 + \check{S}(z;\lambda_{\hat{W}^{(n)}})^2\},
$$

$$
r_{W_R^{(n)}} = \int_1^{z_L}\frac{dz}{z}(1+c_H^2)C(z;\lambda_{W_R^{(n)}})^2,
$$

$$
\hat{S}(z;\lambda) = \frac{C(1;\lambda)}{S(1;\lambda)}S(z;\lambda), \qquad \check{S}(z;\lambda) = \frac{2C'(1;\lambda)C(1;\lambda)}{\lambda}S(z;\lambda). \tag{B1}
$$

Here $c_H = \cos\theta_H$, $s_H = \sin\theta_H$, and functions $C(z;\lambda)$, $S(z;\lambda)$, etc. are defined in (A1). It is instructive to express the wave functions of W, $\hat{W}$, and W_R towers in the original gauge by making use of (6.6). For $\theta_H = \pi$, $A_\mu(x,z)$ in the original gauge is expanded as

$$
\begin{aligned}
2A_\mu(x,z) \Rightarrow & \sum_{n=0}^{\infty} W_\mu^{(n)}(x)\frac{1}{\sqrt{r_{W^{(n)}}}}C(z;\lambda_{W^{(n)}})\{(1-\cos\theta(z))(T^{1_L}+iT^{2_L}) + (1+\cos\theta(z))(T^{1_R}+iT^{2_R}) - \sin\theta(z)\sqrt{2}(T^{\hat{1}}+iT^{\hat{2}})\} \\
& + \sum_{n=1}^{\infty} \hat{W}_\mu^{(n)}(x)\frac{1}{\sqrt{r_{\hat{W}^{(n)}}}}\check{S}(z;\lambda_{\hat{W}^{(n)}})\{\cos\theta(z)\sqrt{2}(T^{\hat{1}}+iT^{\hat{2}}) - \sin\theta(z)(T^{1_L}+iT^{2_L}-T^{1_R}-iT^{2_R})\} \\
& + \sum_{n=1}^{\infty} W_{R\mu}^{(n)}(x)\frac{1}{\sqrt{r_{W_R^{(n)}}}}C(z;\lambda_{W_R^{(n)}})\{(1+\cos\theta(z))(T^{1_L}+iT^{2_L}) \\
& + (1-\cos\theta(z))(T^{1_R}+iT^{2_R}) + \sin\theta(z)\sqrt{2}(T^{\hat{1}}+iT^{\hat{2}})\}.
\end{aligned} \tag{B2}
$$

For the $W(=W^{(0)})$ boson $\lambda_{W^{(0)}} = 0$ and $C(z;\lambda_{W^{(0)}})/\sqrt{r_{W^{(0)}}} = 1/\sqrt{2kL}$. It is seen that W boson in the $\theta_H = \pi$ state is $SU(2)_L$-like at $z=1$, continuously changes in the group space $SO(5)$ in the bulk, and becomes $SU(2)_R$-like at $z = z_L$. The W_R tower, on the other hand, is $SU(2)_R$-like at $z=1$ and becomes $SU(2)_L$-like at $z = z_L$. The brane interaction $\delta(y)(D_\mu\Phi_S)^\dagger D^\mu\Phi_S$ with the brane scalar Φ_S yields brane mass terms $\delta(y)\frac{1}{4}g_A^2|w|^2(A_\mu^{1_R}A^{1_R\mu} + A_\mu^{2_R}A^{2_R\mu})$ after Φ_S spontaneously develops vacuum expectation value $\langle\Phi_{[1,2]}\rangle = (0,w)^t \neq 0$. Notice that this affects only W_R tower in (B2). W boson becomes massless in the $\theta_H = \pi$ state.

Similarly in the sector of neutral gauge bosons, $(A_\mu^{3_L}, A_\mu^{3_R}, A_\mu^{\hat{3}}, B_\mu)$, one finds in the twisted gauge that

$$\begin{bmatrix} \tilde{A}_\mu^{3_L} \\ \tilde{A}_\mu^{3_R} \\ \tilde{A}_\mu^{\hat{3}} \\ B_\mu \end{bmatrix} = \sqrt{\frac{1+s_\phi^2}{2}} \begin{bmatrix} (1+c_H)\overset{\circ}{Z}_\mu + s_H(1+c_H)\overset{\circ}{\hat{Z}}_\mu \\ (1-c_H)\overset{\circ}{Z}_\mu + s_H(1-c_H)\overset{\circ}{\hat{Z}}_\mu \\ -\sqrt{2}s_H \overset{\circ}{Z}{}_\mu^S + \sqrt{2}(1+s_\phi^2)^{-1}\overset{\circ}{\hat{Z}}{}_\mu^S \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} (1-c_H)c_\phi \\ -(1+c_H)c_\phi \\ 0 \\ 2c_H s_\phi \end{bmatrix} \overset{\circ}{Z}_{R\mu} + \frac{1}{\sqrt{1+s_\phi^2}} \begin{bmatrix} s_\phi \\ s_\phi \\ 0 \\ c_\phi \end{bmatrix} \{\overset{\circ}{A}{}_\mu^\gamma - \sqrt{2}s_\phi(\overset{\circ}{Z}_\mu + s_H \overset{\circ}{\hat{Z}}_\mu)\},$$

$$\begin{aligned}
\begin{bmatrix} \overset{\circ}{Z}_\mu(x,z) \\ \overset{\circ}{Z}{}_\mu^S(x,z) \end{bmatrix} &= \sqrt{k}\sum_{n=0}^\infty Z_\mu^{(n)}(x)\frac{1}{\sqrt{r_{Z^{(n)}}}} \begin{bmatrix} C(z;\lambda_{Z^{(n)}}) \\ \hat{S}(z;\lambda_{Z^{(n)}}) \end{bmatrix}, \\
\begin{bmatrix} \overset{\circ}{\hat{Z}}_\mu(x,z) \\ \overset{\circ}{\hat{Z}}{}_\mu^S(x,z) \end{bmatrix} &= \sqrt{k}\sum_{n=1}^\infty \hat{Z}_\mu^{(n)}(x)\frac{1}{\sqrt{r_{\hat{Z}^{(n)}}}} \begin{bmatrix} C(z;\lambda_{\hat{Z}^{(n)}}) \\ \check{S}(z;\lambda_{\hat{Z}^{(n)}}) \end{bmatrix}, \\
\overset{\circ}{Z}_{R\mu}(x,z) &= \sqrt{k}\sum_{n=1}^\infty Z_{R\mu}^{(n)}(x)\frac{1}{\sqrt{r_{Z_R^{(n)}}}} C(z;\lambda_{Z_R^{(n)}}), \\
\overset{\circ}{A}{}_\mu^\gamma(x,z) &= \sqrt{k}\sum_{n=0}^\infty A_\mu^{\gamma(n)}(x)\frac{1}{\sqrt{r_{\gamma^{(n)}}}} C(z;\lambda_{\gamma^{(n)}}), \\
r_{Z^{(n)}} &= \int_1^{z_L}\frac{dz}{z}\{[c_\phi^2 + (1+s_\phi^2)c_H^2]C(z;\lambda_{Z^{(n)}})^2 + (1+s_\phi^2)s_H^2\hat{S}(z;\lambda_{Z^{(n)}})^2\}, \\
r_{\hat{Z}^{(n)}} &= \int_1^{z_L}\frac{dz}{z}\left\{s_H^2[c_\phi^2 + (1+s_\phi^2)c_H^2]C(z;\lambda_{\hat{Z}^{(n)}})^2 + \frac{1}{1+s_\phi^2}\check{S}(z;\lambda_{\hat{Z}^{(n)}})^2\right\}, \\
r_{Z_R^{(n)}} &= \int_1^{z_L}\frac{dz}{z}[c_\phi^2 + (1+s_\phi^2)c_H^2]C(z;\lambda_{Z_R^{(n)}})^2, \\
r_{\gamma^{(n)}} &= \int_1^{z_L}\frac{dz}{z}C(z;\lambda_{Z_R^{(n)}})^2.
\end{aligned} \tag{B3}$$

APPENDIX C: QUARKS, LEPTONS, AND DARK FERMIONS

Gauge couplings of quarks, leptons, and dark fermions in the $\theta_H = \pi$ state are quite different from those in the $\theta_H = 0$ state. For a fermion field $\Psi(x,z)$ it is most convenient to express its KK expansion for $\check{\Psi}(x,z) = z^{-2}\Psi(x,z)$.

1. Quark sector

Wave functions of KK modes of up-type quarks in (6.13) are given by

$$\begin{aligned}
u^{(0)} &= \frac{\hat{u}_L^{(0)}(x)}{\sqrt{r_0}} \begin{bmatrix} \bar{c}_H C_L(z;\lambda_{u^{(0)}}, c_u) \\ i\bar{s}_H \hat{S}_L(z;\lambda_{u^{(0)}}, c_u) \end{bmatrix} + \frac{\hat{u}_R^{(0)}(x)}{\sqrt{r_0}} \begin{bmatrix} i\bar{s}_H \hat{S}_R(z;\lambda_{u^{(0)}}, c_u) \\ \bar{c}_H C_R(z;\lambda_{u^{(0)}} c_u) \end{bmatrix}, \\
u^{(n)} &= \frac{\hat{u}_L^{(n)}(x)}{\sqrt{r_n}} \begin{bmatrix} \bar{c}_H C_L(z;\lambda_{u^{(n)}}, c_u) \\ i\bar{s}_H \hat{S}_L(z;\lambda_{u^{(n)}}, c_u) \end{bmatrix} + \frac{\hat{u}_R^{(n)}(x)}{\sqrt{r_n}} \begin{bmatrix} \bar{c}_H S_R(z;\lambda_{u^{(n)}}, c_u) \\ i\bar{s}_H \hat{C}_R(z;\lambda_{u^{(n)}}, c_u) \end{bmatrix}, \\
u'^{(n)} &= \frac{\hat{u}_L'^{(n)}(x)}{\sqrt{r_n}} \begin{bmatrix} i\bar{s}_H \hat{C}_L(z;\lambda_{u'^{(n)}}, c_u) \\ \bar{c}_H S_L(z;\lambda_{u'^{(n)}}, c_u) \end{bmatrix} + \frac{\hat{u}_R'^{(n)}(x)}{\sqrt{r_n}} \begin{bmatrix} i\bar{s}_H \hat{S}_R(z;\lambda_{u'^{(n)}}, c_u) \\ \bar{c}_H C_R(z;\lambda_{u'^{(n)}}, c_u) \end{bmatrix},
\end{aligned} \tag{C1}$$

where

$$\begin{aligned}
&\bar{c}_H = \cos\frac{1}{2}\theta_H, \qquad \bar{s}_H = \sin\frac{1}{2}\theta_H, \\
&\hat{S}_L(z;\lambda,c) = N_L(\lambda,c)S_L(z;\lambda,c), \qquad \hat{C}_R(z;\lambda,c) = N_L(\lambda,c)C_R(z;\lambda,c), \\
&\hat{S}_R(z;\lambda,c) = N_R(\lambda,c)S_R(z;\lambda,c), \qquad \hat{C}_L(z;\lambda,c) = N_R(\lambda,c)C_L(z;\lambda,c), \\
&N_L(\lambda,c) = \frac{C_L(1;\lambda,c)}{S_L(1;\lambda,c)}, \qquad N_R(\lambda,c) = \frac{C_R(1;\lambda,c)}{S_R(1;\lambda,c)},
\end{aligned} \tag{C2}$$

and a normalization factor r_n should be understood in each term as

$$r_n = \int_1^{z_L} dz\{|f_n(z)|^2 + |g_n(z)|^2\} \quad \text{in} \ \frac{1}{\sqrt{r_n}}\begin{bmatrix} f_n(z) \\ g_n(z) \end{bmatrix}. \tag{C3}$$

Here $\hat{u}_L^{(n)}(x)$ and $\hat{u}_L^{\prime(n)}(x)$ [$\hat{u}_R^{(n)}(x)$ and $\hat{u}_R^{\prime(n)}(x)$] are the left-handed (right-handed) components of $4D$ fields $\hat{u}^{(n)}(x)$ and $\hat{u}^{\prime(n)}(x)$, respectively.

By making use of (6.14) the expansion (C1) can be written as

$$\begin{aligned}
u^{(0)} &= \frac{\hat{u}_L^{(0)}(x)}{\sqrt{r_0}}\begin{bmatrix} \bar{s}_H C_L(z;\lambda_{u^{(0)}},c_u) \\ -i\bar{c}_H \check{S}_L(z;\lambda_{u^{(0)}},c_u) \end{bmatrix} + \frac{\hat{u}_R^{(0)}(x)}{\sqrt{r_0}}\begin{bmatrix} i\bar{s}_H S_R(z;\lambda_{u^{(0)}},c_u) \\ \bar{c}_H \check{C}_R(z;\lambda_{u^{(0)}} c_u) \end{bmatrix}, \\
u^{(n)} &= \frac{\hat{u}_L^{(n)}(x)}{\sqrt{r_n}}\begin{bmatrix} \bar{s}_H C_L(z;\lambda_{u^{(n)}},c_u) \\ -i\bar{c}_H \check{S}_L(z;\lambda_{u^{(n)}},c_u) \end{bmatrix} + \frac{\hat{u}_R^{(n)}(x)}{\sqrt{r_n}}\begin{bmatrix} \bar{s}_H S_R(z;\lambda_{u^{(n)}},c_u) \\ -i\bar{c}_H \check{C}_R(z;\lambda_{u^{(n)}},c_u) \end{bmatrix}, \\
u^{\prime(n)} &= \frac{\hat{u}_L^{\prime(n)}(x)}{\sqrt{r_n}}\begin{bmatrix} -i\bar{c}_H \check{C}_L(z;\lambda_{u^{\prime(n)}},c_u) \\ \bar{s}_H S_L(z;\lambda_{u^{\prime(n)}},c_u) \end{bmatrix} + \frac{\hat{u}_R^{\prime(n)}(x)}{\sqrt{r_n}}\begin{bmatrix} -i\bar{c}_H \check{S}_R(z;\lambda_{u^{\prime(n)}},c_u) \\ \bar{s}_H C_R(z;\lambda_{u^{\prime(n)}},c_u) \end{bmatrix},
\end{aligned} \tag{C4}$$

where

$$\begin{aligned}
&\check{S}_L(z;\lambda,c) = N_R(\lambda,c)^{-1}S_L(z;\lambda,c), \qquad \check{C}_R(z;\lambda,c) = N_R(\lambda,c)^{-1}C_R(z;\lambda,c), \\
&\check{S}_R(z;\lambda,c) = N_L(\lambda,c)^{-1}S_R(z;\lambda,c), \qquad \check{C}_L(z;\lambda,c) = N_L(\lambda,c)^{-1}C_L(z;\lambda,c).
\end{aligned} \tag{C5}$$

The expression in (C4) is more suitable at $\theta_H = \pi$ than that in (C1). For the u tower, for instance, $C_L(1;\lambda_{u^{(n)}},c_u) = 0$ so that $\hat{S}_L(z;\lambda_{u^{(n)}},c_u)$ and $\hat{C}_L(z;\lambda_{u^{(n)}},c_u)$ also vanish there.

The spectrum alternates as $\lambda_{u^{(0)}} < \lambda_{u^{\prime(1)}} < \lambda_{u^{(1)}} < \lambda_{u^{\prime(2)}} < \lambda_{u^{(2)}} < \cdots$. Wave functions are given, up to normalization constants, by

$$\begin{aligned}
\text{for } \theta_H = 0,\ \hat{u}_L^{(0)}&: \begin{bmatrix} C_L(z;\lambda_{u^{(0)}},c_u) \\ 0 \end{bmatrix}, \qquad \hat{u}_R^{(0)}: \begin{bmatrix} 0 \\ C_R(z;\lambda_{u^{(0)}},c_u) \end{bmatrix}, \\
\hat{u}_L^{(n)}&: \begin{bmatrix} C_L(z;\lambda_{u^{(n)}},c_u) \\ 0 \end{bmatrix}, \qquad \hat{u}_R^{(n)}: \begin{bmatrix} S_R(z;\lambda_{u^{(n)}},c_u) \\ 0 \end{bmatrix}, \qquad (n \geq 1) \\
\hat{u}_L^{\prime(n)}&: \begin{bmatrix} 0 \\ S_L(z;\lambda_{u^{\prime(n)}},c_u) \end{bmatrix}, \qquad \hat{u}_R^{\prime(n)}: \begin{bmatrix} 0 \\ C_R(z;\lambda_{u^{\prime(n)}},c_u) \end{bmatrix}, \qquad (n \geq 1)
\end{aligned} \tag{C6}$$

and

$$
\begin{aligned}
&\text{for } \theta_H = \pi, \qquad \hat{u}_L^{(0)}: \begin{bmatrix} C_L(z;\lambda_{u^{(0)}}, c_u) \\ 0 \end{bmatrix}, \qquad \hat{u}_R^{(0)}: \begin{bmatrix} iS_R(z;\lambda_{u^{(0)}}, c_u) \\ 0 \end{bmatrix}, \\
&\hat{u}_L^{(n)}: \begin{bmatrix} C_L(z;\lambda_{u^{(n)}}, c_u) \\ 0 \end{bmatrix}, \qquad \hat{u}_R^{(n)}: \begin{bmatrix} S_R(z;\lambda_{u^{(n)}}, c_u) \\ 0 \end{bmatrix}, \qquad (n \geq 1) \\
&\hat{u}_L^{\prime(n)}: \begin{bmatrix} 0 \\ S_L(z;\lambda_{u^{\prime(n)}}, c_u) \end{bmatrix}, \qquad \hat{u}_R^{\prime(n)}: \begin{bmatrix} 0 \\ C_R(z;\lambda_{u^{\prime(n)}}, c_u) \end{bmatrix}, \qquad (n \geq 1).
\end{aligned} \tag{C7}
$$

Wave functions of KK modes of the four KK towers, $\mathbf{d} = (d, d', D^+, D^-)$, in (6.16) are given by

$$
\mathbf{d}^{(n)} = \hat{\mathbf{d}}_L^{(n)}(x) \begin{pmatrix} \alpha_d C_L(z;\lambda_{\mathbf{d}^{(n)}}) \\ \alpha_{d'} S_L(z;\lambda_{\mathbf{d}^{(n)}}) \\ a_d \mathcal{C}_{L2}(z;\lambda_{\mathbf{d}^{(n)}}) + b_d \mathcal{C}_{L1}(z;\lambda_{\mathbf{d}^{(n)}}) \\ a_d \mathcal{S}_{L1}(z;\lambda_{\mathbf{d}^{(n)}}) + b_d \mathcal{S}_{L2}(z;\lambda_{\mathbf{d}^{(n)}}) \end{pmatrix} + \hat{\mathbf{d}}_R^{(n)}(x) \begin{pmatrix} \alpha_d S_R(z;\lambda_{\mathbf{d}^{(n)}}) \\ \alpha_{d'} C_R(z;\lambda_{\mathbf{d}^{(n)}}) \\ a_d \mathcal{S}_{R2}(z;\lambda_{\mathbf{d}^{(n)}}) + b_d \mathcal{S}_{R1}(z;\lambda_{\mathbf{d}^{(n)}}) \\ a_d \mathcal{C}_{R1}(z;\lambda_{\mathbf{d}^{(n)}}) + b_d \mathcal{C}_{R2}(z;\lambda_{\mathbf{d}^{(n)}}) \end{pmatrix}, \tag{C8}
$$

where $C_L(z;\lambda_{\mathbf{d}^{(n)}}) = C_L(z;\lambda_{\mathbf{d}^{(n)}}, c_u)$, $\mathcal{C}_{Lj}(z;\lambda_{\mathbf{d}^{(n)}}) = \mathcal{C}_{Lj}(z;\lambda_{\mathbf{d}^{(n)}}, c_{D_d}, \tilde{m}_{D_d})$, and so on. Coefficients $(\alpha_d, \alpha_{d'}, a_d, b_d)$ in each term satisfy

$$
\begin{pmatrix} \bar{c}_H S_R^Q & -i\bar{s}_H C_R^Q & 0 & 0 \\ -i\bar{s}_H C_L^Q & \bar{c}_H S_L^Q & \mu_1 \mathcal{C}_{L2}^D & \mu_1 \mathcal{C}_{L1}^D \\ -i\mu_1^* \bar{s}_H S_R^Q & \mu_1^* \bar{c}_H C_R^Q & -\mathcal{S}_{R2}^D & -\mathcal{S}_{R1}^D \\ 0 & 0 & \mathcal{S}_{L1}^D & \mathcal{S}_{L2}^D \end{pmatrix} \begin{pmatrix} \alpha_d \\ \alpha_{d'} \\ a_d \\ b_d \end{pmatrix} = 0, \tag{C9}
$$

and an appropriate normalization condition, where $S_R^Q = S_R(1;\lambda)$, $\mathcal{S}_{Lj}^D = \mathcal{S}_{Lj}(1;\lambda)$, etc.

In the present paper it suffices to know the wave functions in the KK expansion for $\theta_H = 0$ and π. For $\theta_H = 0$ the condition matrix in (C9) becomes block diagonal, and (6.15) and (C9) become

$$
\begin{aligned}
&S_R^Q K^0 = 0, \qquad K^0 = S_L^Q(\mathcal{S}_{L1}^D \mathcal{S}_{R1}^D - \mathcal{S}_{L2}^D \mathcal{S}_{R2}^D) + |\mu_1|^2 C_R^Q (\mathcal{S}_{L1}^D \mathcal{C}_{L1}^D - \mathcal{S}_{L2}^D \mathcal{C}_{L2}^D), \\
&S_R^Q \alpha_d = 0, \begin{pmatrix} S_L^Q & \mu_1 \mathcal{C}_{L2}^D & \mu_1 \mathcal{C}_{L1}^D \\ \mu_1^* C_R^Q & -\mathcal{S}_{R2}^D & -\mathcal{S}_{R1}^D \\ 0 & \mathcal{S}_{L1}^D & \mathcal{S}_{L2}^D \end{pmatrix} \begin{pmatrix} \alpha_{d'} \\ a_d \\ b_d \end{pmatrix} = 0.
\end{aligned} \tag{C10}
$$

The KK tower specified by $S_R^Q = 0$, $\{\lambda_{d^{(n)}}\}$, contains a massless mode $\lambda_{d^{(0)}} = 0$, which allows $\alpha_d \neq 0$ and a nontrivial left-handed mode. $\lambda_{d^{(0)}} = 0$ implies $K^0 = 0$ as well, and a nontrivial right-handed mode is contained in the (d', D^+, D^-) components. The spectrum determined by $S_R^Q \neq 0$ and $K^0 = 0$ consists of three KK towers. All modes are massive and $\alpha_d = 0$. Their wave functions are contained in the (d', D^+, D^-) components. One finds that

$\underline{\theta_H = 0;}$

$$d^{(0)} = \hat{d}_L^{(0)}(x)\begin{pmatrix} \alpha_d C_L(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \hat{d}_R^{(0)}(x)\begin{pmatrix} 0 \\ \bar{\alpha}_{d'} C_R(z) \\ \bar{a}_d \mathcal{S}_{R2}(z) \\ \bar{a}_d \mathcal{C}_{R1}(z) \end{pmatrix}, \qquad \bar{a}_d = \mu_1^* \frac{C_R^Q}{\mathcal{S}_{R2}^D} \bar{\alpha}_{d'},$$

$$d^{(n)} = \hat{d}_L^{(n)}(x)\begin{pmatrix} \alpha_d C_L(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \hat{d}_R^{(n)}(x)\begin{pmatrix} \alpha_d S_R(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (n \ge 1),$$

for $\mathbf{d} = d', D^+, D^-$

$$\mathbf{d}^{(n)} = \hat{\mathbf{d}}_L^{(n)}(x)\begin{pmatrix} 0 \\ \alpha_{d'} S_L(z) \\ a_d \mathcal{C}_{L2}(z) + b_d \mathcal{C}_{L1}(z) \\ a_d \mathcal{S}_{L1}(z) + b_d \mathcal{S}_{L2}(z) \end{pmatrix} + \hat{\mathbf{d}}_R^{(n)}(x)\begin{pmatrix} 0 \\ \alpha_{d'} C_R(z) \\ a_d \mathcal{S}_{R2}(z) + b_d \mathcal{S}_{R1}(z) \\ a_d \mathcal{C}_{R1}(z) + b_d \mathcal{C}_{R2}(z) \end{pmatrix}. \tag{C11}$$

For brevity $\lambda_{\mathbf{d}^{(n)}}$ in $C_L(z; \lambda_{\mathbf{d}^{(n)}})$, etc. has been suppressed in the above formulas.

For $\theta_H = \pi$ the condition matrix in (C9) becomes, in place of (C10),

$$C_R^Q K^\pi = 0, \qquad K^\pi = C_L^Q(\mathcal{S}_{L1}^D \mathcal{S}_{R1}^D - \mathcal{S}_{L2}^D \mathcal{S}_{R2}^D) + |\mu_1|^2 S_R^Q (\mathcal{S}_{L1}^D \mathcal{C}_{L1}^D - \mathcal{S}_{L2}^D \mathcal{C}_{L2}^D),$$

$$C_R^Q \alpha_{d'} = 0, \qquad \begin{pmatrix} -iC_L^Q & \mu_1 \mathcal{C}_{L2}^D & \mu_1 \mathcal{C}_{L1}^D \\ -i\mu_1^* S_R^Q & -\mathcal{S}_{R2}^D & -\mathcal{S}_{R1}^D \\ 0 & \mathcal{S}_{L1}^D & \mathcal{S}_{L2}^D \end{pmatrix} \begin{pmatrix} \alpha_d \\ a_d \\ b_d \end{pmatrix} = 0. \tag{C12}$$

There is no massless mode. The lowest mode $d^{(0)}$ is contained in one of the three KK towers determined by the conditions $C_R^Q \neq 0$ and $K^\pi = 0$, for which $\alpha_{d'} = 0$. (Recall that when $\mu_1 = 0$, the lowest mode satisfies $C_L^Q = 0$ just as in the up-type quark spectrum.) The spectrum of d' tower is determined by $C_R^Q = 0$ for which $K^\pi \neq 0$ and $\alpha_d = a_d = b_d = 0$. One finds that

$\underline{\theta_H = \pi;}$

$$d'^{(n)} = \hat{d}_L'^{(n)}(x)\begin{pmatrix} 0 \\ \alpha_{d'} S_L(z) \\ 0 \\ 0 \end{pmatrix} + \hat{d}_R'^{(n)}(x)\begin{pmatrix} 0 \\ \alpha_{d'} C_R(z) \\ 0 \\ 0 \end{pmatrix},$$

for $\mathbf{d} = d, D^+, D^-$

$$\mathbf{d}^{(n)} = \hat{\mathbf{d}}_L^{(n)}(x)\begin{pmatrix} \alpha_d C_L(z) \\ 0 \\ a_d \mathcal{C}_{L2}(z) + b_d \mathcal{C}_{L1}(z) \\ a_d \mathcal{S}_{L1}(z) + b_d \mathcal{S}_{L2}(z) \end{pmatrix} + \hat{\mathbf{d}}_R^{(n)}(x)\begin{pmatrix} \alpha_d S_R(z) \\ 0 \\ a_d \mathcal{S}_{R2}(z) + b_d \mathcal{S}_{R1}(z) \\ a_d \mathcal{C}_{R1}(z) + b_d \mathcal{C}_{R2}(z) \end{pmatrix}. \tag{C13}$$

It is easy to find W and Z couplings of quarks. At $\theta_H = 0$ the spectrum of both u and d towers is determined by $S_R(1; \lambda_n, c_u) = 0$ so that $\lambda_{u^{(n)}} = \lambda_{d^{(n)}}$. Couplings with $W_\mu^{(0)}$, $Z_\mu^{(0)}$, and $A_\mu^{\gamma(0)}$ are obtained by inserting (C6) and (C11) into

$$\int_1^{z_L} dz \Bigg[(\bar{\check{u}}, \bar{\check{d}})\gamma^\mu \bigg\{ \frac{g_w}{\sqrt{2}} W_\mu^{(0)} (T^{1_L} + iT^{2_L}) + \frac{g_w}{\sqrt{2}} W_\mu^{(0)\dagger} (T^{1_L} - iT^{2_L}) + \frac{g_w}{\cos\theta_W^0} Z_\mu^{(0)} (T^{3_L} - \sin^2\theta_W^0 Q_{\rm EM}) + eA_\mu^{\gamma(0)} Q_{\rm EM} \bigg\} \begin{pmatrix} \check{u} \\ \check{d} \end{pmatrix} + \bigg\{ -g_w \frac{\sin^2\theta_W^0}{\cos\theta_W^0} Z_\mu^{(0)} + eA_\mu^{\gamma(0)} \bigg\} (\bar{\check{u}}\,', \bar{\check{d}}\,', \bar{\check{D}}\,^+, \bar{\check{D}}\,^-)\gamma^\mu Q_{\rm EM} \begin{pmatrix} \check{u}\,' \\ \check{d}\,' \\ \check{D}^+ \\ \check{D}^- \end{pmatrix} \Bigg], \tag{C14}$$

with the fact that wave functions of gauge bosons are constant. It leads to the expression in (6.17) for $\theta_H = 0$. The couplings of the zero modes in (C14) are the same as in the SM with θ_W^0 replaced by θ_W.

At $\theta_H = \pi$, $SU(2)_R$ doublet components become relevant. Notice that the spectrum of both u' and d' towers is determined by $C_R(1, \lambda_n, c_u) = 0$ and $\lambda_{u'^{(n)}} = \lambda_{d'^{(n)}}$. Further $SU(2)_R$ components of the wave functions of $\hat{u}^{(n)}$ and $\hat{d}^{(n)}$ vanish. It follows from (6.11) gauge couplings are obtained by inserting (C7) and (C13) into

$$\int_1^{z_L} dz \Bigg[(\bar{\tilde{\check{u}}}\,', \bar{\tilde{\check{d}}}\,')\gamma^\mu \bigg\{ \frac{g_w}{\sqrt{2}} W_\mu^{(0)} (T^{1_R} + iT^{2_R}) + \frac{g_w}{\sqrt{2}} W_\mu^{(0)\dagger} (T^{1_R} - iT^{2_R}) + \frac{g_w}{\cos\theta_W^0} Z_\mu^{(0)} (T^{3_R} - \sin^2\theta_W^0 Q_{\rm EM}) + eA_\mu^{\gamma(0)} Q_{\rm EM} \bigg\} \begin{pmatrix} \tilde{\check{u}}\,' \\ \tilde{\check{d}}\,' \end{pmatrix} + \bigg\{ -g_w \frac{\sin^2\theta_W^0}{\cos\theta_W^0} Z_\mu^{(0)} + eA_\mu^{\gamma(0)} \bigg\} (\bar{\tilde{\check{u}}}, \bar{\tilde{\check{d}}}, \bar{\tilde{\check{D}}}\,^+, \bar{\tilde{\check{D}}}\,^-)\gamma^\mu Q_{\rm EM} \begin{pmatrix} \tilde{\check{u}} \\ \tilde{\check{d}} \\ \tilde{\check{D}}^+ \\ \tilde{\check{D}}^- \end{pmatrix} \Bigg]. \tag{C15}$$

It leads to the expression in (6.17) for $\theta_H = \pi$.

2. Lepton sector

Charged lepton towers have the same form of KK expansions as up-type quark towers. For the first generation

$$\begin{bmatrix} \tilde{\check{e}} \\ \tilde{\check{e}}\,' \end{bmatrix} = \sqrt{k} \bigg\{ e^{(0)} + \sum_{n=1}^\infty e^{(n)} + \sum_{n=1}^\infty e'^{(n)} \bigg\}. \tag{C16}$$

The spectrum is determined by the same formula as (6.14) where c_u is replaced by c_e. The expansions have the same form as (C1), (C4), (C6), and (C7) where the replacement $(u, u') \to (e, e')$ should be made.

In the neutrino sector brane fermion χ satisfying the Majorana condition couples to ν and ν' through brane interactions. In the two-component basis

$$\begin{bmatrix} \tilde{\check{\nu}}_L \\ \tilde{\check{\nu}}_L{}' \\ \eta \end{bmatrix} = \sqrt{k} \bigg\{ \sum_{n=0}^\infty (\nu_{+L}^{(n)} + \nu_{-L}^{(n)}) + \sum_{n=1}^\infty (\nu_{+L}'^{(n)} + \nu_{-L}'^{(n)}) \bigg\},$$

$$\begin{bmatrix} \tilde{\check{\nu}}_R \\ \tilde{\check{\nu}}_R{}' \\ \eta^c \end{bmatrix} = \sqrt{k} \bigg\{ \sum_{n=0}^\infty (\nu_{+R}^{(n)} + \nu_{-R}^{(n)}) + \sum_{n=1}^\infty (\nu_{+R}'^{(n)} + \nu_{-R}'^{(n)}) \bigg\}, \tag{C17}$$

where $\xi^c \equiv e^{i\delta_C} \sigma^2 \xi^*$ and $\chi = (\eta^c, \eta)$. The spectrum is determined by

$$(k\lambda \mp M) \bigg\{ S_L^L S_R^L + \sin^2 \frac{\theta_H}{2} \bigg\} + \frac{m_B^2}{k} S_R^L C_R^L = 0 \tag{C18}$$

for $\nu_\pm$ and $\nu'_\pm$ fields. Here $S_R^L = S_R(1, \lambda_{\nu^{(n)}}, c_e)$, etc., M is a Majorana mass for χ, and m_B comes from a brane interaction among ν', χ, and Φ_S. For $\theta_H \neq 0$ a tiny neutrino mass is generated by gauge-Higgs seesaw mechanism [36] similar to the inverse seesaw mechanism [37];

$m_\nu \sim m_e^2 M/(2|c_e|-1)m_B^2$ for $c_e < -\frac{1}{2}$. Moderate values $M \sim 5$ GeV, $m_B \sim 1$ TeV yield $m_\nu \sim 1$ meV.

Wave functions of each mode in the expansion (C17) in the neutrino sector are given, with $\boldsymbol{\nu} = (\nu, \nu')$,

$$\boldsymbol{\nu}_{\pm L}^{(n)} = \hat{\boldsymbol{\nu}}_{\pm L}^{(n)}(x) \begin{pmatrix} \alpha_\nu C_L(z, \lambda_{\boldsymbol{\nu}^{(n)}}, c_e) \\ i\alpha_{\nu'} S_L(z, \lambda_{\boldsymbol{\nu}^{(n)}}, c_e) \\ i\alpha_\eta/\sqrt{k} \end{pmatrix},$$
$$\boldsymbol{\nu}_{\pm R}^{(n)} = \hat{\boldsymbol{\nu}}_{\pm R}^{(n)}(x) \begin{pmatrix} \alpha_\nu S_R(z, \lambda_{\boldsymbol{\nu}^{(n)}}, c_e) \\ i\alpha_{\nu'} C_R(z, \lambda_{\boldsymbol{\nu}^{(n)}}, c_e) \\ \mp i\alpha_\eta^*/\sqrt{k} \end{pmatrix},$$
$$\hat{\boldsymbol{\nu}}_{\pm L}^{(n)}(x)^c = \pm \hat{\boldsymbol{\nu}}_{\pm R}^{(n)}(x), \tag{C19}$$

where $\xi^c \equiv e^{i\delta_C}\sigma^2\xi^*$. Coefficients $(\alpha_\nu, \alpha_{\nu'}, \alpha_\eta)$ in each term satisfy

$$\begin{pmatrix} \bar{c}_H S_R^L & \bar{s}_H C_R^L & 0 \\ -\bar{s}_H C_L^L & \bar{c}_H S_L^L & m_B/k \\ m_B \bar{s}_H S_R^L & -m_B \bar{c}_H C_R^L & k\lambda \mp M \end{pmatrix} \begin{pmatrix} \alpha_\nu \\ \alpha_{\nu'} \\ \alpha_\eta \end{pmatrix} = 0, \tag{C20}$$

which leads to the spectrum-determining equation (C18).

For $\theta_H = 0$ the condition (C18) reduces to $S_R^L\{(k\lambda \mp M)S_L^L + k^{-1}m_B^2 C_R^L\} = 0$. $\hat{\nu}$ tower with the spectrum determined by $S_R^L = 0$ contains a massless left-handed neutrino ($\lambda_{\nu^{(0)}} = 0$).

$\underline{\theta_H = 0}$;

$$\nu_{+L}^{(0)} = \hat{\nu}_{+L}^{(0)}(x) \begin{pmatrix} \alpha_\nu C_L(z) \\ 0 \\ 0 \end{pmatrix}, \qquad \nu_{+R}^{(0)} = 0,$$
$$\nu_{\pm L}^{(n)} = \hat{\nu}_{\pm L}^{(n)}(x) \begin{pmatrix} \alpha_\nu C_L(z) \\ 0 \\ 0 \end{pmatrix}, \qquad \nu_{\pm R}^{(n)} = \hat{\nu}_{\pm R}^{(n)}(x) \begin{pmatrix} \alpha_\nu S_R(z) \\ 0 \\ 0 \end{pmatrix} \quad (n \geq 1),$$
$$\nu_{\pm L}^{\prime(n)} = \hat{\nu}_{\pm L}^{\prime(n)}(x) \begin{pmatrix} 0 \\ i\alpha_{\nu'} S_L(z) \\ i\alpha_\eta/\sqrt{k} \end{pmatrix}, \qquad \nu_{\pm R}^{\prime(n)} = \hat{\nu}_{\pm R}^{\prime(n)}(x) \begin{pmatrix} 0 \\ i\alpha_{\nu'} C_R(z) \\ \mp i\alpha_\eta^*/\sqrt{k} \end{pmatrix}, \qquad S_L^L \alpha_{\nu'} + m_B k^{-1}\alpha_\eta = 0, \qquad (n \geq 1). \tag{C21}$$

For brevity $\lambda_{\boldsymbol{\nu}^{(n)}}$ and c_e in $C_L(z, \lambda_{\nu^{(n)}}, c_e)$, etc., have been suppressed. There is no $\nu_-^{(0)}$ mode. For $\theta_H = \pi$ the condition (C18) reduces to $C_R^L\{(k\lambda \mp M)C_L^L + k^{-1}m_B^2 S_R^L\} = 0$. There is no massless mode. The spectrum of ν' tower is given by $C_R^L = 0$.

$\underline{\theta_H = \pi}$;

$$\nu_{\pm L}^{(n)} = \hat{\nu}_{\pm L}^{(n)}(x) \begin{pmatrix} \alpha_\nu C_L(z) \\ 0 \\ i\alpha_\eta/\sqrt{k} \end{pmatrix}, \qquad \nu_{\pm R}^{(n)} = \hat{\nu}_{\pm R}^{(n)}(x) \begin{pmatrix} \alpha_\nu S_R(z) \\ 0 \\ \mp i\alpha_\eta^*/\sqrt{k} \end{pmatrix},$$
$$-C_L^L \alpha_\nu + m_B k^{-1}\alpha_\eta = 0,$$
$$\nu_{\pm L}^{\prime(n)} = \hat{\nu}_{\pm L}^{\prime(n)}(x) \begin{pmatrix} 0 \\ i\alpha_{\nu'} S_L(z) \\ 0 \end{pmatrix}, \qquad \nu_{\pm R}^{\prime(n)} = \hat{\nu}_{\pm R}^{\prime(n)}(x) \begin{pmatrix} 0 \\ i\alpha_{\nu'} C_R(z) \\ 0 \end{pmatrix}, \tag{C22}$$

where $n \geq 0$ for ν_+ and $n \geq 1$ for others.

We note that $\lambda_{\nu_+^{(n)}} = \lambda_{\nu_-^{(n)}}$ $(n \geq 1)$ for $\theta_H = 0$, and that $\lambda_{\nu_+^{\prime(n)}} = \lambda_{\nu_-^{\prime(n)}}$ $(n \geq 1)$ for $\theta_H = \pi$. Couplings with $W_\mu^{(0)}$, $Z_\mu^{(0)}$ and $A_\mu^{\gamma(0)}$ are given by for $\theta_H = 0$;

$$\frac{g_w}{\sqrt{2}}\left\{W_\mu^{(0)}\left(\bar{\hat{\nu}}_{eL}^{(0)}\gamma^\mu\hat{e}_L^{(0)} + \sum_{n=1}^\infty \bar{\hat{\nu}}_e^{(n)}\gamma^\mu\hat{e}^{(n)}\right) + W_\mu^{(0)\dagger}\left(\bar{\hat{e}}_L^{(0)}\gamma^\mu\hat{\nu}_{eL}^{(0)} + \sum_{n=1}^\infty \bar{\hat{e}}^{(n)}\gamma^\mu\hat{\nu}_e^{(n)}\right)\right\}$$
$$+ \frac{g_w}{2\cos\theta_W^0}Z_\mu^{(0)}\left\{(\bar{\hat{\nu}}_{eL}^{(0)}\gamma^\mu\hat{\nu}_{eL}^{(0)} - \bar{\hat{e}}_L^{(0)}\gamma^\mu\hat{e}_L^{(0)}) + \sum_{n=1}^\infty(\bar{\hat{\nu}}_e^{(n)}\gamma^\mu\hat{\nu}_e^{(n)} - \bar{\hat{e}}^{(n)}\gamma^\mu\hat{e}^{(n)})\right\} + \left[-g_w\frac{\sin^2\theta_W^0}{\cos\theta_W^0}Z_\mu^{(0)} + eA_\mu^{\gamma(0)}\right]J_{\rm EM}^\mu,$$

for $\theta_H = \pi$;

$$\frac{g_w}{\sqrt{2}}\left\{W_\mu^{(0)}\sum_{n=1}^\infty \bar{\hat{\nu}}_e^{\prime(n)}\gamma^\mu\hat{e}^{\prime(n)} + W_\mu^{(0)\dagger}\sum_{n=1}^\infty \bar{\hat{e}}^{\prime(n)}\gamma^\mu\hat{\nu}_e^{\prime(n)}\right\} + \frac{g_w}{2\cos\theta_W^0}Z_\mu^{(0)}\sum_{n=1}^\infty(\bar{\hat{\nu}}_e^{\prime(n)}\gamma^\mu\hat{\nu}_e^{\prime(n)} - \bar{\hat{e}}^{\prime(n)}\gamma^\mu\hat{e}^{\prime(n)})$$
$$+ \left[-g_w\frac{\sin^2\theta_W^0}{\cos\theta_W^0}Z_\mu^{(0)} + eA_\mu^{\gamma(0)}\right]J_{\rm EM}^\mu,$$

$$J_{\rm EM}^\mu = -\sum_{n=0}^\infty \bar{\hat{e}}^{(n)}\gamma^\mu\hat{e}^{(n)} - \sum_{n=1}^\infty \bar{\hat{e}}^{\prime(n)}\gamma^\mu\hat{e}^{\prime(n)}. \tag{C23}$$

3. Dark fermion (darkF $\Psi_F^\beta, \Psi_{F_e}^\beta$) sector

Wave functions of each mode in the KK expansion of darkF fermions with $c_F > 0$, (6.21), are given by

$$F_j^{(n)} = \frac{\hat{F}_{jL}^{(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}\bar{c}_H C_L(z;\lambda_{F^{(n)}},c_F)\\ i\bar{s}_H\check{S}_L(z;\lambda_{F^{(n)}},c_F)\end{bmatrix} + \frac{\hat{F}_{jR}^{(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}\bar{c}_H S_R(z;\lambda_{F^{(n)}},c_F)\\ i\bar{s}_H\check{C}_R(z;\lambda_{F^{(n)}},c_F)\end{bmatrix},$$
$$F_j^{\prime(n)} = \frac{\hat{F}_{jL}^{\prime(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}i\bar{s}_H\check{C}_L(z;\lambda_{F^{\prime(n)}},c_F)\\ \bar{c}_H S_L(z;\lambda_{F^{\prime(n)}},c_F)\end{bmatrix} + \frac{\hat{F}_{jR}^{\prime(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}i\bar{s}_H\check{S}_R(z;\lambda_{F^{\prime(n)}},c_F)\\ \bar{c}_H C_R(z;\lambda_{F^{\prime(n)}},c_F)\end{bmatrix}. \tag{C24}$$

Making use of (6.20), one can write the wave functions in (C24) as

$$F_j^{(n)} = \frac{\hat{F}_{jL}^{(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}\bar{s}_H C_L(z;\lambda_{F^{(n)}},c_F)\\ -i\bar{c}_H\hat{S}_L(z;\lambda_{F^{(n)}},c_F)\end{bmatrix} + \frac{\hat{F}_{jR}^{(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}\bar{s}_H S_R(z;\lambda_{F^{(n)}},c_F)\\ -i\bar{c}_H\hat{C}_R(z;\lambda_{F^{(n)}},c_F)\end{bmatrix},$$
$$F_j^{\prime(0)} = \frac{\hat{F}_{jL}^{\prime(0)}(x)}{\sqrt{r_0}}\begin{bmatrix}i\bar{s}_H C_L(z;\lambda_{F^{\prime(0)}},c_F)\\ \bar{c}_H\hat{S}_L(z;\lambda_{F^{\prime(0)}},c_F)\end{bmatrix} + \frac{\hat{F}_{jR}^{\prime(0)}(x)}{\sqrt{r_0}}\begin{bmatrix}-i\bar{c}_H\hat{S}_R(z;\lambda_{F^{\prime(0)}},c_F)\\ \bar{s}_H C_R(z;\lambda_{F^{\prime(0)}},c_F)\end{bmatrix},$$
$$F_j^{\prime(n)} = \frac{\hat{F}_{jL}^{\prime(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}-i\bar{c}_H\hat{C}_L(z;\lambda_{F^{\prime(n)}},c_F)\\ \bar{s}_H S_L(z;\lambda_{F^{\prime(n)}},c_F)\end{bmatrix} + \frac{\hat{F}_{jR}^{\prime(n)}(x)}{\sqrt{r_n}}\begin{bmatrix}-i\bar{c}_H\hat{S}_R(z;\lambda_{F^{\prime(n)}},c_F)\\ \bar{s}_H C_R(z;\lambda_{F^{\prime(n)}},c_F)\end{bmatrix}, \tag{C25}$$

where $n \geq 1$. The expression (C25) is appropriate to use at $\theta_H = \pi$. In particular, notice that $\lambda_{F^{\prime(0)}} = 0$, $N_L(\lambda_{F^{\prime(0)}},c_F)^{-1} = 0$, and $\hat{S}_L(z;\lambda_{F^{\prime(0)}},c_F)$ is finite there.

The spectrum alternates as $\lambda_{F^{\prime(0)}} < \lambda_{F^{(1)}} < \lambda_{F^{\prime(1)}} < \lambda_{F^{(2)}} < \lambda_{F^{\prime(2)}} < \cdots$. Wave functions are given, up to normalization constants, by

$$\text{for } \theta_H = 0, \quad \hat{F}_{jL}^{(n)}: \begin{bmatrix}C_L(z;\lambda_{F^{(n)}},c_F)\\ 0\end{bmatrix}, \quad \hat{F}_{jR}^{(n)}: \begin{bmatrix}S_R(z;\lambda_{F^{(n)}},c_F)\\ 0\end{bmatrix}, \quad (n \geq 1)$$
$$\hat{F}_{jL}^{\prime(n)}: \begin{bmatrix}0\\ S_L(z;\lambda_{F^{\prime(n)}},c_F)\end{bmatrix}, \quad \hat{F}_{jR}^{\prime(n)}: \begin{bmatrix}0\\ C_R(z;\lambda_{F^{\prime(n)}},c_F)\end{bmatrix}, \quad (n \geq 0) \tag{C26}$$

and

$$\text{for } \theta_H = \pi, \quad \hat{F}_{jL}^{(n)}: \begin{bmatrix} C_L(z;\lambda_{F^{(n)}},c_F) \\ 0 \end{bmatrix}, \quad \hat{F}_{jR}^{(n)}: \begin{bmatrix} S_R(z;\lambda_{F^{(n)}},c_F) \\ 0 \end{bmatrix}, \quad (n \geq 1)$$
$$\hat{F}_{jL}^{\prime(0)}: \begin{bmatrix} iC_L(z;\lambda_{F^{\prime(0)}},c_F) \\ 0 \end{bmatrix}, \quad \hat{F}_{jR}^{\prime(0)}: \begin{bmatrix} 0 \\ C_R(z;\lambda_{F^{\prime(0)}},c_F) \end{bmatrix},$$
$$\hat{F}_{jL}^{\prime(n)}: \begin{bmatrix} 0 \\ S_L(z;\lambda_{F^{\prime(n)}},c_F) \end{bmatrix}, \quad \hat{F}_{jR}^{\prime(n)}: \begin{bmatrix} 0 \\ C_R(z;\lambda_{F^{\prime(n)}},c_F) \end{bmatrix}, \quad (n \geq 1). \tag{C27}$$

Remember that $\lambda_{F^{\prime(0)}}|_{\theta_H=0} > 0$ but $\lambda_{F^{\prime(0)}}|_{\theta_H=\pi} = 0$. Wave functions of the massless (zero) modes at $\theta_H = \pi$ are given, up to normalization factors, by

$$\check{\tilde{\Psi}}_F(x,z) \Rightarrow \begin{bmatrix} \hat{F}_{1L}^{\prime(0)}(x) \\ \hat{F}_{2L}^{\prime(0)}(x) \\ 0 \\ 0 \end{bmatrix} iC_L(z;0,c_F),$$
$$\begin{bmatrix} 0 \\ 0 \\ \hat{F}_{1R}^{\prime(0)}(x) \\ \hat{F}_{2R}^{\prime(0)}(x) \end{bmatrix} C_R(z;0,c_F). \tag{C28}$$

For $c_F < 0$, zero modes appear in the F tower, and the KK expansion is given by (6.22). The zero mode $F_j^{(0)}$ can be written as

$$F_j^{(0)} = \frac{\hat{F}_{jL}^{(0)}(x)}{\sqrt{r_0}} \begin{bmatrix} \bar{s}_H C_L(z;\lambda_{F^{(0)}},c_F) \\ -i\bar{c}_H \hat{S}_L(z;\lambda_{F^{(0)}},c_F) \end{bmatrix} + \frac{\hat{F}_{jR}^{(0)}(x)}{\sqrt{r_0}} \begin{bmatrix} \bar{c}_H \hat{S}_R(z;\lambda_{F^{(0)}},c_F) \\ i\bar{s}_H C_R(z;\lambda_{F^{(0)}},c_F) \end{bmatrix}, \tag{C29}$$

and at $\theta_H = \pi$

$$\check{\tilde{\Psi}}_F(x,z) \Rightarrow \begin{bmatrix} \hat{F}_{1L}^{(0)}(x) \\ \hat{F}_{2L}^{(0)}(x) \\ 0 \\ 0 \end{bmatrix} C_L(z;0,c_F),$$
$$\begin{bmatrix} 0 \\ 0 \\ \hat{F}_{1R}^{(0)}(x) \\ \hat{F}_{2R}^{(0)}(x) \end{bmatrix} iC_R(z;0,c_F), \tag{C30}$$

which takes the same form as (C28).

Gauge couplings at $\theta_H = \pi$ are given, for $c_F > 0$, by

$$\int_1^{z_L} dz \Big[(\bar{\check{\tilde{F}}}_1{}', \bar{\check{\tilde{F}}}_2{}')\gamma^\mu \Big\{ \frac{g_w}{\sqrt{2}} W_\mu^{(0)} (T^{1_R} + iT^{2_R}) + \frac{g_w}{\sqrt{2}} W_\mu^{(0)\dagger} (T^{1_R} - iT^{2_R}) + \frac{g_w}{\cos\theta_W^0} Z_\mu^{(0)} (T^{3_R} - \sin^2\theta_W^0 Q_{\rm EM}) + eA_\mu^{\gamma(0)} Q_{\rm EM} \Big\} \begin{pmatrix} \check{\tilde{F}}_1{}' \\ \check{\tilde{F}}_2{}' \end{pmatrix} + \Big\{ -g_w \frac{\sin^2\theta_W^0}{\cos\theta_W^0} Z_\mu^{(0)} + eA_\mu^{\gamma(0)} \Big\} (\bar{\check{\tilde{F}}}_1, \bar{\check{\tilde{F}}}_2)\gamma^\mu Q_{\rm EM} \begin{pmatrix} \check{\tilde{F}}_1 \\ \check{\tilde{F}}_2 \end{pmatrix} \Big], \tag{C31}$$

which leads to (6.23).

4. Dark fermion (darkV Ψ_V^γ) sector

DarkV field $\Psi_V^\pm = (\Psi_1^\pm, \ldots, \Psi_5^\pm)$ is in the representation $(\mathbf{1},\mathbf{5})_0$. Let us denote

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_4^\pm + i\Psi_3^\pm & \Psi_2^\pm + i\Psi_1^\pm \\ -\Psi_2^\pm + i\Psi_1^\pm & \Psi_4^\pm - i\Psi_3^\pm \end{pmatrix} = \begin{pmatrix} N^\pm & -E^{\prime\pm} \\ E^\pm & -N^{\prime\pm} \end{pmatrix}, \qquad \Psi_5^\pm = S^\pm. \tag{C32}$$

$N^\pm, N^{\prime\pm}, S^\pm$, are neutral ($Q_{\rm EM} = 0$). $E^\pm$ and $E^{\prime\pm}$ have charges $Q_{\rm EM} = -1$ and $+1$. Under $SU(2)_L$ and $SU(2)_R$ rotations

$$SU(2)_L: \begin{pmatrix} N^\pm \\ E^\pm \end{pmatrix}, \begin{pmatrix} E^{\prime\pm} \\ N^{\prime\pm} \end{pmatrix},$$
$$SU(2)_R: \begin{pmatrix} N^{\prime\pm} \\ E^\pm \end{pmatrix}, \begin{pmatrix} E^{\prime\pm} \\ N^\pm \end{pmatrix}, \tag{C33}$$

transform as doublets. $S^\pm$ fields are singlets. We assume that bulk-mass parameters of Ψ_V^+ and Ψ_V^- are the same; $c_{V^+} = c_{V^-} = c_V$.

The mass spectrum is determined by

$$\mathcal{S}^V_{L1}\mathcal{S}^V_{R1} - \mathcal{S}^V_{L2}\mathcal{S}^V_{R2} = 0 \quad \text{for } E^\pm, E'^\pm,$$
$$\mathcal{S}^V_{L1}\mathcal{S}^V_{R1} - \mathcal{S}^V_{L2}\mathcal{S}^V_{R2} + 4\sin^2\theta_H = 0 \quad \text{for } N^\pm, N'^\pm, S^\pm, \tag{C34}$$

where $\mathcal{S}^V_{L1} = \mathcal{S}_{L1}(1;\lambda, c_V, \tilde{m}_V)$, $\tilde{m}_V = m_V/k$, etc. There are no zero modes. As seen in Fig. 1, the lowest mode has a mass ∼3 TeV. For $\theta_H = 0$ and π, the neutral and charged sectors become degenerate in masses.

E^+ (E'^+) fields couple to E^- (E'^-) fields by Dirac mass terms. The KK expansions are given, with the notation $\boldsymbol{E} = (E, E')$ and $\boldsymbol{N} = (N, N')$, by

$$\begin{bmatrix} \check{\tilde{\boldsymbol{E}}}^+ \\ \check{\tilde{\boldsymbol{E}}}^- \end{bmatrix} = \sqrt{k}\sum_{n=1}^\infty \boldsymbol{E}^{(n)},$$
$$\begin{bmatrix} \check{\tilde{\boldsymbol{N}}}^+ \\ \check{\tilde{\boldsymbol{N}}}^- \end{bmatrix} = \sqrt{k}\sum_{n=1}^\infty \boldsymbol{N}^{(n)}, \quad \begin{bmatrix} \check{\tilde{S}}^+ \\ \check{\tilde{S}}^- \end{bmatrix} = \sqrt{k}\sum_{n=1}^\infty S^{(n)}. \tag{C35}$$

Wave functions of each KK mode in (C35) are given by

$$\boldsymbol{E}^{(n)} = \frac{\hat{\boldsymbol{E}}^{(n)}_L(x)}{\sqrt{r_n}}\begin{bmatrix} a\mathcal{C}_{L2}(z;\lambda_{E^{(n)}}) + b\mathcal{C}_{L1}(z;\lambda_{E^{(n)}}) \\ a\mathcal{S}_{L1}(z;\lambda_{E^{(n)}}) + b\mathcal{S}_{L2}(z;\lambda_{E^{(n)}}) \end{bmatrix} + \frac{\hat{\boldsymbol{E}}^{(n)}_R(x)}{\sqrt{r_n}}\begin{bmatrix} a\mathcal{S}_{R2}(z;\lambda_{E^{(n)}}) + b\mathcal{S}_{R1}(z;\lambda_{E^{(n)}}) \\ a\mathcal{C}_{R1}(z;\lambda_{E^{(n)}}) + b\mathcal{C}_{R2}(z;\lambda_{E^{(n)}}) \end{bmatrix},$$
$$\mathcal{C}_{Lj}(z;\lambda_{E^{(n)}}) = \mathcal{C}_{Lj}(z;\lambda_{E^{(n)}}, c_V, \tilde{m}_V), \text{etc.}, \tag{C36}$$

where $\mathcal{S}^V_{L1}a + \mathcal{S}^V_{L2}b = \mathcal{S}^V_{R2}a + \mathcal{S}^V_{R1}b = 0$ in each term. Note that $\lambda_{E^{(n)}} = \lambda_{E'^{(n)}}$.

In the neutral sector all fields $N^\pm, N'^\pm, S^\pm$, couple with each other for general θ_H, but they split into pairs (N^+, N^-), (N'^+, N'^-), and (S^+, S^-) for $\theta_H = 0$ and π. The spectrum becomes degenerate; $\lambda_{N^{(n)}} = \lambda_{N'^{(n)}} = \lambda_{S^{(n)}} = \lambda_{E^{(n)}}$. Wave functions are given by

$$\boldsymbol{N}^{(n)} = \frac{\hat{\boldsymbol{N}}^{(n)}_L(x)}{\sqrt{r_n}}\begin{bmatrix} a\mathcal{C}_{L2}(z;\lambda_{N^{(n)}}) + b\mathcal{C}_{L1}(z;\lambda_{N^{(n)}}) \\ a\mathcal{S}_{L1}(z;\lambda_{N^{(n)}}) + b\mathcal{S}_{L2}(z;\lambda_{N^{(n)}}) \end{bmatrix} + \frac{\hat{\boldsymbol{N}}^{(n)}_R(x)}{\sqrt{r_n}}\begin{bmatrix} a\mathcal{S}_{R2}(z;\lambda_{N^{(n)}}) + b\mathcal{S}_{R1}(z;\lambda_{N^{(n)}}) \\ a\mathcal{C}_{R1}(z;\lambda_{N^{(n)}}) + b\mathcal{C}_{R2}(z;\lambda_{N^{(n)}}) \end{bmatrix},$$
$$S^{(n)} = \frac{\hat{S}^{(n)}_L(x)}{\sqrt{r_n}}\begin{bmatrix} a\mathcal{S}_{L1}(z;\lambda_{S^{(n)}}) + b\mathcal{S}_{L2}(z;\lambda_{S^{(n)}}) \\ a\mathcal{C}_{L2}(z;\lambda_{S^{(n)}}) + b\mathcal{C}_{L1}(z;\lambda_{S^{(n)}}) \end{bmatrix} + \frac{\hat{S}^{(n)}_R(x)}{\sqrt{r_n}}\begin{bmatrix} a\mathcal{C}_{R1}(z;\lambda_{S^{(n)}}) + b\mathcal{C}_{R2}(z;\lambda_{S^{(n)}}) \\ a\mathcal{S}_{R2}(z;\lambda_{S^{(n)}}) + b\mathcal{S}_{R1}(z;\lambda_{S^{(n)}}) \end{bmatrix}, \tag{C37}$$

where $\mathcal{S}^V_{L1}a + \mathcal{S}^V_{L2}b = \mathcal{S}^V_{R2}a + \mathcal{S}^V_{R1}b = 0$ in each term.

Gauge couplings of darkV fields at $\theta_H = 0$ and π are easily found. We note that the mass spectrum and wave functions of $\boldsymbol{E}^{(n)}$ are the same as those of $\boldsymbol{N}^{(n)}$. $S^\pm$ fields do not couple to W, Z, and A^γ fields. One finds that

$$\underline{\text{for } \theta_H = 0;}$$
$$\frac{g_w}{\sqrt{2}}\left\{W^{(0)}_\mu \sum_{n=1}^\infty (\bar{\hat{N}}^{(n)}\gamma^\mu \hat{E}^{(n)} + \bar{\hat{E}}'^{(n)}\gamma^\mu \hat{N}'^{(n)}) + \text{H.c.}\right\}$$
$$+ \frac{g_w}{2\cos\theta^0_W} Z^{(0)}_\mu \sum_{n=1}^\infty (\bar{\hat{N}}^{(n)}\gamma^\mu \hat{N}^{(n)} - \bar{\hat{E}}^{(n)}\gamma^\mu \hat{E}^{(n)} + \bar{\hat{E}}'^{(n)}\gamma^\mu \hat{E}'^{(n)} - \bar{\hat{N}}'^{(n)}\gamma^\mu \hat{N}'^{(n)}) + \left\{-g_w \frac{\sin^2\theta^0_W}{\cos\theta^0_W} Z^{(0)}_\mu + eA^{\gamma(0)}_\mu\right\} J^\mu_{\text{EM}},$$
$$\underline{\text{for } \theta_H = \pi;}$$
$$\frac{g_w}{\sqrt{2}}\left\{W^{(0)}_\mu \sum_{n=1}^\infty (\bar{\hat{N}}'^{(n)}\gamma^\mu \hat{E}^{(n)} + \bar{\hat{E}}'^{(n)}\gamma^\mu \hat{N}^{(n)}) + \text{H.c.}\right\}$$
$$+ \frac{g_w}{2\cos\theta^0_W} Z^{(0)}_\mu \sum_{n=1}^\infty (\bar{\hat{N}}'^{(n)}\gamma^\mu \hat{N}'^{(n)} - \bar{\hat{E}}^{(n)}\gamma^\mu \hat{E}^{(n)} + \bar{\hat{E}}'^{(n)}\gamma^\mu \hat{E}'^{(n)} - \bar{\hat{N}}^{(n)}\gamma^\mu \hat{N}^{(n)}) + \left\{-g_w \frac{\sin^2\theta^0_W}{\cos\theta^0_W} Z^{(0)}_\mu + eA^{\gamma(0)}_\mu\right\} J^\mu_{\text{EM}},$$
$$J^\mu_{\text{EM}} = \sum_{n=1}^\infty (-\bar{\hat{E}}^{(n)}\gamma^\mu \hat{E}^{(n)} + \bar{\hat{E}}'^{(n)}\gamma^\mu \hat{E}'^{(n)}). \tag{C38}$$

All couplings are vectorlike.

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