# Revisiting semileptonic $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ decays

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We systematically revisit the baryonic four-body semileptonic decays of $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ by the perturbative QCD counting rules with $\mathbf{B}$ representing octet baryons and $\ell = e, \mu$. We study the transition form factors of $B^- \to \mathbf{B}\bar{\mathbf{B}}'$ in the limit of $(p_\mathbf{B} + p_{\bar{\mathbf{B}}'})^2 \to \infty$ with the three-body $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ and $B^- \to p\bar{p}\mu^-\bar{\nu}_\mu$ data along with $SU(3)_f$ flavor symmetry. We calculate the decay branching ratios and angular asymmetries as well as the differential decay branching fractions of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$. In particular, we find that our new result of $\mathcal{B}(B^- \to p\bar{p}\ell^-\bar{\nu}_\ell) = (5.21 \pm 0.34) \times 10^{-6}$, which is about one order of magnitude lower than the previous theoretical prediction of $(10.4 \pm 2.9) \times 10^{-5}$, agrees well with both experimental measurements of $(5.8^{+2.6}_{-2.3}) \times 10^{-6}$ and $(5.3 \pm 0.4) \times 10^{-6}$ by the Belle and LHCb Collaborations, respectively. We also evaluate the branching ratios and angular asymmetries in other channels of $B^- \to \mathbf{B}\bar{\mathbf{B}}\ell^-\bar{\nu}_\ell$, which can be tested by the ongoing experiments at LHCb and Belle-II.

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## I. INTRODUCTION

In 2011, the baryonic four-body semileptonic decay of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ ($\ell = e$ or $\mu$) was studied with its decay branching ratio predicted to be $(10.4 \pm 2.6 \pm 1.2) \times 10^{-5}$ in Ref. [1]. Both $e$ and $\mu$ modes were indeed measured by the Belle Collaboration [2] in 2014 with $\mathcal{B}(B^- \to p\bar{p}e^-\bar{\nu}_e) = (8.2^{+3.7}_{-3.2} \pm 0.6) \times 10^{-6}$ and $\mathcal{B}(B^- \to p\bar{p}\mu^-\bar{\nu}_\mu) = (3.1^{+3.1}_{-2.4} \pm 0.7) \times 10^{-6}$ along with the combined value of $\mathcal{B}(B^- \to p\bar{p}\ell^-\bar{\nu}_\ell) = (5.8^{+2.4}_{-2.1} \pm 0.9) \times 10^{-6}$, which is about one order of magnitude lower than the theoretical prediction. Recently, the LHCb Collaboration has published the observation of $B^- \to p\bar{p}\mu^-\bar{\nu}_\mu$ with its decay branching ratio determined to be $(5.27^{+0.23}_{-0.24} \pm 0.21 \pm 0.15) \times 10^{-6}$ [3], where the first and second uncertainties correspond to statistical and systematic uncertainties, and the third one is from the branching fraction of the normalization channel, respectively. Both statistical and systematic uncertainties of the LHCb data have significant improvements compared with the previous ones by Belle [2].

These decay modes are useful for determining the value of $|V_{ub}|$ as the works in the other baryonic modes [4], as well as the underlying new $CPT$-violating effects. The main difficulty in both extracting $|V_{ub}|$ from $B^- \to p\bar{p}\ell\bar{\nu}_\ell$ and constraining the new $CPT$-violating effects is figuring out how to obtain their hadronic transition amplitude of $B^- \to p\bar{p}$, as it is hard to calculate it via the usual QCD methods, such as the factorizations and sum rules which have been widely used in the mesonic decays of $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$ ($B_{\ell 4}$) [5–7]. Nevertheless, these modes should be considered in the fit for the extraction of $V_{ub}$. Qualitatively speaking, to reduce the theoretical values for the decay branching ratios of $B^- \to p\bar{p}\ell\bar{\nu}_\ell$, a smaller value of $|V_{ub}|$ is needed besides the form factors. It is similar to the extractions from the exclusive $B$ and $\Lambda_b$ decays, but lower than that from the inclusive B decays. Clearly, as a baryonic complementary version of $B_{\ell 4}$ decays, both theoretical and experimental studies of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ may shed light on the baryonic transition amplitude of $B^- \to p\bar{p}$, uncover the nature of the QCD dynamics, and improve the measurement of $|V_{ub}|$.

Because of the rarity of four-body $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ decays with $\mathbf{B}^{(\prime)}$ representing octet baryons, people have concentrated on the three-body $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ decays to extract the baryonic transition from factors in the $\mathbf{B}\bar{\mathbf{B}}'$ transitions, where $\mathbf{B}(\bar{\mathbf{B}}')$ and $M$ are octet (anti)baryons and pseudoscalar or vector mesons, respectively. There have been several theoretical analyses on the baryonic three-body $B \to \mathbf{B}\bar{\mathbf{B}}'M$ decays based on the factorization assumptions [8–14]. These baryonic $B$ decays can be basically classified into current production $\mathcal{C}$, transition $\mathcal{T}$, and hybrid $\mathcal{C} + \mathcal{T}$ types [14], with the quark flow diagrams

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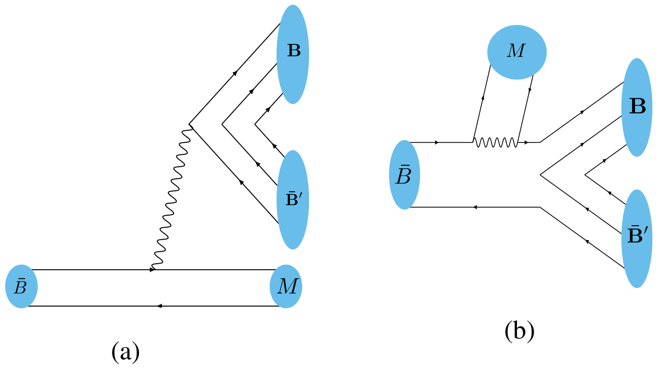


FIG. 1. Quark flow diagrams for three-body baryonic B decays $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ with (a) current and (b) transition types.

shown in Fig. 1. Among them, the transition channel is the only channel directly related to the $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'$ baryonic transition amplitudes. In Ref. [1], perturbative QCD counting rules combined with the available data of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ decays at the time were used to fit the form factors and predict the $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ decays. Although the prediction of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ motivated its active search, it is clearly disproved by the experiments of both Belle [2] and LHCb [3]. The main reason for such a large prediction is that there was a shortage of relevant data as well as a lack of understanding of the underlined QCD dynamics for the baryonic transition of $B^- \to p\bar{p}$. In this work, we would like to reanalyze the semileptonic decays of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ with the same strategy as that in Ref. [1] with the updated data. In addition, we shall use the flavor symmetry to extend our results to other $B^- \to \mathbf{B}\bar{\mathbf{B}}\ell^-\bar{\nu}_\ell$ decays. This work is the first step to knowing the properties of $\bar{B} \to \mathbf{B}\mathbf{B}'$ transition matrix elements. After getting a better understanding of these elements, we can use them for not only improving the measurement of $|V_{ub}|$, but probing or constraining the new physics effects, such as the $T$-violating triple momentum correlations due to the rich kinematic structure in the four-body decays of $\bar{B} \to \mathbf{B}\mathbf{B}'\ell\bar{\nu}$.

This paper is organized as follows. In Sec. II we present our formalism, which contains the effective Hamiltonians and generalized transition form factors. In Sec. III, we show our numerical results of the form factors fitted by three-body $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ processes and the latest $B^- \to p\bar{p}\mu^-\bar{\nu}_\mu$ result, and present our predictions of the branching ratios and angular asymmetries in $B^- \to \mathbf{B}\bar{\mathbf{B}}\ell^-\bar{\nu}_\ell$. We also compare our results of the $p\bar{p}$ invariant mass spectrum in the $B^- \to p\bar{p}\mu^-\bar{\nu}_\mu$ decay compared with the one measured by the LHCb Collaboration. We give our conclusions in Sec. IV.

## II. FORMALISM

The effective Hamiltonian for $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ at the quark level is given by

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ub}\bar{u}\gamma^\mu(1-\gamma_5)b\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell, \qquad (1)$$

where $G_F$ is the Fermi constant and $V_{ub}$ represents the element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The transition amplitude of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ can be easily factorized into hadronic and leptonic parts, written as

$$\begin{aligned}\mathcal{A}(\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell) \\ = \frac{G_F}{\sqrt{2}} V_{ub}\langle \mathbf{B}\bar{\mathbf{B}}'|\bar{u}\gamma^\mu(1-\gamma_5)|\bar{B}\rangle\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell,\end{aligned} \qquad (2)$$

where $\bar{\ell}$ and $\nu_\ell$ are the usual Dirac spinors and $\langle \mathbf{B}\bar{\mathbf{B}}'|\bar{u}\gamma^\mu(1-\gamma_5)|\bar{B}\rangle$ is the unknown hadronic transition amplitude. The most general Lorentz invariant forms of the hadronic transitions for the vector and axial-vector currents can be parametrized by [1,14]

$$\begin{aligned}\langle \mathbf{B}\bar{\mathbf{B}}'|\bar{u}\gamma^\mu b|\bar{B}\rangle &= i\bar{u}(p_{\mathbf{B}})[g_1\gamma^\mu + ig_2\sigma^{\mu\nu}p_\nu + g_3p^\mu \\ &\quad + g_4(p_{\mathbf{B}}+p_{\bar{\mathbf{B}}'})^\mu + g_5(p_{\mathbf{B}}-p_{\bar{\mathbf{B}}'})^\mu]\gamma_5 v(p_{\bar{\mathbf{B}}'}), \\ \langle \mathbf{B}\bar{\mathbf{B}}'|\bar{u}\gamma^\mu\gamma_5 b|\bar{B}\rangle &= i\bar{u}(p_{\mathbf{B}})[f_1\gamma^\mu + if_2\sigma^{\mu\nu}p_\nu + f_3p^\mu \\ &\quad + f_4(p_{\mathbf{B}}+p_{\bar{\mathbf{B}}'})^\mu + f_5(p_{\mathbf{B}}-p_{\bar{\mathbf{B}}'})^\mu]v(p_{\bar{\mathbf{B}}'}),\end{aligned} \qquad (3)$$

respectively, where $f_i$ and $g_i$ ($i = 1, 2, \ldots, 5$) are the form factors and $p^\mu = (p_{B^-} - p_{\mathbf{B}} - p_{\bar{\mathbf{B}}'})^\mu$. Inspired by the threshold effects [15], which have been observed in three-body $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ decays [16–18] and the perturbative QCD (pQCD) counting rules [19–21], the momentum dependencies of $f_i$ and $g_i$ can be assumed to be

$$f_i = \frac{C_{f_i}}{t^n}, \qquad g_i = \frac{C_{g_i}}{t^n}, \qquad (4)$$

with $n = 3$, where $C_{f_i,g_i}$ are constants determined by the branching ratios of the input channels. Note that $n$ relates to the number of hard gluon propagators as shown in Fig. 2.

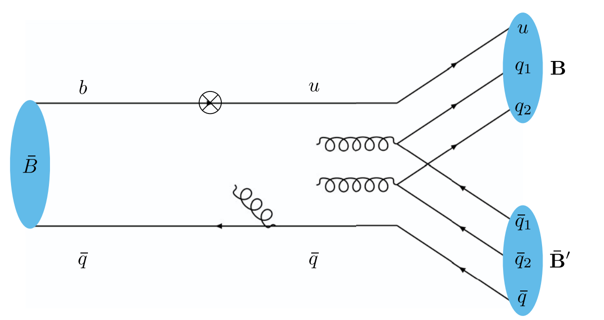


FIG. 2. Diagram for the $\bar{B} \to \bar{\mathbf{B}}\mathbf{B}'$ transition, where the curl lines stand for hard gluons and the symbol of $\otimes$ denotes the weak vertex, while each hard gluon contributes a $1/t$ in the form factors.

TABLE I. Electroweak coefficients of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'$ under $t \to \infty$ and heavy quark limits.

| Channel | $e_{RR}$ | $e_{LL}$ | $e_{LR}$ |
|---|---|---|---|
| $B^- \to p\bar{p}$ | $\frac{1}{3}$ | $\frac{5}{3}$ | $-\frac{4}{3}$ |
| $B^- \to n\bar{n}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $B^- \to \Sigma^+\bar{\Sigma}^+$ | $\frac{1}{3}$ | $\frac{5}{3}$ | $-\frac{4}{3}$ |
| $B^- \to \Sigma^0\bar{\Sigma}^0$ | $\frac{1}{6}$ | $\frac{5}{6}$ | $-\frac{2}{3}$ |
| $B^- \to \Xi^0\bar{\Xi}^0$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $B^- \to \Lambda^0\bar{\Lambda}^0$ | $\frac{1}{2}$ | $\frac{1}{3}$ | 0 |
| $B^- \to \Lambda^0\bar{\Sigma}^0$ | $-\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ |
| $B^- \to \Sigma^0\bar{\Lambda}^0$ | $-\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ |
| $B^- \to \Lambda^0\bar{p}$ | 0 | $\sqrt{\frac{2}{3}}$ | $-\sqrt{\frac{2}{3}}$ |
| $\bar{B}^0 \to \Lambda^0\bar{\Lambda}^0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| $\bar{B}^0 \to p\bar{p}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

In the $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'$ transition, two hard gluons produce the valance quarks in the $\mathbf{B}\bar{\mathbf{B}}'$ pair separately, and one more hard gluon is needed to speed up the spectator quark in $\bar{B}$ [14]. As a result, we can use $C_{f_i}$ and $C_{g_i}$ to describe the hadronic form factors in both transition-type three-body and semileptonic four-body decays. With the help of $SU(3)$ flavor and $SU(2)$ spin symmetries in $t \to \infty$, and the heavy quark limit, $C_{f_i}$ and $C_{g_i}$ are related by only two chiral-conserving parameters, $C_{RR}$ and $C_{LL}$, and one chiral-flipping parameter, $C_{LR}$. Consequently, we have

$$\begin{aligned} C_{f_1} &= m_{\bar{B}}(e_{LL}C_{LL}+e_{RR}C_{RR})+(m_{\mathbf{B}}+m_{\bar{\mathbf{B}}'})e_{LR}C_{LR} \\ C_{g_1} &= m_{\bar{B}}(e_{LL}C_{LL}-e_{RR}C_{RR})+(m_{\mathbf{B}}-m_{\bar{\mathbf{B}}'})e_{LR}C_{LR} \\ C_{f_2} &= -C_{g_2}=e_{LR}C_{LR}, \quad C_{f_i}=C_{g_i}=-e_{LR}C_{LR} \quad \text{for}\, i=3,4,5, \end{aligned} \tag{5}$$

where $e_{RR}$, $e_{LL}$, and $e_{LR}$ are the electroweak coefficients determined by the spin-flavor structure of $\bar{B}$ and $\mathbf{B}\bar{\mathbf{B}}'$, and $m_{\mathbf{B},\bar{\mathbf{B}}'}$ corresponds to baryon and antibaryon masses, respectively. The detailed derivations of Eq. (5) are presented in the Appendix. We list the coefficients of relevant channels in Table I.

Following the same formalism in the literature of $B_{\ell 4}$, $D_{\ell 4}$, and $K_{\ell 4}$ analyses [22,23], we examine the $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ system in the $\bar{B}$ rest frame with five kinematic variables, $s=(p_\ell+p_{\bar{\nu}_\ell})^2$, $t$, $\theta_\mathbf{B}$, $\theta_\ell$, and $\phi$, where $\sqrt{s}$ and $\sqrt{t}$ are the invariant masses of lepton and $\mathbf{B}\bar{\mathbf{B}}'$ pairs, respectively, and three kinematic angles are shown in Fig. 3. The differential decay width is given by

$$d\Gamma = \frac{|\bar{\mathcal{A}}|^2}{4(4\pi)^6 m_{\bar{B}}^3} X\beta_\mathbf{B}\beta_\mathbf{L}\, ds\,dt\,d\cos\theta_\mathbf{B}\,d\cos\theta_\ell\,d\phi, \tag{6}$$

where $|\bar{\mathcal{A}}|^2$ is the spin-averaged amplitude and $X$, $\beta_\mathbf{B}$, $\beta_\mathbf{L}$ are given by

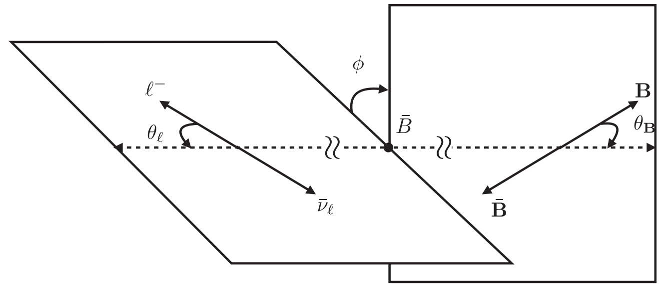


FIG. 3. $\theta_\mathbf{B}$, $\theta_\ell$, and $\phi$ in $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}\ell^-\bar{\nu}_\ell$ decays.

$$\begin{aligned} X &= \frac{\sqrt{(m_{\bar{B}}^2-s-t)^2-4st}}{2} \\ \beta_\mathbf{B} &= \frac{1}{t}\lambda^{\frac{1}{2}}(t, m_\mathbf{B}^2, m_{\bar{\mathbf{B}}'}^2) \\ \beta_\mathbf{L} &= \frac{1}{s}\lambda^{\frac{1}{2}}(s, m_\ell^2, m_{\bar{\nu}}^2), \end{aligned} \tag{7}$$

with $\lambda(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz$. We can also define the integrated $\theta_\mathbf{B}$ and $\theta_\ell$ asymmetries of $\mathbf{B}\bar{\mathbf{B}}'$ and lepton pairs as follows:

$$\langle\alpha_{\theta_f}\rangle \equiv \frac{\int_0^1 \frac{d\Gamma}{d\cos\theta_f} d\cos\theta_f - \int_{-1}^0 \frac{d\Gamma}{d\cos\theta_f} d\cos\theta_f}{\int_0^1 \frac{d\Gamma}{d\cos\theta_f} d\cos\theta_f + \int_{-1}^0 \frac{d\Gamma}{d\cos\theta_f} d\cos\theta_f}, \tag{8}$$

with $f=\mathbf{B}$ and $\ell$, respectively.

## III. NUMERICAL RESULTS

In our numerical analysis, we use the Wolfenstein parametrization for the CKM matrix with the corresponding parameters, taken to be [24]

$$\lambda=0.22650, \quad A=0.790, \quad \rho=0.141, \quad \eta=0.357, \tag{9}$$

and leading to $|V_{ub}|=3.8\times10^{-3}$. To extract the form factors, we use the factorization assumption and follow the formula in Ref. [14] to calculate the branching ratios of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$. The full analysis of three-body kinematics and the detailed derivations of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ factorization amplitudes can be found in Ref. [14]. Based on Refs. [14,25,26], the effective Wilson coefficients of $a_2^{D^{(*)}}$ and $a_2^{J^{(*)}}$ should include nonfactorizable effects and can be parametrized by the effective color number ($N_c^{\rm eff}$), $a_2^M=c_2^M+c_1^M/N_c^{\rm eff}$, where $N_c^{\rm eff}$ will be fitted. We present the numerical inputs of hadron masses, lifetimes, meson decay constants, and Wilson coefficients in Table II [1,24,27]. By performing the minimum $\chi^2$ method with six data points, the free parameters of $C_{RR,LL,LR}$ in Eq. (5) and the effective color number of $N_c^{\rm eff}$ are fitted to be

TABLE II. Input values of hadron masses, lifetimes, meson decay constants, and Wilson coefficients, where the masses (decay constants) and lifetimes are in units of GeV and fs, while the Wilson coefficients are dimensionless.

| $m_{B^-}$ | $m_{B^0}$ | $m_D$ | $m_{D^*}$ | $m_{J/\Psi}$ | $m_p$ | $m_n$ | $m_{\Lambda^0}$ | $m_{\Xi^0}$ | $m_{\Sigma^0}$ |
|---|---|---|---|---|---|---|---|---|---|
| 5.28 | 5.28 | 1.87 | 2.01 | 3.10 | 0.94 | 0.94 | 1.12 | 1.32 | 1.19 |

| $m_{\Sigma^+}$ | $\tau_{B^-}$ | $\tau_{B^0}$ | $f_D$ | $f_{D^*}$ | $f_{J/\Psi}$ | $c_1^{D^{(*)}}$ | $c_2^{D^{(*)}}$ | $c_1^{J/\Psi}$ | $c_2^{J/\Psi}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.19 | 164 | 152 | 0.22 | 0.23 | 0.41 | −0.367 | 1.169 | −0.185 | 1.082 |

$$\begin{aligned}(C_{RR}, C_{LL}, C_{LR}) = (&-11.67 \pm 1.97, 17.78 \pm 0.83,\\ &6.41 \pm 1.62)\ \mathrm{GeV}^4,\\ N_c^{\mathrm{eff}} = &\ 0.51 \pm 0.03,\end{aligned} \tag{10}$$

respectively, with $\chi^2/\mathrm{d.o.f} = 0.28$. Our fitting results, along with the input data for the transition-type three-body decays of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ and $B^- \to p\bar{p}\mu^-\bar{\nu}_\mu$, are presented in Table III. In Table IV, we show our predictions of other four-body $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ decays. In Tables III and IV, we only consider the errors caused by the data inputs and $\chi^2$ fitting; the other uncertainties are not listed in our results due to the lack of a comprehensive model to describe $\bar{B} \to \mathbf{B}\mathbf{B}'$. The most important source of the theoretical errors is the assumption of the heavy quark limit as shown around Eq. (A9) in the Appendix. Without this assumption, we need to totally fit ten parameters, which much exceed the number of the current data points. However, from the kinematical point of view, we expect that the error from the heavy quark approximation should be the same order as that in $\Lambda_c \to \Lambda$ because of the similar mass ratio of $2m_{\mathbf{B}}/m_{\bar{B}} \simeq m_\Lambda/m_{\Lambda_c}$ between the two types of channels. On the other hand, we are confident with our momentum behaviors of the form factors given by the QCD counting rules, which have been used to explain the threshold effects in three-body $\bar{B} \to \mathbf{B}\mathbf{B}'M$ decays. Moreover, these momentum behaviors can match the newest $B^- \to p\bar{p}\mu^+\nu_\mu$ differential decay width measured by the LHCb. It is interesting to see that the $SU(3)_f$ flavor symmetry guarantees that all observables in $B^- \to \Lambda^0\bar{\Sigma}^0\ell^-\bar{\nu}_\ell$ are the same as those in $B^- \to \Sigma^0\bar{\Lambda}^0\ell^-\bar{\nu}_\ell$. We note that the angular distribution asymmetries in $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ mainly depend on the electroweak coefficients, which are associated with the spin-flavor structures of the $\mathbf{B}\bar{\mathbf{B}}'$ pairs. Interestingly, the angular asymmetries of $B^- \to \Lambda\bar{\Lambda}\ell^-\bar{\nu}_\ell$ vanish because only the chiral-conserving interaction participates in $B^- \to \Lambda\bar{\Lambda}$. As a result, the physical observables in $B^- \to \Lambda\bar{\Lambda}\ell^-\bar{\nu}_\ell$ are sensitive enough to test the availability of pQCD counting rules as well as the asymptotic relations in the limit of $t \to \infty$.

TABLE III. Results for the transition-type decays of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ and $B^- \to p\bar{p}\mu^-\bar{\nu}_e$.

| Channel | Data | Our results |
|---|---|---|
| $10^5\mathcal{B}(B^- \to \Lambda^0\bar{p}J/\Psi)$ | $1.46 \pm 0.12$ | $1.47 \pm 0.12$ |
| $10^5\mathcal{B}(B^0 \to \Lambda^0\bar{\Lambda}^0 D)$ | $1.00 \pm 0.30$ | $1.23 \pm 0.10$ |
| $10^5\mathcal{B}(B^0 \to p\bar{p}D)$ | $10.4 \pm 0.70$ | $10.42 \pm 0.28$ |
| $10^5\mathcal{B}(B^0 \to p\bar{p}D^*)$ | $9.9 \pm 1.1$ | $9.04 \pm 0.49$ |
| $10^7\mathcal{B}(B^0 \to p\bar{p}J/\Psi)$ | $4.50 \pm 0.60$ | $4.83 \pm 0.34$ |
| $10^6\mathcal{B}(B^- \to p\bar{p}\mu^-\bar{\nu}_\mu)$ | $5.27 \pm 0.35$ | $5.21 \pm 0.34$ |

TABLE IV. Our numerical results of four-body $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ decays, where the errors come from the $\chi^2$.

| Channel | $10^6\mathcal{B}$ | $10^2\langle\alpha_{\theta_\mathbf{B}}\rangle$ | $10^2\langle\alpha_{\theta_\ell}\rangle$ |
|---|---|---|---|
| $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ | $5.21 \pm 0.34$ | $-6.51 \pm 1.51$ | $-2.74 \pm 0.40$ |
| $B^- \to n\bar{n}\ell^-\bar{\nu}_\ell$ | $0.68 \pm 0.10$ | $4.42 \pm 1.66$ | $0.41 \pm 0.95$ |
| $B^- \to \Lambda^0\bar{\Lambda}^0\ell^-\bar{\nu}_\ell$ | $0.08 \pm 0.01$ | 0.00 | 0.00 |
| $B^- \to \Sigma^+\bar{\Sigma}^+\ell^-\bar{\nu}_\ell$ | $0.24 \pm 0.02$ | $-6.91 \pm 1.62$ | $-2.83 \pm 0.50$ |
| $B^- \to \Sigma^0\bar{\Sigma}^0\ell^-\bar{\nu}_\ell$ | $0.06 \pm 0.01$ | $-6.91 \pm 1.62$ | $-2.83 \pm 0.49$ |
| $B^- \to \Xi^0\bar{\Xi}^0\ell^-\bar{\nu}_\ell$ | $0.008 \pm 0.001$ | $4.82 \pm 1.85$ | $0.28 \pm 0.81$ |
| $B^- \to \Lambda^0\bar{\Sigma}^0\ell^-\bar{\nu}_\ell$ | $0.014 \pm 0.004$ | $-5.65 \pm 2.05$ | $-7.88 \pm 0.64$ |
| $B^- \to \Sigma^0\bar{\Lambda}^0\ell^-\bar{\nu}_\ell$ | $0.014 \pm 0.004$ | $-5.65 \pm 2.05$ | $-7.88 \pm 0.64$ |

In Table V, we summarize our results of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ along with the previous theoretical ones [1], as well as the experimental data [2,3]. We note that the theoretical calculations are insensitive to the lepton mass for the $\ell = e$ and $\mu$ channels. As seen from Table V, our new result of $\mathcal{B}(B^- \to p\bar{p}\ell^-\bar{\nu}_\ell) = (5.21 \pm 0.34) \times 10^{-6}$ is about one order of magnitude lower than the previous theoretical prediction of $(10.4 \pm 2.9) \times 10^{-5}$ in Ref. [1], but the

TABLE V. Our results of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ along with those in Ref. [1] and the data.

| | $10^6\mathcal{B}$ | $10^2\langle\alpha_{\theta_\mathbf{B}}\rangle$ | $10^2\langle\alpha_{\theta_\ell}\rangle$ |
|---|---|---|---|
| Our results | $5.21 \pm 0.34$ | $-6.51 \pm 1.51$ | $-2.74 \pm 0.40$ |
| Ref. [1] | $104 \pm 29$ | $6 \pm 2$ | $59 \pm 2$ |
| LHCb ($\ell = \mu$) [3] | $5.27^{+0.23}_{-0.24} \pm 0.21 \pm 0.15$ | ... | ... |
| Belle ($\ell = e$) [2] | $8.2^{+3.7}_{-3.2} \pm 0.6$ | ... | ... |
| Belle ($\ell = \mu$) [2] | $3.1^{+3.1}_{-2.4} \pm 0.7$ | ... | ... |
| Belle (Combined) [2] | $5.8^{+2.4}_{-2.1} \pm 0.9$ | ... | ... |

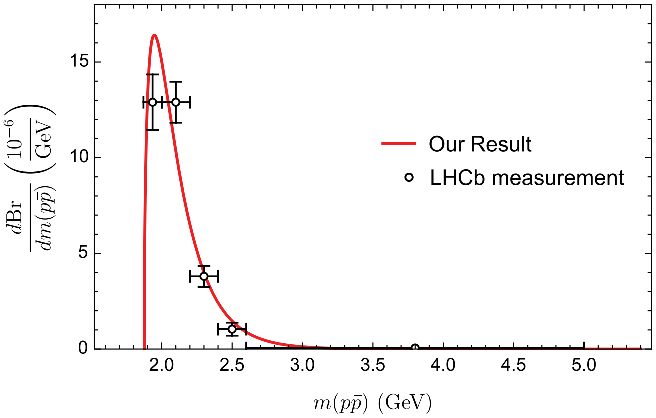


FIG. 4. Differential branching fraction of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ as a function of the $p\bar{p}$ invariant mass $[m(p\bar{p})]$, where the red solid line is our results, and the hollow dots are the data from the LHCb measurements [3].

effective color number in the $\bar{B} \to \mathbf{B}\mathbf{B}'M$ channel has a magnificent change. Our fitting result is consistent with the Belle measurements of $\mathcal{B}(B^- \to p\bar{p}e^-\bar{\nu}_e) = (8.2^{+3.8}_{-3.3}) \times 10^{-6}$ [2] and $\mathcal{B}(B^- \to p\bar{p}\mu^-\bar{\nu}_\mu) = (3.1^{+3.2}_{-2.5}) \times 10^{-6}$ [2] and agrees well with the combined measurement of $(5.8^{+2.6}_{-2.3}) \times 10^{-6}$ by Belle [2], as well as the recent $\mu$-channel data of $(5.3 \pm 0.4) \times 10^{-6}$ by LHCb [3], which is one of our input channels. Clearly, more precise modeling and explanations are needed to find the QCD origin of the effective color number being $N_c^{\mathrm{eff}} = 0.51 \pm 0.03$, indicating that the nonperturbative effects in three-body $\bar{B} \to \mathbf{B}\mathbf{B}'M$ channels are much stronger than those in two-body $\bar{B} \to M_1 M_2$ ones. In Fig. 4, we plot the $p\bar{p}$ invariant mass spectrum in $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$. By comparing our results with the LHCb measurement [3], we find that our spectrum is consistent with the observed one. We further show the differential branching fractions of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ as functions of $\sqrt{s} \equiv m_{\ell\bar{\nu}}$, $\cos_{\theta_\mathbf{B}}$, and $\cos_{\theta_\ell}$ in Fig. 5, respectively, which can provide us not only the information of the leptonic sector but also the spin-flavor relations in the $t \to \infty$ asymptotic limit. These differential branching fractions could be tested by the ongoing experiments.

## IV. CONCLUSIONS

We have systematically revisited the baryonic four-body semileptonic decays of $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ with $\ell = e, \mu$. We have reduced the ten form factors in the hadronic transition of $B^- \to \mathbf{B}\bar{\mathbf{B}}'$ into three free parameters in the heavy quark limit and $t \to \infty$. We have performed the minimum $\chi^2$ method to fit the three parameters and the effective color number of $N_c^{\mathrm{eff}}$ with $\chi^2/\mathrm{d.o.f} = 0.28$ by using five three-body decays of $\bar{B} \to \mathbf{B}\bar{\mathbf{B}}'M$ along with the $B^- \to p\bar{p}\mu^-\bar{\nu}_\mu$ measurement. We have obtained a consistent fitting result of $\mathcal{B}(B^- \to p\bar{p}\ell^-\bar{\nu}_\ell) = (5.21 \pm 0.34) \times 10^{-6}$, as well as other input channels. Our $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ decay branching ratio is about one order of magnitude lower than the previous theoretical prediction of $(10.4 \pm 2.9) \times 10^{-5}$ in Ref. [1], and agrees well with the experimental data of $(5.8^{+2.6}_{-2.3}) \times 10^{-6}$ and $(5.3 \pm 0.4) \times 10^{-6}$ by Belle [2] and LHCb [3], respectively. In addition, our evaluation of the $m_{p\bar{p}}$ invariant mass spectrum is also consistent with that by the LHCb measurement [3], demonstrating that the threshold effect and the $t^{-3}$ dependence of the form factors from the QCD counting rules are still dominant in the baryonic four-body semileptonic decays, while the other Lorentz invariant variables, such as $(p_{\bar{B}} + p_\mathbf{B})^2$ (as well as the resonant states), are highly suppressed. Furthermore, we have plotted the differential branching fractions with respect to the kinematic variables of $m_{\ell\bar{\nu}}$ and $\cos\theta_{\mathbf{B},\ell}$ in $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ to provide the information in the lepton sector and angular distributions, respectively. We have also used the flavor symmetry to explore the physical observables in other $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ decays. In particular, we have found that the angular asymmetries of $B^- \to \Lambda\bar{\Lambda}\ell^-\bar{\nu}_\ell$ vanish due to the absence of the chiral-flipping interaction $(e_{LR} = 0)$. On the theoretical side, the nonobserved

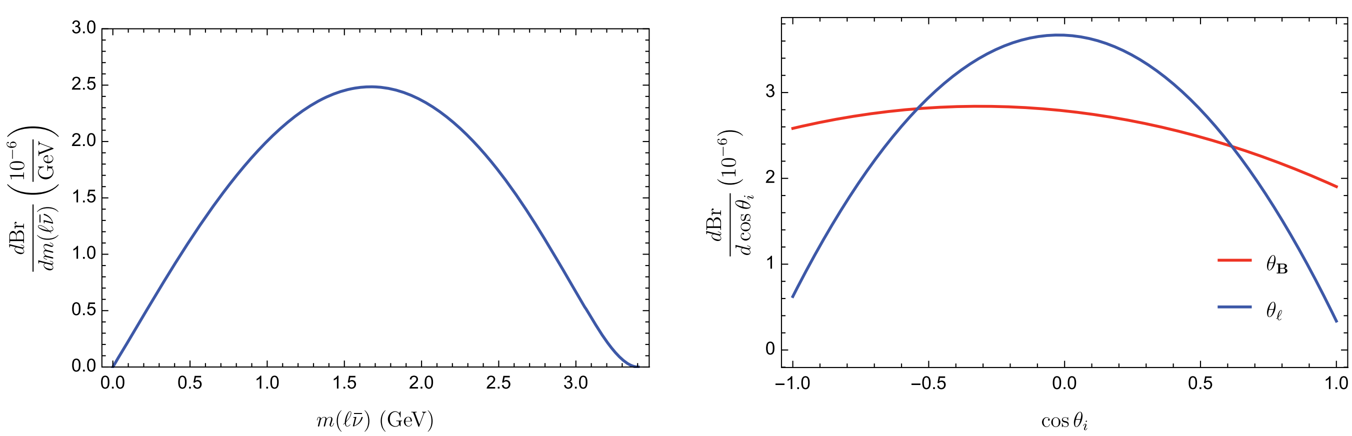


FIG. 5. Differential branching fractions of $B^- \to p\bar{p}\ell^-\bar{\nu}_\ell$ as functions of the $\ell\bar{\nu}$ invariant mass, $\cos\theta_\mathbf{B}$ and $\cos\theta_\ell$, respectively.

transition modes of the three-body $\bar{B} \to \mathbf{B}\mathbf{B}'M$ and four-body $\bar{B} \to \mathbf{B}\mathbf{B}'\ell\bar{\nu}$ decays can help us to relax the assumption of the heavy quark limit once they are measured. Otherwise, the lattice simulation would currently be the most trustworthy theoretical method to reliably extract the hadronic form factors. On the experimental side, some of our results in $B^- \to \mathbf{B}\bar{\mathbf{B}}'\ell^-\bar{\nu}_\ell$ can be tested by the ongoing experiments at Belle-II and LHCb. Finally, we remark that the theoretical determination of the four-body decays of $\bar{B} \to \mathbf{B}\mathbf{B}'\ell\bar{\nu}$ would provide a valuable opportunity to search for $T$-violating effects from the triple momentum correlations, and improve the measurement of $|V_{ub}|$ as the works in exclusive $B$ and $\Lambda_b$ decays.

## ACKNOWLEDGMENTS

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## APPENDIX: FORM FACTORS

Starting with transition matrix elements of $\langle \mathbf{B}\bar{\mathbf{B}}'|J_V^\mu - J_A^\mu|\bar{B}\rangle$ with $J^\mu_{V(A)} = \bar{q}\gamma^\mu(\gamma^5)b$, we assume that the $\bar{B}$ meson state can be approximately expressed by the field operator of free quarks, $|\bar{B}\rangle \sim \bar{b}\gamma^5 q'|0\rangle$. Therefore, the matrix elements become

$$\begin{aligned}
&\langle \mathbf{B}\bar{\mathbf{B}}'|J_V^\mu - J_A^\mu|\bar{B}(\bar{q}'b)\rangle \\
&\quad \simeq \langle \mathbf{B}\bar{\mathbf{B}}'|\bar{q}\gamma^\mu(1-\gamma^5)b\bar{b}\gamma^5 q'|0\rangle \\
&\quad = \langle \mathbf{B}\bar{\mathbf{B}}'|\bar{q}\gamma^\mu(1-\gamma^5)(\not{p}_b + m_b)\gamma^5 q'|0\rangle \\
&\quad = \langle \mathbf{B}\bar{\mathbf{B}}'|\bar{q}\gamma^\mu(1-\gamma^5)(\not{p}_b - m_b) q'|0\rangle \\
&\quad = 2\langle \mathbf{B}\bar{\mathbf{B}}'|\bar{q}_L\gamma^\mu\not{p}_b q'_R|0\rangle - 2m_b\langle \mathbf{B}\bar{\mathbf{B}}'|\bar{q}_L\gamma^\mu q'_L|0\rangle \\
&\quad = \langle \mathbf{B}\bar{\mathbf{B}}'|J'^\mu|0\rangle - \langle \mathbf{B}\bar{\mathbf{B}}'|\tilde{J}^\mu|0\rangle, \qquad \text{(A1)}
\end{aligned}$$

where $J'^\mu = 2\bar{q}_L\gamma^\mu\not{p}_b q'_R$ and $\tilde{J}^\mu = 2m_b\bar{q}_L\gamma^\mu q'_L$, with $q_{L(R)} = (1\mp\gamma^5)/2q$ and $\bar{q}_{L(R)} = \bar{q}(1\pm\gamma^5)/2$. Note that by inserting QCD (gluon-quark-antiquark) vertices in the corresponding diagrams, the Dirac structure in Eq. (A1) could be altered. However, because of the asymptotic freedom in QCD, we can treat these alterations from QCD vertices as small perturbations, which are negligible in the limit of $(p_\mathbf{B} + p_{\mathbf{B}'})^2 \to \infty$. In terms of the crossing symmetry (c.s.), the final state antibaryon ($\bar{\mathbf{B}}'$) is transformed into the initial baryon ($\mathbf{B}'$) in the initial state with opposite four-momentum $\tilde{p}_{\mathbf{B}'} = -p_{\bar{\mathbf{B}}'}$, resulting in

$$\langle \mathbf{B}(p_\mathbf{B})\bar{\mathbf{B}}'(p_{\bar{\mathbf{B}}'})|J'^\mu(\tilde{J}^\mu)|0\rangle \xrightarrow{c.s.} \langle \mathbf{B}(p_\mathbf{B})|J'^\mu(\tilde{J}^\mu)|\mathbf{B}'(\tilde{p}_{\mathbf{B}'})\rangle. \qquad \text{(A2)}$$

According to Refs. [11,21], the amplitude can be parametrized as

$$\begin{aligned}
\langle \mathbf{B}|J'^\mu|\mathbf{B}'\rangle &= 2i\bar{u}_\mathbf{B}\gamma^\mu\not{p}_b\left(\frac{1+\gamma^5}{2}F'^+ + \frac{1-\gamma^5}{2}F'^-\right)u_{\mathbf{B}'} \\
&= 2i\bar{u}^L_\mathbf{B}\gamma^\mu\not{p}_b u^R_{\mathbf{B}'}F'^+ + 2i\bar{u}^R_\mathbf{B}\gamma^\mu\not{p}_b u^L_{\mathbf{B}'}F'^- \\
\langle \mathbf{B}|\tilde{J}^\mu|\mathbf{B}'\rangle &= 2m_b i\bar{u}_\mathbf{B}\gamma^\mu\left(\frac{1+\gamma^5}{2}\tilde{F}^+ + \frac{1-\gamma^5}{2}\tilde{F}^-\right)u_{\mathbf{B}'}, \\
&= 2m_b i\bar{u}^R_\mathbf{B}\gamma^\mu u^R_{\mathbf{B}'}\tilde{F}^+ + 2m_b i\bar{u}^L_\mathbf{B}\gamma^\mu u^L_{\mathbf{B}'}\tilde{F}^-. \qquad \text{(A3)}
\end{aligned}$$

In the asymptotic limit of $(p_\mathbf{B} - \tilde{p}_{\mathbf{B}'})^2 \to \infty$, the helicity of a particle can be approximately treated as its chirality, so that the amplitudes with a specific chirality can be written as

$$\begin{aligned}
\langle \mathbf{B},L|J'^\mu|\mathbf{B}',R\rangle &= 2i\bar{u}^L_\mathbf{B}\gamma^\mu\not{p}_b(e_{LR}F_{LR})u^R_{\mathbf{B}'} \\
\langle \mathbf{B},R(L)|\tilde{J}^\mu|\mathbf{B}',R(L)\rangle &= 2m_b i\bar{u}^{R(L)}_\mathbf{B}\gamma^\mu(e_{RR(LL)}F_{RR(LL)})u^{R(L)}_{\mathbf{B}'}, \qquad \text{(A4)}
\end{aligned}$$

where

$$\begin{aligned}
e_{LR} &= \langle \mathbf{B},L|(a^L_q)^\dagger a^R_{q'}|\mathbf{B}',R\rangle, \\
e_{RR} &= \langle \mathbf{B},R|(a^L_q)^\dagger a^L_{q'}|\mathbf{B}',R\rangle, \\
e_{LL} &= \langle \mathbf{B},L|(a^L_q)^\dagger a^L_{q'}|\mathbf{B}',L\rangle, \qquad \text{(A5)}
\end{aligned}$$

with the corresponding particle creation and annihilation operators $(a^s_q)^\dagger$ and $a^s_q$, and where other combinations are zero due to the angular momentum conservation. From Eqs. (A3)–(A5) we find that

$$\begin{aligned}
F'^+ &= e_{LR}F_{LR}, \qquad F'^- = 0, \\
\tilde{F}^+ &= e_{RR}F_{RR}, \qquad \tilde{F}^- = e_{LL}F_{LL}. \qquad \text{(A6)}
\end{aligned}$$

Consequently, the transition amplitude in $(p_\mathbf{B} - \tilde{p}_{\mathbf{B}'})^2 \to \infty$ is given by

$$\langle \mathbf{B}|J'^\mu - \tilde{J}^\mu|\mathbf{B}'\rangle = 2i\bar{u}_\mathbf{B}\gamma^\mu\left(\not{p}_b\frac{1+\gamma^5}{2}e_{LR}F_{LR} - m_b\frac{1+\gamma^5}{2}e_{RR}F_{RR} - m_b\frac{1-\gamma^5}{2}e_{LL}F_{LL}\right)u_{\mathbf{B}'}. \qquad \text{(A7)}$$

After applying the crossing symmetry again, we get that

$$
\begin{aligned}
\langle \mathbf{B}\bar{\mathbf{B}}'|J_V^\mu|\bar{B}\rangle &= i\bar{u}_{\mathbf{B}}\gamma^\mu(\not{p}_b e_{LR}F_{LR} + m_b(e_{LL}F_{LL} - e_{RR}F_{RR}))\gamma^5 v_{\bar{\mathbf{B}}'}\\
&= i\bar{u}_{\mathbf{B}}\gamma^\mu(\not{p}_{\bar{B}} e_{LR}F_{LR} + m_{\bar{B}}(e_{LL}F_{LL} - e_{RR}F_{RR}))\gamma^5 v_{\bar{\mathbf{B}}'}\\
\langle \mathbf{B}\bar{\mathbf{B}}'|J_A^\mu|\bar{B}\rangle &= i\bar{u}_{\mathbf{B}}\gamma^\mu(-\not{p}_b e_{LR}F_{LR} + m_b(e_{LL}F_{LL} + e_{RR}F_{RR}))v_{\bar{\mathbf{B}}'}\\
&= i\bar{u}_{\mathbf{B}}\gamma^\mu(-\not{p}_{\bar{B}} e_{LR}F_{LR} + m_{\bar{B}}(e_{LL}F_{LL} + e_{RR}F_{RR}))v_{\bar{\mathbf{B}}'},
\end{aligned}
\tag{A8}
$$

where we have used the approximations of $p_b \simeq p_{\bar{B}}$ and $m_b \simeq m_{\bar{B}}$ in the heavy quark limit. With the help of the equations of motion

$$
\bar{u}_{\mathbf{B}}\not{p}_{\mathbf{B}} = \bar{u}_{\mathbf{B}}m_{\mathbf{B}}, \qquad \not{p}_{\bar{\mathbf{B}}'}v_{\bar{\mathbf{B}}'} = -m_{\bar{\mathbf{B}}'}v_{\bar{\mathbf{B}}'},
\tag{A9}
$$

and the Dirac algebra,

$$
\begin{aligned}
\gamma^\mu\not{p}_{\bar{B}} &= \gamma^\mu\not{p} + \gamma^\mu(\not{p}_{\mathbf{B}} + \not{p}_{\bar{\mathbf{B}}'}) = p^\mu - i\sigma^{\mu\nu}p_\nu + 2p_{\mathbf{B}}^\mu - \not{p}_{\mathbf{B}}\gamma^\mu + \gamma^\mu\not{p}_{\bar{\mathbf{B}}'}\\
&= -\not{p}_{\mathbf{B}}\gamma^\mu + \gamma^\mu\not{p}_{\bar{\mathbf{B}}'} - i\sigma^{\mu\nu}p_\nu + p^\mu + (p_{\mathbf{B}} - p_{\mathbf{B}'})^\mu + (p_{\mathbf{B}} + p_{\bar{\mathbf{B}}'})^\mu,
\end{aligned}
\tag{A10}
$$

we finally obtain

$$
\begin{aligned}
\langle \mathbf{B}\bar{\mathbf{B}}'|J_V^\mu|\bar{B}\rangle &= i\bar{u}_{\mathbf{B}}\gamma^\mu[-(m_{\mathbf{B}} - m_{\bar{\mathbf{B}}'})e_{LR}F_{LR} + m_{\bar{B}}(e_{LL}F_{LL} - e_{RR}F_{RR})\\
&\quad + e_{LR}F_{LR}(-i\sigma^{\mu\nu}p_\nu + p^\mu + (p_{\mathbf{B}} - p_{\mathbf{B}'})^\mu + (p_{\mathbf{B}} + p_{\bar{\mathbf{B}}'})^\mu)]\gamma^5 v_{\bar{\mathbf{B}}'},\\
\langle \mathbf{B}\bar{\mathbf{B}}'|J_A^\mu|\bar{B}\rangle &= i\bar{u}_{\mathbf{B}}\gamma^\mu[(m_{\mathbf{B}} + m_{\bar{\mathbf{B}}'})e_{LR}F_{LR} + m_{\bar{B}}(e_{LL}F_{LL} - e_{RR}F_{RR})\\
&\quad - e_{LR}F_{LR}(-i\sigma^{\mu\nu}p_\nu + p^\mu + (p_{\mathbf{B}} - p_{\mathbf{B}'})^\mu + (p_{\mathbf{B}} + p_{\bar{\mathbf{B}}'})^\mu)]v_{\bar{\mathbf{B}}'},
\end{aligned}
\tag{A11}
$$

which clearly leads to

$$
\begin{aligned}
f_1 &= m_{\bar{B}}(e_{LL}F_{LL} + e_{RR}F_{RR}) + (m_{\mathbf{B}} + m_{\bar{\mathbf{B}}'})e_{LR}F_{LR},\\
g_1 &= m_{\bar{B}}(e_{LL}F_{LL} - e_{RR}F_{RR}) - (m_{\mathbf{B}} - m_{\bar{\mathbf{B}}'})e_{LR}F_{LR},\\
f_2 &= -g_2 = e_{LR}F_{LR}, \qquad f_i = -g_i = -e_{LR}F_{LR}, \qquad (i = 3, 4, 5).
\end{aligned}
\tag{A12}
$$

As a result, the constant parts of form factors $C_{f_i(g_i)}$, $C_{RR(LL)}$, and $C_{LR}$ in Eq. (A12) directly imply the relations in Eq. (5).


[1] C. Q. Geng and Y. K. Hsiao, Phys. Lett. B **704**, 495 (2011).
[2] K. J. Tien *et al.* (Belle Collaboration), Phys. Rev. D **89**, 011101 (2014).
[3] R. Aaij *et al.* (LHCb Collaboration), J. High Energy Phys. 03 (2020) 146.
[4] Y. K. Hsiao and C. Q. Geng, Phys. Lett. B **782**, 728 (2018).
[5] S. Cheng, A. Khodjamirian, and J. Virto, J. High Energy Phys. 05 (2017) 157.
[6] P. Böer, T. Feldmann, and D. van Dyk, J. High Energy Phys. 02 (2017) 133.
[7] T. Feldmann, D. Van Dyk, and K. K. Vos, J. High Energy Phys. 10 (2018) 030.
[8] H. Y. Cheng and K. C. Yang, Phys. Rev. D **66**, 014020 (2002).
[9] H. Y. Cheng and K. C. Yang, Phys. Rev. D **65**, 054028 (2002); **65**, 099901(E) (2002).
[10] H. Y. Cheng and K. C. Yang, Phys. Rev. D **66**, 094009 (2002).
[11] C. K. Chua and W. S. Hou, Eur. Phys. J. C **29**, 27 (2003).
[12] C. K. Chua, W. S. Hou, and S. Y. Tsai, Phys. Rev. D **66**, 054004 (2002).
[13] C. Q. Geng and Y. K. Hsiao, Phys. Lett. B **619**, 305 (2005).
[14] C. H. Chen, H. Y. Cheng, C. Q. Geng, and Y. K. Hsiao, Phys. Rev. D **78**, 054016 (2008).
[15] W. S. Hou and A. Soni, Phys. Rev. Lett. **86**, 4247 (2001).
[16] K. Abe *et al.* (Belle Collaboration), Phys. Rev. Lett. **89**, 151802 (2002).
[17] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. D **74**, 051101 (2006).
[18] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **119**, 041802 (2017).
[19] G. P. Lepage and S. J. Brodsky, Phys. Rev. Lett. **43**, 545 (1979); **43**, 1625(E) (1979).

[20] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[21] S. J. Brodsky, G. P. Lepage, and S. A. A. Zaidi, Phys. Rev. D **23**, 1152 (1981).
[22] C. L. Y. Lee, M. Lu, and M. B. Wise, Phys. Rev. D **46**, 5040 (1992).
[23] A. Pais and S. B. Treiman, Phys. Rev. **168**, 1858 (1968).
[24] P. A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).
[25] H. Y. Cheng and K. C. Yang, Phys. Rev. D **59**, 092004 (1999).
[26] H. Y. Cheng, Phys. Rev. D **65**, 094012 (2002).
[27] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996).