Analytic solutions of relativistic dissipative spin hydrodynamics with Bjorken expansion

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We have studied analytically the longitudinally boost-invariant motion of a relativistic dissipative fluid with spin. We have derived the analytic solutions of spin density and spin chemical potential as a function of proper time τ in the presence of the viscous tensor and the second order relaxation time corrections for spin. Interestingly, analogous to the ordinary particle number density and chemical potential, we find that the spin density and spin chemical potential decay as $\sim\tau^{-1}$ and $\sim\tau^{-1/3}$, respectively. These solutions can serve both to gain insight on the dynamics of spin polarization in relativistic heavy-ion collisions and as test beds for further numerical codes.

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I. INTRODUCTION

In the noncentral relativistic heavy-ion collisions, huge orbital angular momenta are generated and polarize the particles in the quark gluon plasma (QGP) through the spin-orbital couplings [1,2]. In 2017, the global polarization of Λ and $\bar{\Lambda}$ hyperons led by the initial huge orbital angular momentum has been measured by STAR experiments and been well understood by various phenomenological models [3–15]. The experimental results also indicate that QGP is the most vortical fluid [16] so far.

The experimental data in Au + Au collisions at 200 GeV for the local spin polarization, which is the azimuthal angle dependent spin polarization of the Λ and $\bar{\Lambda}$ hyperons along the beam and the out-plane directions [17], disagree with many phenomenological models, e.g., the relativistic hydrodynamics model [14,18] and transport models [4,12,19]. Later on, people find that this disagreement cannot be solved by considering the feed-down effects [20,21]. Although the kinetic theory of massless fermions in Ref. [22] and results from a simple phenomenological model in Ref. [23] can show the similar azimuthal angle dependence as experimental data, it is still a puzzle in the relativistic heavy-ion community.

Solving this puzzle requires a deeper understanding of spin effects in the QGP from both microscopic and macroscopic theories. One microscopic theory for massive fermions is the quantum kinetic theory for massive fermions, which is a natural extension of the chiral kinetic theory for massless fermions [24–38]. Recently, the quantum kinetic theory for massive fermions with the collisional kernel has been developed [39–47]. Also, see the early works on the statistical model for the relativistic particles with spin [48–50] and Ref. [51] for a microscopic model for spin polarization through particle collisions.

Very recently, many studies have shown that the polarization can also been induced by the shear viscous tensor, e.g., from early studies of massless fermions [52], the recent studies for massive fermions [53,54], and the statistic model [55]. Those studies [55–57] show quantitative agreement with experimental data by adding the shear induced polarization to the original studies [50,58].

The macroscopic theory for the spinness particles is the relativistic spin hydrodynamics, which is a theory with the relativistic hydrodynamics equations coupled to the conservation equation of total angular momentum. The expression of spin hydrodynamics has been derived from the entropy principle [59–63], the Lagrangian effective theory [64,65], kinetic approaches [9,11,66–69], and the general discussion from field theory [70]. The ideal spin hydrodynamics has been discussed in Refs. [9,11,59,62,64–69] and also see the early works on spin hydrodynamics in Refs. [5,6,9–11,71,72]. In the gradient expansion, the dissipative spin hydrodynamics are derived in canonical [59,69,73–75] and Belinfante forms [60–62]. Also, see recent reviews [76–80] and the references therein.

Analogous to works on the relativistic magnetohydrodynamics [81–86], it is necessary to derive the analytic solutions for the dissipative spin dynamics. These analytic solutions provide the power law behavior of the quantities related to the spin as a function of proper time τ, such as the spin density and spin chemical potential. On the other hand, the numerical codes for the dissipative spin hydrodynamics have not been developed yet in our community. It is worthwhile to search for analytic solutions for spin hydrodynamics in some simple, but nevertheless realistic, test cases.

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In this work, we study analytically the canonical relativistic spin hydrodynamics in the time-honored longitudinal boost-invariant Bjorken flow [87]. We also consider the viscous effects coupled to the spin hydrodynamics. Since relativistic hydrodynamics in the first order of the gradient expansion are unstable and violate the causality [88–94], we solve the equations of spin hydrodynamics coupled to the simplest relaxation time corrections for the spin dynamics. To see the spin corrections to the ordinary terms, we take the pseudogauge transformation and check the results in the Belinfante form of spin dynamics.

We emphasize that in the current work we are searching for the self-consistent analytic solutions of dissipative spin hydrodynamics, which are different from the studies of spin hydrodynamics in Bjorken [71,95,96] and Gubser [97] expanding backgrounds.

The structure of this work is as follows. In Sec. II, we will briefly review the conservation equations of dissipative spin hydrodynamics in both canonical and Belinfante forms. In Sec. III A, we study the canonical spin hydrodynamics with the nonvanishing viscous tensor in a longitudinally boost-invariant fluid and obtain the analytic solutions. Next, we consider the relaxation time equations for spin dynamics and derive the analytic solutions in the gradient expansion in Sec. III B. We also take the pseudogauge transformation and compute the Belinfante form spin hydrodynamics in Sec. III C and discuss our results in Sec. III D. We summarize in Sec. IV.

Throughout this work, we adopt the metric $g^{\mu\nu} = \mathrm{diag}\{+,-,-,-\}$ and the fluid velocity u^μ satisfying $u^2 = 1$. We define the projector $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. For simplicity, for an arbitrary tensor $A^{\mu\nu}$, we introduce the symmetric, antisymmetric, and traceless parts as $A^{(\mu\nu)} = (A^{\mu\nu} + A^{\nu\mu})/2$, $A^{[\mu\nu]} = (A^{\mu\nu} - A^{\nu\mu})/2$, and $A^{\langle\mu\nu\rangle} = \frac{1}{2}[\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\nu\alpha}\Delta^{\mu\beta}]A_{\alpha\beta} - \frac{1}{3}\Delta^{\mu\nu}(A^{\rho\sigma}\Delta_{\rho\sigma})$.

II. MAIN EQUATIONS IN RELATIVISTIC DISSIPATIVE SPIN HYDRODYNAMICS

In this section, we briefly review the main equations of dissipative spin hydrodynamics in both canonical and Belinfante forms. In the canonical form of relativistic spin hydrodynamics, the conserved total angular momentum (TAM) can be decomposed as

$$J^{\alpha\mu\nu} = x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu} + \Sigma^{\alpha\mu\nu}, \tag{1}$$

where $T^{\mu\nu}$ is the canonical energy momentum tensor (EMT) and $\Sigma^{\alpha\mu\nu}$ is a rank-three spin tensor corresponding to the classical spin part of TAM. Up to $\mathcal{O}(\partial^1)$ in the gradient expansion, the canonical EMT can be further decomposed as [59,60,62]

$$T^{\mu\nu} = eu^\mu u^\nu - p\Delta^{\mu\nu} + 2h^{(\mu}u^{\nu)} + 2q^{[\mu}u^{\nu]} + \pi^{\mu\nu} + \phi^{\mu\nu}, \tag{2}$$

where e, p, u^μ, h^μ, and $\pi^{\mu\nu}$ denote energy density, pressure, fluid velocity, heat flow, and viscous tensor, respectively. The q^μ and $\phi^{\mu\nu}$ are the antisymmetric EMT parts. Note that $h^\mu, q^\mu, \pi^{\mu\nu}$, and $\phi^{\mu\nu}$ are perpendicular to the fluid velocity u^μ.

Both EMT and TAM are conserved, i.e.,

$$\partial_\mu T^{\mu\nu} = 0,$$
$$\partial_\alpha J^{\alpha\mu\nu} = 0, \tag{3}$$

which lead to

$$\partial_\alpha \Sigma^{\alpha\mu\nu} = -2T^{[\mu\nu]}. \tag{4}$$

It means the antisymmetric part of EMT acts as a source or absorption term for spin current. Then we make a tensor decomposition for $\Sigma^{\alpha\mu\nu}$,

$$\Sigma^{\alpha\mu\nu} = u^\alpha S^{\mu\nu} + \Sigma^{\alpha\mu\nu}_{(1)}, \tag{5}$$

where $S^{\mu\nu}$ is named as spin density and $\Sigma^{\alpha\mu\nu}_{(1)}$ refers to the higher order correction with $u_\alpha\Sigma^{\alpha\mu\nu}_{(1)} = 0$. Inserting Eqs. (2) and (5) into Eq. (4) yields

$$\partial_\alpha(u^\alpha S^{\mu\nu}) = -2(q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}). \tag{6}$$

There are two different types of $\Sigma^{\alpha\mu\nu}$. For example, if we implement the Nöther theorem to the Lagrangian density of the noninteracting Dirac field, $\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial - m)\psi$, we will get $L^{\lambda\mu\nu} = \bar{\psi}i(\gamma^\lambda x^\mu\partial^\nu - \gamma^\lambda x^\nu\partial^\mu)\psi$ and $\Sigma^{\lambda\mu\nu} = \frac{1}{4}\bar{\psi}i\gamma^\lambda[\gamma^\mu,\gamma^\nu]\psi$. This $\Sigma^{\alpha\mu\nu}$, corresponding to Eq. (5), refers to a non-anti-symmetric gauge which has been used in Refs. [6,59,61–63,65,70] and also in spin hydrodynamics for massless fermions [98]. On the other hand, one can also use the symmetrized Lagrangian $\mathcal{L} = \frac{1}{2}\bar{\psi}(i\gamma\cdot\overrightarrow{\partial} - m)\psi - \frac{1}{2}\bar{\psi}(i\gamma\cdot\overleftarrow{\partial} - m)\psi$, and get a total antisymmetric spin tensor $\Sigma^{\lambda\mu\nu} = \frac{1}{8}\bar{\psi}i\{\gamma^\lambda,[\gamma^\mu,\gamma^\nu]\}\psi$. This kind of choice has been used in Refs. [68,75].

After introducing the spin degree of freedom, it is necessary to modify the thermodynamic relations. We treat spin density $S^{\mu\nu}$ as the particle number density; then, we also need to introduce the spin potential $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Now, the new thermodynamic relations become

$$e + p = Ts + \omega_{\mu\nu}S^{\mu\nu},$$
$$de = Tds + \omega_{\mu\nu}dS^{\mu\nu},$$
$$dp = sdT + S^{\mu\nu}d\omega_{\mu\nu}, \tag{7}$$

where s is the entropy density and T is the temperature. In general, we can also have the charge currents in a system. For simplicity, we neglect these currents throughout this work. For more general discussions, one can also see Refs. [59,60,62].

The second law of thermodynamics requires that the entropy production rate should always be positive definite. It gives the general expression for those dissipative terms. The q^μ and $\phi^{\mu\nu}$ are given by

$$q^\mu = \lambda\left[\frac{1}{T}\Delta^{\mu\alpha}\partial_\alpha T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu\right],$$
$$\phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - 2\frac{1}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right), \tag{8}$$

where $\Omega^{\mu\nu} = -\Delta^{\mu\rho}\Delta^{\nu\sigma}\frac{1}{2}[\partial_\rho(\frac{u_\sigma}{T}) - \partial_\sigma(\frac{u_\rho}{T})]$ is related to the thermal vorticity [50,58] and λ, $\gamma \geq 0$ are the transport coefficients. The ordinary viscous tensor $\pi^{\mu\nu}$ is given by

$$\pi^{\mu\nu} = \eta_s\partial^{\langle\mu}u^{\nu\rangle} - \Pi\Delta^{\mu\nu},$$
$$\Pi = -\zeta(\partial\cdot u), \tag{9}$$

where η_s and ζ are the shear and bulk viscosities, respectively.

Note that all of the above results based on the assumption that the $\omega^{\mu\nu}$ and/or $S^{\mu\nu}$ are at $\mathcal{O}(\partial^1)$. If both $\omega^{\mu\nu}$ and $S^{\mu\nu}$ are at the leading order of the gradient expansion, then the decomposition of EMT can be very different; also see Ref. [63].

The canonical EMT is gauge dependent at the operator level and not symmetric. One can choose the Belinfante form EMT $\mathcal{T}^{\mu\nu}$, which is gauge invariant and symmetric, through the following pseudogauge transformation:

$$\mathcal{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}, \tag{10}$$

with

$$K^{\lambda\mu\nu} = \frac{1}{2}(\Sigma^{\lambda\mu\nu} - \Sigma^{\mu\lambda\nu} + \Sigma^{\nu\mu\lambda}). \tag{11}$$

According to Ref. [61], up to $\mathcal{O}(\partial^1)$ we get

$$\begin{aligned}\mathcal{T}^{\mu\nu} &= eu^\mu u^\nu - p\Delta^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\\ &= (e+\delta e)u^\mu u^\nu - p\Delta^{\mu\nu} + 2(h^{(\mu} + \delta h^{(\mu})u^{\nu)}\\ &\quad + (\pi^{\mu\nu} + \delta\pi^{\mu\nu}),\end{aligned} \tag{12}$$

Here we have introduced the spin corrections to the energy density, δe, the heat flow, δh^μ, and the viscous tensor, $\delta\pi^{\mu\nu}$,

$$\delta e = u_\mu\partial_\lambda S^{\mu\lambda},$$
$$\delta h^\mu = \frac{1}{2}(\Delta^\mu_\beta\partial_\lambda S^{\beta\lambda} + u_\beta S^{\beta\lambda}\partial_\lambda u^\mu),$$
$$\delta\pi^{\mu\nu} = \partial_\lambda(u^{\langle\mu}S^{\nu\rangle\lambda}) + \frac{1}{3}\Delta_{\rho\sigma}\partial_\lambda(u^\rho S^{\sigma\lambda})\Delta^{\mu\nu}. \tag{13}$$

The Belinfante TAM becomes

$$\mathcal{J}^{\alpha\mu\nu} = J^{\alpha\mu\nu} + \partial_\rho(x^\mu K^{\rho\alpha\nu} - x^\nu K^{\rho\alpha\mu}) = x^\mu\mathcal{T}^{\alpha\nu} - x^\nu\mathcal{T}^{\alpha\mu}. \tag{14}$$

Both Belinfante TAM and EMT are conserved. Also see Refs. [60,61] for the discussion about the connection between canonical and Belinfante form spin hydrodynamics.

III. ANALYTIC SOLUTIONS IN BJORKEN EXPANSION

In this section, we consider the time-honored Bjorken expansion of ordinary relativistic hydrodynamics to the dissipative spin hydrodynamics. The basic idea of Bjorken expansion is as follows. The fluid velocity is given by [87]

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right), \tag{15}$$

where $\tau = \sqrt{t^2 - z^2}$ is the proper time. The system is assumed to be homogenous in the transverse plane. As a consequence, all of the macroscopic quantities only depend on the proper time τ and are independent of the space rapidity $\eta = \frac{1}{2}\ln[(t+z)/(t-z)]$.

To close the system, the equations of state (EOS) are essential. We choose the energy density e and $\omega^{\mu\nu}$ as two kinds of thermodynamic variables. In this work, we follow the simplest EOS for the relativistic fluid in the high temperature limit,

$$e = 3p. \tag{16}$$

Although Eq. (16) looks like an EOS for an ideal fluid or ideal gas, it does not mean the fluid must be conformal. We emphasize that it is a widely used simplest EOS for both hydrodynamic simulations and theoretical studies. One can also consider the EOS (16) as the leading order term in the high temperature limit.

In analogy to number density and chemical potential in the high temperature limit, we assume the EOS for $S^{\mu\nu}$ is

$$S^{\mu\nu} = a_1 T^2\omega^{\mu\nu}, \tag{17}$$

where a_1 is constant.

Similar to our previous works on searching for the analytic solutions of relativistic magnetohydrodynamics [81–86], our strategy is as follows. As initial conditions, we assume that the initial fluid velocity is given by Eq. (15) and all the thermodynamic quantities are independent on x, y and η. Next, we will compute the dynamical evolution equations for fluid velocity u^μ, energy density e, and spin density $S^{\mu\nu}$. At last, we search for the special configurations of e and $\omega^{\mu\nu}$, which can hold the initial Bjorken velocity (15). To keep the whole system boost invariant, we therefore assume that only ω^{xy} is nonzero initially. Then the EOS (17) reduces to

$$S^{xy} = a_1 T^2 \omega^{xy}. \tag{18}$$

In general, one can also consider other components of ω^{ij}, e.g., as discussed in Ref. [71]. Here, we emphasize that the main point of the current work is to search for the simplest self-consistent solutions for the relativistic dissipative spin hydrodynamics in a Bjorken flow. We need to solve all the differential equations in a self-consistent way instead of taking the Bjorken flow as a background field. In this sense, following the similar strategy used in the relativistic magnetohydrodynamics [81–86], we have to reduce our variables as much as possible. In fact, we also have found that all other components of ω^{ij} will break the Bjorken flow [i.e., the acceleration equation (19) cannot be satisfied].

Furthermore, to simplify the calculations and to highlight the spin effect, we choose the Landau frame in which heat flow h^μ is always zero.

A. Canonical form spin hydrodynamics in the Bjorken flow

Now, we discuss the canonical EMT and the corresponding conservation equations for the Bjorken flow. The main equations are Eqs. (3) and (6).

First, let us consider the acceleration equation $\Delta_{\nu\alpha}\partial_\mu T^{\mu\nu} = 0$, which provides the dynamical evolution of fluid velocity,

$$(u\cdot\partial)u_\alpha = \frac{1}{(e+p)}[\Delta^\mu_\alpha \partial_\mu p - (q\cdot\partial)u_\alpha + q_\alpha(\partial\cdot u) + \Delta_{\nu\alpha}(u\cdot\partial)q^\nu - \Delta_{\nu\alpha}\partial_\mu \phi^{\mu\nu} - \Delta_{\nu\alpha}\partial_\mu \pi^{\mu\nu}]. \tag{19}$$

If ω^{xy} depends on the proper time τ only, with the Bjorken velocity (15), the q^μ and $\phi^{\mu\nu}$ become

$$q^\mu = -4\lambda\omega^{\mu\nu}u_\nu,$$
$$\phi^{\mu\nu} = \frac{2\gamma}{T}[\omega^{\mu\nu} + 2u^{[\mu}\omega^{\nu]\beta}u_\beta]. \tag{20}$$

Inserting Eq. (20) into Eq. (19), we find that the Bjorken velocity (15) will not be modified under our assumption.

Next, let us consider the energy conservation equation $u_\nu\partial_\mu T^{\mu\nu} = 0$, i.e.,

$$(u\cdot\partial)e + (e+p)\partial\cdot u + \partial\cdot q + q^\nu(u\cdot\partial)u_\nu + u_\nu\partial_\mu\phi^{\mu\nu} - \pi^{\mu\nu}\partial_\mu u_\nu = 0. \tag{21}$$

Using Eq. (20), the above equation becomes

$$\frac{d}{d\tau}e + \frac{4}{3}e\frac{1}{\tau} - s\left(\frac{2}{3}\frac{\eta_s}{s} + \frac{\zeta}{s}\right)\frac{1}{\tau^2} = 0, \tag{22}$$

where the s is the entropy. Note that we assume that the dimensionless quantities η_s/s and ζ/s are constant.

Third, from Eq. (20), the spin density in Eq. (6) reads

$$\frac{dS^{xy}}{d\tau} + S^{xy}\frac{1}{\tau} = -\frac{4\gamma}{T}\omega^{xy}. \tag{23}$$

Note that, in this case, the q^μ does not contribute to the S^{xy}.

To solve Eqs. (22) and (23), we can express the s and e as functions of T and ω^{xy} by using EOS (16) and (18) and the thermodynamic relations (7),

$$e(T,\omega^{xy}) = c_1T^4 + 3a_1T^2\omega_{xy}^2,$$
$$s(T,\omega^{xy}) = \frac{4}{3}c_1T^3 + 2a_1T\omega_{xy}^2, \tag{24}$$

where the constant c_1 can be determined by the initial conditions, i.e.,

$$c_1 = \left[\frac{e_0}{T_0^4} - 3a_1\left(\frac{\omega_0^{xy}}{T_0}\right)^2\right], \tag{25}$$

with $e_0 = e(\tau_0)$, $T_0 = T(\tau_0)$, and ω_0^{xy} being the energy density, temperature, and spin chemical potential at the initial proper time τ_0, respectively.

After inserting Eq. (24) into Eq. (22), we find that the solutions cannot be written into a compact form. On the other hand, as mentioned in Sec. II, we have always assume that the $\omega^{xy} \sim \mathcal{O}(\partial^1)$ in the gradient expansion in the current formalism and it implies that

$$\omega^{xy}/T \ll 1. \tag{26}$$

Therefore, we can consider a power expansion of ω^{xy}/T, which is equivalent to the gradient expansion. Equations (22) and (23) with Eq. (24) becomes

$$\frac{d}{d\tau}T + \frac{1}{3}\frac{T}{\tau} - \frac{1}{3}\left(\frac{2}{3}\frac{\eta_s}{s} + \frac{\zeta}{s}\right)\frac{1}{\tau^2} + \mathcal{O}((\omega_{xy}/T)^2) = 0,$$
$$T\frac{d}{d\tau}\omega^{xy} + 2\omega^{xy}\frac{d}{d\tau}T + T\omega^{xy}\frac{1}{\tau} + \frac{4\gamma}{a_1T^2}\omega^{xy} + \mathcal{O}((\omega_{xy}/T)^2) = 0. \tag{27}$$

The solutions are

$$T(\tau) = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3} - \frac{1}{2\tau}\left(\frac{2}{3}\frac{\eta_s}{s} + \frac{\zeta}{s}\right)\left[1 - \left(\frac{\tau}{\tau_0}\right)^{2/3}\right] + \mathcal{O}((\omega_0^{xy}/T_0)^2),$$
$$\omega^{xy}(\tau) = \omega_0^{xy}\left(\frac{\tau_0}{\tau}\right)^{1/3}\exp\left[-\frac{2\gamma\tau_0}{a_1T_0^3}\left(\frac{\tau^2}{\tau_0^2} - 1\right)\right]\left\{1 + \left(\frac{2}{3}\frac{\eta_s}{s} + \frac{\zeta}{s}\right)\frac{1}{T_0^4}\right.$$
$$\left.\times\left[\frac{T_0^3}{\tau_0}\left(\left(\frac{\tau_0}{\tau}\right)^{2/3} - 1\right) + \frac{\gamma}{a_1}\left(3\left(\frac{\tau}{\tau_0}\right)^2 - \frac{9}{2}\left(\frac{\tau}{\tau_0}\right)^{4/3} + \frac{3}{2}\right)\right]\right\}$$
$$+ \mathcal{O}((\omega_0^{xy}/T_0)^2, (\eta_s/s)^2, (\zeta/s)^2, (\eta_s\zeta/s^2)), \tag{28}$$

where for $\omega^{xy}(\tau)$, we only keep the linear terms of η_s/s and ζ/s.

Inserting solutions (28) into Eq. (24) and EOS (17) yields

$$e(\tau) = e_0\left(\frac{\tau_0}{\tau}\right)^{4/3} - 2\frac{e_0\tau_0}{T_0\tau^2}\left(\frac{2}{3}\frac{\eta_s}{s} + \frac{\zeta}{s}\right)\left[1 - \left(\frac{\tau}{\tau_0}\right)^{2/3}\right]$$
$$+ \mathcal{O}((\omega_0^{xy}/T_0)^2, (\eta_s/s)^2, (\zeta/s)^2, (\eta_s\zeta/s^2)), \tag{29}$$

and

$$S^{xy}(\tau) = a_1\omega_0^{xy}T_0^2\left(\frac{\tau_0}{\tau}\right)\exp\left[-\frac{2\gamma\tau_0}{a_1T_0^3}\left(\frac{\tau^2}{\tau_0^2} - 1\right)\right]$$
$$\times\left\{1 + \left(\frac{2}{3}\frac{\eta_s}{s} + \frac{\zeta}{s}\right)\frac{3\gamma}{2a_1T_0^4}\left[2\left(\frac{\tau}{\tau_0}\right)^2 - 3\left(\frac{\tau}{\tau_0}\right)^{\frac{4}{3}} + 1\right]\right\}$$
$$+ \mathcal{O}((\omega_0^{xy}/T_0)^2, (\eta_s/s)^2, (\zeta/s)^2, (\eta_s\zeta/s^2)). \tag{30}$$

The dissipative effects from $\phi^{\mu\nu}$ contribute an extra factor $\exp[-2\gamma(\tau^2 - \tau_0^2)/(a_1T_0^3\tau_0)]$ to Eqs. (28) and (30). Both ω^{xy} and S^{xy} decay more rapidity in a finite $\phi^{\mu\nu}$ system than in a vanishing $\phi^{\mu\nu}$ system. Note that the factor related to γ looks similar to the factor $\sim\exp[-\sigma(\tau - \tau_0)]$ caused by the electric conducting flow in magnetohydrodynamics with σ the electric conductivity [85]. From Eq. (30), we find that the viscous corrections to $S^{xy}(\tau)$ are always positive when $\tau/\tau_0 \geq 1$ and increase when τ_0 grows. It means that the viscous effects accelerate the decay of spin density.

B. Corrections from relaxation time for the spin transport

In this section, we compute the higher order effects for the spin transport. For simplicity, we neglect $\pi^{\mu\nu}$ and concentrate on the spin effects. It is well known that the relativistic dissipative hydrodynamics in $\mathcal{O}(\partial^1)$ violate causality and are unstable; e.g., see Refs. [88–94] and the references therein. Therefore, one needs to introduce the second order corrections. In this work, we only consider a standard second order correction to the dissipative terms in Eq. (8),

$$\tau_\phi\frac{d}{d\tau}\phi^{\mu\nu} + \phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - 2\frac{1}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right), \tag{31}$$

where τ_ϕ is the relaxation time for $\phi^{\mu\nu}$. The relaxation time τ_ϕ describes how fast the system could reach to the equilibrium again after taking some perturbations $\phi^{\mu\nu}$ to an equilibrium system. It is similar to the relaxation time equations for viscous terms. Also see Refs. [62,63,70,99] for other possible second order corrections of spin hydrodynamics.

In this case, the conservation equations become

$$\frac{d}{d\tau}e + \frac{4}{3}e\frac{1}{\tau} = 0, \tag{32}$$

and

$$(u\cdot\partial)u_\alpha = \frac{1}{(e+p)}[\Delta_\alpha^\mu\partial_\mu p - \Delta_{\nu\alpha}\partial_\mu\phi^{\mu\nu}]. \tag{33}$$

The evolution equations for the spin density become

$$\frac{dS^{xy}}{d\tau} + S^{xy}\frac{1}{\tau} = -2\phi^{xy},$$
$$\tau_\phi\frac{d}{d\tau}\phi^{xy} + \phi^{xy} = \frac{2\gamma}{T}\omega^{xy}. \tag{34}$$

We find that the fluid velocity will not be modified when ϕ^{xy} only depends on the τ. It is straightforward to get the solution for Eq. (32),

$$e(\tau) = e_0\left(\frac{\tau_0}{\tau}\right)^{4/3},$$
$$T(\tau) = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3} + \mathcal{O}((\omega_0^{xy}/T_0)^2). \tag{35}$$

Then, we can solve the ϕ^{xy}, S^{xy}, and ω^{xy},

$$\phi^{xy}(\tau) = e^{-\frac{\tau}{2\tau_\phi}} f(\tau) + \mathcal{O}((\omega_0^{xy}/T_0)^2),$$
$$\omega^{xy}(\tau) = \left(\frac{\tau_0}{\tau}\right)^{\frac{1}{3}} \frac{T_0}{2\gamma} e^{-\frac{\tau}{2\tau_\phi}} \left[\frac{1}{2} f(\tau) - \left(\frac{4\gamma\tau_\phi^2}{a_1\tau_0 T_0^3}\right)^{\frac{1}{3}} g(\tau)\right] + \mathcal{O}((\omega_0^{xy}/T_0)^2),$$
$$S^{xy}(\tau) = a_1 \left(\frac{\tau_0}{\tau}\right) \frac{T_0^3}{2\gamma} e^{-\frac{\tau}{2\tau_\phi}} \left[\frac{1}{2} f(\tau) - \left(\frac{4\gamma\tau_\phi^2}{a_1\tau_0 T_0^3}\right)^{\frac{1}{3}} g(\tau)\right] + \mathcal{O}((\omega_0^{xy}/T_0)^2), \quad (36)$$

where $f(\tau)$ and $g(\tau)$ are shown in the Appendix.

In the $\tau_\phi \to 0$ limit, we prove that the solutions (36) reduce to Eqs. (28) and (30). In the $\tau_\phi \to \infty$ limit, the solutions (36) become

$$\lim_{\tau_\phi \to \infty} \omega^{xy}(\tau) = \omega_0^{xy} (\tau_0/\tau)^{1/3},$$
$$\lim_{\tau_\phi \to \infty} S^{xy}(\tau) = a_1 T_0^2 \omega_0^{xy} (\tau_0/\tau). \quad (37)$$

Therefore, the new term proportional to τ_ϕ in Eq. (31) slows down the decay caused by the $\phi^{\mu\nu}$ as expected.

C. Results for the Belinfante form EMT

By using the pseudogauge transformation (11), we can obtain the Belinfante form EMT $\mathcal{T}^{\mu\nu}$. With the solutions (30) in Sec. III A and solutions (36) in Sec. III B, we find that

$$\frac{1}{2} \partial_\lambda (u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda}) = \mathcal{O}(\partial^2), \quad (38)$$

up to the order of $\mathcal{O}(\partial^2)$. Therefore, the Belinfante EMT reduces to the ordinary EMT without spin effects. We have checked these spin corrections in Eq. (13) and found that all of them vanish under our assumption.

We emphasize that it does not mean that there are no the spin effects in the Belinfante form of dissipative spin hydrodynamics. In order to observe these spin corrections in Eq. (13), we have to consider the inhomogeneity in the transverse plane.

D. Discussion

Let us take a close look to the leading order of ω^{xy} and S^{xy} in Eqs. (28) and (30). We find that ω^{xy} and S^{xy} decay as $\sim\tau^{-1/3}$ and $\sim 1/\tau$, respectively. From Refs. [50,58], one can compute the spin polarization of Λ hyperons caused by the thermal vorticity by the modified Cooper-Frye formula,

$$\mathcal{S}^\mu(\mathbf{p}) = \frac{\int d\Sigma \cdot p f (1-f) \epsilon^{\mu\nu\alpha\beta} p_\nu [\partial_\alpha (\frac{u_\beta}{T}) + \cdots]}{8 m_\Lambda \int d\Sigma \cdot p f}, \quad (39)$$

where the m_Λ is the mass of Λ hyperon, Σ^μ is the freeze-out hypersurface, $f = f(x, p)$ is the distribution functions of particles, and … denotes the other possible corrections from the shear viscous tensor and others [33,53–57].

In general, the spin density $S^{\mu\nu}$ and $\omega^{\mu\nu}$ can contribute to the distribution function $f(x, p)$ and play a role as dissipative corrections like $\partial_\alpha (u_\beta/T)$ in $\mathcal{S}^\mu(\mathbf{p})$. Here, we refer the recent studies alone this direction by Refs. [54,69,74]. Our results (28) and (30) imply that the initial spin density decays rapidly. So far, we cannot conclude that the corrections from $S^{\mu\nu}$ and $\omega^{\mu\nu}$ in $\mathcal{S}^\mu(\mathbf{p})$ are negligible. Therefore, the systematical studies from spin hydrodynamics are essential to clarify it.

IV. CONCLUSION

In this work, we have derived the analytic solutions of a longitudinally boost-invariant dissipative spinness fluid with the finite viscous tensor and second order relaxation time corrections for the spin. Our main results are shown in Eqs. (28), (30), and (36).

We find that the spin density and spin chemical potential decay as $\sim\tau^{-1}$ and $\tau^{-1/3}$, respectively. Although the huge initial orbital angular momentum could transfer to the spin, it may be difficult to accumulate the net spin density in such an expanding system. Therefore, the systematical studies from spin hydrodynamics are necessary to clarify the contributions from $S^{\mu\nu}$ and $\omega^{\mu\nu}$ to the spin polarization vector $\mathcal{S}^\mu(\mathbf{p})$.

We have also computed the Belinfante EMT and have not found any corrections from spin in this work. To observe the possible spin corrections to the ordinary terms in Eq. (13), one needs to consider the inhomogeneity in the transverse plane, e.g., similar to the Ref. [83] for the magnetohydrodynamics.

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APPENDIX: EXPRESSION FOR EQ. (36)

The $f(\tau)$ and $g(\tau)$ in Eq. (36) are

$$f(\tau) = C_1 \mathrm{Ai}[\mathcal{H}(\tau)] + C_2 \mathrm{Bi}[\mathcal{H}(\tau)],$$
$$g(\tau) = C_1 \mathrm{Ai}'[\mathcal{H}(\tau)] + C_2 \mathrm{Bi}'[\mathcal{H}(\tau)], \quad (A1)$$

where

$$\mathcal{H}(\tau) = 2^{-\frac{4}{3}} \left(\frac{1}{4\tau_\phi^2} - \frac{4\gamma\tau}{a_1 \tau_0 \tau_\phi T_0^3}\right) \left(-\frac{\gamma}{a_1 \tau_0 T_0^3 \tau_\phi}\right)^{-\frac{2}{3}}. \quad (A2)$$

The $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are named as Airy functions, which are the solutions of the differential equation $y''(x) \pm x y(x) = 0$, respectively. Note that here both $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are entire

Airy functions of z with no branch cut discontinuities. We also use the notations $\mathrm{Ai}'(x) = \frac{d}{dx}\mathrm{Ai}(x)$ and $\mathrm{Bi}'(x) = \frac{d}{dx}\mathrm{Bi}(x)$.

The coefficients $C_{1,2}$ are determined by the initial conditions $\phi^{xy}(\tau_0) = \phi_0^{xy}$ and $\omega^{xy}(\tau_0) = \omega_0^{xy}$,

$$C_1 = \mathcal{A}^{-1} e^{\frac{\tau_0}{2\tau_\phi}} [\mathrm{Bi}(\mathcal{U})\mathcal{C} - \phi_0^{xy}\mathrm{Bi}'(\mathcal{U})],$$
$$C_2 = \mathcal{A}^{-1} e^{\frac{\tau_0}{2\tau_\phi}} [\mathrm{Ai}'(\mathcal{U})\phi_0^{xy} - \mathrm{Ai}(\mathcal{U})\mathcal{C}], \tag{A3}$$

where

$$\mathcal{A} = \mathrm{Ai}'(\mathcal{U})\mathrm{Bi}(\mathcal{U}) - \mathrm{Ai}(\mathcal{U})\mathrm{Bi}'(\mathcal{U}),$$
$$\mathcal{C} = \left(\frac{a_1 \tau_0 T_0^3}{4\gamma\tau_\phi^2}\right)^{\frac{1}{3}} \left(\frac{1}{2}\phi_0^{xy} - \frac{2\gamma}{T_0}\omega_0^{xy}\right),$$
$$\mathcal{U} = \left(\frac{a_1 \tau_0 T_0^3}{4\gamma}\right)^{\frac{2}{3}} \left(\frac{1}{4}\tau_\phi^{-\frac{4}{3}} - \frac{4\gamma}{a_1 T_0^3}\tau_\phi^{-\frac{1}{3}}\right). \tag{A4}$$

Note that here we only take the real solutions and neglect imaginary solutions.

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