Flipped $SU(5)$ with modular A_4 symmetry

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We study flipped $SU(5) \times U(1)$ grand unified theories (GUTs) with $\Gamma_3 \simeq A_4$ modular symmetry. We propose two models with different modular weights assignments, where the fermion mass hierarchy can arise from weighton fields. In order to relax the constraint on the Dirac neutrino Yukawa matrix we appeal to mechanisms which allow incomplete GUT representations, allowing a good fit to quark and charged lepton masses and quark mixing for a single modulus field τ, with the neutrino masses and lepton mixing well determined by the type-I seesaw mechanism, at the expense of some tuning. We also discuss the double seesaw possibility allowed by the extra singlets generically predicted in such string inspired theories.

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I. INTRODUCTION

The existence of three fermion families and the origin of flavor mixing are long-standing questions in particle physics. Regarding the first issue, there is no theoretical explanation why there are three families for the otherwise successful standard model and its field theory extensions such as grand unified theories (GUT) and their supersymmetric analogues. A possible interpretation gaining attention these days lies in effective models derived within the context of string theory. In a wide class of such constructions the number of fermion generations is attributed to the topological properties—and in particular the Euler characteristic—of the compactification manifold. The origin of flavor mixing among the three families, however, is still unclear. Various mechanisms have been implemented, including those based on the geometry of the compactification manifold, Abelian and discrete symmetries, fluxes and nonperturbative effects, but they are still debatable.

Over the last few decades Abelian and non-Abelian discrete symmetries have gained an increasing interest in the particle physics literature, with special focus on their role in model building. They have been introduced to predict viable fermion and particularly neutrino mass textures as well as to suppress processes leading to baryon and lepton number violating interactions. More recently, modular invariance has been suggested as a novel candidate for predicting viable fermion mass textures [1].[1] This is an intrinsic symmetry in theories with ultraviolet completion, such as string theory, and can exist in parallel with some discrete group of a different origin. Indeed, modular invariance is a fundamental concept in string theory and is naturally expected to leave its trace in the effective field theory model (possibly based on some GUT). Among other implications, it governs the structure of the potential and particularly the Yukawa couplings. For example, in orientifold compactifications of type-II strings the Yukawa couplings are functions with specific modular properties and in orbifold compactifications of heterotic strings Yukawa couplings between twisted states are subject to restrictions from modular invariance and recent attempts to exploit these properties have already appeared (see for example [5,6]). In general, depending on the details of the derivation of the superstring model, Yukawa couplings are expected to be expressed in terms of certain modular forms exhibiting certain transformation properties under the modular group.

Modular forms depend on a positive integer called the level N, together with an integer weight k, and are

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[1]For early work on mass matrices and modular invariance in string motivated models see [2–4].

manifested as modular multiplets of the homogeneous finite modular group $\Gamma'_N \equiv \Gamma/\Gamma(N)$ [7]. If k is an even number [1], they may be organized into modular multiplets of the inhomogeneous finite modular group $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$. Realistic models have been constructed based on Γ_N for the levels $N = 2$ [8–11], $N = 3$ [1,8,9,12–37], $N = 4$ [38–43], [25,44–46], $N = 5$ [42,47,48] and $N = 7$ [49]. The modular invariance approach may also be extended to incorporate several factorizable [40] and nonfactorizable moduli [50]. Modular invariance can also address the origin of mass hierarchies without introducing an additional Froggatt-Nielsen (FN) $U(1)$ [51] symmetry. The role of the FN flavon is played by a singlet field called the weighton [30], which carries a nonzero modular weight, but no other charges.

Grand unified theories (GUTs) are well motivated theories which reduce the three gauge interactions of the SM group into a simpler structure such as $SU(5)$ [52]. In GUTs, the quark and lepton fields are embedded into fewer gauge multiplets, resulting in relations between quark and lepton mass matrices. There is good motivation for including also family symmetry together with GUTs in order to account for large lepton mixing [53]. The discrete group A_4 is the minimal choice which admits triplet representations [54]. Without modular symmetry, combining A_4 family symmetry with $SU(5)$ GUTs [55] requires vacuum alignment of the flavons in order to break the A_4, and this provides a further motivation for including modular symmetry. Modular symmetry was first combined with $SU(5)$ GUTs in an $(\Gamma_3 \simeq A_4) \times SU(5)$ model in [14,56], and subsequently modular $SU(5)$ GUT models have been constructed based on $(\Gamma_2 \simeq S_3) \times SU(5)$ [10,57], and $(\Gamma_4 \simeq S_4) \times SU(5)$ [58–60]. Recently $SO(10)$ GUTs have been studied based on $\Gamma_3 \simeq A_4$ modular symmetry [61].

The above studied GUTs depend on a Higgs in an adjoint representation in order to break the gauge symmetry, unless an extra-dimensional mechanism is invoked such as Wilson lines or F-theory flux breaking. On the other hand, in some string theories the adjoint representation is not available to break GUT symmetry, for example the viable GUT models in the context of heterotic compactifications, are only those which do not rely on such Higgs fields. The most popular ones are the flipped $SU(5)$ [62–66] and the Pati-Salam models [67,68].

In this paper, motivated by the above considerations, we study flipped $SU(5) \times U(1)$ GUTs with $\Gamma_3 \simeq A_4$ modular symmetry. To illustrate the approach, we propose two models with different modular weights assignments, where the fermion mass hierarchy can arise from weighton fields, where one of the models is studied in detail using a numerical χ^2 analysis. In such models the neutrino sector can be tightly constrained by the up type quark mass matrix, in particular the up-quarks and Dirac neutrino mass matrices satisfy the relation $m_D^T = m_u$ at the GUT scale. In order to avoid this constraint we appeal to F-theory constructions where the components of the GUT multiplets may lie on different matter curves. With this constraint relaxed, we can fit the quark and lepton mass matrices and quark mixing for a single modulus field τ, with the neutrino masses and lepton mixing determined by the type-I seesaw mechanism. We also discuss the double seesaw possibility allowed by the extra singlets possible in flipped $SU(5)$.

The layout of the remainder of the paper is as follows. In Sec. II we present a short introduction to modular transformations, mainly focusing on the modular symmetry A_4. In Sec. III we start a brief account of the field theory version of the flipped $SU(5)$ model. Next, we proceed with a modular invariant version proposed in the present work. Considering that Yukawa couplings are certain modular forms, we derive the modular invariant superpotential and the induced mass matrices for up and down quarks, charged leptons and neutrinos. We perform an detailed numerical investigation and show that the proposed construction is in agreement with all low energy data regarding the fermion masses and their mixing. In Sec. IV we discuss the predictions of the suggested models and summarise the main results. We study the contribution of singlet neutrinos to the flavor structure in the Appendix A. A variant of this model, based on a different choice of the modular properties is presented in the Appendix B.

II. MODULAR SYMMETRIES

In this section, we give a brief review of the modular symmetry and the tetrahedral group A_4 as a finite modular group.

A. The infinite modular symmetry

A modular transformation γ is defined as a linear fractional transformation on the complex modulus τ varying in the upper-half plane $\mathcal{H} = \mathrm{Im}(\tau) > 0$,

$$\gamma : \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{1}$$

where a, b, c, d are integers and $ad - bc = 1$. Each modular transformation can be represented by a 2×2 matrix with integer entries and the determinant equal to one, i.e.,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(\gamma) = 1. \tag{2}$$

The modular group $\bar{\Gamma}$ is defined as a group of these transformations, i.e.,

$$\bar{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm \mathbf{1}) \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}. \tag{3}$$

It includes infinite elements. All elements can be generated by S and T, given by

$$S:\tau \mapsto -\frac{1}{\tau}, \qquad T:\tau \mapsto \tau + 1. \tag{4}$$

They are represented by 2×2 matrices as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

The actions of S and T in $\mathcal{H}$ are given by:

$$S:\tau \mapsto -\frac{1}{\tau}, \qquad T:\tau \mapsto \tau + 1. \tag{6}$$

One can prove that the identities $S^2 = (ST)^3 = \mathbb{I}$ are satisfied, namely, $S^2\tau = (ST)^3\tau = \tau$.

A subgroup of $\bar{\Gamma}$ is obtained by restricting $a, d = 1 (\mathrm{mod}\ N)$ and $b, c = 0 (\mathrm{mod}\ N)$,

$$\bar{\Gamma}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\mathrm{mod}\ N) \right\}, \tag{7}$$

where N is a positive integer. $\bar{\Gamma}(N)$ is also an infinite group. The quotient group $\bar{\Gamma}/\bar{\Gamma}(N)$ is finite and labeled as Γ_N. It is equivalently obtained by requiring $a, b, c, d \in \mathbb{Z}_N$, namely

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm \mathbf{1}) \middle| a, b, c, d \in \mathbb{Z}_N, ad - bc = 1 \right\}. \tag{8}$$

For $N = 2$, 3, 4, 5, 7, $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, $\Gamma_5 \simeq A_5$ and $\Gamma_7 \simeq \Sigma(168)$.

In $\mathcal{N} = 1$ supersymmetric theories, a modular-invariant superpotential is expanded as series of polynomials in powers of supermultiplets ϕ^I,

$$\mathcal{W} = \sum_n Y_{I_1 I_2 \cdots I_n}(\tau) \phi^{I_1} \phi^{I_2} \cdots \phi^{I_n}. \tag{9}$$

Here, $Y_{I_1 I_2 \cdots I_n}(\tau)$ are called modular forms keeping the superpotential term invariant under any modular transformation. A modular form $Y_i(\tau)$ of level N and weight $2k$ is defined as a holomorphic function of the modulus τ with the modular transformations under $\Gamma(N)$,

$$\gamma \in \Gamma(N):\ Y_i(\tau) \to Y_i(\gamma\tau) = (c\tau + d)^{2k} Y_i(\tau). \tag{10}$$

Under the quotient group Γ_N, they are transformed not as holomorphic functions but linear superposition of a series of modular forms $\{Y_1(\tau), Y_2(\tau), \cdots\}$ which take the same weight and level,

$$\gamma \in \Gamma_N : Y_i(\tau) \to Y_i(\gamma\tau) = (c\tau + d)^{2k} \sum_j \rho_{ij}(\gamma) Y_j(\tau), \tag{11}$$

where j runs as an index of the series $\{Y_1(\tau), Y_2(\tau), \cdots\}$, $\rho(\gamma)$ is a unitary representation matrix of $\gamma \in \Gamma_N$. For a given finite Γ_N, one can choose a basis, where the representation ρ is decomposed to a few irreducible representations. In this basis, modular forms and fields appear as a series of irreducible representations of Γ_N. Assigning this basis as the flavor basis of matter fields, the restriction of the modular invariance could strongly constrain the flavor structure. In this work, we will take $\Gamma_3 \simeq A_4$ as an example to discuss the flavor mixing in flipped $SU(5)$ framework.

B. The finite modular symmetry A_4

Γ_3 is a finite subgroup of $\bar{\Gamma}$, referring to $N = 3$. Due to the requirement $a, b, c, d \in \mathbb{Z}_3$, the generator T satisfies one more condition $T^3 = \mathbb{I}$, leading to its isomorphism to the tetrahedral group A_4.

A_4 contains 12 elements. All can be written as products of S and T and are shown below,

$$I, T, ST, TS, STS, T^2, ST^2, T^2S, TST, S, T^2ST, TST^2. \tag{12}$$

It has three singlet ($\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$) and one triplet ($\mathbf{3}$) irreducible representations. The generators S and T in these representations are given by

$$\begin{aligned}
\mathbf{1}&: \quad S = 1, \quad T = 1, \\
\mathbf{1}'&: \quad S = 1, \quad T = \omega, \\
\mathbf{1}''&: \quad S = 1, \quad T = \omega^2, \\
\mathbf{3}&: \quad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\end{aligned} \tag{13}$$

where $\omega = e^{2i\pi/3}$. Tensor products of two irreducible representations are decomposed as,

$$\begin{aligned}
&\mathbf{1}\otimes\mathbf{r} = \mathbf{r}, \quad \mathbf{1}'\otimes\mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}''\otimes\mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}'\otimes\mathbf{1}'' = \mathbf{1}, \\
&\mathbf{1}'\otimes\mathbf{3} = \mathbf{3}, \quad \mathbf{1}''\otimes\mathbf{3} = \mathbf{3}, \quad \mathbf{3}\otimes\mathbf{3}' = \mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''\oplus\mathbf{3}_S\oplus\mathbf{3}_A,
\end{aligned} \tag{14}$$

where $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{3}$, and "$S$" and "$A$" in the subscript denote symmetric and antisymmetric combinations, respectively. In particular, the decomposition of the tensor product of two triplets $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ is explicitly written as:

$$(ab)_{\mathbf{1}} = a_1b_1 + a_2b_3 + a_3b_2,$$
$$(ab)_{\mathbf{1}'} = a_3b_3 + a_1b_2 + a_2b_1,$$
$$(ab)_{\mathbf{1}''} = a_2b_2 + a_1b_3 + a_3b_1,$$
$$(ab)_{\mathbf{3}_S} = \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix},$$
$$(ab)_{\mathbf{3}_A} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}. \tag{15}$$

Modular forms of level $N = 3$ and weight $2k$ form a linear space of dimension $2k + 1$. All of them have been explicitly obtained in terms of the Dedekind eta-function $\eta(\tau)$ [1]:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i \tau} \tag{16}$$

(i) For $2k = 2$, there are 3 linearly independent modular forms, transforming as a triplet of A_4. Given the triplet basis in Eq. (13), this triplet modular form is written as $Y^{(2)}_{\mathbf{3}} = (Y_1, Y_2, Y_3)^T \sim \mathbf{3}$ with

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right], \\ Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right], \\ Y_3(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right]. \end{aligned} \tag{17}$$

Modular forms of higher weights are derived from products of those of lower weights.

(ii) For $2k = 4$, there are 5 linearly independent modular forms, derived from products of two modular forms of weight 2 and written as

$$\begin{aligned} Y^{(4)}_{\mathbf{3}} &= (Y^{(2)}_{\mathbf{3}} Y^{(2)}_{\mathbf{3}})_{\mathbf{3}} = \begin{pmatrix} Y^{(4)}_1 \\ Y^{(4)}_2 \\ Y^{(4)}_3 \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}, \\ Y^{(4)}_{\mathbf{1}} &= (Y^{(2)}_{\mathbf{3}} Y^{(2)}_{\mathbf{3}})_{\mathbf{1}} = Y_1^2 + 2Y_2Y_3, \\ Y^{(4)}_{\mathbf{1}'} &= (Y^{(2)}_{\mathbf{3}} Y^{(2)}_{\mathbf{3}})_{\mathbf{1}'} = Y_3^2 + 2Y_1Y_2. \end{aligned} \tag{18}$$

(iii) For $2k = 6$, the 7 linearly independent modular forms are given by

$$\begin{aligned} Y^{(6)}_{\mathbf{1}} &= (Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{3}})_{\mathbf{1}} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3, \\ Y^{(6)}_{\mathbf{3}_1} &= Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{1}} = \begin{pmatrix} Y^{(6)}_{1_1} \\ Y^{(6)}_{2_1} \\ Y^{(6)}_{3_1} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix}, \\ Y^{(6)}_{\mathbf{3}_2} &= Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{1}'} = \begin{pmatrix} Y^{(6)}_{1_2} \\ Y^{(6)}_{2_2} \\ Y^{(6)}_{3_2} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix}. \end{aligned} \tag{19}$$

(iv) For $2k = 8$, the 9 linearly independent modular forms are

$$
\begin{aligned}
Y^{(8)}_{\mathbf{1}} &= (Y^{(2)}_{\mathbf{3}} Y^{(6)}_{\mathbf{3}_1})_{\mathbf{1}} = (Y_1^2 + 2Y_2Y_3)^2, \\
Y^{(8)}_{\mathbf{1}'} &= (Y^{(2)}_{\mathbf{3}} Y^{(6)}_{\mathbf{3}_1})_{\mathbf{1}'} = (Y_1^2 + 2Y_2Y_3)(Y_3^2 + 2Y_1Y_2), \\
Y^{(8)}_{\mathbf{1}''} &= (Y^{(2)}_{\mathbf{3}} Y^{(6)}_{\mathbf{3}_2})_{\mathbf{1}''} = (Y_3^2 + 2Y_1Y_2)^2, \\
Y^{(8)}_{\mathbf{3}_1} &= Y^{(2)}_{\mathbf{3}} Y^{(6)}_{\mathbf{1}} = \begin{pmatrix} Y^{(8)}_{1_1} \\ Y^{(8)}_{2_1} \\ Y^{(8)}_{3_1} \end{pmatrix} = (Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \\
Y^{(8)}_{\mathbf{3}_2} &= (Y^{(2)}_{\mathbf{3}} Y^{(6)}_{\mathbf{3}_2})_{\mathbf{3}_A} = \begin{pmatrix} Y^{(8)}_{1,_2} \\ Y^{(8)}_{2_2} \\ Y^{(8)}_{3_2} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2) \begin{pmatrix} (Y_2^2 - Y_1Y_3) \\ (Y_1^2 - Y_2Y_3) \\ (Y_3^2 - Y_1Y_2) \end{pmatrix}.
\end{aligned}
\tag{20}
$$

We further list some singlet modular forms of higher weights ($2k = 10$ and 12), which may be useful for the rest of the work,

$$
\begin{aligned}
Y^{(10)}_{\mathbf{1}} &= (Y^{(2)}_{\mathbf{3}} Y^{(8)}_{\mathbf{3}_1})_1 = (Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3)(Y_1^2 + 2Y_2Y_3), \\
Y^{(10)}_{\mathbf{1}'} &= (Y^{(2)}_{\mathbf{3}} Y^{(8)}_{\mathbf{3}_1})_{1'} = (Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3)(Y_3^2 + 2Y_1Y_2), \\
Y^{(10)}_{\mathbf{1}''} &= (Y^{(2)}_{\mathbf{3}} Y^{(8)}_{\mathbf{3}_2})_{1''} = (Y_3^2 + 2Y_1Y_2)[Y_3(Y_2^2 - Y_1Y_{\mathbf{3}}) + Y_2(Y_1^2 - Y_2Y_3) + Y_1(Y_3^2 - Y_1Y_2)], \\
Y^{(12)}_{\mathbf{1}} &= (Y^{(4)}_{\mathbf{3}} Y^{(8)}_{\mathbf{3}_1})_{\mathbf{1}} = (Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3)^2, \\
Y^{(12)}_{\mathbf{1}} &= (Y^4_{\mathbf{1}})^3 = (Y_1^2 + 2Y_2Y_3)^3, \\
Y^{(12)}_{\mathbf{1}'} &= (Y^{(4)}_{\mathbf{1}})^2 Y^{(4)}_{\mathbf{1}'} = (Y_1^2 + 2Y_2Y_3)^2(Y_3^2 + 2Y_1Y_2), \\
Y^{(12)}_{\mathbf{1}''} &= (Y^{(4)}_{\mathbf{3}} Y^{(8)}_{\mathbf{3}_2})_{\mathbf{1}''} = (Y_3^2 + 2Y_1Y_2)[2(Y_3^2 - Y_1Y_2)(Y_1^2 - Y_2Y_3) + (Y_2^2 - Y_1Y_3)^2].
\end{aligned}
\tag{21}
$$

III. MODEL BUILDING

A. The flipped $SU(5)$ framework

The flipped $SU(5)$ model has been proposed long time ago [64,65] as an alternative symmetry breaking pattern of the $SO(10)$ gauge group. It is based on the $SU(5) \times U(1)_\chi$ gauge symmetry and has been reconsidered as a possible superstring alternative to Georgi-Glashow $SU(5)$ due to the fact that its spontaneous breaking to SM symmetry requires only a pair of $\mathbf{10} + \overline{\mathbf{10}}$ Higgs representations and does not need any adjoint Higgs representation. In fact, this is a welcome property since in many string derived effective models the Higgs adjoint representation does not appear in the massless spectrum. Among other virtues the model admits a doublet-triplet mass splitting for the color triplets, and in the presence of additional neutral singlets, an extended seesaw mechanism for neutrino masses is naturally realized. The hypercharge generator is a linear combination of the $U(1)$ inside $SU(5)$ and the external Abelian factor $U(1)_\chi$ and it is no longer fully embedded in $SU(5)$. This way flipped $SU(5)$ representations accommodate the SM matter fields differently. To start with, the following quantum numbers follow from $SO(10) \to SU(5) \times U(1)_\chi$ decompositions

$$
\begin{aligned}
\mathbf{16} &\to \left(\mathbf{10}, -\frac{1}{2}\right) + \left(\bar{\mathbf{5}}, \frac{3}{2}\right) + \left(\mathbf{1}, -\frac{5}{2}\right), \\
\mathbf{10} &\to (\mathbf{5}, 1) + (\bar{\mathbf{5}}, -1).
\end{aligned}
\tag{22}
$$

The flipped gauge symmetry $SU(5) \times U(1)_\chi$, can be broken to the SM gauge symmetry via a two-step symmetry breaking, $SU(5) \times U(1)_\chi \to SU(3)_c \times SU(2)_L \times U(1)_y \times U(1)_\chi \to SU(3)_c \times SU(2)_L \times U(1)_Y$. In the first step of symmetry breaking, representation decompositions follow as,

$$\left(\mathbf{10},-\frac{1}{2}\right)\to\left(\mathbf{3},\mathbf{2},\frac{1}{6},-\frac{1}{2}\right)+\left(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3},-\frac{1}{2}\right)+\left(\mathbf{1},\mathbf{1},1,-\frac{1}{2}\right),$$
$$\left(\bar{\mathbf{5}},\frac{3}{2}\right)\to\left(\bar{\mathbf{3}},\mathbf{1},\frac{1}{3},\frac{3}{2}\right)+\left(\mathbf{1},\mathbf{2},-\frac{1}{2},\frac{3}{2}\right),$$
$$(\bar{\mathbf{5}},-1)\to\left(\bar{\mathbf{3}},\mathbf{1},\frac{1}{3},-1\right)+\left(\mathbf{1},\mathbf{2},-\frac{1}{2},-1\right). \tag{23}$$

In the second step of symmetry breaking, the two $U(1)$ symmetries are broken to $U(1)_Y$ with the hypercharge defined by

$$Y=-\frac{1}{5}(y+2\chi). \tag{24}$$

Each representations then gain hypercharges as

$$\left(\mathbf{10},-\frac{1}{2}\right)\to Y=\left\{\frac{1}{6},\frac{1}{3},0\right\}\to\{Q,d^c,\nu^c\},$$
$$\left(\bar{\mathbf{5}},+\frac{3}{2}\right)\to Y=\left\{-\frac{2}{3},-\frac{1}{2}\right\}\to\{u^c,L\},$$
$$\left(\mathbf{1},-\frac{5}{2}\right)\to Y=\{+1\}\to e^c,$$
$$(\bar{\mathbf{5}},-1)\to Y=\left\{\frac{1}{3},\frac{1}{2}\right\}\to\{D^c,h_u\},$$
$$(\mathbf{5},+1)\to Y=\left\{-\frac{1}{3},-\frac{1}{2}\right\}\to\{D,h_d\}, \tag{25}$$

where $Q=(u,d)$ and $L=(\nu,e)$. After the symmetry breaking, SM matter fields Q,L,u^c,d^c,e^c, as well as the right-handed neutrino ν^c and MSSM Higgses h_u and h_d, are separated as shown on the right-hand side of the above formula. The definition of the hypercharge includes a component of the external $U(1)_\chi$ in such a way that flips the positions of $u^c\leftrightarrow d^c$ and $e^c\leftrightarrow\nu^c$ within these representations, while leaves the remaining unaltered.

In summary, we obtain the following "flipped" embedding of the SM representations. The chiral matter fields are

$$F_i=\left(\mathbf{10},-\frac{1}{2}\right)=\{Q_i,d_i^c,\nu_i^c\},$$
$$\bar{f}_i=\left(\bar{\mathbf{5}},+\frac{3}{2}\right)=\{u_i^c,L_i\},$$
$$\ell_i^c=\left(\mathbf{1},-\frac{5}{2}\right)=e_i^c. \tag{26}$$

The Higgs fields breaking GUT and SM symmetries reside in the following flipped $SU(5)$ representations

$$H\equiv\left(\mathbf{10},-\frac{1}{2}\right)=\{Q_H,D_H^c,\nu_H^c\},$$
$$\bar{H}\equiv\left(\overline{\mathbf{10}},+\frac{1}{2}\right)=\{\bar{Q}_H,\bar{d}_H^c,\bar{\nu}_H^c\},$$
$$h\equiv(\mathbf{5},+1)=\{D_h,h_d\},\quad \bar{h}\equiv(\bar{\mathbf{5}},-1)=\{\bar{D}_h,h_u\}. \tag{27}$$

A remarkable fact in the flipped model, is that the $\bar{\mathbf{5}}$ matter field is completely distinguished from the $\bar{\mathbf{5}}$ Higgs field. Indeed, due to their different $U(1)_\chi$ charge which is involved in the hypercharge definition, their SM components do not contain exactly the same type of SM-fields (the $\bar{\mathbf{5}}$ matter field contains u^c, while the $\bar{\mathbf{5}}$ Higgs field contains the down-type D_h). Several R-parity violating terms will not be allowed because of this distinction.

The fermion masses arise from the following $SU(5)\times U(1)_\chi$ invariant couplings

$$\mathcal{W}_d=\left(\mathbf{10},-\frac{1}{2}\right)\cdot\left(\mathbf{10},-\frac{1}{2}\right)\cdot(\mathbf{5},1)\to Qd^ch_d,$$
$$\mathcal{W}_u=\left(\mathbf{10},-\frac{1}{2}\right)\cdot\left(\bar{\mathbf{5}},\frac{3}{2}\right)\cdot(\bar{\mathbf{5}},-1)\to Qu^ch_u+\nu^cLh_u,$$
$$\mathcal{W}_l=\left(\mathbf{1},-\frac{5}{2}\right)\cdot\left(\bar{\mathbf{5}},\frac{3}{2}\right)\cdot(\mathbf{5},1)\to e^cLh_d. \tag{28}$$

Also, a higher order term providing Majorana masses for the right-handed neutrinos can be written

$$\mathcal{W}_{\nu^c}=\lambda_{ij}^{\nu^c}\frac{1}{M_S}\bar{H}\bar{H}F_iF_j\to\lambda_{ij}^{\nu^c}\frac{\langle\bar{\nu}_H^c\rangle^2}{M_S}\nu_i^c\nu_j^c. \tag{29}$$

If additional singlet fields ν_S, Φ_i are present (which is the usual case in string derived models), then—depending on their specific properties—the following couplings could be generated

$$F\bar{H}\nu_S+\bar{h}h\Phi_i+\lambda_{ijk}\Phi_i\Phi_j\Phi_k+\cdots \tag{30}$$

In addition to SM representations, the Higgs sector contains dangerous color triplets $D_h, D_H+c.c.$. They become massive through the following terms

$$HHh+\bar{H}\bar{H}\bar{h}\to\langle\nu_H^c\rangle D_H^cD_h+\langle\bar{\nu}_H^c\rangle\bar{D}_H^c\bar{D}_h. \tag{31}$$

Note that, if no other symmetry exists the terms such as $HF_ih, H\bar{f}_j\bar{h}$ could be possible. Such terms would generate dangerous mixing between Higgs and matter:

$$(aF+bH)\bar{f}_j\bar{h}+\cdots \tag{32}$$

Remarkably, a Z_2 symmetry [66] which is odd only for $H\to-H$ excludes all these couplings from the Lagrangian, while all the previous (useful) terms are left intact. Similar symmetries have been discussed in [69].

For rank one mass textures the couplings in Eq. (28) predict $m_t = m_{\nu_\tau}$ at the GUT scale. However, in contrast to the standard $SU(5)$ model, down quark and lepton mass matrices are not related, since at the $SU(5)\times U(1)_\chi$ level they originate from different Yukawa couplings. This is an important difference with the ordinary $SU(5)$. We know that in order to obtain the observed lepton and down quark mass spectrum at low energies, at the GUT scale the following relations should hold [70]

$$m_\tau = m_b, \qquad m_\mu = 3m_s. \tag{33}$$

In the ordinary $SU(5)$, the masses are related and the relations can be attributed to the Higgs adjoint which couples differently. This mechanism though is not operative in flipped $SU(5)$ due to the absence of the adjoint, as noticed above, thus the mass matrices are not related and Yukawas can be adjusted accordingly.

We further review the derivation of the matching condition between gauge couplings of $U(1)_Y$ and those of $SU(5)\times U(1)_\chi$. Computing the traces and finding normalization constants so that final trace is 2 give

$$C_y^2 y^2 = C_y^2 \frac{10}{3} = 2 \to C_y = \sqrt{\frac{3}{5}},$$
$$C_\chi^2 \chi^2 = C_\chi^2 20 = 2 \to C_\chi = \frac{1}{\sqrt{10}}. \tag{34}$$

In terms of normalized generators, $\tilde{Y} = \frac{1}{5C_y}(\tilde{y} + \kappa\tilde{\chi})$ where the ratio is $\kappa \equiv 2\frac{C_y}{C_\chi} = 2\sqrt{6}$. Finally $Y = \sqrt{\frac{3}{5}}\tilde{Y}$ implies

$$Y = \frac{1}{5}(\tilde{y} + 2\sqrt{6}\tilde{\chi}) \tag{35}$$

and for the $U(1)_Y$ gauge coupling

$$(1+\kappa^2)\frac{1}{a_Y} = \frac{1}{a_5} + \kappa^2\frac{1}{a_\chi} \tag{36}$$

or equivalently,

$$\frac{1}{a_Y} = \frac{1}{25}\frac{1}{a_5} + \frac{24}{25}\frac{1}{a_\chi}. \tag{37}$$

For initial values $a_\chi = a_5$, we obtain the standard relation of $SU(5)$. In general, however, $a_\chi \neq a_5$, and there is more flexibility.

B. Modular-invariant flipped $SU(5)$

Working in the flipped $SU(5)$ framework, we introduce modular invariance in the flavor space with chiral matter fields arranged as multiplets of A_4. Assignments of matter and Higgs fields in $SU(5)\times U(1)_\chi$ and A_4 are shown in Table I. In addition, a weighton ξ, which is a singlet scalar with non-trivial modular weight, is introduced to generate fermion mass hierarchies [30,59].

We discuss the constraint of the modular symmetry to the flavor structure. For the up quarks, the Yukawa superpotential, as shown in Eq. (28), takes the form $F\bar{f}\tilde{\Phi}$. Now including the flavor indices and constraining the superpotential by the modular invariance, we obtain the most general modular-invariant superpotential terms generating quark masses

$$\begin{aligned}\mathcal{W}_u \supset & \sum_{i=1,2,3}\lambda_{3i}^u\tilde{\xi}^{3-i}Y_{\mathbf{3}}^{(2)}F_i\bar{f}\tilde{\Phi} + \sum_{i=1,2}\lambda_{2i}^u\tilde{\xi}^{5-i}Y_{\mathbf{3}}^{(4)}F_i\bar{f}\tilde{\Phi} \\ & + \lambda_{11}^u\tilde{\xi}^6 Y_{\mathbf{3},1}^{(6)}F_1\bar{f}\tilde{\Phi} + \lambda_{12}^u\tilde{\xi}^6 Y_{\mathbf{3},2}^{(6)}F_1\bar{f}\tilde{\Phi}, \\ \mathcal{W}_d \supset & \sum_{i,j=1,2,3}\lambda_{ij}^d\tilde{\xi}^{8-i-j}F_iF_j\Phi,\end{aligned} \tag{38}$$

where λ_{ij}^u and $\lambda_{ij}^d = \lambda_{ji}^d$ are free parameters and $\tilde{\xi} \equiv \xi/\Lambda$. Subdominant terms such as $\tilde{\xi}^5 Y_3^{(6)}F_2\bar{f}\tilde{\Phi}$ and $\tilde{\xi}^6 Y_3^{(6)}F_3\bar{f}\tilde{\Phi}$ are possible in superpotential. However, these terms do not lead to significant deviations and thus we did not write them explicitly. These terms generate hierarchical Yukawa structures for quarks after the weighton ξ acquires the VEV v_ξ. We write out Y_d and Y_u up to ϵ^6 (with $\epsilon \equiv v_\xi/\Lambda$) as

TABLE I. Transformation properties of leptons, Yukawa couplings and right-handed neutrino masses in $SU(5)\times U(1)_\chi \times A_4$, where $2k$ is the modular weight. $\bar{f}$ in the flavor space is arranged as $\bar{f} = \{\bar{f}_1, \bar{f}_3, \bar{f}_2\}$. Apart from the fermions and Higgs superfields, we also include a weighton superfield ξ.

Fields	$SU(5)\times U(1)_\chi$	A_4	$2k$
$F_1 = \{Q_1, d_1^c, \nu_1^c\}$	$(\mathbf{10}, -\frac{1}{2})$	$\mathbf{1}$	+3
$F_2 = \{Q_2, d_2^c, \nu_2^c\}$	$(\mathbf{10}, -\frac{1}{2})$	$\mathbf{1}$	+2
$F_3 = \{Q_3, d_3^c, \nu_3^c\}$	$(\mathbf{10}, -\frac{1}{2})$	$\mathbf{1}$	+1
$\bar{f} = \{u^c, L\}$	$(\bar{\mathbf{5}}, +\frac{3}{2})$	$\mathbf{3}$	−2
$e_1^c = e^c$	$(\mathbf{1}, -\frac{5}{2})$	$\mathbf{1}'$	+6
$e_2^c = \mu^c$	$(\mathbf{1}, -\frac{5}{2})$	$\mathbf{1}''$	+4
$e_3^c = \tau^c$	$(\mathbf{1}, -\frac{5}{2})$	$\mathbf{1}$	+2
ν_S	$(\mathbf{1}, 0)$	$\mathbf{3}$	0
H	$(\mathbf{10}, -\frac{1}{2})$	$\mathbf{1}$	0
$\bar{H}$	$(\overline{\mathbf{10}}, +\frac{1}{2})$	$\mathbf{1}$	−1
Φ	$(\mathbf{5}, +1)$	$\mathbf{1}$	0
$\tilde{\Phi}$	$(\bar{\mathbf{5}}, -1)$	$\mathbf{1}$	−1
ξ	$(\mathbf{1}, 0)$	$\mathbf{1}$	−1
$Y_{\mathbf{r}}^{2k}$	$(\mathbf{1}, 0)$	$\mathbf{r}$	$2k$

$$Y_u = \begin{pmatrix} \epsilon^6 Y_1^{u(6)} + \epsilon^4 \lambda_{21}^u Y_1^{(4)} + \epsilon^2 \lambda_{31}^u Y_1 & \epsilon^3 \lambda_{22}^u Y_1^{(4)} + \epsilon \lambda_{32}^u Y_1 & \lambda_{33}^u Y_1 \\ \epsilon^6 Y_2^{u(6)} + \epsilon^4 \lambda_{21}^u Y_2^{(4)} + \epsilon^2 \lambda_{31}^u Y_2 & \epsilon^3 \lambda_{22}^u Y_2^{(4)} + \epsilon \lambda_{32}^u Y_2 & \lambda_{33}^u Y_2 \\ \epsilon^6 Y_3^{u(6)} + \epsilon^4 \lambda_{21}^u Y_3^{(4)} + \epsilon^2 \lambda_{31}^u Y_3 & \epsilon^3 \lambda_{22}^u Y_3^{(4)} + \epsilon \lambda_{32}^u Y_3 & \lambda_{33}^u Y_3 \end{pmatrix}^\dagger,$$

$$Y_d = \begin{pmatrix} \lambda_{11}^d \epsilon^6 & \lambda_{12}^d \epsilon^5 & \lambda_{13}^d \epsilon^4 \\ \lambda_{12}^d \epsilon^5 & \lambda_{22}^d \epsilon^4 & \lambda_{23}^d \epsilon^3 \\ \lambda_{13}^d \epsilon^4 & \lambda_{23}^d \epsilon^3 & \lambda_{33}^d \epsilon^2 \end{pmatrix}^*, \tag{39}$$

where $Y_i^{(4)}$ for $i = 1, 2, 3$ represent the three components of modular form $Y_{\mathbf{3}}^{(4)}$ of weight 4, and $Y_i^{u(6)}$ represent three components of the linear combination of modular forms $\lambda_{11}^u Y_{\mathbf{3}_1}^{(6)} + \lambda_{12}^u Y_{\mathbf{3}_2}^{(6)}$ of weight 6. Here, we have written the Yukawa matrices in the left-right notation, where "$*$" and "$\dagger$" represent the complex and Hermitian conjugations, respectively. Y_u can be diagonalized via $Y_u = V_u \mathrm{diag}\{\tilde{y}_u, \tilde{y}_c, \tilde{y}_t\} V_u'^\dagger$, where both V_u and V_u' are unitary matrices. Y_d, as a complex and symmetric matrix, can be diagonalized via $Y_d = V_d \mathrm{diag}\{\tilde{y}_d, \tilde{y}_s, \tilde{y}_b\} V_d^T$. The quark mixing matrix is given by $V_{\mathrm{CKM}} = V_u^\dagger V_d$. The complexity of Y_u can be further addressed in the case of small ϵ. Then, Y_u is approximatively written to be

$$Y_u \approx \left[\begin{pmatrix} 1 & \epsilon \frac{\lambda_{11}^u}{\lambda_{22}^u} & \epsilon^2 \frac{\lambda_{31}^u}{\lambda_{33}^u} \\ -\epsilon \frac{\lambda_{11}^u}{\lambda_{22}^u} & 1 & \epsilon \frac{\lambda_{32}^u}{\lambda_{33}^u} \\ -\epsilon^2 \frac{\lambda_{31}^u}{\lambda_{33}^u} & -\epsilon \frac{\lambda_{32}^u}{\lambda_{33}^u} & 1 \end{pmatrix} \begin{pmatrix} \epsilon^6 Y_1^{(6)} & \epsilon^6 Y_2^{(6)} & \epsilon^6 Y_3^{(6)} \\ \epsilon^3 \lambda_{22}^u Y_1^{(4)} & \epsilon^3 \lambda_{22}^u Y_2^{(4)} & \epsilon^3 \lambda_{22}^u Y_3^{(4)} \\ \lambda_{33}^u Y_1 & \lambda_{33}^u Y_2 & \lambda_{33}^u Y_3 \end{pmatrix} \right]^*. \tag{40}$$

Eigenvalues of Y_u, i.e., Yukawa couplings of u, c and t, can be analytically derived accordingly,

$$\begin{aligned} \tilde{y}_u &\approx \epsilon^6 \left[Y^{u(6)} \cdot Y^{(6)} - \frac{|Y^{u(6)} \cdot Y^{(2)}|^2}{Y^{(2)} \cdot Y^{(2)}} \right. \\ &\quad \left. - \frac{(Y^{(2)} \cdot Y^{(2)})(Y^{u(6)} \cdot Y^{(4)}) - (Y^{u(6)} \cdot Y^{(2)})(Y^{(2)} \cdot Y^{(4)})}{(Y^{(2)} \cdot Y^{(2)})(Y^{(4)} \cdot Y^{(4)}) - |Y^{(4)} \cdot Y^{(2)}|^2} \right]^{1/2}, \\ \tilde{y}_c &\approx \epsilon^3 |\lambda_{22}^u| \left[Y^{(4)} \cdot Y^{(4)} - \frac{|Y^{(4)} \cdot Y^{(2)}|^2}{Y^{(2)} \cdot Y^{(2)}} \right]^{1/2}, \\ \tilde{y}_t &\approx |\lambda_{33}^u| \sqrt{Y^{(2)} \cdot Y^{(2)}}, \end{aligned} \tag{41}$$

where $Y^{(2)} = (Y_1, Y_2, Y_3)^T$, and the dot between two vectors a and b denotes the product $\sum_i a_i b_i^*$. This expression shows that Yukawa couplings of the first, second and third generation up quarks are determined by modular forms of weights $2k = 6, 4, 2$, respectively.

In the charged lepton sector, the superpotential terms to generate charged lepton masses are given by

$$\mathcal{W}_l = \lambda_e \tilde{\xi}^6 Y_{\mathbf{3}}^{(2)} \bar{f} e^c \Phi + \lambda_\mu \tilde{\xi}^4 Y_{\mathbf{3}}^{(2)} \bar{f} \mu^c \Phi + \lambda_\tau \tilde{\xi}^2 Y_{\mathbf{3}}^{(2)} \bar{f} \tau^c \Phi. \tag{42}$$

Here, λ_e, λ_μ and λ_τ are dimensionless coefficients which can always be kept real by rotating phases of e^c, μ^c and τ^c. The Yukawa matrix is given by

$$Y_e = \begin{pmatrix} \epsilon^6 \lambda_e Y_3 & \epsilon^4 \lambda_\mu Y_2 & \epsilon^2 \lambda_\tau Y_1 \\ \epsilon^6 \lambda_e Y_1 & \epsilon^4 \lambda_\mu Y_3 & \epsilon^2 \lambda_\tau Y_2 \\ \epsilon^6 \lambda_e Y_2 & \epsilon^4 \lambda_\mu Y_1 & \epsilon^2 \lambda_\tau Y_3 \end{pmatrix}^*. \tag{43}$$

Approximatively, the three eigenvalues are given by

$$\begin{aligned} \tilde{y}_e &\approx \epsilon^6 \lambda_e \left[\frac{|Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3|^2}{|Y^{(2)} \cdot Y^{(2)}|^2 - |Y^{(2)\prime} \cdot Y^{(2)}|^2} \right]^{1/2}, \\ \tilde{y}_\mu &\approx \epsilon^4 \lambda_\mu \left[Y^{(2)} \cdot Y^{(2)} - \frac{|Y^{(2)\prime} \cdot Y^{(2)}|^2}{Y^{(2)} \cdot Y^{(2)}} \right]^{1/2}, \\ \tilde{y}_\tau &\approx \epsilon^2 \lambda_\tau \sqrt{Y^{(2)} \cdot Y^{(2)}}, \end{aligned} \tag{44}$$

where $Y^{(2)\prime} = (Y_2, Y_3, Y_1)^T$. Different from the up quarks, Yukawa couplings of charged leptons are determined by only the modular forms of weight $2k = 2$.

In the neutrino sector, the Yukawa matrix Y_D between ν and ν^c is obtained from the terms $\mathcal{W}_u$, which lead to the Yukawa matrix relation $Y_D = Y_u^T$. Majorana masses for ν^c are generated via

$$\mathcal{W}_{\nu^c} = \sum_{i,j=1,2,3} \frac{\lambda^c_{ij}}{\Lambda_c} \tilde{\xi}^{6-i-j} F_i F_j \bar{H} \bar{H} + \cdots, \tag{45}$$

where the dots represent negligible terms such as $\tilde{\xi}^{10-i-j} F_i F_j \bar{H} \bar{H}$, which are also allowed by modular invariance but further suppressed by at least ϵ^4. $\mathcal{W}_{\nu^c}$ leads to the Majorana mass matrix for ν^c

$$M_R = \frac{\langle \bar{\tilde{\nu}}^c_H \rangle^2}{\Lambda_c} \begin{pmatrix} \lambda^c_{11}\epsilon^4 & \lambda^c_{12}\epsilon^3 & \lambda^c_{13}\epsilon^2 \\ \lambda^c_{12}\epsilon^3 & \lambda^c_{22}\epsilon^2 & \lambda^c_{23}\epsilon \\ \lambda^c_{13}\epsilon^2 & \lambda^c_{23}\epsilon & \lambda^c_{33} \end{pmatrix}^*. \tag{46}$$

The light neutrino masses, after integrating out ν^c, are generated via the type-I seesaw formula, i.e.,

$$M_\nu = -M_D M_R^{-1} M_D^T. \tag{47}$$

Note that nonrenormalizable superpotential terms as

$$\sum_{i=0}^{\infty} \tilde{\xi}^{3+2i} Y_{\mathbf{1}}^{(4+2i)} \Lambda \Phi \tilde{\Phi} \tag{48}$$

cannot be forbidden by the modular symmetry. These terms, they lead to Higgs mass $\mu \sim \tilde{\xi}^3 \Lambda$ if $Y_{\mathbf{1}}^{(4)} \neq 0$ and at some stabilizers [25,71], apply, $Y_{\mathbf{1}}^{(4)} = 0$ and may be satisfied, $\mu \sim \tilde{\xi}^5 \Lambda$. These suppressions are not enough to restrict $\mu \sim$ TeV scale. It is possible to introduce an additional Z_2 symmetry (with $\tilde{\Phi}$, $\bar{f}$ and e^c_i be Z_2-odd and the other particles Z_2-even) beyond the modular symmetry which can forbid these terms and does not lead to additional corrections to the flavor structure.

C. A mechanism for incomplete representations

As already noted, spontaneous symmetry breaking of the plain field theory flipped $SU(5)$ model down to the SM symmetry, entails certain fermion mass relations emanating from their common origin in the original $SU(5) \times U(1)$-invariant Yukawa lagrangian. Indeed, recall that in the present model, the up-quark and Dirac neutrinos satisfy the relation $m_D^T = m_u$ at the GUT scale. When seeking an ultraviolet completion of the model, however, this is not always true. In string theory constructions, such as the heterotic and F-theory models, quite often the various GUT representations accommodating the MSSM fields are truncated by stringy projection mechanisms, magnetic fluxes etc.,[2] and as a result, such strict relations among the mass matrices are no longer true. In the case of F-theory constructions, in particular, this observation can be illustrated with the following example. Within a generic context of F-theory constructions, matter representations are trapped on the various intersections of the GUT 'surface' wrapped by the appropriate number of 7-branes, with other 7-branes perpendicular to the GUT divisor. As a consequence, an equal number of two-dimensional Riemann surfaces usually called 'matter curves' is formed where the GUT symmetry is further enhanced. In the simplest scenario, each one of the matter curves and the GUT representation residing on them, are characterized by distinct $U(1)$ 'charges' associated with the Cartan algebra of some covering group. For the sake of the argument, therefore, let us assume now that there are M_{10} copies of chiral ten-plets on an appropriate matter curve, i.e., $\#(\mathbf{10}_{-\frac{1}{2}} - \overline{\mathbf{10}}_{\frac{1}{2}}) = M_{10}$ and analogously M_1, M_2 copies of five-plets on two other intersections, $\#(\bar{\mathbf{5}}^{(1)}_{\frac{3}{2}} - \mathbf{5}^{(1)}_{-\frac{3}{2}})$, $\#(\bar{\mathbf{5}}^{(2)}_{-\frac{3}{2}} - \mathbf{5}^{(2)}_{\frac{3}{2}})$ with all of them accommodating fermion generations. Turing on a hypercharge flux of N, N_1, N_2 units respectively, we obtain:

$$\mathbf{10}_i = \begin{cases} (\mathbf{3},2)_i, & M_{10} \\ (\bar{\mathbf{3}},1)_i, & M_{10}+N\,, \\ (\mathbf{1},1)_i, & M_{10}-N \end{cases} \quad \bar{\mathbf{5}}_1 = \begin{cases} (\bar{\mathbf{3}},1)_1, & M_1 \\ (\mathbf{1},2)_1, & M_1 - N_1 \end{cases},$$
$$\bar{\mathbf{5}}_2 = \begin{cases} (\bar{\mathbf{3}},1)_2, & M_2 \\ (\mathbf{1},2)_2, & M_2 - N_2 \end{cases}. \tag{49}$$

As an example we take $M_{10} = 3, N = 0, M_1 = 1, M_2 = 0, N_1 = -N_2 = 1$, thence $\bar{\mathbf{5}}_1 \to (u^c, 0)$, $\bar{\mathbf{5}}_2 \to (0, L)$, and

$$\mathbf{10}_i \bar{\mathbf{5}}_1 \bar{\mathbf{5}}_h \to Q_i u^c_1 h_u, \qquad \mathbf{10}_i \bar{\mathbf{5}}_2 \bar{\mathbf{5}}_h \to \nu^c_i L_2 h_u. \tag{50}$$

Thus, this way the Dirac neutrino Yukawa coupling is splitted from the up quark Yukawa coupling.

Furthermore, it is worth to note that including multiplicities of matter field representations F, $\bar{f}$ and e^c can be embedded into different **16**'s of $SO(10)$, and thus the modular-invariant flipped $SU(5)$ model can be embedded into a modular-invariant $SO(10)$.

In any case, we can write out the superpotential terms $\mathcal{W}_D$ with coefficients independent of from those in $\mathcal{W}_u$,

$$\mathcal{W}_D = \sum_{i=1,2,3} \lambda^D_{3i} \tilde{\xi}^{3-i} Y_{\mathbf{3}}^{(2)} F_i \bar{f}\, \tilde{\Phi} + \sum_{i=1,2} \lambda^D_{2i} \tilde{\xi}^{5-i} Y_{\mathbf{3}}^{(4)} F_i \bar{f}\, \tilde{\Phi}$$
$$+ \lambda^D_{11} \tilde{\xi}^6 Y^{(6)}_{\mathbf{3},1} F_1 \bar{f}\, \tilde{\Phi} + \lambda^D_{12} \tilde{\xi}^6 Y^{(6)}_{\mathbf{3},2} F_1 \bar{f}\, \tilde{\Phi}\,. \tag{51}$$

[2]See for example [72].

Then, we arrive at a more generalized matrix where the dimensionless coefficients could be different from those in Y_u, i.e.,

$$Y_D = \begin{pmatrix} \epsilon^6 Y_1^{D(6)} + \epsilon^4 \lambda_{21}^D Y_1^{(4)} + \epsilon^2 \lambda_{31}^D Y_1 & \epsilon^3 \lambda_{22}^D Y_1^{(4)} + \epsilon \lambda_{32}^D Y_1 & \lambda_{33}^D Y_1 \\ \epsilon^6 Y_2^{D(6)} + \epsilon^4 \lambda_{21}^D Y_2^{(4)} + \epsilon^2 \lambda_{31}^D Y_2 & \epsilon^3 \lambda_{22}^D Y_2^{(4)} + \epsilon \lambda_{32}^D Y_2 & \lambda_{33}^D Y_2 \\ \epsilon^6 Y_3^{D(6)} + \epsilon^4 \lambda_{21}^D Y_3^{(4)} + \epsilon^2 \lambda_{31}^D Y_3 & \epsilon^3 \lambda_{22}^D Y_3^{(4)} + \epsilon \lambda_{32}^D Y_3 & \lambda_{33}^D Y_3 \end{pmatrix}^*, \tag{52}$$

where $Y_i^{D(6)}$ represent the linear combination of modular forms $\lambda_{11}^D Y_{\mathbf{3}_1}^{(6)} + \lambda_{12}^D Y_{\mathbf{3}_2}^{(6)}$ of weight 6. We consider an economical case that the first two generations of u^c and L are split but the third generation does not. Thus, λ_{ij}^D are independent of λ_{ij}^u except λ_{33}^D. Although Y_D is still hierarchical, the relaxing of the coefficients makes the model easier to fit the numerical data, as will be discussed in the next subsection. M_ν in this case is estimated to be

$$M_\nu \sim \begin{pmatrix} \mathcal{O}(\epsilon^4) & \mathcal{O}(\epsilon^3) & \mathcal{O}(\epsilon^2) \\ \mathcal{O}(\epsilon^3) & \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^1) \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^1) & \mathcal{O}(\epsilon^0) \end{pmatrix}. \tag{53}$$

This kind of mass structure predicts masses $m_1 : m_2 : m_3 \sim \epsilon^4 : \epsilon^2 : 1$ and therefore, normal ordering ($m_1 < m_2 < m_3$) for neutrino masses. However, in order to generate the correct mass difference ratio $\Delta m_{21}^2 / |\Delta m_{31}^2| \sim 0.03$, a fine-tuning of order 10^{-2} is required.

D. Numerical analysis

We perform a χ^2 analysis at the GUT scale to show how well the model fits the data.

In quark and charged lepton sectors, we apply the following best-fit and $\pm 1\sigma$ values as inputs of Yukawa couplings at the GUT scale

$$\begin{aligned} &\tilde{y}_u = (2.92 \pm 1.81) \times 10^{-6}, \quad \tilde{y}_c = (1.43 \pm 0.100) \times 10^{-3}, \quad \tilde{y}_t = 0.534 \pm 0.0341, \\ &\tilde{y}_d = (4.81 \pm 1.06) \times 10^{-6}, \quad \tilde{y}_s = (9.52 \pm 1.03) \times 10^{-5}, \quad \tilde{y}_b = (6.95 \pm 0.175) \times 10^{-3}, \\ &\tilde{y}_e = (1.97 \pm 0.0236) \times 10^{-6}, \quad \tilde{y}_\mu = (4.16 \pm 0.0497) \times 10^{-4}, \quad \tilde{y}_\tau = (7.07 \pm 0.0727) \times 10^{-3}. \end{aligned} \tag{54}$$

These data were derived from a minimal SUSY breaking scenario, with $\tan\beta = 5$ [18,30,55,73]. They are insensitive to the exact values of $\tan\beta$ unless a very large $\tan\beta$ is taken. Three mixing angles and one CP-violating phase in the CKM mixing matrix are applied from the same literature,

$$\begin{aligned} &\theta_{12}^q = 13.027° \pm 0.0814°, \quad \theta_{23}^q = 2.054° \pm 0.384°, \quad \theta_{13}^q = 0.1802° \pm 0.0281°, \\ &\delta^q = 69.21° \pm 6.19°. \end{aligned} \tag{55}$$

For neutrino masses and lepton mixing, we take global best-fit values (without including SK atmospheric data) from NuFIT 5.0 [74,75] and average the positive and negative 1σ errors.

$$\begin{aligned} &\Delta m_{21}^2 = (7.42 \pm 0.21) \times 10^{-5}\ \mathrm{eV}^2, \quad \Delta m_{31}^2 = (2.514 \pm 0.028) \times 10^{-3}\ \mathrm{eV}^2 \\ &\theta_{12} = 33.44° \pm 0.77°, \quad \theta_{23} = 49.0° \pm 1.3°, \quad \theta_{13} = 8.57° \pm 0.13°, \end{aligned} \tag{56}$$

for the normal ordering (NO, i.e., $m_1 < m_2 < m_3$) of neutrino masses. For $\tan\beta \lesssim 10$, the RG running effect mainly leads to an small overall enhancement to the neutrino mass scale but has negligible correction to the flavor structure due to the suppression of the charged lepton Yukawa couplings [76]. Thus, in this paper, we will directly use the measured values of lepton mixing angles and mass squared differences in the numerical fit.

We define the following two χ^2 functions in the quark sector and lepton sector, respectively,

$$\begin{aligned} \chi_q^2 &= \sum_{i \in O_q} \left(\frac{p_i(P_q) - b_i}{\sigma_i} \right)^2, \\ \chi_l^2 &= \sum_{i \in O_l} \left(\frac{p_i(P_l) - b_i}{\sigma_i} \right)^2, \end{aligned} \tag{57}$$

where p_i are the model predictions, b_i the current best-fit values and the errors σ_i correspond here to the average of the 1σ ranges for each observable.

The relevant free parameters and observables are listed in sets

$$P_q = \{\lambda^u_{11}, \lambda^u_{12}, \lambda^u_{21}, \lambda^u_{22}, \lambda^u_{31}, \lambda^u_{32}, \lambda^u_{33}, \lambda^d_{11}, \lambda^d_{12}, \lambda^d_{13}, \lambda^d_{22}, \lambda^d_{23}, \lambda^d_{33}, \epsilon, \tau\},$$
$$O_q = \{\tilde{y}_u, \tilde{y}_c, \tilde{y}_t, \tilde{y}_d, \tilde{y}_s, \tilde{y}_b, \theta^q_{12}, \theta^q_{23}, \theta^q_{13}, \delta^q\}, \quad (58)$$

and

$$P_l = \{\lambda^D_{11}, \lambda^D_{12}, \lambda^D_{21}, \lambda^D_{22}, \lambda^D_{31}, \lambda^D_{32}, \lambda^D_{33}, \lambda_e, \lambda_\mu, \lambda_\tau, \lambda^c_{11}, \lambda^c_{12}, \lambda^c_{13}, \lambda^c_{22}, \lambda^c_{23}, \lambda^c_{33}, \epsilon, \tau\},$$
$$O_l = \{\tilde{y}_e, \tilde{y}_\mu, \tilde{y}_\tau, \Delta m^2_{21}, \Delta m^2_{31}, \theta_{12}, \theta_{23}, \theta_{13}\}, \quad (59)$$

respectively. Here, we ignore the contribution of ν_S to the neutrino masses. All coefficients in P_q and P_l, i.e., λ^u_{ij}, λ^d_{ij}, λ^D_{ij}, λ^c_{ij}, and $\lambda_{e,\mu,\tau}$, are scanned with absolute values in the range (0,1) with arbitrary phases in the range $(0, 2\pi)$. In the set P_q and P_l, same values of ϵ and τ are respectively used. ϵ is scanned in (0,0.2) and τ is scanned in the fundamental domain of $\bar{\Gamma}(3)$. The latter is achieved by first scanning in the fundamental domain of Γ and then shifted to the rest region following modular transformations of the finite modular group Γ_3, i.e., $\tau \to \gamma\tau$ for all γ in Γ_3. Furthermore, $\lambda^D_{33} \equiv \lambda^u_{33}$ is always fixed in the scan as discussed before.

Due to the large parameter space, a full scan in all parameter space is hard to be performed. Instead, we listed two benchmark points in Tables II and III, respectively, as representatives. Both benchmarks fit the numerical values very well $\chi^2_q + \chi^2_l < 10$. In both cases, the second octant of θ_{23}, i.e., $\theta_{23} > 45°$ is predicted and the leptonic *CP*-violation is small $|\delta| < 10°$. In the second benchmark,

TABLE II. Inputs and predictions in Benchmark 1.

χ^2_q	3.18401	χ^2_l	5.06176
λ^u_{11}	$-0.767327 + 0.349061i$	λ^D_{11}	$0.339745 + 0.27744i$
λ^u_{12}	$0.263207 + 0.950138i$	λ^D_{12}	$0.399021 - 0.508852i$
λ^u_{21}	$-0.00316804 + 0.0173877i$	λ^D_{21}	$0.00175927 - 0.00351132i$
λ^u_{22}	$0.0567532 - 0.0424349i$	λ^D_{22}	$0.0388083 + 0.968816i$
λ^u_{31}	$0.0104356 - 0.0229778i$	λ^D_{31}	$0.239276 - 0.906576i$
λ^u_{32}	$-0.116958 - 0.00487694i$	λ^D_{32}	$0.0908708 + 0.0193008i$
λ^u_{33}	$-0.0061682 - 0.123591i$	λ^D_{33}	$\equiv\lambda^u_{33}$
λ^d_{11}	$0.159578 + 0.124667i$	λ^c_{11}	$0.283262 + 0.102305i$
λ^d_{12}	$-1.0342 + 0.0286299i$	λ^c_{12}	$0.102422 + 0.805828i$
λ^d_{13}	$0.891457 - 0.105821i$	λ^c_{13}	$0.35523 - 0.179145i$
λ^d_{22}	$-0.314388 - 0.432018i$	λ^c_{22}	$0.0475048 + 0.162506i$
λ^d_{23}	$0.0512689 + 0.365322i$	λ^c_{23}	$-0.791368 - 0.16351i$
λ^d_{33}	$0.0823574 + 0.614062i$	λ^c_{33}	$0.537351 + 0.829515i$
τ	$1.48709 + 0.310071i$	λ_e	0.343493
ϵ	0.10566	λ_μ	0.805984
		λ_τ	0.15144
$\tilde{y}_d$	2.9996×10^{-6}	$\tilde{y}_e$	1.974×10^{-6}
$\tilde{y}_s$	0.0000958	$\tilde{y}_\mu$	0.0004173
$\tilde{y}_b$	0.0069456	$\tilde{y}_\tau$	0.0070656
$\tilde{y}_u$	2.7243×10^{-6}	m_1	0.00086566 eV
$\tilde{y}_c$	0.00143656	m_2	0.00866672 eV
$\tilde{y}_t$	0.519732	m_3	0.0501839 eV
θ^q_{12}	13.0452°	θ_{12}	33.272°
θ^q_{23}	2.06442°	θ_{23}	46.1526°
θ^q_{13}	0.17796°	θ_{13}	8.60762°
δ^q	68.4553°	δ	−7.97559°
		M_1	4.66397×10^{8} GeV
		M_2	1.25924×10^{11} GeV
		M_3	1.5415×10^{13} GeV

TABLE III. Inputs and predictions in Benchmark 2.

χ^2_q	8.55801	χ^2_l	1.24054
λ^u_{11}	$-0.23631 + 0.0206202i$	λ^D_{11}	$-0.0370347 - 0.133943i$
λ^u_{12}	$-0.457106 - 0.212065i$	λ^D_{12}	$0.00890465 - 0.0645904i$
λ^u_{21}	$0.0547413 - 0.00985699i$	λ^D_{21}	$-0.587964 - 0.698841i$
λ^u_{22}	$0.20306 - 0.213503i$	λ^D_{22}	$-1.5934 + 3.98198i$
λ^u_{31}	$0.0535951 - 0.00835941i$	λ^D_{31}	$0.032806 + 0.101563i$
λ^u_{32}	$0.143194 - 0.0289485i$	λ^D_{32}	$-0.137639 + 0.0403966i$
λ^u_{33}	$0.147185 - 0.213406i$	λ^D_{33}	$\equiv\lambda^u_{33}$
λ^d_{11}	$0.872726 - 0.199401i$	λ^c_{11}	$-0.258699 - 0.149259i$
λ^d_{12}	$-0.022182 - 0.107737i$	λ^c_{12}	$0.644902 - 0.835538i$
λ^d_{13}	$0.555058 + 0.563634i$	λ^c_{13}	$0.786701 + 0.656483i$
λ^d_{22}	$0.165182 - 0.00517376i$	λ^c_{22}	$0.00748405 - 0.161169i$
λ^d_{23}	$0.0328347 + 0.434846i$	λ^c_{23}	$0.0404458 + 0.229882i$
λ^d_{33}	$0.224434 + 0.415195i$	λ^c_{33}	$-0.0466412 + 0.0379145i$
τ	$0.653628 + 1.25817i$	λ_e	0.340752
ϵ	0.121925	λ_μ	0.971434
		λ_τ	0.225573
$\tilde{y}_d$	3.40662×10^{-6}	$\tilde{y}_e$	1.96672×10^{-6}
$\tilde{y}_s$	0.000101448	$\tilde{y}_\mu$	0.000413715
$\tilde{y}_b$	0.00679233	$\tilde{y}_\tau$	0.00705526
$\tilde{y}_u$	3.67168×10^{-6}	m_1	0.00086566 eV
$\tilde{y}_c$	0.00140942	m_2	0.00866672 eV
$\tilde{y}_t$	0.571741	m_3	0.0501839 eV
θ^q_{12}	13.0549°	θ_{12}	33.8375°
θ^q_{23}	2.37476°	θ_{23}	49.9436°
θ^q_{13}	0.204838°	θ_{13}	8.5851°
δ^q	79.1935°	δ	−7.97559°
		M_1	2.26121×10^{10} GeV
		M_2	2.21381×10^{11} GeV
		M_3	1.24968×10^{12} GeV

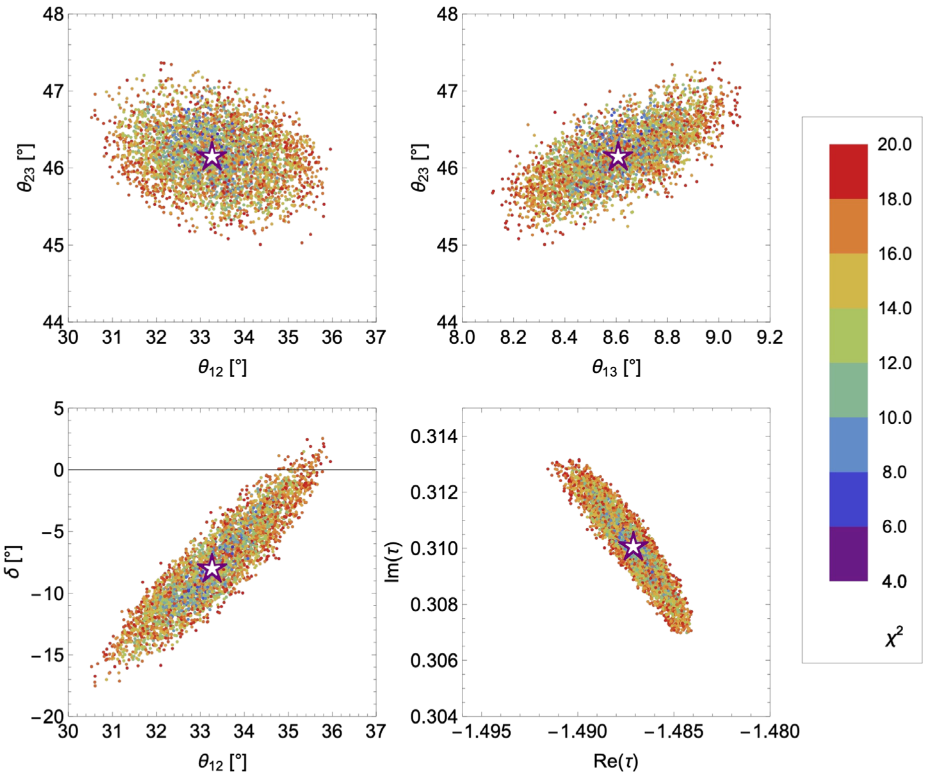

FIG. 1. Predictions of lepton mixing angles and the Dirac CP-violating oscillation phase from the scan around the first benchmark, where stars refer to predictions in the first benchmark.

hierarchical right-handed neutrinos masses with the lightest one $M_1 \sim 2 \times 10^{10}$ GeV is predicted. It can be applied for baryogenesis via thermal leptogenesis.[3] This benchmark has a relatively small coefficient $|\lambda_{33}^u| \sim 0.1$ as an input to predict the top Yukawa coupling $\tilde{y}_t \sim 0.5$. It is achieved due to the enhancement of a large value of the modular form $Y^{(2)} = (-0.0405235 - 0.0260606i, 0.205766 + 0.397288i, 0.180747 - 4.15085i)^T$ at $\tau = -1.48709 + 0.310071i$ (recall that $\tilde{y}_t \approx |\lambda_{33}^u| \sqrt{Y^{(2)} \cdot Y^{(2)}}$). We further make a scan around the first benchmark with all free parameters deviated by less than 1%. Predictions of mixing angles and δ are shown in Fig. 1.

IV. CONCLUSION

In this work, the long-standing problem of the origin of flavor mixing among the fermion families has been investigated within the framework of modular symmetries. Inspired by a wide class of string theory effective models endowed with modular invariance and simplicity of the Higgs sector to implement the symmetry breaking, we have constructed a flipped $SU(5)$ model with A_4 modular symmetry, assigning specific modular weights to the fermion and Higgs fields. In this context, Yukawa couplings are modular forms, with the three families (anti-) five-plets of $SU(5)$ transforming as a triplet under the A_4 modular symmetry, while the fermion ten-plets transform as singlet A_4 representations, distinguished by different modular weights, and the charged lepton electroweak singlets transform as nontrivial one-dimensional A_4 representations with different modular weights. The hierarchy of charged fermion masses is then achieved via higher dimensional operators coupled to a singlet weighton field which carries unit modular weight.

Flipped $SU(5)$ models exhibit many interesting features which make them attractive string motivated candidates as compared to standard $SU(5)$ GUTs. Among other merits, only a pair of $\mathbf{10} + \overline{\mathbf{10}}$ Higgs representations suffices to break the GUT symmetry. At the same time the down-type color triplets of this Higgs pair combine with those of $\mathbf{5} + \bar{\mathbf{5}}$ Higgs representations to realise the doublet-triplet splitting in an elegant manner. As for the mass matrix textures, because charged right-handed lepton fields are $SU(5)$ singlets, the charged lepton mass matrix is unrelated to that of the down quarks. This way the restrictive mass constraints of the ordinary $SU(5)$ are avoided and modular invariance is the main symmetry left over to organize the

[3]If the neutrino is a Majorana fermion, the neutrino and the antineutrino are the same particle. In this case, it becomes possible to convert it matter to antimatter and vice versa. Therefore, the existence of neutrino masses makes possible to create an imbalance between matter and antimatter in the early universe. A successful thermal leptogenesis requires $M_1 \gtrsim 10^9$ GeV.

charged fermion mass matrices. Regarding the neutral fermion sector, a notable property, not shared by the standard $SU(5)$, is that the right-handed neutrinos are contained in the ten-plet representation together with the quark doublets and the down-type color triplets. This implies the relation $Y_D = Y_u^T$ between Dirac neutrino and up quark Yukawa matrices and, in the simplest version of the model, this restrictive relation makes it difficult to fit the neutrino oscillation data.

In order to overcome this problem and avoid fine tuning issues, we appeal to a string-inspired mechanism according to which magnetic fluxes turned on along the Abelian subgroup of $SU(5)$ split the $SU(5)$-representations and disentangle the neutrino and up quark Yukawa couplings. Notwithstanding this mechanism, the Dirac neutrino mass matrix has a hierarchical structure, which can be partially canceled by the hierarchical Majorana mass matrix for right-handed neutrinos, resulting in a normally ordered and hierarchical pattern of light neutrino masses, with m_1 an order of magnitude smaller than m_2. This way, we are able to fit the neutrino oscillation data, only with a slight fine tuning of parameters in the neutrino sector. For the considered model, we find a good fit to charged quark and lepton masses, with a relatively low value of χ^2. The leptonic CP-violating oscillation phase is predicted to be $\delta = -8° \pm 8°$. A by-product of our approach is the prediction of hierarchical heavy neutrino masses. The lightest one may around 10^{10} GeV, which is around the correct value for standard thermal leptogenesis, which however we do not pursue further here.

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APPENDIX A: CONTRIBUTION OF SINGLET NEUTRINOS

At this point, we will study the case where, we take into account the presence of additional singlet neutrino superfields ν_S, as predicted string derived flipped $SU(5)$ models. In this case, more neutrinos mass terms exist,

$$\mathcal{W} \supset \sum_{i=1,2,3} \lambda_{3i}^S \xi^{3-i} Y_{\mathbf{3}}^{(2)} F_i \nu_S \bar{H} + \sum_{i=1,2} \lambda_{2i}^S \tilde{\xi}^{5-i} Y_{\mathbf{3}}^{(4)} F_i \nu_S \bar{H} + \lambda_{11}^S \tilde{\xi}^6 Y_{\mathbf{3},1}^{(6)} F_1 \nu_S \bar{H} + \lambda_{12}^S \tilde{\xi}^6 Y_{\mathbf{3},2}^{(6)} F_1 \nu_S \bar{H} + m_S \nu_S \nu_S. \tag{A1}$$

These terms, together with those in $\mathcal{W}_u$ superpotential, generate a general 9×9 mass matrix for neutrinos. In the basis (ν, ν^c, ν_S), this mass matrix is written as

$$\mathcal{M}_\nu^{9\times 9} = \begin{pmatrix} \mathbf{0} & M_D & \mathbf{0} \\ M_D^T & M_R & M_D' \\ \mathbf{0} & M_D'^T & M_S \end{pmatrix}, \tag{A2}$$

where all subblocks are 3×3 matrices. In particular,

$$M_D' = \langle \bar{\tilde{\nu}}_H^c \rangle \begin{pmatrix} \epsilon^6 Y_1^{S(6)} + \epsilon^4 \lambda_{21}^S Y_1^{(4)} + \epsilon^2 \lambda_{31}^S Y_1 & \epsilon^3 \lambda_{22}^S Y_1^{(4)} + \epsilon \lambda_{32}^S Y_1 & \lambda_{33}^S Y_1 \\ \epsilon^6 Y_2^{S(6)} + \epsilon^4 \lambda_{21}^S Y_2^{(4)} + \epsilon^2 \lambda_{31}^S Y_2 & \epsilon^3 \lambda_{22}^S Y_2^{(4)} + \epsilon \lambda_{32}^S Y_2 & \lambda_{33}^S Y_2 \\ \epsilon^6 Y_3^{S(6)} + \epsilon^4 \lambda_{21}^S Y_3^{(4)} + \epsilon^2 \lambda_{31}^S Y_3 & \epsilon^3 \lambda_{22}^S Y_3^{(4)} + \epsilon \lambda_{32}^S Y_3 & \lambda_{33}^S Y_3 \end{pmatrix}^\dagger,$$
$$M_S = m_S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{A3}$$

and $Y_i^{S(6)}$ for $i = 1, 2, 3$ are three components of $\lambda_{11}^S Y_{\mathbf{3},1}^{(6)} + \lambda_{12}^S Y_{\mathbf{3},2}^{(6)}$. The light neutrino mass matrix in this case is modified into [77]

$$M_\nu = -M_D (M_R - M_D' M_S^{-1} M_D'^T)^{-1} M_D^T. \tag{A4}$$

Without considering the flavor structure and assuming order-one coefficients, from the relation that applies to the double seesaw formula, $-M_D (M_R - M_D' M_S^{-1} M_D'^T)^{-1} M_D^T$, the heaviest eigenvalues of M_R and $M_D' M_S^{-1} M_D'^T$ are of order $\langle \bar{\tilde{\nu}}_H^c \rangle^2 / \Lambda_c$ and $\langle \bar{\tilde{\nu}}_H^c \rangle^2 / m_S$, respectively. Therefore, the light neutrino masses are a competition of the two scales Λ_c and m_S.

In our numerical analysis as discussed in Sec. III D, we have focused only in the scenario $m_S \gg \Lambda_c \sim 10^{16}$ GeV. Below, we give a brief discussion on the opposite scenario $m_S \ll \Lambda_c$. Namely, the light neutrino masses is given by $M_\nu = M_D (M_D' M_S^{-1} M_D'^T)^{-1} M_D^T$, which is also called the double seesaw formula. At leading order of ϵ, one can check that M_ν approximates to

$$M_\nu \sim \epsilon^{-4}\frac{m_S v_u^2}{\langle\bar{\tilde{\nu}}_H^c\rangle^2}\begin{pmatrix} Y_1^2 & Y_1Y_2 & Y_1Y_3 \\ Y_1Y_2 & Y_2^2 & Y_2Y_3 \\ Y_1Y_3 & Y_2Y_3 & Y_3^2 \end{pmatrix}^* \tag{A5}$$

up to an overall factor. It partially determines the flavor structure, but has only one nonzero eigenstate $\sim\epsilon^{-4}\frac{m_S v_u^2}{\langle\bar{\tilde{\nu}}_H^c\rangle^2}$. Including the next-to-leading correction, one obtain the second lightest neutrino has mass $\frac{m_S v_u^2}{\langle\bar{\tilde{\nu}}_H^c\rangle^2}$. Their hierarchy is too large to explain the ratio $|r| \equiv \Delta m_{21}^2/|\Delta m_{31}^2| \sim 0.03$ unless fine tuning between coefficients are considered.

APPENDIX B: THE SECOND MODEL

In this Appendix we present a second flipped $SU(5)$ model which differs from the first with respect to the representations of the A_4 symmetry and the modular weights assigned to the fields. The transformation properties of the spectrum are shown in Table IV.

The superpotential Yukawa couplings for the up and down quark are,

$$\begin{aligned}\mathcal{W}_u &= \lambda_1^u F_1\tilde{\Phi}\bar{f}\,Y_{\mathbf{3}}^{(4)}\tilde{\xi}^5 + \lambda_2^u F_2\tilde{\Phi}\bar{f}\,Y_{\mathbf{3}}^{(4)}\tilde{\xi} + \lambda_3^u F_3\tilde{\Phi}\bar{f}\,Y_{\mathbf{3}}^{(4)},\\ \mathcal{W}_d &= \lambda_{12}^d F_1F_2\Phi Y_{\mathbf{1}'}^{(4)}\tilde{\xi}^3 + \lambda_{13}^d F_1F_3\Phi Y_{\mathbf{1}}^{(4)}\tilde{\xi}^2\\ &\quad + \lambda_{22}^d F_2F_2\Phi Y_{\mathbf{1}}^{(8)}\tilde{\xi}^3 + \lambda_{23}^d F_2F_3\Phi Y_{\mathbf{1}''}^{(8)}\tilde{\xi}^2\\ &\quad + \lambda_{33}^d F_3F_3\Phi Y_{\mathbf{1}'}^{(8)}\tilde{\xi},\end{aligned} \tag{B1}$$

where λ_i^u and λ_{ij}^d are free parameters and $\tilde{\xi} \equiv \xi/\Lambda$, with Λ a dimensionful cutoff flavor scale. The corresponding Yukawa matrices Y_u and Y_d are

TABLE IV. The representations of the second model and their transformation properties under $SU(5)\times U(1)_\chi\times A_4$.

Fields	$SU(5)\times U(1)_\chi$	A_4	$2k$
$F_1=\{Q_1,d_1^c,\nu_1^c\}$	$(\mathbf{10},-\frac{1}{2})$	$\mathbf{1}$	$+1$
$F_2=\{Q_2,d_2^c,\nu_2^c\}$	$(\mathbf{10},-\frac{1}{2})$	$\mathbf{1}'$	-3
$F_3=\{Q_3,d_3^c,\nu_3^c\}$	$(\mathbf{10},-\frac{1}{2})$	$\mathbf{1}''$	-4
$\bar{f}=\{u^c,L\}$	$(\bar{\mathbf{5}},+\frac{3}{2})$	$\mathbf{3}$	-4
$e_1^c=e^c$	$(\mathbf{1},-\frac{5}{2})$	$\mathbf{1}''$	$+4$
$e_2^c=\mu^c$	$(\mathbf{1},-\frac{5}{2})$	$\mathbf{1}$	$+3$
$e_3^c=\tau^c$	$(\mathbf{1},-\frac{5}{2})$	$\mathbf{1}'$	$+1$
ν_S	$(\mathbf{1},0)$	$\mathbf{3}$	0
H	$(\mathbf{10},-\frac{1}{2})$	$\mathbf{1}'$	0
$\bar{H}$	$(\overline{\mathbf{10}},+\frac{1}{2})$	$\mathbf{1}$	-2
Φ	$(\mathbf{5},+1)$	$\mathbf{1}'$	$+1$
$\tilde{\Phi}$	$(\bar{\mathbf{5}},-1)$	$\mathbf{1}$	$+4$
ξ	$(\mathbf{1},0)$	$\mathbf{1}$	-1
$Y_{\mathbf{r}}^{2k}$	$(\mathbf{1},0)$	$\mathbf{r}$	$2k$

$$Y_u = \begin{pmatrix} \lambda_1^u Y_1^{(4)}\tilde{\xi}^5 & \lambda_2^u Y_3^{(4)}\tilde{\xi} & \lambda_3^u Y_2^{(4)} \\ \lambda_1^u Y_3^{(4)}\tilde{\xi}^5 & \lambda_2^u Y_2^{(4)}\tilde{\xi} & \lambda_3^u Y_1^{(4)} \\ \lambda_1^u Y_2^{(4)}\tilde{\xi}^5 & \lambda_2^u Y_1^{(4)}\tilde{\xi} & \lambda_3^u Y_3^{(4)} \end{pmatrix}^\dagger,$$

$$Y_d = \begin{pmatrix} 0 & \lambda_{12}^d Y_{\mathbf{1}'}^{(4)}\tilde{\xi}^3 & \lambda_{13}^d Y_{\mathbf{1}}^{(4)}\tilde{\xi}^2 \\ \lambda_{12}^d Y_{\mathbf{1}'}^{(4)}\tilde{\xi}^3 & \lambda_{22}^d Y_{\mathbf{1}}^{(8)}\tilde{\xi}^3 & \lambda_{23}^d Y_{\mathbf{1}''}^{(8)}\tilde{\xi}^2 \\ \lambda_{13}^d Y_{\mathbf{1}}^{(4)}\tilde{\xi}^2 & \lambda_{23}^d Y_{\mathbf{1}''}^{(8)}\tilde{\xi}^2 & y_{33}^d Y_{\mathbf{1}'}^{(8)}\tilde{\xi} \end{pmatrix}^*, \tag{B2}$$

where $Y_i^{(4)}$ for $i = 1, 2, 3$ represent the three components of modular form $Y_{\mathbf{3}}^{(4)}$ of weight 4 and $Y_{\mathbf{r}}^{(2k)}$ are modular forms with weights $2k = 4$, 6, 8 and the corresponding representations of the A_4 group, $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$.

In the charged lepton sector, the superpotential terms generating the charged lepton masses,

$$\mathcal{W}_l = \lambda^e\bar{f}\Phi e^c Y_{\mathbf{3}}^{(4)}\tilde{\xi}^5 + \lambda^\mu\bar{f}\Phi\mu^c Y_{\mathbf{3}}^{(4)}\tilde{\xi}^3 + \lambda_\tau\bar{f}\Phi\tau^c Y_{\mathbf{3}}^{(4)}\tilde{\xi}^2, \tag{B3}$$

where, λ^e, λ^μ, λ^τ are dimensionless coefficients. So, we have the matrix,

$$Y_l = \begin{pmatrix} \lambda^e Y_1^{(4)}\tilde{\xi}^5 & \lambda_\mu Y_3^{(4)}\tilde{\xi}^4 & \lambda_\tau Y_2^{(4)}\tilde{\xi}^2 \\ \lambda_e Y_3^{(4)}\tilde{\xi}^5 & \lambda_\mu Y_2^{(4)}\tilde{\xi}^4 & \lambda_\tau Y_1^{(4)}\tilde{\xi}^2 \\ \lambda_e Y_2^{(4)}\tilde{\xi}^5 & \lambda_\mu Y_1^{(4)}\tilde{\xi}^4 & \lambda_\tau Y_3^{(4)}\tilde{\xi}^2 \end{pmatrix}. \tag{B4}$$

The superpotential terms in the neutrino sector are

$$\begin{aligned}\mathcal{W}_\nu &= \lambda_1^{u'} F_1\tilde{\Phi}\bar{f}\,Y_{\mathbf{3}}^{(4)}\tilde{\xi}^5 + \lambda_2^{u'} F_2\tilde{\Phi}\bar{f}\,Y_{\mathbf{3}}^{(4)}\tilde{\xi}\\ &\quad + \lambda_3^{u'} F_3\tilde{\Phi}\bar{f}\,Y_{\mathbf{3}}^{(4)} + \lambda_1^H F_1\bar{H}\nu_S Y_{\mathbf{3}}^{(6)}\tilde{\xi}^5\\ &\quad + \lambda_1^{H'} F_1\bar{H}\nu_S Y_{\mathbf{3}}^{(4)}\tilde{\xi}^3 + \lambda_2^H F_2\bar{H}\nu_S Y_{\mathbf{3}}^{(6)}\tilde{\xi}\\ &\quad + \lambda_3^H F_3\bar{H}\nu_S Y_{\mathbf{3}}^{(6)} + M_S\nu_S\nu_S.\end{aligned} \tag{B5}$$

From this superpotential, we find the matrices,

$$Y_D = \frac{y_u v_u}{\sqrt{2}}\begin{pmatrix} \lambda_1^u Y_1^{(4)}\tilde{\xi}^5 & \lambda_2^{u'} Y_3^{(4)}\tilde{\xi} & \lambda_3^u Y_2^{(4)} \\ \lambda_1^u Y_3^{(4)}\tilde{\xi}^5 & \lambda_2^{u'} Y_2^{(4)}\tilde{\xi} & \lambda_3^u Y_1^{(4)} \\ \lambda_1^u Y_2^{(4)}\tilde{\xi}^5 & \lambda_2^{u'} Y_1^{(4)}\tilde{\xi} & \lambda_3^u Y_3^{(4)} \end{pmatrix}^*,$$

$$M_D' = \langle\bar{\tilde{\nu}}_H^c\rangle\begin{pmatrix} \lambda_1 Y_1^{(6)H}\tilde{\xi}^5 + \lambda_1' Y_1^{(4)H}\tilde{\xi}^3 & \lambda_2 Y_3^{(6)H}\tilde{\xi} & \lambda_3 Y_2^{(6)H} \\ \lambda_1 Y_3^{(6)H}\tilde{\xi}^5 + \lambda_1' Y_3^{(4)H}\tilde{\xi}^3 & \lambda_2 Y_2^{(6)H}\tilde{\xi} & \lambda_3 Y_1^{(6)H} \\ \lambda_1 Y_2^{(6)H}\tilde{\xi}^5 + \lambda_1' Y_2^{(4)H}\tilde{\xi}^3 & \lambda_2 Y_1^{(6)H}\tilde{\xi} & \lambda_3 Y_3^{(6)H} \end{pmatrix}^\dagger,$$

$$M_S = m_S\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{B6}$$

where $Y_r^{(2k)H}$ with $r = 1, 2, 3$ and $2k = 4, 6$ represent three components of the linear combination of modular forms $Y_{\mathbf{3}_1}^{(2k)} + Y_{\mathbf{3}_2}^{(2k)}$.

In this model, we consider case that, only the second generation of u^c and L is splitted but the first and third generations do not. So, as we see in Eq. (B6) the free parameters λ_1^u and λ_3^u are the same as those for the domain of up quarks, while only the second parameter is different.

Also, Majorana masses for ν^c are generated via

$$\lambda_{ij}^{\nu} F_i F_j \bar{H} \bar{H} Y_{\mathbf{r}}^{(2k)} \tilde{\xi}^n, \tag{B7}$$

where λ_{ij}^{ν} are free parameters, with $i,\ j = 1,\ 2,\ 3$. Furthermore, $Y_{\mathbf{r}}^{2k}$ are modular forms with $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$ representations of A_4 symmetry and $2k$ are modular weights. We consider the limit $m_S \ll \Lambda_c$. In this case, light neutrino masses are given by the double seesaw formula,

$$M_\nu = Y_D (M'_D M_S^{-1} M_D'^{T})^{-1} Y_D^T. \tag{B8}$$

Note that nonrenormalizable superpotential terms as,

$$\Phi\tilde{\Phi} Y_{\mathbf{1}''}^{(8+2k)} \tilde{\xi}^{12+2k} \tag{B9}$$

for $2k = 2, 4, \ldots$, are suppressed due to the large power of $\tilde{\xi}$. This could also be forbidden by introducing additional Z_2 as discussed in the end of Sec. III B.

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