Three-point functions of a superspin-2 current multiplet in 3D, $\mathcal{N} = 1$ superconformal theory

Evgeny I. Buchbinder* and Benjamin J. Stone†

School of Physics M013, The University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009, Australia

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We consider $\mathcal{N} = 1$ superconformal field theories in three dimensions possessing a conserved current multiplet $\mathcal{F}_{(\alpha_1\alpha_2\alpha_3\alpha_4)}$, which we refer to as the superspin-2 current multiplet. At the component level it contains a conserved spin-2 current different from the energy-momentum tensor and a conserved fermionic higher-spin current of spin $5/2$. Using a superspace formulation, we calculate correlation functions involving $\mathcal{F}$, focusing particularly on the three-point function $\langle \mathcal{F}\mathcal{F}\mathcal{F}\rangle$. After imposing the constraints arising from conservation equations and invariance under permutation of superspace points, we find that the parity-even and parity-odd sectors of this three-point function are each fixed up to a single coefficient. The presence of the parity-odd contribution is rather nontrivial, as there is an apparent tension between supersymmetry and the existence of parity-odd structures.

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I. INTRODUCTION

A peculiar feature of three-dimensional conformal field theories is the presence of parity violating, or parity-odd, structures in the three-point functions of conserved currents such as the energy-momentum tensor and vector currents [1]. These structures were not considered in the systematic studies of [2,3], which utilized a group-theoretic approach to solve for the correlation functions of conserved currents in a generic number of spacetime dimensions.[1] Parity-odd structures are not present in free theories but have been shown to arise in Chern-Simons theories interacting with parity violating matter. In various approaches and contexts they were studied in [14–25].

In general, besides the energy-momentum tensor and vector currents, conformal field theories also possess currents of higher spin. In [17] Maldacena and Zhiboedov proved under certain assumptions (see below) that all correlation functions of higher-spin currents in three-dimensional conformal field theory are equal to that of a free theory. In particular, it implies that they do not have parity-odd contributions. This theorem was later generalized to higher-dimensional cases in [26–28]. These results can be viewed as the analog of the Coleman-Mandula theorem [29] for conformal field theories.

In this paper we will be interested in $\mathcal{N} = 1$ superconformal field theories in three dimensions. The general formalism to construct the two- and three-point functions of conserved currents in three-dimensional superconformal field theories was developed in [30–33] (a similar formalism in four dimensions was developed in [34–36] and in six dimensions in [37]). In supersymmeric theories, conserved currents are contained within supermultiplets. The energy-momentum tensor lies in the supercurrent multiplet [38], which in three dimensions also contains a fermionic supersymmetry current. On the other hand, a vector current becomes a component of the flavor current multiplet. As was pointed out in [31,39] there is an apparent tension between supersymmetry and the existence of parity violating structures in the three-point functions of conserved currents. In particular, three-point functions containing the supercurrent and flavor current multiplets admit only parity-even contributions. Combining this with the Maldacena-Zhiboedov theorem, it follows that supersymmetric conformal field theories do not admit parity-odd contributions to the three-point functions of conserved currents for any spin unless the assumptions of the theorem are violated.

The strongest assumption of the Maldacena-Zhiboedov theorem is that the conformal field theory under consideration possesses a unique conserved spin-2 current—the energy-momentum tensor. However, in the same article [17] Maldacena and Zhiboedov showed that the existence

*evgeny.buchbinder@uwa.edu.au
†benjamin.stone@research.uwa.edu.au

[1]For earlier work concerning correlation functions of conserved currents in conformal field theory, the reader may consult Refs. [4–13].

of a conserved fermionic higher-spin current implies that there is more than one conserved current of spin 2. In supersymmetric theories conserved currents belong to supermultiplets that contain both bosonic and fermionic currents. This implies that a supersymmetric conformal field theory possessing a bosonic higher-spin current also possesses a fermionic higher-spin current (and vice versa); thus it is conceivable that there exists another conserved current of spin 2. This, in turn, implies that in three-dimensional superconformal field theories the assumptions of [17] might be violated and the properties of correlation functions of higher-spin currents might be more subtle.

In this paper, we will assume that the $\mathcal{N}=1$ superconformal field theory under consideration possesses a spin-2 conserved current different from the energy-momentum tensor. It naturally sits in the supermultiplet

$$\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \mathcal{F}_{(\alpha_1\alpha_2\alpha_3\alpha_4)} = \mathcal{F}_{\alpha(4)} \tag{1.1}$$

and satisfies the conservation equation

$$D^{\alpha_1}\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0. \tag{1.2}$$

The superfield $\mathcal{F}_{\alpha(4)}$ contains two independent conserved currents (see Sec. III)

$$\begin{aligned} J_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) &= \mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)|, \\ Q_{\alpha_1\alpha_2\alpha_3\alpha_4,\alpha}(x) &= D_\alpha\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)|, \end{aligned} \tag{1.3}$$

where, as usual, bar projection means setting all Grassmann odd variables to zero. We will refer to $\mathcal{F}_{\alpha(4)}$ as to the "superspin-2 current multiplet." The component current $J_{\alpha(4)}$ is a conserved spin-2 current different from the energy momentum tensor, though it satisfies similar properties (the latter belongs to the supercurrent multiplet $\mathcal{J}_{\alpha(3)}$), while $Q_{\alpha_1\alpha_2\alpha_3\alpha_4,\alpha_5} = Q_{\alpha(5)}$ is a conserved fermionic current of spin $5/2$. We will not discuss here particular realizations of superconformal theories possessing a conserved superspin-2 multiplet; our interest here is to explore how the $\mathcal{N}=1$ superconformal symmetry constrains the three-point correlation functions involving $\mathcal{F}_{\alpha(4)}$.

Our main result is that the three-point function,

$$\langle \mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle, \tag{1.4}$$

is fixed by the $\mathcal{N}=1$ superconformal symmetry up to one parity-even and one parity-odd structure. Our analysis is technically quite involved; the analytic superfield consideration turns out to be quite intractable and we were required to complete both superfield and component analysis with the aid of the xAct package [40] for *Mathematica*, which contains an advanced suite of tools designed for tensor analysis. The three-point function (1.4) contains two independent component correlators [all others can be found in terms of these two by virtue of the conservation law (1.2)],

$$\begin{aligned} &\langle J_{\alpha(4)}(x_1)J_{\beta(4)}(x_2)J_{\gamma(4)}(x_3)\rangle, \\ &\langle Q_{\alpha(5)}(x_1)J_{\beta(4)}(x_2)Q_{\gamma(5)}(x_3)\rangle. \end{aligned} \tag{1.5}$$

These two correlators were analyzed analytically; however, to provide a complete check that all the necessary conditions are satisfied, we had to also perform some numerical analysis. We also discuss some basic mixed three-point functions involving $\mathcal{F}_{\alpha(4)}$.[2] In particular, we analyze the three-point function $\langle \mathcal{O}(z_1)\mathcal{F}_{\alpha(4)}(z_2)\mathcal{O}(z_3)\rangle$, where $\mathcal{O}(z)$ is a scalar superfield of dimension Δ. We found that it is fixed up to a single parity-even tensor structure. We also compute the three-point function

$$\langle \mathcal{F}_{\alpha(4)}(z_1)L^{\bar{a}}_{\beta}(z_2)L^{\bar{b}}_{\gamma}(z_3)\rangle, \tag{1.6}$$

where $L^{\bar{a}}_{\alpha}(z)$ is the non-Abelian flavor current multiplet. We found that this three-point function is also fixed up to a single parity-even tensor structure, which is in disagreement with the result previously reported in [20], which used a different approach (see Sec. V B for details). In our approach the analysis of this correlation function is relatively straightforward as it can be studied analytically, so we are confident in our result.

The paper is organized as follows. In Sec. II we introduce the superconformal building blocks that are essential to the construction of two- and three-point correlation functions of primary operators. In Sec. III we analyze the structure of the supermultiplet $\mathcal{F}$; in particular, we define the component fields in the multiplet and determine the constraints on them resulting from the superfield conservation equations. Section IV is devoted to studying the three-point function $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$. First, we impose the constraints resulting from the conservation of $\mathcal{F}$ and invariance under permutation of superspace points z_1 and z_2; we show that these constraints are sufficient to fix the parity-even and parity-odd sectors each up to a single coefficient. Next we check invariance under permutation of superspace points z_1 and z_3, which is technically quite involved and involves a combination of both analytic and numerical methods. As a result, we show that the three-point function is fixed by the $\mathcal{N}=1$ superconformal symmetry up to two independent tensor structures: one is parity even while the other is parity odd. Section V is devoted to the study of mixed correlation functions involving the superfield $\mathcal{F}$. We compute the three-point function of $\mathcal{F}$ with two scalar superfield insertions, and the three-point function of $\mathcal{F}$ with two non-Abelian flavor current multiplets. In Sec. VI we

[2]A more detailed study of the mixed three-point functions involving the superspin-2 multiplet, the supercurrent, and the flavor current multiplet will be presented elsewhere.

provide a brief summary of the work and some future directions. Appendixes A, B, and C are devoted to our conventions, technical details, and some consistency checks.

II. SUPERCONFORMAL BUILDING BLOCKS

In this section we will review the pertinent details of the group theoretic formalism used to compute correlation functions of primary superfields. For a more detailed review of our conventions the reader may consult [31,39].

A. Superconformal transformations

Consider three-dimensional (3D), $\mathcal{N}=1$ Minkowski superspace $\mathbb{M}^{3|2}$, parametrized by coordinates $z^A = (x^a, \theta^\alpha)$, where $a = 0, 1, 2$ and $\alpha = 1, 2$ are Lorentz and spinor indices, respectively. Under infinitesimal superconformal transformations, the superspace coordinates transform as

$$\delta z^A = \xi z^A \Leftrightarrow \delta x^a = \xi^a(z) + \mathrm{i}(\gamma^a)_{\alpha\beta}\xi^\alpha(z)\theta^\beta, \quad \delta\theta^\alpha = \xi^\alpha(z), \tag{2.1}$$

where $\xi^a(z)$ is a conformal Killing supervector

$$\xi = \xi^A(z)\partial_A = \xi^a(z)\partial_a + \xi^\alpha(z)D_\alpha, \tag{2.2}$$

which satisfies the master equation $[\xi, D_\alpha] \propto D_\beta$. From the master equation we find

$$\xi^\alpha = \frac{\mathrm{i}}{6}D_\beta\xi^{\alpha\beta}, \tag{2.3}$$

which, in particular, implies the conformal Killing equation

$$\partial_a\xi_b + \partial_b\xi_a = \frac{2}{3}\eta_{ab}\partial_c\xi^c. \tag{2.4}$$

The solutions to the master equation are called the conformal Killing supervector fields of Minkowski superspace [41,42], which span a Lie algebra isomorphic to the superconformal algebra $\mathfrak{osp}(1|2;\mathbb{R})$.

Now consider a generic tensor superfield $\Phi_{\mathcal{A}}(z)$ transforming in a representation T of the Lorentz group with respect to the index $\mathcal{A}$.[3] Such a superfield is called primary with dimension q if its superconformal transformation law is

$$\delta\Phi_{\mathcal{A}} = -\xi\Phi_{\mathcal{A}} - q\sigma(z)\Phi_{\mathcal{A}} + \lambda^{\alpha\beta}(z)(M_{\alpha\beta})_{\mathcal{A}}{}^{\mathcal{B}}\Phi_{\mathcal{B}}, \tag{2.5}$$

where ξ is the superconformal Killing vector and the matrix $M_{\alpha\beta}$ is the Lorentz generator. The z-dependent parameters $\sigma(z)$ and $\lambda^{\alpha\beta}(z)$ associated with ξ are defined as follows:

$$\lambda_{\alpha\beta}(z) = -D_{(\alpha}\xi_{\beta)}, \qquad \sigma(z) = D_\alpha\xi^\alpha. \tag{2.6}$$

[3]We assume that the representation T is irreducible.

B. Two-point and three-point building blocks

1. Two-point building blocks

Given two superspace points z_1 and z_2, we can define the two-point functions

$$\boldsymbol{x}_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_1^{(\alpha}\theta_2^{\beta)} - \mathrm{i}\theta_{12}^\alpha\theta_{12}^\beta, \qquad \theta_{12}^\alpha = \theta_1^\alpha - \theta_2^\alpha. \tag{2.7}$$

Note that $\boldsymbol{x}_{21}^{\alpha\beta} = -\boldsymbol{x}_{12}^{\beta\alpha}$. It is convenient to split the two-point function (2.7) into symmetric and antisymmetric parts as follows:

$$\boldsymbol{x}_{12}^{\alpha\beta} = y_{12}^{\alpha\beta} + \frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\theta_{12}^2, \qquad \theta_{12}^2 = \theta_{12}^\alpha\theta_{12\alpha}, \tag{2.8}$$

where $y_{12}^{\alpha\beta}$ is the symmetric part of $\boldsymbol{x}_{12}^{\alpha\beta}$,

$$y_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_1^{(\alpha}\theta_2^{\beta)}. \tag{2.9}$$

It can also be represented by the three-vector $y_{12}^m = -\frac{1}{2}(\gamma^m)_{\alpha\beta}y_{12}^{\alpha\beta}$. Next we introduce the two-point objects

$$\boldsymbol{x}_{12}^2 = -\frac{1}{2}\boldsymbol{x}_{12}^{\alpha\beta}\boldsymbol{x}_{12\alpha\beta}, \tag{2.10a}$$

$$\hat{\boldsymbol{x}}_{12}^{\alpha\beta} = \frac{\boldsymbol{x}_{12}^{\alpha\beta}}{\sqrt{\boldsymbol{x}_{12}^2}}, \qquad \hat{\boldsymbol{x}}_{12\alpha}{}^\gamma\hat{\boldsymbol{x}}_{12\gamma}{}^\beta = \delta_\alpha{}^\beta. \tag{2.10b}$$

Hence, we find

$$(\boldsymbol{x}_{12}^{-1})^{\alpha\beta} = -\frac{\boldsymbol{x}_{12}^{\beta\alpha}}{\boldsymbol{x}_{12}^2}. \tag{2.11}$$

These objects are essential in the construction of correlation functions of primary superfields. We also have the useful differential identities

$$D_{(1)\gamma}\boldsymbol{x}_{12}^{\alpha\beta} = -2\mathrm{i}\theta_{12}^\beta\delta_\gamma^\alpha, \qquad D_{(1)\alpha}\boldsymbol{x}_{12}^{\alpha\beta} = -4\mathrm{i}\theta_{12}^\beta, \tag{2.12}$$

where $D_{(i)\alpha}$ is the standard covariant spinor derivative (A16) acting on the superspace point z_i.

2. Three-point building blocks

Given three superspace points z_i, $i = 1, 2, 3$, one can define the following three-point building blocks,

$$\boldsymbol{X}_{1\alpha\beta} = -(\boldsymbol{x}_{21}^{-1})_{\alpha\gamma}\boldsymbol{x}_{23}^{\gamma\delta}(\boldsymbol{x}_{13}^{-1})_{\delta\beta}, \tag{2.13a}$$

$$\Theta_{1\alpha} = (\boldsymbol{x}_{21}^{-1})_{\alpha\beta}\theta_{12}^\beta - (\boldsymbol{x}_{31}^{-1})_{\alpha\beta}\theta_{13}^\beta, \tag{2.13b}$$

and, similarly, $(\boldsymbol{X}_2, \Theta_2)$ and $(\boldsymbol{X}_3, \Theta_3)$, which can be found from (2.13) by cyclic permutation. Next we define

$$\boldsymbol{X}_1^2 = -\frac{1}{2}\boldsymbol{X}_1^{\alpha\beta}\boldsymbol{X}_{1\alpha\beta} = \frac{\boldsymbol{x}_{23}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{12}^2}, \qquad \Theta_1^2 = \Theta_1^\alpha\Theta_{1\alpha}. \tag{2.14}$$

We also define the normalized building block, $\hat{\boldsymbol{X}}_1$, and the inverse of $\boldsymbol{X}_1$,

$$\hat{\boldsymbol{X}}_{1\alpha\beta} = \frac{\boldsymbol{X}_{1\alpha\beta}}{\sqrt{\boldsymbol{X}_1^2}}, \qquad (\boldsymbol{X}_1^{-1})^{\alpha\beta} = -\frac{\boldsymbol{X}_1^{\beta\alpha}}{\boldsymbol{X}_1^2}. \tag{2.15}$$

There are also useful identities involving $\boldsymbol{X}_i$ and Θ_i at different superspace points, e.g.,

$$\boldsymbol{x}_{13}^{\alpha\alpha'}\boldsymbol{X}_{3\alpha'\beta'}\boldsymbol{x}_{31}^{\beta'\beta} = -(\boldsymbol{X}_1^{-1})^{\beta\alpha}, \tag{2.16a}$$

$$\Theta_{1\gamma}\boldsymbol{x}_{13}^{\gamma\delta}\boldsymbol{X}_{3\delta\beta} = \Theta_{3\beta}. \tag{2.16b}$$

The three-point objects (2.13a) and (2.13b) have many properties similar to those of the two-point building blocks. Now if we decompose $\boldsymbol{X}_1$ into symmetric and antisymmetric parts similar to (2.8), we have

$$\boldsymbol{X}_{1\alpha\beta} = X_{1\alpha\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\Theta_1^2, \qquad X_{1\alpha\beta} = X_{1\beta\alpha}, \tag{2.17}$$

where the symmetric spinor $X_{1\alpha\beta}$ can be equivalently represented by the three-vector $X_{1m} = -\frac{1}{2}(\gamma_m)^{\alpha\beta}X_{1\alpha\beta}$. Now let us introduce analogs of the covariant spinor derivative and supercharge operators involving the three-point objects,

$$\mathcal{D}_{(1)\alpha} = \frac{\partial}{\partial\Theta_1^\alpha} + \mathrm{i}(\gamma^m)_{\alpha\beta}\Theta_1^\beta\frac{\partial}{\partial X_1^m},$$
$$\mathcal{Q}_{(1)\alpha} = \mathrm{i}\frac{\partial}{\partial\Theta_1^\alpha} + (\gamma^m)_{\alpha\beta}\Theta_1^\beta\frac{\partial}{\partial X_1^m}, \tag{2.18}$$

which obey the standard anticommutation relations

$$\{\mathcal{D}_{(i)\alpha}, \mathcal{D}_{(i)\beta}\} = \{\mathcal{Q}_{(i)\alpha}, \mathcal{Q}_{(i)\beta}\} = 2\mathrm{i}(\gamma^m)_{\alpha\beta}\frac{\partial}{\partial X_i^m}. \tag{2.19}$$

Some useful identities involving (2.18) are

$$\mathcal{D}_{(1)\gamma}\boldsymbol{X}_{1\alpha\beta} = -2\mathrm{i}\varepsilon_{\gamma\beta}\Theta_{1\alpha}, \quad \mathcal{Q}_{(1)\gamma}\boldsymbol{X}_{1\alpha\beta} = -2\varepsilon_{\gamma\alpha}\Theta_{1\beta}. \tag{2.20}$$

We must also account for the fact that various primary superfields obey certain differential equations. Using (2.12) we arrive at the following:

$$D_{(1)\gamma}\boldsymbol{X}_{3\alpha\beta} = 2\mathrm{i}(\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}\Theta_{3\beta}, \quad D_{(1)\alpha}\Theta_{3\beta} = -(\boldsymbol{x}_{13}^{-1})_{\beta\alpha}, \tag{2.21a}$$

$$D_{(2)\gamma}\boldsymbol{X}_{3\alpha\beta} = 2\mathrm{i}(\boldsymbol{x}_{23}^{-1})_{\beta\gamma}\Theta_{3\beta}, \quad D_{(2)\alpha}\Theta_{3\beta} = (\boldsymbol{x}_{23}^{-1})_{\beta\alpha}. \tag{2.21b}$$

Now given a function $f(\boldsymbol{X}_3, \Theta_3)$, there are the following differential identities that arise as a consequence of (2.20), (2.21a), and (2.21b):

$$D_{(1)\gamma}f(\boldsymbol{X}_3, \Theta_3) = (\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}\mathcal{D}_{(3)}^\alpha f(\boldsymbol{X}_3, \Theta_3), \tag{2.22a}$$

$$D_{(2)\gamma}f(\boldsymbol{X}_3, \Theta_3) = \mathrm{i}(\boldsymbol{x}_{23}^{-1})_{\alpha\gamma}\mathcal{Q}_{(3)}^\alpha f(\boldsymbol{X}_3, \Theta_3). \tag{2.22b}$$

These identities are essential for imposing differential constraints on correlation functions.

3. Building blocks in components

For future reference we will also review the nonsupersymmetric conformal blocks detailed in [2]. These objects will appear in component reduction of superspace correlation functions. The two-point and three-point structures are defined as follows:

$$x_{ij} = x_i - x_j, \quad X_{ij} = \frac{x_{ik}}{x_{ik}^2} - \frac{x_{jk}}{x_{jk}^2}, \quad i,j,k = 1,2,3. \tag{2.23}$$

These objects may be obtained by bar projection of the superspace variables defined in Sec. II as follows:

$$(x_{ij})_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}(\boldsymbol{x}_{ij})_{\alpha\beta}|, \quad (X_{ij})_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}(\boldsymbol{X}_k)_{\alpha\beta}|. \tag{2.24}$$

Here (i,j,k) is a cyclic permutation of (1, 2, 3). That is,

$$X_{12} = \frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}, \qquad (X_{12})_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}(\boldsymbol{X}_3)_{\alpha\beta}|, \text{ etc.} \tag{2.25}$$

In addition, we introduce the inversion tensor, $I_{a_1a_2}$, and its representation acting on rank-2 symmetric traceless tensors, $\mathcal{I}_{a_1a_2,m_1m_2}$,

$$I_{a_1a_2}(X) = \eta_{a_1a_2} - \frac{2X_{a_1}X_{a_2}}{X^2}, \tag{2.26a}$$

$$\mathcal{I}_{a_1a_2,m_1m_2}(X) = I_{a_1n_1}(X)I_{a_2n_2}(X)\mathcal{E}^{n_1n_2}{}_{m_1m_2}, \tag{2.26b}$$

where we have introduced the projection operator

$$\mathcal{E}_{m_1m_2,n_1n_2} = \frac{1}{2}(\eta_{m_1n_1}\eta_{m_2n_2} + \eta_{m_1n_2}\eta_{m_2n_1}) - \frac{1}{3}\eta_{m_1m_2}\eta_{n_1n_2}. \tag{2.27}$$

C. Correlation functions of primary superfields

The two-point correlation function of a primary superfield $\Phi_{\mathcal{A}}$ and its conjugate $\bar{\Phi}^{\mathcal{B}}$ is fixed by the superconformal symmetry as follows:

$$\langle\Phi_{\mathcal{A}}(z_1)\bar{\Phi}^{\mathcal{B}}(z_2)\rangle = c\frac{T_{\mathcal{A}}{}^{\mathcal{B}}(\hat{\boldsymbol{x}}_{12})}{(\boldsymbol{x}_{12}^2)^q}, \tag{2.28}$$

where c is a constant coefficient. The denominator of the two-point function is determined by the conformal dimension q of $\Phi_{\mathcal{A}}$, which guarantees that the correlation function transforms with the appropriate weight under scale transformations.

Concerning the three-point functions, let Φ, Ψ, and Π be primary superfields with conformal dimensions q_1, q_2, and q_3, respectively. The three-point function may be constructed using the general expression

$$\langle \Phi_{\mathcal{A}_1}(z_1)\Psi_{\mathcal{A}_2}(z_2)\Pi_{\mathcal{A}_3}(z_3)\rangle = \frac{T^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}'_1}(\hat{\boldsymbol{x}}_{13})T^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}'_2}(\hat{\boldsymbol{x}}_{23})}{(\boldsymbol{x}_{13}^2)^{q_1}(\boldsymbol{x}_{23}^2)^{q_2}}\mathcal{H}_{\mathcal{A}'_1\mathcal{A}'_2\mathcal{A}_3}(\boldsymbol{X}_3,\Theta_3,U_3), \tag{2.29}$$

where the tensor $\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}$ is highly constrained by the superconformal symmetry as follows:

(i) Under scale transformations of superspace $z^A = (x^a,\theta^\alpha) \mapsto z'^A = (\lambda^{-2}x^a, \lambda^{-1}\theta^\alpha)$, the three-point building blocks transform as $\mathcal{Z} = (\boldsymbol{X},\Theta) \mapsto \mathcal{Z}' = (\lambda^2\boldsymbol{X}, \lambda\Theta)$. As a consequence, the correlation function transforms as

$$\langle \Phi_{\mathcal{A}_1}(z'_1)\Psi_{\mathcal{A}_2}(z'_2)\Pi_{\mathcal{A}_3}(z'_3)\rangle = (\lambda^2)^{q_1+q_2+q_3}\langle \Phi_{\mathcal{A}_1}(z_1)\Psi_{\mathcal{A}_2}(z_2)\Pi_{\mathcal{A}_3}(z_3)\rangle, \tag{2.30}$$

which implies that $\mathcal{H}$ obeys the scaling property

$$\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(\lambda^2\boldsymbol{X},\lambda\Theta,U) = (\lambda^2)^{q_3-q_2-q_1}\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(\boldsymbol{X},\Theta,U), \quad \forall\,\lambda\in\mathbb{R}\backslash\{0\}. \tag{2.31}$$

This guarantees that the correlation function transforms correctly under conformal transformations.

(ii) If any of the fields Φ, Ψ, Π obey differential equations, such as conservation laws in the case of conserved current multiplets, then the tensor $\mathcal{H}$ is also constrained by differential equations. Such constraints may be derived with the aid of identities (2.22a) and (2.22b).

(iii) If any (or all) of the superfields Φ, Ψ, Π coincide, the correlation function possesses symmetries under permutations of superspace points, e.g.,

$$\langle \Phi_{\mathcal{A}_1}(z_1)\Phi_{\mathcal{A}_2}(z_2)\Pi_{\mathcal{A}_3}(z_3)\rangle = (-1)^{\epsilon(\Phi)}\langle \Phi_{\mathcal{A}_2}(z_2)\Phi_{\mathcal{A}_1}(z_1)\Pi_{\mathcal{A}_3}(z_3)\rangle, \tag{2.32}$$

where $\epsilon(\Phi)$ is the Grassmann parity of Φ. As a consequence, the tensor $\mathcal{H}$ obeys constraints that will be referred to as "point-switch identities."

The constraints above fix the functional form of $\mathcal{H}$ (and therefore the correlation function) up to finitely many parameters. Hence, the procedure described above reduces the problem of computing three-point correlation functions to deriving the tensor $\mathcal{H}$ subject to the above constraints.

III. COMPONENT STRUCTURE OF A SUPERSPIN-2 CURRENT MULTIPLET

In this paper we will be interested in three-point functions of a superspin-2 current multiplet described by the totally symmetric superfield $\mathcal{F}_{\alpha(4)} \coloneqq \mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)$, satisfying the conservation equation

$$D^{\alpha_1}\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z) = 0. \tag{3.1}$$

In three dimensions this superfield admits the following Taylor expansion:

$$\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z) = J_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) + Q_{\alpha_1\alpha_2\alpha_3\alpha_4,\alpha}(x)\theta^\alpha + \theta_{(\alpha_1}S_{\alpha_2\alpha_3\alpha_4)}(x) + \theta^2 B_{\alpha_1\alpha_2\alpha_3\alpha_4}(x). \tag{3.2}$$

It can be convenient to express some of these fields in vector notation as follows:

$$J_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) = (\gamma^{a_1})_{\alpha_1\alpha_2}(\gamma^{a_2})_{\alpha_3\alpha_4}J_{a_1a_2}(x), \tag{3.3}$$

where $J_{a_1a_2}$ is symmetric and traceless; a similar treatment follows for the other fields in the multiplet. Imposing the conservation equation is then tantamount to the following constraints on the component fields:

$$\partial^{a_1}J_{a_1a_2} = 0, \qquad \partial^{a_1}Q_{a_1a_2,\alpha} = 0, \qquad (\gamma^{a_1})_\delta{}^\alpha Q_{a_1a_2,\alpha} = 0, \tag{3.4a}$$

$$B_{a_1a_2} = \frac{\mathrm{i}}{2}\epsilon_{(a_1}{}^{mn}\partial_m J_{a_2)n}, \qquad S_{\alpha_1\alpha_2\alpha_3} = 0. \tag{3.4b}$$

Hence, we see this multiplet contains only two independent component currents: a conserved spin-2 field $J_{a_1a_2}$ satisfying the same properties as the energy momentum tensor, and a conserved spin-5/2 field $Q_{a_1a_2,\alpha}$ which is conserved and gamma traceless (the latter guarantees that Q is totally symmetric in spinor notation). Let us stress that $J_{a_1a_2}$ is different from the energy-momentum tensor $T_{a_1a_2}$, the latter is a component of the supercurrent multiplet $\mathcal{J}_{\alpha(3)}$. The independent components of $\mathcal{F}_{\alpha(4)}$ may be extracted by bar projection,

$$J_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) = \mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)|, \qquad Q_{\alpha_1\alpha_2\alpha_3\alpha_4,\alpha}(x) = D_\alpha\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)|. \tag{3.5}$$

In addition, under infinitesimal superconformal transformations, the superfield $\mathcal{F}$ transforms as

$$\delta\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z) = -\xi\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z) - q\sigma(z)\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z) + 4\lambda_{(\alpha_1}{}^{\delta}(z)\mathcal{F}_{\alpha_2\alpha_3\alpha_4)\delta}(z), \tag{3.6}$$

where q is the scaling dimension of $\mathcal{F}$. The conservation equation (3.1) then uniquely fixes the dimension of the field as follows: if we compute $\delta D^{\alpha_1}F_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)$ and use the definitions (2.5) and (2.6), we obtain

$$\delta(D^{\alpha_1}\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)) = \frac{1}{2}(q-3)D^2\xi^{\delta}\mathcal{F}_{\delta\alpha_2\alpha_3\alpha_4}(z). \tag{3.7}$$

Hence, we see that we require $q = 3$ for covariant conservation of $\mathcal{F}$.

IV. CORRELATION FUNCTION $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$

In this section we will derive an explicit solution for the three-point function $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$. In Sec. IV A we impose the constraints that arise due to the superfield conservation equations and invariance under permutation of superspace points z_1 and z_2. This is already too sufficient to fix the three-point function up to one parity-even and one parity-odd structure. In Sec. IV B we computationally analyze the constraints arising from the invariance of the three-point function under permutation of superspace points z_1 and z_3; this is done by considering the independent component correlators contained within $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$: $\langle JJJ\rangle$ and $\langle QJQ\rangle$. This is followed by a numerical analysis of the point-switch identity for consistency. Most of the tensor expressions are too large to be manipulated efficiently by hand, so we make use of *Mathematica* to do most of the lengthy calculations.

A. Superfield analysis

The ansatz for the correlation function $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ is

$$\langle\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle = \frac{\prod_{i=1}^4 \hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3), \tag{4.1}$$

where the tensor $\mathcal{H}$ is independently totally symmetric in the α_i, β_i, and γ_i, and is required to satisfy covariant constraints which arise due to conservation equations and invariance under permutations of superspace points. The constraints are summarized below:

(i) *Homogeneity constraint*

Covariance of the correlation function under scale transformations of superspace results in the following constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha(4)\beta(4)\gamma(4)}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-3}\mathcal{H}_{\alpha(4)\beta(4)\gamma(4)}(\boldsymbol{X},\Theta), \tag{4.2}$$

which implies that $\mathcal{H}$ is a homogeneous tensor field of degree -3. This constraint ensures conformal covariance of the three-point function.

(ii) *Differential constraints*

The conservation equation (3.1) implies that the correlation function must satisfy the following constraint:

$$D^{\sigma}_{(1)}\langle\mathcal{F}_{\sigma\alpha(3)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle = 0. \tag{4.3}$$

Application of the identities (2.22a) results in the following differential constraint on $\mathcal{H}$:

$$\mathcal{D}^{\sigma}\mathcal{H}_{\sigma\alpha(3)\beta(4)\gamma(4)}(\boldsymbol{X},\Theta) = 0. \tag{4.4}$$

(iii) *Point-switch identities*

Invariance under permutation of the superspace points z_1 and z_2 results in the following constraint on the correlation function:

$$\langle\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle = \langle\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\gamma(4)}(z_3)\rangle, \tag{4.5}$$

which results in the condition

$$\mathcal{H}_{\alpha(4)\beta(4)\gamma(4)}(\boldsymbol{X},\Theta) = \mathcal{H}_{\beta(4)\alpha(4)\gamma(4)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta). \tag{4.6}$$

There is an additional point-switch identity obtained from imposing invariance under permutation of the points z_1 and z_3; however, it is considerably more complicated so we will discuss it in detail later.

To make subsequent calculations more tractable, it is often convenient to express $\mathcal{H}$ in terms of its vector equivalent by factoring out gamma matrices as follows:

$$\mathcal{H}_{\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4\gamma_1\gamma_2\gamma_3\gamma_4}(\boldsymbol{X},\Theta) = (\gamma^{a_1})_{\alpha_1\alpha_2}(\gamma^{a_2})_{\alpha_3\alpha_4}(\gamma^{b_1})_{\beta_1\beta_2}(\gamma^{b_2})_{\beta_3\beta_4} \times(\gamma^{c_1})_{\gamma_1\gamma_2}(\gamma^{c_2})_{\gamma_3\gamma_4}\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta). \tag{4.7}$$

This equality holds provided that $\mathcal{H}$ (in vector notation) is symmetric and traceless in the pairs a_i, b_i, and c_i, respectively. This is seen by requiring that the components antisymmetric in α_2 and α_3 (and other combinations involving β_i and γ_i) vanish. Further, since $\mathcal{H}$ is Grassmann even it admits the Taylor expansion

$$\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta) = F_{a_1a_2b_1b_2c_1c_2}(X) + \Theta^2 G_{a_1a_2b_1b_2c_1c_2}(X). \tag{4.8}$$

At this step it is more convenient to view F and G as functions of the three-vector X^m rather than of $\boldsymbol{X}^{\alpha\beta}$. The point-switch identity (4.6) then implies the following constraints on F and G:

$$F_{a_1a_2b_1b_2c_1c_2}(X) = F_{b_1b_2a_1a_2c_1c_2}(-X), \tag{4.9a}$$

$$G_{a_1a_2b_1b_2c_1c_2}(X) = G_{b_1b_2a_1a_2c_1c_2}(-X). \tag{4.9b}$$

On the other hand, the differential constraint (4.4) results in[4]

$$\begin{aligned} &\partial^{a_1} F_{a_1a_2b_1b_2c_1c_2} = 0, \\ &G_{a_1a_2b_1b_2c_1c_2} = \frac{\mathrm{i}}{2}\epsilon_{(a_1}{}^{mn}\partial_{\underline{m}} F_{\underline{n}a_2)b_1b_2c_1c_2}. \end{aligned} \tag{4.10}$$

In the next subsections we will computationally solve for the tensor $\mathcal{H}$ subject to the constraints listed above using the xAct package.

1. Parity-even sector

In the parity-even sector, we begin by constructing a solution for F that is an even function of X; hence, Eq. (4.9a) implies

$$F_{a_1a_2b_1b_2c_1c_2}(X) = F_{b_1b_2a_1a_2c_1c_2}(X). \tag{4.11}$$

A general expansion for F consistent with the symmetry property (4.11) may be obtained by introducing the symmetric and traceless basis tensors found in [2]. Explicit expressions for the elements of the tensor basis $\{\Upsilon^i\}$, $i = 1, \ldots, 8$, are as follows:

$$\Upsilon^1_{a_1a_2}(\hat{X}) = \hat{X}_{a_1}\hat{X}_{a_2} - \frac{1}{3}\eta_{a_1a_2}, \qquad \hat{X}_a = \frac{X_a}{\sqrt{X^2}}, \tag{4.12a}$$

$$\begin{aligned} \Upsilon^2_{a_1a_2b_1b_2}(\hat{X}) = {}& \hat{X}_{a_1}\hat{X}_{b_1}\eta_{a_2b_2} + (a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2) \\ & -\frac{4}{3}\hat{X}_{a_1}\hat{X}_{a_2}\eta_{b_1b_2} - \frac{4}{3}\hat{X}_{b_1}\hat{X}_{b_2}\eta_{a_1a_2} \\ & +\frac{4}{9}\eta_{a_1a_2}\eta_{b_1b_2}, \end{aligned} \tag{4.12b}$$

$$\Upsilon^3_{a_1a_2b_1b_2} = \eta_{a_1b_1}\eta_{a_2b_2} + \eta_{a_1b_2}\eta_{a_2b_1} - \frac{2}{3}\eta_{a_1a_2}\eta_{b_1b_2}, \tag{4.12c}$$

$$\begin{aligned} \Upsilon^4_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = {}& \Upsilon^3_{a_1a_2b_1c_1}\hat{X}_{b_2}\hat{X}_{c_2} + (b_1 \leftrightarrow b_2, c_1 \leftrightarrow c_2) \\ & -\frac{2}{3}\eta_{b_1b_2}\Upsilon^2_{a_1a_2c_1c_2}(\hat{X}) - \frac{2}{3}\eta_{c_1c_2}\Upsilon^2_{a_1a_2b_1b_2}(\hat{X}) \\ & -\frac{8}{9}\eta_{b_1b_2}\eta_{c_1c_2}\Upsilon^1_{a_1a_2}(\hat{X}), \end{aligned} \tag{4.12d}$$

$$\begin{aligned} \Upsilon^5_{a_1a_2b_1b_2c_1c_2} = {}& \eta_{a_1b_1}\eta_{a_2c_1}\eta_{b_2c_2} \\ & + (a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2, c_1 \leftrightarrow c_2) \\ & -\frac{4}{3}\eta_{a_1a_2}\Upsilon^3_{b_1b_2c_1c_2} - \frac{4}{3}\eta_{b_1b_2}\Upsilon^3_{a_1a_2c_1c_2} \\ & -\frac{4}{3}\eta_{c_1c_2}\Upsilon^3_{a_1a_2b_1b_2} - \frac{8}{9}\eta_{a_1a_2}\eta_{b_1b_2}\eta_{c_1c_2}. \end{aligned} \tag{4.12e}$$

These tensors each possess a variety of symmetry properties; in particular, they are symmetric and traceless in pairs of indices. Using this basis we can construct the following set of rank-6 tensors

$$t^1_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^5_{a_1a_2b_1b_2c_1c_2}, \tag{4.13a}$$

$$t^2_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^4_{c_1c_2a_1a_2b_1b_2}(\hat{X}), \tag{4.13b}$$

$$t^3_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^4_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + \Upsilon^4_{b_1b_2a_1a_2c_1c_2}(\hat{X}), \tag{4.13c}$$

$$t^4_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^3_{a_1a_2b_1b_2}\Upsilon^1_{c_1c_2}(\hat{X}), \tag{4.13d}$$

$$t^5_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^3_{b_1b_2c_1c_2}\Upsilon^1_{a_1a_2}(\hat{X}) + \Upsilon^3_{a_1a_2c_1c_2}\Upsilon^1_{b_1b_2}(\hat{X}), \tag{4.13e}$$

$$t^6_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^2_{a_1a_2b_1b_2}(\hat{X})\Upsilon^1_{c_1c_2}(\hat{X}), \tag{4.13f}$$

$$\begin{aligned} t^7_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = {}& \Upsilon^2_{a_1a_2c_1c_2}(\hat{X})\Upsilon^1_{b_1b_2}(\hat{X}) \\ & + \Upsilon^2_{b_1b_2c_1c_2}(\hat{X})\Upsilon^1_{a_1a_2}(\hat{X}), \end{aligned} \tag{4.13g}$$

$$t^8_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = \Upsilon^1_{a_1a_2}(\hat{X})\Upsilon^1_{b_1b_2}(\hat{X})\Upsilon^1_{c_1c_2}(\hat{X}). \tag{4.13h}$$

The $t^i_{a_1a_2b_1b_2c_1c_2}$ each possess the symmetry property (4.11); hence, the ansatz for the tensor F is a linear combination of these tensor structures,

$$\begin{aligned} F_{a_1a_2b_1b_2c_1c_2}(X) &= \frac{1}{X^3}t_{a_1a_2b_1b_2c_1c_2}(\hat{X}), \\ t_{a_1a_2b_1b_2c_1c_2}(\hat{X}) &= \sum_{i=1}^{8} k_i t^i_{a_1a_2b_1b_2c_1c_2}(\hat{X}), \end{aligned} \tag{4.14}$$

where we have used the homogeneity constraint (4.2). It now remains to impose the differential constraint (4.4), which results in the following relations:

$$k_3 = -2k_1 - k_2, \qquad k_5 = k_4, \qquad k_6 = 15k_1 + 5k_2 - 5k_4, \tag{4.15a}$$

$$k_7 = -7k_1 - k_2 + 3k_4, \qquad k_8 = 28k_1 + 14k_2 - 7k_4. \tag{4.15b}$$

[4] The underlined indices are excluded from the symmetrization.

Hence, we see that the differential constraint immediately fixes the parity-even sector down to three independent coefficients. It is at this step where the linear dependence of the first five tensor structures can be noticed, as the k_1 dependence can be removed by shifting the variables as follows: $k_2 \to k_2 - k_1$, $k_3 \to k_3 - k_1$, $k_4 \to k_4 + 2k_1$, $k_5 \to k_5 + 2k_1$. Alternatively it may be shown that the following linear dependence relation holds:

$$t^1_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - t^2_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - t^3_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + 2t^4_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + 2t^5_{a_1a_2b_1b_2c_1c_2}(\hat{X}) = 0. \tag{4.16}$$

It is now clear that the k_1 term is redundant; hence, it can be completely removed from our analysis. This reduces our system of equations to

$$k_3 = -k_2, \quad k_5 = k_4, \quad k_6 = 5k_2 - 5k_4, \tag{4.17a}$$

$$k_7 = -k_2 + 3k_4, \qquad k_8 = 14k_2 - 7k_4. \tag{4.17b}$$

Therefore, the parity-even sector of the three-point function is fixed at this stage up to two independent coefficients, k_2 and k_4, and the explicit solution for F is

$$\begin{aligned} F_{a_1a_2b_1b_2c_1c_2}(X) &= \frac{k_2}{X^3}\{t^2_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - t^3_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + 5t^6_{a_1a_2b_1b_2c_1c_2}(\hat{X}) \\ &\quad - t^7_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + 14t^8_{a_1a_2b_1b_2c_1c_2}(\hat{X})\} \\ &\quad + \frac{k_4}{X^3}\{t^4_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + t^5_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - 5t^6_{a_1a_2b_1b_2c_1c_2}(\hat{X}) \\ &\quad + 3t^7_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - 7t^8_{a_1a_2b_1b_2c_1c_2}(\hat{X})\}. \end{aligned} \tag{4.18}$$

The tensor G is then determined in terms of F using (4.10). However, we have not yet imposed the condition (4.9b). Since G is an odd function of X by virtue of (4.10), the constraint (4.9b) implies

$$G_{a_1a_2b_1b_2c_1c_2}(X) = -G_{b_1b_2a_1a_2c_1c_2}(X). \tag{4.19}$$

After some calculations one can show that this results in an additional relation between the coefficients k_2 and k_4:

$$k_2 = -2k_4. \tag{4.20}$$

Thus, the conservation equations and the proper transformation under the $z_1 \leftrightarrow z_2$ exchange fix the parity-even sector up to a single overall coefficient. Note that so far we have not imposed the $z_1 \leftrightarrow z_3$ point-switch identity. It will be imposed later.

2. Parity-odd sector

Let us now construct the parity-odd sector of the correlation function, where we begin by assuming that the tensor $\tilde{F}$ is an odd function of X. Due to (4.9a), this implies that $\tilde{F}$ must satisfy

$$\tilde{F}_{a_1a_2b_1b_2c_1c_2}(X) = -\tilde{F}_{b_1b_2a_1a_2c_1c_2}(X). \tag{4.21}$$

Now let us construct an explicit solution for the tensor F; it must be an odd function of X, and each term must contain at most one instance of the Levi-Civita tensor (as products of the latter may be expressed in terms of the metric). We may decompose $\tilde{F}$ as follows:

$$\begin{aligned} \tilde{F}_{a_1a_2b_1b_2c_1c_2}(X) = \frac{1}{X^3}\{&\epsilon_{a_1b_1}{}^m P^1_{m,a_2b_2c_1c_2}(\hat{X}) + \epsilon_{a_1b_2}{}^m P^2_{m,a_2b_1c_1c_2}(\hat{X}) \\ &+ \epsilon_{a_2b_1}{}^m P^3_{m,a_1b_2c_1c_2}(\hat{X}) + \epsilon_{a_2b_2}{}^m P^4_{m,a_1b_1c_1c_2}(\hat{X})\}, \end{aligned} \tag{4.22}$$

where each P^i must have the symmetry property $P^i_{m,a_1a_2b_1b_2}(X) = P^i_{m,(a_1a_2)(b_1b_2)}(X)$. Requiring that the expansion (4.22) is consistent with the properties of pairwise index symmetry and (4.21) implies that the P^i must be identical. Hence, we need to find a general expansion for a tensor $P_{m,a_1a_2b_1b_2}$, which is homogeneous degree 0 and is composed solely of $\hat{X}$ and the metric tensor. Using *Mathematica* we can generate an ansatz consistent with the symmetry properties:

$$\begin{aligned} P_{m,a_1a_2b_1b_2}(\hat{X}) = {} & c_1\hat{X}^{a_1}\hat{X}^{a_2}\hat{X}^{b_1}\hat{X}^{b_2}\hat{X}^m + c_2\hat{X}^{b_1}\hat{X}^{b_2}\hat{X}^m\eta^{a_1a_2} + c_3\hat{X}^{a_1}\hat{X}^{a_2}\hat{X}^m\eta^{b_1b_2} \\ & + c_4\{\hat{X}^{a_2}\hat{X}^{b_1}\hat{X}^{b_2}\eta^{a_1m} + \hat{X}^{a_1}\hat{X}^{b_1}\hat{X}^{b_2}\eta^{a_2m}\} + c_5\{\hat{X}^{a_1}\hat{X}^{a_2}\hat{X}^{b_2}\eta^{b_1m} + \hat{X}^{a_1}\hat{X}^{a_2}\hat{X}^{b_1}\eta^{b_2m}\} \\ & + c_6\{\hat{X}^{a_2}\hat{X}^{b_2}\hat{X}^m\eta^{a_1b_1} + \hat{X}^{a_2}\hat{X}^{b_1}\hat{X}^m\eta^{a_1b_2} + \hat{X}^{a_1}\hat{X}^{b_2}\hat{X}^m\eta^{a_2b_1} + \hat{X}^{a_1}\hat{X}^{b_1}\hat{X}^m\eta^{a_2b_2}\} \\ & + c_7\hat{X}^m\eta^{a_1a_2}\eta^{b_1b_2} + c_8\{\hat{X}^m\eta^{a_1b_2}\eta^{a_2b_1} + \hat{X}^m\eta^{a_1b_1}\eta^{a_2b_2}\} \\ & + c_9\{\hat{X}^{b_2}\eta^{a_1m}\eta^{a_2b_1} + \hat{X}^{b_1}\eta^{a_1m}\eta^{a_2b_2} + \hat{X}^{b_2}\eta^{a_1b_1}\eta^{a_2m} + \hat{X}^{b_1}\eta^{a_1b_2}\eta^{a_2m}\} \\ & + c_{10}\{\hat{X}^{a_2}\eta^{a_1m}\eta^{b_1b_2} + \hat{X}^{a_1}\eta^{a_2m}\eta^{b_1b_2}\} + c_{11}\{\hat{X}^{b_2}\eta^{a_1a_2}\eta^{b_1m} + \hat{X}^{b_1}\eta^{a_1a_2}\eta^{b_2m}\} \\ & + c_{12}\{\hat{X}^{a_2}\eta^{a_1b_2}\eta^{b_1m} + \hat{X}^{a_1}\eta^{a_2b_2}\eta^{b_1m} + \hat{X}^{a_2}\eta^{a_1b_1}\eta^{b_2m} + \hat{X}^{a_1}\eta^{a_2b_1}\eta^{b_2m}\}. \end{aligned} \tag{4.23}$$

Only nine of these structures contribute when substituted into (4.22); in particular, the terms with c_9, c_4, and c_{10} may be neglected. Imposing tracelessness on each pair of indices is tantamount to the following constraints on the coefficients:

$$c_5 = c_6, \qquad c_{12} = c_8, \qquad c_1 = -6c_6 - 3c_3, \tag{4.24a}$$

$$c_7 = -\frac{2}{3}c_8 - \frac{2}{3}c_{11} - \frac{1}{3}c_2. \tag{4.24b}$$

It remains to impose the differential constraint for $\tilde{F}$ in (4.10), from which we find the additional relations

$$c_8 = -\frac{1}{4}c_6, \qquad c_3 = 0, \qquad c_{11} = \frac{1}{2}c_6, \qquad c_2 = -2c_6. \tag{4.25}$$

Hence, the solution for $\tilde{F}$ is fixed up to a single coefficient, $b = c_6$.[5] The solution for $\tilde{F}$ becomes

$$\begin{aligned}\tilde{F}_{a_1a_2b_1b_2c_1c_2}(X) = \frac{b}{X^3}\{&\epsilon_{a_1b_1}{}^m P_{m,a_2b_2c_1c_2}(\hat{X}) + \epsilon_{a_1b_2}{}^m P_{m,a_2b_1c_1c_2}(\hat{X}) \\ &+ \epsilon_{a_2b_1}{}^m P_{m,a_1b_2c_1c_2}(\hat{X}) + \epsilon_{a_2b_2}{}^m P_{m,a_1b_1c_1c_2}(\hat{X})\},\end{aligned} \tag{4.26}$$

where the explicit solution for P is

$$\begin{aligned}P_{m,a_1a_2b_1b_2}(\hat{X}) = &-6\hat{X}_{a_1}\hat{X}_{a_2}\hat{X}_{b_1}\hat{X}_{b_2}\hat{X}_m - 2\hat{X}_{b_1}\hat{X}_{b_2}\hat{X}_m\eta_{a_1a_2} + \hat{X}_{a_2}\hat{X}_{b_2}\hat{X}_m\eta_{a_1b_1} + \hat{X}_{a_2}\hat{X}_{b_1}\hat{X}_m\eta_{a_1b_2} + \hat{X}_{a_1}\hat{X}_{b_2}\hat{X}_m\eta_{a_2b_1} \\ &+ \hat{X}_{a_1}\hat{X}_{b_1}\hat{X}_m\eta_{a_2b_2} + \hat{X}_{a_1}\hat{X}_{a_2}\hat{X}_{b_1}\eta_{b_2m} + \hat{X}_{a_1}\hat{X}_{a_2}\hat{X}_{b_2}\eta_{b_1m} - \frac{1}{4}\hat{X}_m\eta_{a_1b_1}\eta_{a_2b_2} + \frac{1}{2}\hat{X}_m\eta_{a_1a_2}\eta_{b_1b_2} \\ &- \frac{1}{4}\hat{X}_m\eta_{a_1b_2}\eta_{a_2b_1} + \frac{1}{2}\hat{X}_{b_2}\eta_{a_1a_2}\eta_{b_1m} - \frac{1}{4}\hat{X}_{a_2}\eta_{a_1b_2}\eta_{b_1m} - \frac{1}{4}\hat{X}_{a_1}\eta_{a_2b_2}\eta_{b_1m} + \frac{1}{2}\hat{X}_{b_1}\eta_{a_1a_2}\eta_{b_2m} \\ &- \frac{1}{4}\hat{X}_{a_2}\eta_{a_1b_1}\eta_{b_2m} - \frac{1}{4}\hat{X}_{a_1}\eta_{a_2b_1}\eta_{b_2m}.\end{aligned} \tag{4.27}$$

The tensor $\tilde{G}$ is found using Eq. (4.10). However, we still need to impose the symmetry property (4.9b). Since $\tilde{G}$ is an even function of X, Eq. (4.9b) implies

$$\tilde{G}_{a_1a_2b_1b_2c_1c_2}(X) = \tilde{G}_{b_1b_2a_1a_2c_1c_2}(X). \tag{4.28}$$

After some calculations we find that Eq. (4.28) is satisfied automatically and does not result in any restrictions on b. Thus, the conservation equations and the proper transformation under the $z_1 \leftrightarrow z_2$ exchange fix the parity-odd sector up to a single overall coefficient.

B. Point-switch identity

The last constraint to be imposed on the correlation function $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ is invariance under the permutation of points z_1 and z_3; i.e., we must have

$$\langle\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle = \langle\mathcal{F}_{\gamma(4)}(z_3)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\alpha(4)}(z_1)\rangle. \tag{4.29}$$

This results in the following constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha(4)\beta(4)\gamma(4)}(\boldsymbol{X}_3, \Theta_3) = \frac{1}{\boldsymbol{x}_{13}^6\boldsymbol{X}_3^6}\prod_{i=1}^{4}\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{13\gamma_i}{}^{\gamma'_i}\hat{\boldsymbol{x}}_{13}{}^{\beta'_i\delta_i}\hat{\boldsymbol{X}}_{3\delta_i\beta_i}\mathcal{H}_{\gamma'(4)\beta'(4)\alpha'(4)}(-\boldsymbol{X}_1^{\mathrm{T}}, -\Theta_1). \tag{4.30}$$

[5]To account for linear dependence of the tensor structures, each constraint is checked by computationally analyzing every element of the tensor for an arbitrary building block vector $X = (X_0, X_1, X_2)$.

It is clear that direct calculation of (4.30) is inefficient due to (i) the large number of tensor structures in the solution for $\mathcal{H}$, and (ii) the linear dependence between the structures. Therefore, we will need to consider some alternative approaches, which will be explored in the next subsections.

The superfield correlator $\langle \mathcal{F}\mathcal{F}\mathcal{F}\rangle$ contains only two independent component correlators,

$$\langle J_{a_1a_2}(x_1)J_{b_1b_2}(x_2)J_{c_1c_2}(x_3)\rangle, \qquad \langle Q_{a_1a_2,\alpha}(x_1)J_{b_1b_2}(x_2)Q_{c_1c_2,\gamma}(x_3)\rangle. \tag{4.31}$$

These may be obtained by bar projection of the three-point function $\langle \mathcal{F}\mathcal{F}\mathcal{F}\rangle$ as follows[6]:

$$\langle J_{\alpha(4)}(x_1)J_{\beta(4)}(x_2)J_{\gamma(4)}(x_3)\rangle = \langle \mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle|, \tag{4.32a}$$

$$\langle Q_{\alpha(4),\alpha}(x_1)J_{\beta(4)}(x_2)Q_{\gamma(4),\gamma}(x_3)\rangle = D_{(3)\gamma}D_{(1)\alpha}\langle \mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle|. \tag{4.32b}$$

All correlators involving the components $S_{\alpha(3)}$ and $B_{\alpha(4)}$ in Eq. (3.2) either vanish or are expressed in terms of (4.31) by virtue of (3.4b). From Eq. (4.29) it follows that the component correlators (4.31) satisfy the following point-switch identities:

$$\langle J_{a_1a_2}(x_1)J_{b_1b_2}(x_2)J_{c_1c_2}(x_3)\rangle = \langle J_{c_1c_2}(x_3)J_{b_1b_2}(x_2)J_{a_1a_2}(x_1)\rangle, \tag{4.33a}$$

$$\langle Q_{a_1a_2,\alpha}(x_1)J_{b_1b_2}(x_2)Q_{c_1c_2,\gamma}(x_3)\rangle = -\langle Q_{c_1c_2,\gamma}(x_3)J_{b_1b_2}(x_2)Q_{a_1a_2,\alpha}(x_1)\rangle. \tag{4.33b}$$

These relations will be studied analytically (though with extensive use of *Mathematica*) in Secs. IV B 1 and IV B 2. However, proving Eqs. (4.33) is not sufficient to prove Eq. (4.29). The reason is that we cannot use Eqs. (3.4b) because we have not yet proven that the conservation law on the third point is satisfied. In fact, it will follow once we prove Eq. (4.29). Hence, to prove Eq. (4.29) at the component level we must consider all component correlators obtained from (4.29) by the action of the superspace covariant derivatives followed by bar projection. This is, clearly, impractical. Therefore, our approach will be to study Eq. (4.29) at higher orders in θ_i numerically, which we do in IV B 3. For this we will keep θ_i arbitrary but use various numeric values for the spacetime points x_1, x_2, x_3. Then the components of $\langle \mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle$ will be polynomials in θ_i with numeric coefficients. Since these polynomials are quite complicated, we are confident in our results despite the proof not being fully analytic.

1. Component correlator $\langle JJJ\rangle$

The computation of the component correlator $\langle JJJ\rangle$ is relatively straightforward, explicitly we have

$$\begin{aligned}\langle J_{\alpha(4)}(x_1)J_{\beta(4)}(x_2)J_{\gamma(4)}(x_3)\rangle &= \langle \mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle| \\ &= \frac{\prod_{i=1}^4 \hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)|.\end{aligned}$$

Since bar projections of any objects involving Θ vanish, combined with the result

$$\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)| = F_{\alpha'(4)\beta'(4)\gamma(4)}(X_{12}), \tag{4.34}$$

we obtain

$$\langle J_{\alpha(4)}(x_1)J_{\beta(4)}(x_2)J_{\gamma(4)}(x_3)\rangle = \frac{\prod_{i=1}^4 \hat{x}_{13\alpha_i}{}^{\alpha'_i}\hat{x}_{23\beta_i}{}^{\beta'_i}}{(x_{13}^2)^3(x_{23}^2)^3}F_{\alpha'(4)\beta'(4)\gamma(4)}(X_{12}). \tag{4.35}$$

If we convert this result into vector notation by combining pairs of spinor indices and apply the identity

$$I_{a_1a'_1}(x) = -\frac{1}{2}(\gamma_{a_1})^{\alpha_1\alpha_2}(\gamma_{a'_1})^{\alpha'_1\alpha'_2}\hat{x}_{\alpha_1\alpha'_1}\hat{x}_{\alpha_2\alpha'_2}, \tag{4.36}$$

[6] To express each of these correlators in the form (4.31), we combine symmetric pairs of spinor indices into a vector index as in (4.7) and use Eq. (2.24).

we obtain the result

$$\langle J_{a_1a_2}(x_1)J_{b_1b_2}(x_2)J_{c_1c_2}(x_3)\rangle = \frac{\mathcal{I}_{a_1a_2,a'_1a'_2}(x_{13})\mathcal{I}_{b_1b_2,b'_1b'_2}(x_{23})}{(x_{13}^2)^3(x_{23}^2)^3}F_{a'_1a'_2b'_1b'_2c_1c_2}(X_{12}). \tag{4.37}$$

If we now use (4.14), then this component correlator can be put in the covariant canonical form

$$\langle J_{a_1a_2}(x_1)J_{b_1b_2}(x_2)J_{c_1c_2}(x_3)\rangle = \frac{\mathcal{I}_{a_1a_2,a'_1a'_2}(x_{13})\mathcal{I}_{b_1b_2,b'_1b'_2}(x_{23})}{x_{13}^3x_{23}^3x_{12}^3}t_{a'_1a'_2b'_1b'_2c_1c_2}(X_{12}). \tag{4.38}$$

It then follows that the constrained tensors $t_{a'_1a'_2b'_1b'_2c_1c_2}(X_{12})$ appearing in $\langle JJJ\rangle$ satisfy all the same properties as those present in the energy-momentum tensor three-point function $\langle TTT\rangle$, so we can simply use the known results.

It was shown in [2] that the parity-even contribution to the three-point function $\langle TTT\rangle$ in general dimensions is fixed up to three independent coefficients. However, in three-dimensional theories there is linear dependence between the tensor structures due to the identity (4.16). This reduces the number of independent structures down to two. The solution for $t_{a_1a_2b_1b_2c_1c_2}(X)$ found in [2] is the same as given in our Eqs. (4.14) and (4.18). It was also shown in [2] that this solution satisfies the $z_1 \leftrightarrow z_3$ point-switch identity. Since the solution for the correlator $\langle JJJ\rangle$ is identical to that of $\langle TTT\rangle$, it follows that the three-point function $\langle JJJ\rangle$ in (4.35) with $F_{\alpha(4)\beta(4)\gamma(4)}$ defined in (4.18) is compatible with the point-switch identity (4.33a) for arbitrary k_2 and k_4. In this case there is a further relation between k_2 and k_4 in Eq. (4.20); however, it does not affect the $z_1 \leftrightarrow z_3$ point-switch identity.

The parity-odd sector of the energy-momentum tensor three-point function was obtained in [1]. It was shown that it is fixed up to one independent structure given in Eqs. (4.26) and (4.27).[7] Hence, Eqs. (4.26) and (4.27) are also compatible with the point-switch identity (4.33a). In the remaining subsections we will consider the relation (4.29) at higher orders in θ_i.

2. Component correlator $\langle QJQ\rangle$

The correlator $\langle QJQ\rangle$ can be computed as follows:

$$\begin{aligned}\langle Q_{\alpha(4),\alpha}(x_1)J_{\beta(4)}(x_2)Q_{\gamma(4),\gamma}(x_3)\rangle &= D_{(3)\gamma}D_{(1)\alpha}\langle\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle| \\ &= D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)\right\}\Bigg| \\ &= A+B.\end{aligned} \tag{4.39}$$

After evaluating the derivatives, one finds that the calculation is broken up into two relevant parts: the A contribution is due to the derivatives hitting the prefactor,

$$A = D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\right\}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)\Big|, \tag{4.40}$$

while the B contribution arises due to the derivatives hitting $\mathcal{H}$,

$$B = \frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}D_{(3)\gamma}D_{(1)\alpha}\{\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)\}|. \tag{4.41}$$

Other combinations in which either derivative acts on the prefactor and $\mathcal{H}$ result in terms that are at least linear in θ_1, θ_3, or Θ_3, which vanish upon bar projection, so they may be neglected. The A contribution can be written in the form

$$\begin{aligned}A = &\frac{1}{(x_{13}^2)^{7/2}(x_{23}^2)^3}\hat{x}_{13\alpha}{}^{\alpha'}\hat{x}_{13\alpha_1}{}^{\alpha'_1}\hat{x}_{13\alpha_2}{}^{\alpha'_2}\hat{x}_{13\alpha_3}{}^{\alpha'_3}\hat{x}_{13\alpha_4}{}^{\alpha'_4} \\ &\times\hat{x}_{23\beta_1}{}^{\beta'_1}\hat{x}_{23\beta_2}{}^{\beta'_2}\hat{x}_{23\beta_3}{}^{\beta'_3}\hat{x}_{23\beta_4}{}^{\beta'_4}\mathcal{T}^A_{\alpha',\alpha'(4)\beta'(4)\gamma,\gamma(4)}(X_{12}),\end{aligned} \tag{4.42}$$

with $\mathcal{T}^A$ defined as

$$\mathcal{T}^A_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) = -10\mathrm{i}\varepsilon_{\gamma(\alpha}F_{\alpha_1\alpha_2\alpha_3\alpha_4)\beta(4)\gamma(4)}(X). \tag{4.43}$$

Similarly if we evaluate the B contribution, we find it can be written in the form

[7] We use a different approach and notation than the authors in [1]; however, our results agree.

$$B = \frac{1}{(x_{13}^2)^{7/2}(x_{23}^2)^3} \hat{x}_{13\alpha}{}^{\alpha'} \hat{x}_{13\alpha_1}{}^{\alpha'_1} \hat{x}_{13\alpha_2}{}^{\alpha'_2} \hat{x}_{13\alpha_3}{}^{\alpha'_3} \hat{x}_{13\alpha_4}{}^{\alpha'_4} \hat{x}_{23\beta_1}{}^{\beta'_1} \hat{x}_{23\beta_2}{}^{\beta'_2} \hat{x}_{23\beta_3}{}^{\beta'_3} \hat{x}_{23\beta_4}{}^{\beta'_4} \mathcal{T}^B_{\alpha',\alpha'(4)\beta'(4)\gamma,\gamma(4)}(X_{12}), \tag{4.44}$$

with $\mathcal{T}^B$ given by the expression

$$\mathcal{T}^B_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) = -\mathrm{i}(\gamma^m)_{\alpha\sigma} X^{\sigma}{}_{\gamma} \partial_m F_{\alpha(4)\beta(4)\gamma(4)}(X) - 2X_{\alpha\gamma} G_{\alpha(4)\beta(4)\gamma(4)}(X). \tag{4.45}$$

Hence, we see that the correlation function $\langle QJQ \rangle$ may be written in the following covariant canonical form:

$$\langle Q_{\alpha(4),\alpha}(x_1) J_{\beta(4)}(x_2) Q_{\gamma(4),\gamma}(x_3) \rangle = \frac{1}{(x_{13}^2)^{7/2}(x_{23}^2)^3} \hat{x}_{13\alpha}{}^{\alpha'} \hat{x}_{13\alpha_1}{}^{\alpha'_1} \hat{x}_{13\alpha_2}{}^{\alpha'_2} \hat{x}_{13\alpha_3}{}^{\alpha'_3} \hat{x}_{13\alpha_4}{}^{\alpha'_4} \times \hat{x}_{23\beta_1}{}^{\beta'_1} \hat{x}_{23\beta_2}{}^{\beta'_2} \hat{x}_{23\beta_3}{}^{\beta'_3} \hat{x}_{23\beta_4}{}^{\beta'_4} \mathcal{T}_{\alpha',\alpha'(4)\beta'(4)\gamma,\gamma(4)}(X_{12}), \tag{4.46}$$

with

$$\mathcal{T}_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) = \mathcal{T}^A_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) + \mathcal{T}^B_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X). \tag{4.47}$$

Additional details regarding this calculation are contained in Appendix B. It is worth commenting that it is not immediately obvious that $\mathcal{T}$ is totally symmetric in the α indices; indeed, it may be shown by direct calculations that this symmetry is manifest by virtue of (4.3) [and by extension (4.10)].[8] To make subsequent calculations more tractable, we convert this entire expression into vector notation. The component three-point function may then be written in the following form:

$$\langle Q_{a_1a_2,\alpha}(x_1) J_{b_1b_2}(x_2) Q_{c_1c_2,\gamma}(x_3) \rangle = \frac{\hat{x}_{13}^m}{(x_{13}^2)^{7/2}(x_{23}^2)^3} \mathcal{I}_{a_1a_2,a'_1a'_2}(x_{13}) \mathcal{I}_{b_1b_2,b'_1b'_2}(x_{23}) \times \mathcal{T}_{m,a'_1a'_2b'_1b'_2c_1c_2,\alpha\gamma}(X_{12}). \tag{4.48}$$

It is convenient to decompose the tensor $\mathcal{T}$ into the symmetric and antisymmetric parts

$$\mathcal{T}_{m,a_1a_2b_1b_2c_1c_2,\alpha\gamma}(X) = \varepsilon_{\alpha\gamma} A_{m,a_1a_2b_1b_2c_1c_2}(X) + (\gamma_n)_{\alpha\gamma} S^n{}_{m,a_1a_2b_1b_2c_1c_2}(X). \tag{4.49}$$

We find the following expressions for the tensors A and S:

$$A_{m,a_1a_2b_1b_2c_1c_2}(X) = \mathrm{i}\epsilon_m{}^{pq} X_q \partial_p F_{a_1a_2b_1b_2c_1c_2}(X) + 2X_m G_{a_1a_2b_1b_2c_1c_2}(X) - 2\mathrm{i}\Pi_{m,a_1a_2m_1m_2} F^{m_1m_2}{}_{b_1b_2c_1c_2}(X), \tag{4.50}$$

$$S^n{}_{m,a_1a_2b_1b_2c_1c_2}(X) = \mathrm{i}\mathfrak{D}^n{}_m F_{a_1a_2b_1b_2c_1c_2}(X) + 2\epsilon^n{}_{mp} X^p G_{a_1a_2b_1b_2c_1c_2}(X) - 2\mathrm{i}\Xi^n{}_{m,a_1a_2m_1m_2} F^{m_1m_2}{}_{b_1b_2c_1c_2}(X). \tag{4.51}$$

The differential operator $\mathfrak{D}$ and the constant "projection" tensors Π and Ξ naturally arise when expressing $\langle QJQ \rangle$ in the covariant form (4.48). They have the following definitions:

$$\mathfrak{D}_{nm} = X_n \partial_m - X_m \partial_n + \eta_{nm} X^p \partial_p - \eta_{nm}, \tag{4.52}$$

$$\Pi_{m,a_1a_2,b_1b_2} = \frac{1}{2}\epsilon_{a_2b_2n}\eta_{a_1b_1} + \frac{1}{2}\epsilon_{a_2b_1n}\eta_{a_1b_2} + \frac{1}{2}\epsilon_{a_1b_2n}\eta_{a_2b_1} + \frac{1}{2}\epsilon_{a_1b_1n}\eta_{a_2b_2}, \tag{4.53}$$

$$\begin{aligned}\Xi_{nm,a_1a_2b_1b_2} = {} & \frac{1}{2}\eta_{a_1n}\eta_{a_2b_2}\eta_{b_1m} + \frac{1}{2}\eta_{a_1b_2}\eta_{a_2n}\eta_{b_1m} - \frac{1}{2}\eta_{a_1m}\eta_{a_2b_2}\eta_{b_1n} - \frac{1}{2}\eta_{a_1b_2}\eta_{a_2m}\eta_{b_1n} \\ & + \frac{1}{2}\eta_{a_1n}\eta_{a_2b_1}\eta_{b_2m} + \frac{1}{2}\eta_{a_1b_1}\eta_{a_2n}\eta_{b_2m} - \frac{1}{2}\eta_{a_1m}\eta_{a_2b_1}\eta_{b_2n} - \frac{1}{2}\eta_{a_1b_1}\eta_{a_2m}\eta_{b_2n} \\ & - \eta_{a_1b_2}\eta_{a_2b_1}\eta_{mn} - \eta_{a_1b_1}\eta_{a_2b_2}\eta_{mn} + \frac{2}{3}\eta_{a_1a_2}\eta_{b_1b_2}\eta_{mn}.\end{aligned} \tag{4.54}$$

[8]Recall that in (3.4b) it was shown that the component field Q is totally symmetric after imposing conservation of $\mathcal{F}$. Since we have already imposed conservation of $\langle \mathcal{F}\mathcal{F}\mathcal{F} \rangle$ at z_1, the fact that $\langle QJQ \rangle$ is totally symmetric in α is implicit.

The point-switch identity on $\langle QJQ\rangle$, Eq. (4.33b) can be written in terms of the following two equations involving only vector indices:

$$\mathcal{I}_{b_1b_2}{}^{b_1'b_2'}(X)A_{m,a_1a_2b_1'b_2'c_1c_2}(X)+A_{m,c_1c_2b_1b_2a_1a_2}(-X)=0, \tag{4.55a}$$

$$\mathcal{I}_{b_1b_2}{}^{b_1'b_2'}(X)S^n{}_{m,a_1a_2b_1'b_2'c_1c_2}(X)-S^n{}_{m,c_1c_2b_1b_2a_1a_2}(-X)=0. \tag{4.55b}$$

To recall, here the tensors A and S are given by Eqs. (4.50) and (4.51), where the tensor F is given by Eq. (4.18) in the parity-even case and in Eqs. (4.26) and (4.27) in the parity-odd case, and the tensor G in both cases is obtained from F using Eq. (4.10).

Now the task is to substitute F and G into Eqs. (4.50) and (4.51) in the parity-even and in the parity-odd cases separately to determine if there are additional, different from Eq. (4.20) constraints on the coefficients k_2, k_4, and b in Eqs. (4.18) and (4.26). Since A and S have rather complicated definitions, it is futile to attempt to impose them by hand; however, computation of these identities is possible in *Mathematica* using the xAct package [40]. The package allows for symbolic manipulation of tensors using index notation and contains a suite of "canonicalization" functions which can essentially manipulate tensor structures down to their simplest form. In this way the computations are completely symbolic and are exactly the same as if they were done "by hand." Once a given tensor is canonicalized, we can then convert the expression into an array using in-built functions.

Parity-even sector.—Evaluating (4.55a) using definitions (4.50) and the solution (4.18) results in ≈400 terms after canonicalization. On the other hand, Eq. (4.55b) results in ≈800. The tensor structures in each identity should cancel among each other for some relation between the coefficients k_2 and k_4. However, if we naively just collect all of the tensor structures, one would find that $k_2=k_4=0$, as there is a hidden linear dependence between the terms. A way around this is to go into a coordinate basis and check every component of the left-hand sides (LHS) of (4.55a) and (4.55b). If we carry out this computation, the identities are satisfied for the choice $k_2=-2k_4$. Hence, we do not get any new relations in the parity-even sector, and it is still fixed up to an overall coefficient.

Parity-odd sector.—We now carry out an identical analysis for the parity-odd solution (4.26), which turns out to be more computationally intensive. In this case there are ≈800 tensor structures after canonicalization of the LHS of (4.55a), while there are ≈1600 for (4.55b). If one goes into a coordinate basis, the identities are satisfied for an arbitrary choice of the coefficient b. Hence, the parity-odd sector is also fixed up to a single tensor structure.

3. Numerical analysis

To supplement the results above, we will carry out a numerical analysis of the point-switch identity by substituting in various configurations of points. To do this, first we convert the ansatz (4.1) into vector notation. This can be done by introducing the following $\mathcal{N}=1$ object:

$$\begin{aligned}I_{ab}(\boldsymbol{x}_{12})&=-\frac{1}{2}(\gamma_a)^{\alpha_1\alpha_2}(\gamma_b)^{\alpha_1'\alpha_2'}(\hat{\boldsymbol{x}}_{12})_{\alpha_1\alpha_1'}(\hat{\boldsymbol{x}}_{12})_{\alpha_2\alpha_2'}\\&=I_{ab}(y_{12})-\mathrm{i}\epsilon_{abm}\hat{y}_{12}^m\frac{\theta^2}{y_{12}}.\end{aligned} \tag{4.56}$$

To recall, $\boldsymbol{x}_{12}$ is given in Eq. (2.8), the vector y_{12} is given in (2.9) and $\hat{y}_{12}^m=y_{12}^m/y_{12}$. $I_{ab}(\boldsymbol{x}_{12})$ may be thought of as the supersymmetric generalization of (2.26b). It obeys some useful properties such as

$$I_a{}^m(\boldsymbol{x}_{12})I_{mb}(-\boldsymbol{x}_{12})=\eta_{ab},\qquad I_a{}^m(\boldsymbol{x}_{12})I_{mb}(\boldsymbol{x}_{12})=\eta_{ab}-2\mathrm{i}\epsilon_{abm}\hat{y}_{12}^m\frac{\theta^2}{y_{12}}. \tag{4.57}$$

Using this new object, the ansatz (4.1) can be written in the form

$$\langle\mathcal{F}_{a_1a_2}(z_1)\mathcal{F}_{b_1b_2}(z_2)\mathcal{F}_{c_1c_2}(z_3)\rangle=\frac{I_{a_1a_1'}(\boldsymbol{x}_{13})I_{a_2a_2'}(\boldsymbol{x}_{13})I_{b_1b_1'}(\boldsymbol{x}_{23})I_{b_2b_2'}(\boldsymbol{x}_{23})}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{a_1'a_2'b_1'b_2'c_1c_2}(\boldsymbol{X}_3,\Theta_3). \tag{4.58}$$

Now to check the point-switch identity, we will introduce null vectors λ_1, λ_2, λ_3, and contract them with the ansatz to obtain

$$\langle\mathcal{F}(z_1)\mathcal{F}(z_2)\mathcal{F}(z_3)\rangle=\langle\mathcal{F}_{a_1a_2}(z_1)\mathcal{F}_{b_1b_2}(z_2)\mathcal{F}_{c_1c_2}(z_3)\rangle\lambda_1^{a_1}\lambda_1^{a_2}\lambda_2^{b_1}\lambda_2^{b_2}\lambda_3^{c_1}\lambda_3^{c_2}. \tag{4.59}$$

Essentially our approach is to pick a configuration of points x_1, x_2, x_3 and null vectors λ_1, λ_2, λ_3, and then expand out (4.59) to all orders and combinations of the fermionic superspace coordinates θ_1, θ_2, θ_3. This simplifies the point-switch identity

$$\langle \mathcal{F}(z_1)\mathcal{F}(z_2)\mathcal{F}(z_3)\rangle = \langle \mathcal{F}(z_3)\mathcal{F}(z_2)\mathcal{F}(z_1)\rangle \tag{4.60}$$

to a polynomial expression in the fermionic coordinates. We then check whether the point-switch identity is satisfied for both the parity-even and parity-odd solutions that we found in Sec. IVA. To carry out these computations we must make use of the following expansions for the fermionic two-point (three-point) functions, which follow from the definitions (2.8) and (2.13b):

$$\theta_{13}^2 = \theta_1^2 + \theta_3^2 - 2\theta_1 \cdot \theta_3, \tag{4.61a}$$

$$\Theta_3^2 = \frac{\theta_{13}^2}{y_{13}^2} + \frac{\theta_{23}^2}{y_{23}^2} + \frac{2}{y_{13}^2 y_{23}^2} y_{13}^m y_{23}^n \{\eta_{mn}\theta_{13}\cdot\theta_{23} - \epsilon_{mnc}(\theta_{13}\cdot\gamma\cdot\theta_{23})^c\}, \tag{4.61b}$$

where we have used the notation

$$\theta_i \cdot \theta_j = \theta_i^\alpha \theta_{j\alpha}, \qquad (\theta_{ij}\cdot\gamma\cdot\theta_{jk})^a = (\gamma^a)_{\alpha\beta}\theta_{ij}^\alpha\theta_{jk}^\beta. \tag{4.62}$$

Expansions for the other building blocks are obtained by cyclic permutations of superspace points. Hence, we see that the resulting polynomial from (4.59) will be a function of $\theta_i^2, \theta_i \cdot \theta_j, (\theta_{ij}\cdot\gamma\cdot\theta_{jk})^a$, and combinations/products of these objects.[9] All the θ expansions and numerical calculations are done computationally. We performed a numeric analysis for various configurations of points and null vectors and always found the same result. Below we present one example when the polynomials are relatively simple.

Let us pick the following points and null vectors:

$$x_1 = (0,-1,0), \qquad x_2 = (0,1,0), \qquad x_3 = (0,0,1), \tag{4.63a}$$

$$\lambda_1 = (1,0,1), \qquad \lambda_2 = (1,1,0), \qquad \lambda_3 = (1,-1,0). \tag{4.63b}$$

We now substitute the above values into (4.60). For the parity-even solution (denoted by subscript E) we obtain

$$\begin{aligned}
\langle \mathcal{F}(z_1)\mathcal{F}(z_2)\mathcal{F}(z_3)\rangle_E - \langle \mathcal{F}(z_3)\mathcal{F}(z_2)\mathcal{F}(z_1)\rangle_E = {} & \left(\frac{k_2}{128}+\frac{k_4}{64}\right)\theta_1^2\theta_2^2 + \left(\frac{k_2}{128}+\frac{k_4}{64}\right)\theta_1^2\theta_3^2 + \left(\frac{k_2}{128}+\frac{k_4}{64}\right)\theta_2^2\theta_3^2 \\
& + \mathrm{i}\left(\frac{k_2}{64}+\frac{k_4}{32}\right)\theta_1\cdot\theta_3 + \mathrm{i}\left(\frac{k_2}{64}+\frac{k_4}{32}\right)\theta_2\cdot\theta_3 - \mathrm{i}\left(\frac{k_2}{64}+\frac{k_4}{32}\right)(\theta_{13}\cdot\gamma\cdot\theta_{23})^1 \\
& + \left(\frac{k_2}{32}+\frac{k_4}{16}\right)\theta_1\cdot\theta_3(\theta_{13}\cdot\gamma\cdot\theta_{23})^1 - \left(\frac{k_2}{16}+\frac{k_4}{8}\right)\theta_2\cdot\theta_3(\theta_{13}\cdot\gamma\cdot\theta_{23})^1 \\
& + \theta_1^2\left\{-\frac{\mathrm{i}k_2}{128} - \frac{\mathrm{i}k_4}{64} - \left(\frac{k_2}{64}+\frac{k_4}{32}\right)\theta_2\cdot\theta_3 - \left(\frac{k_2}{64}+\frac{k_4}{32}\right)(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\right\} \\
& + \theta_2^2\left\{-\frac{\mathrm{i}k_2}{128} - \frac{\mathrm{i}k_4}{64} - \left(\frac{k_2}{64}+\frac{k_4}{32}\right)\theta_1\cdot\theta_3 + \left(\frac{k_2}{32}+\frac{k_4}{16}\right)(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\right\} \\
& + \theta_3^2\left\{-\frac{\mathrm{i}k_2}{64} - \frac{\mathrm{i}k_4}{32} - \left(\frac{k_2}{64}+\frac{k_4}{32}\right)\theta_1\cdot\theta_2 + \left(\frac{k_2}{64}+\frac{k_4}{32}\right)(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\right\}.
\end{aligned} \tag{4.64}$$

Clearly, this expression vanishes at each order of θ for the choice $k_2 = -2k_4$, which is the same condition as found previously in Eq. (4.20). Hence, the numerical evaluations agree with our previous calculations and do not give any new relations.

Next we perform the same calculation for the parity-odd solution (denoted by O). Explicit evaluation of $\langle \mathcal{F}(z_1)\mathcal{F}(z_2)\mathcal{F}(z_3)\rangle_O$ yields the following polynomial:

[9]Not all these objects are linearly independent since θ_i are Grassmann odd but one can choose a convenient basis.

$$\begin{aligned}\langle\mathcal{F}(z_1)\mathcal{F}(z_2)\mathcal{F}(z_3)\rangle_O = &-\frac{15}{64}b\theta_1^2\theta_2^2-\frac{15}{64}b\theta_1^2\theta_3^2-\frac{15}{64}b\theta_2^2\theta_3^2\\ &+\frac{3\mathrm{i}}{16}b\theta_1\cdot\theta_3-\frac{3\mathrm{i}}{32}b\theta_2\cdot\theta_3-\frac{9\mathrm{i}}{32}b(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\\ &-\frac{3}{16}b\theta_1\cdot\theta_3(\theta_{13}\cdot\gamma\cdot\theta_{23})^1+\frac{3}{4}b\theta_2\cdot\theta_3(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\\ &+\theta_1^2\left\{-\frac{3\mathrm{i}}{32}b+\frac{15}{32}b\theta_2\cdot\theta_3+\frac{3}{32}b(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\right\}\\ &+\theta_2^2\left\{\frac{3\mathrm{i}}{64}b+\frac{15}{32}b\theta_1\cdot\theta_3-\frac{3}{8}b(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\right\}\\ &+\theta_3^2\left\{-\frac{3\mathrm{i}}{64}b+\frac{15}{32}b\theta_1\cdot\theta_2-\frac{9}{32}b(\theta_{13}\cdot\gamma\cdot\theta_{23})^1\right\}.\end{aligned}\tag{4.65}$$

When we similarly compute $\langle\mathcal{F}(z_3)\mathcal{F}(z_2)\mathcal{F}(z_1)\rangle_O$, we find the same result. Hence, we observe cancellation at every order, and therefore the odd solution also satisfies the point-switch identity for an arbitrary coefficient b.

Note that the polynomials in both parity-even and parity-odd cases are quite nontrivial even for a simple choice of the points and null vectors. We performed a similar numeric analysis for various other choices and obtained the same result as above. However, in all other cases the polynomials are quite large so we will not present them here. The complexity of the polynomials makes any accidental cancellations highly unlikely. Hence, we are confident that the point-switch identity is satisfied for $k_2=-2k_4$ and arbitrary b.

C. Summary of results

Since our analysis is rather technical and involves analytic and numeric computations of the superfield and component expressions, we will collect all the pieces together and summarize our results. We found that the correlation function $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ contains two independent tensor structures after imposing all of the constraints; one of them is parity even, while the other is parity odd. In particular, we found that the parity-odd contribution is present unlike in all cases of three-point functions involving the supercurrent and flavor current multiplets [31,39].

The correlation function found above has the following structure:

$$\langle\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle=\frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha_i'}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta_i'}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3),\tag{4.66}$$

where $\mathcal{H}$ can also be written as follows:

$$\begin{aligned}\mathcal{H}_{\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4\gamma_1\gamma_2\gamma_3\gamma_4}(\boldsymbol{X},\Theta)=&(\gamma^{a_1})_{\alpha_1\alpha_2}(\gamma^{a_2})_{\alpha_3\alpha_4}(\gamma^{b_1})_{\beta_1\beta_2}(\gamma^{b_2})_{\beta_3\beta_4}\\ &\times(\gamma^{c_1})_{\gamma_1\gamma_2}(\gamma^{c_2})_{\gamma_3\gamma_4}\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta).\end{aligned}\tag{4.67}$$

The tensor $\mathcal{H}$ in vector notation then may be split into parity-even and parity-odd sectors

$$\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta)=\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta)_E+\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta)_O,\tag{4.68}$$

where each solution admits the following expansion:

$$\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta)_E=F_{a_1a_2b_1b_2c_1c_2}(X)+\Theta^2G_{a_1a_2b_1b_2c_1c_2}(X),\tag{4.69a}$$

$$\mathcal{H}_{a_1a_2b_1b_2c_1c_2}(\boldsymbol{X},\Theta)_O=\tilde{F}_{a_1a_2b_1b_2c_1c_2}(X)+\Theta^2\tilde{G}_{a_1a_2b_1b_2c_1c_2}(X),\tag{4.69b}$$

with G, $\tilde{G}$ determined in terms of F, $\tilde{F}$ by Eqs. (4.10). After imposing all the constraints, we find that the solution for the tensor F in the even sector is

$$
\begin{aligned}
F_{a_1a_2b_1b_2c_1c_2}(X) = \frac{a}{X^3}\{&-2t^2_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + 2t^3_{a_1a_2b_1b_2c_1c_2}(\hat{X}) + t^4_{a_1a_2b_1b_2c_1c_2}(\hat{X}) \\
&+ t^5_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - 15t^6_{a_1a_2b_1b_2c_1c_2}(\hat{X}) \\
&+ 5t^7_{a_1a_2b_1b_2c_1c_2}(\hat{X}) - 35t^8_{a_1a_2b_1b_2c_1c_2}(\hat{X})\},
\end{aligned} \tag{4.70}
$$

where we have relabeled $k_4 \to a$ and $t^i(\hat{X})$ are given by Eqs. (4.12) and (4.13).

On the other hand, we find the solution in the odd sector to be

$$
\begin{aligned}
\tilde{F}_{a_1a_2b_1b_2c_1c_2}(X) = \frac{b}{X^3}\{&\epsilon_{a_1b_1}{}^m P_{m,a_2b_2c_1c_2}(\hat{X}) + \epsilon_{a_1b_2}{}^m P_{m,a_2b_1c_1c_2}(\hat{X}) \\
&+ \epsilon_{a_2b_1}{}^m P_{m,a_1b_2c_1c_2}(\hat{X}) + \epsilon_{a_2b_2}{}^m P_{m,a_1b_1c_1c_2}(\hat{X})\},
\end{aligned} \tag{4.71}
$$

with P defined as in (4.27).

V. MIXED CORRELATORS

In this section we will discuss some basic examples of three-point functions of $\mathcal{F}$ with other fields such as a scalar superfield $\mathcal{O}$ of dimension Δ, and the non-Abelian flavor current superfield $L^{\bar{a}}_{\alpha}$ of dimension $3/2$. The calculations are straightforward compared to the $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ case, so we will not require computational methods here. These three-point functions were also previously studied in [20]; however, our method is different and more explicit.

A. Correlation function $\langle\mathcal{O}\mathcal{F}\mathcal{O}\rangle$

Let us now compute the correlation function $\langle\mathcal{O}\mathcal{F}\mathcal{O}\rangle$, which admits the general ansatz

$$
\langle\mathcal{O}(z_1)\mathcal{F}_{\alpha(4)}(z_2)\mathcal{O}(z_3)\rangle = \frac{\prod_{i=1}^4 \boldsymbol{x}_{23\alpha_i}{}^{\alpha_i'}}{(\boldsymbol{x}_{13}^2)^{\Delta}(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'(4)}(\boldsymbol{X}_3,\Theta_3). \tag{5.1}
$$

As usual, the tensor $\mathcal{H}$ is required to satisfy covariant constraints arising from conservation equations and point-switch identities. They are summarized below:

(i) *Homogeneity constraint*

Covariance of the correlation function under scale transformations of superspace results in the following constraint on $\mathcal{H}$:

$$
\mathcal{H}_{\alpha(4)}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-3}\mathcal{H}_{\alpha(4)}(\boldsymbol{X},\Theta), \tag{5.2}
$$

which implies that $\mathcal{H}$ is a homogeneous tensor field of degree -3.

(ii) *Differential constraints*

The conservation equation (3.1) implies that the correlation function must satisfy the following constraint:

$$
D^{\sigma}_{(2)}\langle\mathcal{O}(z_1)\mathcal{F}_{\sigma\alpha(3)}(z_2)\mathcal{O}(z_3)\rangle = 0. \tag{5.3}
$$

Application of the identities (2.22a) results in the following differential constraint on $\mathcal{H}$:

$$
\mathcal{Q}^{\sigma}\mathcal{H}_{\sigma\alpha(3)}(\boldsymbol{X},\Theta) = 0. \tag{5.4}
$$

(iii) *Point-switch identity*

Invariance under permutation of the superspace points z_1 and z_3 results in the following constraint on the correlation function:

$$
\langle\mathcal{O}(z_1)\mathcal{F}_{\alpha(4)}(z_2)\mathcal{O}(z_3)\rangle = \langle\mathcal{O}(z_3)\mathcal{F}_{\alpha(4)}(z_2)\mathcal{O}(z_1)\rangle, \tag{5.5}
$$

which results in the following constraint on $\mathcal{H}$:

$$
\mathcal{H}_{\alpha(4)}(\boldsymbol{X}_3,\Theta_3) = \frac{\prod_{i=1}^4 \hat{\boldsymbol{x}}_{13}^{\alpha_i'\delta_i}\hat{\boldsymbol{X}}_{3\delta_i\alpha_i}}{\boldsymbol{x}_{13}^6\boldsymbol{X}_3^6}\mathcal{H}_{\alpha'(4)}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1). \tag{5.6}
$$

Now we must construct an explicit solution; analogous to the $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ case, we combine symmetric pairs of spinor indices into vector ones as follows:

$$
\mathcal{H}_{\alpha_1\alpha_2\alpha_3\alpha_4}(\boldsymbol{X},\Theta) = (\gamma^{a_1})_{\alpha_1\alpha_2}(\gamma^{a_2})_{\alpha_3\alpha_4}\mathcal{H}_{a_1a_2}(\boldsymbol{X},\Theta), \tag{5.7}
$$

where it is required that $\mathcal{H}$ in vector notation is both symmetric and traceless. It has the expansion

$$
\mathcal{H}_{a_1a_2}(\boldsymbol{X},\Theta) = F_{a_1a_2}(X) + \Theta^2 G_{a_1a_2}(X). \tag{5.8}
$$

The component fields F and G are both required to be symmetric and traceless. If we now impose (5.3), we obtain the constraints

$$
\partial^{a_1}F_{a_1a_2} = 0, \qquad G_{a_1a_2} = \frac{\mathrm{i}}{2}\epsilon_{(a_1}{}^{mn}\partial_n F_{a_2)m}. \tag{5.9}
$$

Therefore we need only solve for the field F. A general expansion consistent with the tensor symmetries and homogeneity is

$$F_{a_1a_2} = \frac{c}{X^3}\left\{\eta_{a_1a_2} - \frac{3X_{a_1}X_{a_2}}{X^2}\right\}. \tag{5.10}$$

Note that no parity violating structures are permitted as there is simply not enough indices on the tensor F to allow for such contributions. Substituting this solution into (5.9) shows that it is satisfied for any value of c, while $G = 0$. The final solution for the tensor $\mathcal{H}$ in spinor notation is

$$\begin{aligned}\mathcal{H}_{\alpha_1\alpha_2\alpha_3\alpha_4}(\boldsymbol{X},\Theta) &= c\left\{-\frac{1}{\boldsymbol{X}^3}(\varepsilon_{\alpha_1\alpha_3}\varepsilon_{\alpha_2\alpha_4}+\varepsilon_{\alpha_1\alpha_4}\varepsilon_{\alpha_2\alpha_3}) - \frac{3}{\boldsymbol{X}^5}\boldsymbol{X}_{\alpha_1\alpha_2}\boldsymbol{X}_{\alpha_3\alpha_4}\right.\\ &\left.-\frac{3\mathrm{i}}{2}(\varepsilon_{\alpha_1\alpha_2}\boldsymbol{X}_{\alpha_3\alpha_4}+\varepsilon_{\alpha_3\alpha_4}\boldsymbol{X}_{\alpha_1\alpha_2})\frac{\Theta^2}{\boldsymbol{X}^5}\right\}. \end{aligned}\tag{5.11}$$

Indeed, substitution of this solution into (5.6) demonstrates that it is compatible with the point-switch identity. Hence, this correlation function is determined up to a single parity-even tensor structure. A similar result was obtained in [20].

B. Correlation function $\langle \mathcal{F}LL\rangle$

In this subsection we will compute the correlation function $\langle \mathcal{F}LL\rangle$, where L is the non-Abelian flavor current superfield of dimension $3/2$, which obeys the conservation equation

$$D^{\alpha}L^{\bar a}_{\alpha} = 0. \tag{5.12}$$

The correlation function admits the general ansatz

$$\begin{aligned}&\langle \mathcal{F}_{\alpha(4)}(z_1)L^{\bar a}_{\beta}(z_2)L^{\bar b}_{\gamma}(z_3)\rangle\\ &= \delta^{\bar a\bar b}\frac{(\prod_{i=1}^4 \hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha_i'})\hat{\boldsymbol{x}}_{23\beta}{}^{\beta'}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^{3/2}}\mathcal{H}_{\alpha'(4)\beta'\gamma}(\boldsymbol{X}_3,\Theta_3).\end{aligned}\tag{5.13}$$

The constraints on this three-point function are summarized below:

(i) *Homogeneity constraint*

Covariance under scale transformations of superspace results in the following constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha(4)\beta\gamma}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-3}\mathcal{H}_{\alpha(4)\beta\gamma}(\boldsymbol{X},\Theta), \tag{5.14}$$

which implies that $\mathcal{H}$ is a homogeneous tensor field of degree -3.

(ii) *Differential constraints*

The conservation equations (3.1) and (5.12) imply the following constraints:

$$D^{\sigma}_{(1)}\langle \mathcal{F}_{\sigma\alpha(3)}(z_1)L^{\bar a}_{\alpha}(z_2)L^{\bar b}_{\gamma}(z_3)\rangle = 0, \tag{5.15a}$$

$$D^{\beta}_{(2)}\langle \mathcal{F}_{\alpha(4)}(z_1)L^{\bar a}_{\beta}(z_2)L^{\bar b}_{\gamma}(z_3)\rangle = 0. \tag{5.15b}$$

Application of the identities (2.22a) then gives

$$\mathcal{D}^{\sigma}\mathcal{H}_{\sigma\alpha(3)\beta\gamma}(\boldsymbol{X},\Theta) = 0, \tag{5.16a}$$

$$\mathcal{Q}^{\beta}\mathcal{H}_{\alpha(4)\beta\gamma}(\boldsymbol{X},\Theta) = 0. \tag{5.16b}$$

(iii) *Point-switch identity*

Invariance under permutation of the superspace points z_2 and z_3 is equivalent to the condition

$$\begin{aligned}&\langle \mathcal{F}_{\alpha(4)}(z_1)L^{\bar a}_{\beta}(z_2)L^{\bar b}_{\gamma}(z_3)\rangle\\ &= -\langle \mathcal{F}_{\alpha(4)}(z_1)L^{\bar b}_{\gamma}(z_3)L^{\bar a}_{\beta}(z_2)\rangle,\end{aligned}\tag{5.17}$$

which results in the following constraint on $\mathcal{H}$:

$$\begin{aligned}\mathcal{H}_{\alpha(4)\beta\gamma}(\boldsymbol{X}_3,\Theta_3) &= \frac{\hat{\boldsymbol{x}}_{23\beta}{}^{\beta'}\hat{\boldsymbol{x}}_{23\gamma}{}^{\gamma'}\prod_{i=1}^4\hat{\boldsymbol{x}}_{23}^{\alpha_i'\delta_i}\hat{\boldsymbol{X}}_{3\alpha_i\delta_i}}{\boldsymbol{x}_{23}^6\boldsymbol{X}_3^6}\\ &\times\mathcal{H}_{\alpha'(4)\gamma'\beta'}(-\boldsymbol{X}_2^{\mathrm{T}},-\Theta_2).\end{aligned}\tag{5.18}$$

As before we combine symmetric pairs of spinor indices into vector ones as follows:

$$\mathcal{H}_{\alpha_1\alpha_2\alpha_3\alpha_4\beta\gamma}(\boldsymbol{X},\Theta) = (\gamma^{a_1})_{\alpha_1\alpha_2}(\gamma^{a_2})_{\alpha_3\alpha_4}\mathcal{H}_{a_1a_2,\beta\gamma}(\boldsymbol{X},\Theta). \tag{5.19}$$

The above decomposition holds provided that $\mathcal{H}_{a_1a_2,\beta\gamma}$ is symmetric and traceless in a_1, a_2.[10] We then expand this in irreducible components as follows:

$$\mathcal{H}_{a_1a_2,\beta\gamma}(\boldsymbol{X},\Theta) = \varepsilon_{\beta\gamma}A_{a_1a_2}(\boldsymbol{X},\Theta) + (\gamma^c)_{\beta\gamma}S_{a_1a_2,c}(\boldsymbol{X},\Theta), \tag{5.20}$$

with

$$A_{a_1a_2}(\boldsymbol{X},\Theta) = A^1_{a_1a_2}(X) + \Theta^2A^2_{a_1a_2}(X), \tag{5.21a}$$

$$S_{a_1a_2,c}(\boldsymbol{X},\Theta) = S^1_{a_1a_2,c}(X) + \Theta^2S^2_{a_1a_2,c}(X). \tag{5.21b}$$

Here the A^i and S^i are both symmetric and traceless in a_1, a_2. Imposing the differential relation (5.16) results in the following constraints on the tensors A^i and S^i:

$$\partial^m A^1_{ma_2}(X) = 0, \qquad A^2_{a_1a_2}(X) = -\frac{\mathrm{i}}{2}\epsilon_{(a_1}{}^{mn}\partial_m A^1_{a_2)n}(X), \tag{5.22a}$$

$$\partial^m S^1_{ma_2,c}(X) = 0, \qquad S^2_{a_1a_2,c}(X) = -\frac{\mathrm{i}}{2}\epsilon_{(a_1}{}^{mn}\partial_m S^1_{a_2)n,c}(X), \tag{5.22b}$$

[10] In the right-hand side we require that the antisymmetric part in α_2, α_3 vanishes. Using Eq. (A7a), it can be seen that $\mathcal{H}$ must be symmetric and traceless in a_1 and a_2.

while (5.16b) gives the additional relations

$$A^2_{a_1a_2}(X) = \frac{\mathrm{i}}{2}\partial^m S^1_{a_1a_2,m}(X), \tag{5.23a}$$

$$S^2_{a_1a_2,c}(X) = -\frac{\mathrm{i}}{2}\{\partial_c A^1_{a_1a_2}(X) + \epsilon_c{}^{mn}\partial_m S^1_{a_1a_2,n}(X)\}. \tag{5.23b}$$

Hence, we may treat A^1 and S^1 as independent. The only solution for the tensor A^1 compatible with the symmetries is

$$A^1_{a_1a_2}(X) = \frac{c}{X^3}\left\{\eta_{a_1a_2} - \frac{3X_{a_1}X_{a_2}}{X^2}\right\}, \tag{5.24}$$

while for S we have the general ansatz

$$\begin{aligned} S^1_{a_1a_2,c}(X) = {}& k_1\frac{X_{a_1}X_{a_2}X_c}{X^6} \\ & + k_2\left\{\frac{\epsilon_{a_1c}{}^m X_m X_{a_2}}{X^5} + \frac{\epsilon_{a_2c}{}^m X_m X_{a_1}}{X^5}\right\} \\ & + k_3\left\{\frac{\eta_{ca_1}X_{a_2}}{X^4} + \frac{\eta_{ca_2}X_{a_1}}{X^4}\right\} + k_4\frac{\eta_{a_1a_2}X_c}{X^4}. \end{aligned} \tag{5.25}$$

Imposing tracelessness in a_1, a_2 on this expansion results in the constraint

$$k_3 = -\frac{3}{2}k_4 - \frac{1}{2}k_1. \tag{5.26}$$

The solution then becomes

$$\begin{aligned} S^1_{a_1a_2,c}(X) = {}& k_1\left\{\frac{X_{a_1}X_{a_2}X_c}{X^6} - \frac{1}{2}\frac{\eta_{ca_2}X_{a_1}}{X^4} - \frac{1}{2}\frac{\eta_{ca_1}X_{a_2}}{X^4}\right\} \\ & + k_2\left\{\frac{\epsilon_{a_1c}{}^m X_m X_{a_2}}{X^5} + \frac{\epsilon_{a_2c}{}^m X_m X_{a_1}}{X^5}\right\} \\ & + k_4\left\{\frac{\eta_{a_1a_2}X_c}{X^4} - \frac{3}{2}\frac{\eta_{ca_2}X_{a_1}}{X^4} - \frac{3}{2}\frac{\eta_{ca_1}X_{a_2}}{X^4}\right\}. \end{aligned} \tag{5.27}$$

It remains to impose the differential constraints. In particular, Eqs. (5.22a) and (5.22b) result in

$$k_1 = k_4 = 0, \tag{5.28}$$

while A^2 vanishes. After making the replacement $k_2 \to \tilde{c}$, the solutions for the tensors A^i and S^i now become

$$A^1_{a_1a_2}(X) = \frac{c}{X^3}\left\{\eta_{a_1a_2} - \frac{3X_{a_1}X_{a_2}}{X^2}\right\}, \qquad A^2_{a_1a_2}(X) = 0, \tag{5.29a}$$

$$S^1_{a_1a_2,c}(X) = \tilde{c}\left\{\frac{\epsilon_{a_1c}{}^m X_m X_{a_2}}{X^5} + \frac{\epsilon_{a_2c}{}^m X_m X_{a_1}}{X^5}\right\}, \tag{5.29b}$$

$$\begin{aligned} S^2_{a_1a_2,c}(X) = \tilde{c}\Bigg\{ & -\frac{5\mathrm{i}}{2}\frac{X_{a_1}X_{a_2}X_c}{X^7} + \frac{\mathrm{i}}{2}\frac{\eta_{ca_2}X_{a_1}}{X^5} + \frac{\mathrm{i}}{2}\frac{\eta_{ca_1}X_{a_2}}{X^5} \\ & + \frac{\mathrm{i}}{2}\frac{\eta_{a_1a_2}X_c}{X^5}\Bigg\}. \end{aligned} \tag{5.29c}$$

These solutions are consistent with the remaining constraints (5.23a) and (5.23b) for the choice $\tilde{c} = -3c$. It can also be shown by direct substitution that this solution is consistent with the point-switch identity (5.18). Hence, this correlator is determined up to a single tensor structure.

Let us comment on the absence of parity-odd contributions.[11] They could only potentially come from the following terms contained in S^1:

$$\begin{aligned} S^1_{(\mathrm{odd})a_1a_2,c}(X) = {}& k_1\frac{X_{a_1}X_{a_2}X_c}{X^6} + k_3\left\{\frac{\eta_{ca_1}X_{a_2}}{X^4} + \frac{\eta_{ca_2}X_{a_1}}{X^4}\right\} \\ & + k_4\frac{\eta_{a_1a_2}X_c}{X^4}, \end{aligned} \tag{5.30}$$

which are odd under $X^m \to -X^m$. However, this expression cannot be at the same time traceless and transverse for any choice of the coefficient k_1, k_3, k_4, which can easily be checked.

This result is contrary to the computation carried out using the polarization spinor formalism in [20], where it was shown that a parity violating contribution can exist. A direct comparison with the results obtained in [20] is difficult as our approach and notation are quite different. Our formalism, however, has the benefit that it is analytic and rather explicit.[12] As a consistency check, in Appendix C we reformulate this problem and use the $\langle LL\mathcal{F}\rangle$ ansatz. The evaluation procedure is slightly different but the same conclusion is obtained.

VI. CONCLUSION

In this paper we analyzed various correlation functions involving a conserved superspin-2 current multiplet $\mathcal{F}_{\alpha(4)}$. The case of $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ is particularly challenging due to the proliferation of tensor structures in the solution; indeed, we found that it could only be studied efficiently using

[11] In our formalism, the presence of the antisymmetric ϵ tensor in the tensor $\mathcal{H}$ does not necessarily imply it is parity odd. Instead, one must count the overall number of γ matrices contained in both $\mathcal{H}$ and the prefactor after performing superspace reduction, and then make use of identities such as $\epsilon_{mnp} = -\frac{1}{2}\mathrm{Tr}(\gamma_m\gamma_n\gamma_p)$. This approach was applied to the study of mixed correlators of the supercurrent and flavor current multiplets [39].

[12] The corresponding result in [20] is listed in Table 2 with few details provided. To our knowledge it is based mostly on numerical methods, whereas our result is obtained analytically.

computational methods. We obtained that the three-point function $\langle \mathcal{F}\mathcal{F}\mathcal{F}\rangle$ contains one parity-even and one parity-odd structure.

The appearance of a single parity-even structure can be understood intuitively and is somewhat expected. Indeed, the superfield $\mathcal{F}_{\alpha(4)}$ contains a conserved spin-2 current $J_{\alpha(4)}$ as the lowest component which, though being different from the energy-momentum tensor, satisfies the same conservation equation. Its three-point function has two parity-even structures which can be attributed to contributions from a free boson and a free fermion. Since supersymmetry relates bosons and fermions, it is reasonable to expect that these structures become related, giving rise to a single independent contribution. On the other hand, the existence of the parity-odd structure in $\langle \mathcal{F}\mathcal{F}\mathcal{F}\rangle$ is rather nontrivial because, as was pointed out in the Introduction, there is an apparent tension between parity-odd structures and supersymmetry: all three-point functions involving the energy-momentum tensor and vector currents admit parity-odd structures in the nonsupersymmetric case [1] but not in the supersymmetric one [31,39].

Let us now clarify a possibly confusing point. The three-point function of the energy-momentum tensor T does not allow parity-odd structures in the supersymmetric case, whereas the three-point function of the similar spin-2 current J does. This might look paradoxical because T and J have the same symmetry properties and satisfy the same conservation equation. However, it is important to remember that T and J belong to different supermultiplets and, hence, transform differently under supersymmetry. Therefore, restrictions on their correlation functions due to supersymmetry are different.

A natural extension of our results is to study the three-point functions of higher-spin current multiplets of (arbitrary) higher (super)spin. For nonsupersymmetric conformal field theories, the three-point functions of bosonic higher-spin currents were found in [43–45]. In four-dimensional supersymmetric conformal field theories correlation functions of higher-spin spinor currents were recently studied in [46] (see also [47]). Deriving explicit solutions becomes increasingly difficult for fields with higher spins. It is possible that other approaches, for example, based on supersymmetric generalizations of the embedding formalism [48–51] or of the spinor-helicity formalism [24,52,53], can be more efficient. It would be interesting to explore them as well.

Another natural question is to find explicit realizations of superconformal field theories possessing a conserved superspin-2 current multiplet. Since this multiplet also contains a higher-spin current, one should expect that these theories possess infinitely many conserved higher-(super) spin currents [54].

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APPENDIX A: THREE-DIMENSIONAL CONVENTIONS AND NOTATION

For the Minkowski metric we use the "mostly plus" convention: $\eta_{mn} = \mathrm{diag}(-1, 1, 1)$. Spinor indices are then raised and lowered with the $\mathrm{SL}(2, \mathbb{R})$ invariant antisymmetric ε-tensor

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta, \tag{A1}$$

$$\phi_\alpha = \varepsilon_{\alpha\beta}\phi^\beta, \qquad \phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta. \tag{A2}$$

The γ matrices are chosen to be real and are expressed in terms of the Pauli matrices σ as follows:

$$(\gamma_0)_\alpha{}^\beta = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_1)_\alpha{}^\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A3a}$$

$$(\gamma_2)_\alpha{}^\beta = -\sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A3b}$$

$$(\gamma_m)_{\alpha\beta} = \varepsilon_{\beta\delta}(\gamma_m)_\alpha{}^\delta, \quad (\gamma_m)^{\alpha\beta} = \varepsilon^{\alpha\delta}(\gamma_m)_\delta{}^\beta. \tag{A4}$$

The γ matrices are traceless and symmetric

$$(\gamma_m)^\alpha{}_\alpha = 0, \qquad (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha}, \tag{A5}$$

and also satisfy the Clifford algebra

$$\gamma_m\gamma_n + \gamma_n\gamma_m = 2\eta_{mn}. \tag{A6}$$

Products of γ matrices are then

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\beta = \eta_{mn}\delta_\alpha{}^\beta + \epsilon_{mnp}(\gamma^p)_\alpha{}^\beta, \tag{A7a}$$

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\sigma(\gamma_p)_\sigma{}^\beta = \eta_{mn}(\gamma_p)_\alpha{}^\beta - \eta_{mp}(\gamma_n)_\alpha{}^\beta + \eta_{np}(\gamma_m)_\alpha{}^\beta + \epsilon_{mnp}\delta_\alpha{}^\beta, \tag{A7b}$$

where we have introduced the 3D Levi-Civita tensor ϵ, with $\epsilon^{012} = -\epsilon_{012} = 1$. It satisfies the following identities:

$$\epsilon_{mnp}\epsilon_{m'n'p'} = -\eta_{mm'}(\eta_{nn'}\eta_{pp'} - \eta_{np'}\eta_{pn'}) - (n' \leftrightarrow m') - (m' \leftrightarrow p'), \tag{A8a}$$

$$\epsilon_{mnp}\epsilon^{m}{}_{n'p'} = -\eta_{nn'}\eta_{pp'} + \eta_{np'}\eta_{pn'}, \tag{A8b}$$

$$\epsilon_{mnp}\epsilon^{mn}{}_{p'} = -2\eta_{pp'}, \tag{A8c}$$

$$\epsilon_{mnp}\epsilon^{mnp} = -6. \tag{A8d}$$

We also have the orthogonality and completeness relations for the γ matrices

$$(\gamma^m)_{\alpha\beta}(\gamma_m)^{\rho\sigma} = -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho, \quad (\gamma_m)_{\alpha\beta}(\gamma_n)^{\alpha\beta} = -2\eta_{mn}. \tag{A9}$$

Finally, the γ matrices are used to swap from vector to spinor indices. For example, given some three-vector x_m, it can be expressed equivalently in terms of a symmetric second-rank spinor $x_{\alpha\beta}$ as follows:

$$x^{\alpha\beta} = (\gamma^m)^{\alpha\beta}x_m, \qquad x_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}x_{\alpha\beta}, \tag{A10}$$

$$\det(x_{\alpha\beta}) = \frac{1}{2}x^{\alpha\beta}x_{\alpha\beta} = -x^m x_m = -x^2. \tag{A11}$$

The same conventions are also adopted for the spacetime partial derivatives ∂_m,

$$\partial^{\alpha\beta} = \partial^m(\gamma_m)^{\alpha\beta}, \qquad \partial_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}\partial_{\alpha\beta}, \tag{A12}$$

$$\partial_m x^n = \delta_m^n, \qquad \partial_{\alpha\beta}x^{\rho\sigma} = -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho, \tag{A13}$$

$$\xi^m\partial_m = -\frac{1}{2}\xi^{\alpha\beta}\partial_{\alpha\beta}. \tag{A14}$$

We also define the supersymmetry generators Q^I_α,

$$Q_\alpha = \mathrm{i}\frac{\partial}{\partial\theta^\alpha} + (\gamma^m)_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m}, \tag{A15}$$

and the covariant spinor derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\gamma^m)_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m}, \tag{A16}$$

which anticommute with the supersymmetry generators, $\{Q_\alpha, D_\beta\} = 0$, and obey the standard anticommutation relations

$$\{D_\alpha, D_\beta\} = 2\mathrm{i}(\gamma^m)_{\alpha\beta}\partial_m. \tag{A17}$$

APPENDIX B: COMPONENT REDUCTION: $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle \to \langle QJQ\rangle$

In this appendix we will provide some additional details regarding the component reduction from $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ to $\langle QJQ\rangle$. We recall from Sec. IV B that the component correlation function $\langle QJQ\rangle$ is obtained from $\langle\mathcal{F}\mathcal{F}\mathcal{F}\rangle$ as follows:

$$\begin{aligned}\langle Q_{\alpha(4),\alpha}(x_1)J_{\beta(4)}(x_2)Q_{\gamma(4),\gamma}(x_3)\rangle &= D_{(3)\gamma}D_{(1)\alpha}\langle\mathcal{F}_{\alpha(4)}(z_1)\mathcal{F}_{\beta(4)}(z_2)\mathcal{F}_{\gamma(4)}(z_3)\rangle|\\ &= D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)\right\}\Bigg|\\ &= A + B.\end{aligned} \tag{B1}$$

The calculation is broken up into two relevant parts: the A contribution is due to the derivatives hitting the prefactor,

$$A = D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\right\}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)|, \tag{B2}$$

while the B contribution arises due to the derivatives hitting $\mathcal{H}$,

$$B = \frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}D_{(3)\gamma}D_{(1)\alpha}\{\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3,\Theta_3)\}|. \tag{B3}$$

Let us start with the A term. After distributing the derivatives we obtain

$$\begin{aligned}D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\prod_{i=1}^4\hat{\boldsymbol{x}}_{13\alpha_i}{}^{\alpha'_i}\hat{\boldsymbol{x}}_{23\beta_i}{}^{\beta'_i}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\right\} &= D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\hat{\boldsymbol{x}}_{13\alpha_1}{}^{\alpha'_1}\hat{\boldsymbol{x}}_{13\alpha_2}{}^{\alpha'_2}\hat{\boldsymbol{x}}_{13\alpha_3}{}^{\alpha'_3}\hat{\boldsymbol{x}}_{13\alpha_4}{}^{\alpha'_4}}{(\boldsymbol{x}_{13}^2)^3}\right\}\\ &\quad\times\frac{\hat{\boldsymbol{x}}_{23\beta_1}{}^{\beta'_1}\hat{\boldsymbol{x}}_{23\beta_2}{}^{\beta'_2}\hat{\boldsymbol{x}}_{23\beta_3}{}^{\beta'_3}\hat{\boldsymbol{x}}_{23\beta_4}{}^{\beta'_4}}{(\boldsymbol{x}_{23}^2)^3},\end{aligned} \tag{B4}$$

where we have used the fact that $D_{(3)}$ hitting the objects $\boldsymbol{x}_{23}$ result in θ linear terms; hence, they do not contribute. We then find

$$D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\hat{\boldsymbol{x}}_{13\alpha_1}{}^{\alpha'_1}\hat{\boldsymbol{x}}_{13\alpha_2}{}^{\alpha'_2}\hat{\boldsymbol{x}}_{13\alpha_3}{}^{\alpha'_3}\hat{\boldsymbol{x}}_{13\alpha_4}{}^{\alpha'_4}}{(x_{13}^2)^3}\right\}\bigg| = \frac{2\mathrm{i}}{(\boldsymbol{x}_{13}^2)^5}\{\varepsilon_{\alpha\alpha_1}\delta_\gamma{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_3}{}^{\alpha'_3}x_{13\alpha_4}{}^{\alpha'_4} + \varepsilon_{\alpha\alpha_2}\delta_\gamma{}^{\alpha'_2}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_3}{}^{\alpha'_3}x_{13\alpha_4}{}^{\alpha'_4}$$
$$+ \varepsilon_{\alpha\alpha_3}\delta_\gamma{}^{\alpha'_3}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_4}{}^{\alpha'_4} + \varepsilon_{\alpha\alpha_4}\delta_\gamma{}^{\alpha'_4}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_3}{}^{\alpha'_3}\}$$
$$+ \frac{10\mathrm{i}}{(x_{13}^2)^6}x_{13\alpha\gamma}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_3}{}^{\alpha'_3}x_{13\alpha_4}{}^{\alpha'_4}.$$

Finally, after repeated application of the identity

$$\varepsilon_{\alpha\alpha_1}\varepsilon_{\gamma\alpha'_1} = \frac{x_{13\alpha\alpha'_1}x_{13\gamma\alpha_1} - x_{13\alpha\gamma}x_{13\alpha_1\alpha'_1}}{x_{13}^2}, \tag{B5}$$

we obtain the result

$$D_{(3)\gamma}D_{(1)\alpha}\left\{\frac{\hat{\boldsymbol{x}}_{13\alpha_1}{}^{\alpha'_1}\hat{\boldsymbol{x}}_{13\alpha_2}{}^{\alpha'_2}\hat{\boldsymbol{x}}_{13\alpha_3}{}^{\alpha'_3}\hat{\boldsymbol{x}}_{13\alpha_4}{}^{\alpha'_4}}{(\boldsymbol{x}_{13}^2)^3}\right\}\bigg| = \frac{2\mathrm{i}}{(x_{13}^2)^6}\{x_{13\alpha}{}^{\alpha'_1}x_{13\gamma\alpha_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_3}{}^{\alpha'_3}x_{13\alpha_4}{}^{\alpha'_4} + x_{13\alpha}{}^{\alpha'_2}x_{13\gamma\alpha_2}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_3}{}^{\alpha'_3}x_{13\alpha_4}{}^{\alpha'_4}$$
$$+ x_{13\alpha}{}^{\alpha'_3}x_{13\gamma\alpha_3}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_4}{}^{\alpha'_4} + x_{13\alpha}{}^{\alpha'_4}x_{13\gamma\alpha_4}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_3}{}^{\alpha'_3}$$
$$+ x_{13\alpha\gamma}x_{13\alpha_1}{}^{\alpha'_1}x_{13\alpha_2}{}^{\alpha'_2}x_{13\alpha_3}{}^{\alpha'_3}x_{13\alpha_4}{}^{\alpha'_4}\}.$$

After some additional minor manipulations we obtain

$$A = \frac{1}{(x_{13}^2)^{7/2}(x_{23}^2)^3}\hat{x}_{13\alpha}{}^{\alpha'}\hat{x}_{13\alpha_1}{}^{\alpha'_1}\hat{x}_{13\alpha_2}{}^{\alpha'_2}\hat{x}_{13\alpha_3}{}^{\alpha'_3}\hat{x}_{13\alpha_4}{}^{\alpha'_4} \times \hat{x}_{23\beta_1}{}^{\beta'_1}\hat{x}_{23\alpha_2}{}^{\beta'_2}\hat{x}_{23\beta_3}{}^{\beta'_3}\hat{x}_{23\beta_4}{}^{\beta'_4}\mathcal{T}^A_{\alpha'\alpha'(4)\beta'(4)\gamma\gamma(4)}(X_{12}). \tag{B6}$$

Now consider the B term: in particular, we need to evaluate

$$D_{(3)\gamma}D_{(1)\alpha}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3, \Theta_3). \tag{B7}$$

Using the identities (2.22a) we obtain

$$D_{(3)\gamma}D_{(1)\alpha}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3, \Theta_3) = D_{(3)\gamma}\left\{-\frac{\boldsymbol{x}_{13\alpha}{}^{\alpha'}}{\boldsymbol{x}_{13}^2}\mathcal{D}_{(3)\alpha'}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3, \Theta_3)\right\}. \tag{B8}$$

Evaluating the derivative within the brackets gives

$$\mathcal{D}_{\alpha'}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}, \Theta) = \mathrm{i}(\gamma^m)_{\alpha'\delta}\Theta^\delta\partial_m F_{\alpha'(4)\beta'(4)\gamma(4)}(X) + 2\Theta_{\alpha'}G_{\alpha'(4)\beta'(4)\gamma(4)}(X). \tag{B9}$$

Now in order to compute

$$D_{(3)\gamma}D_{(1)\alpha}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3, \Theta_3)|, \tag{B10}$$

we note that contributions in which the spinor derivative acts on $\boldsymbol{x}_{13}$ or X_{12} produce terms that are linear in θ, so they may be neglected as they vanish after bar projection. On the other hand, the following identity holds:

$$D_{(3)\alpha'}\Theta_3^\delta| = X_{12\alpha'}{}^\delta. \tag{B11}$$

Hence, we obtain

$$D_{(3)\gamma}D_{(1)\alpha}\mathcal{H}_{\alpha'(4)\beta'(4)\gamma(4)}(\boldsymbol{X}_3, \Theta_3)| = -\frac{x_{13\alpha}{}^{\alpha'}}{x_{13}^2}\{\mathrm{i}(\gamma^m)_{\alpha'\delta}X_{12\gamma}^\delta\partial_m F_{\alpha'(4)\beta'(4)\gamma(4)}(X_{12}) + 2X_{12\alpha'\gamma}G_{\alpha'(4)\beta'(4)\gamma(4)}(X_{12})\}. \tag{B12}$$

Therefore the B contribution may be expressed in the form

$$B = \frac{1}{(x_{13}^2)^{7/2}(x_{23}^2)^3} \hat{x}_{13\alpha}{}^{\alpha'} \hat{x}_{13\alpha_1}{}^{\alpha'_1} \hat{x}_{13\alpha_2}{}^{\alpha'_2} \hat{x}_{13\alpha_3}{}^{\alpha'_3} \hat{x}_{13\alpha_4}{}^{\alpha'_4} \times \hat{x}_{23\beta_1}{}^{\beta'_1} \hat{x}_{23\beta_2}{}^{\beta'_2} \hat{x}_{23\beta_3}{}^{\beta'_3} \hat{x}_{23\beta_4}{}^{\beta'_4} \mathcal{T}^B_{\alpha',\alpha'(4)\beta'(4)\gamma,\gamma(4)}(X_{12}), \tag{B13}$$

with $\mathcal{T}^B$ given by the expression

$$\mathcal{T}^B_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) = -\mathrm{i}(\gamma^m)_{\alpha\sigma} X^\sigma{}_\gamma \partial_m F_{\alpha(4)\beta(4)\gamma(4)}(X) - 2X_{\alpha\gamma} G_{\alpha(4)\beta(4)\gamma(4)}(X). \tag{B14}$$

Combining both the A and B terms we obtain the component correlation function

$$\langle Q_{\alpha(4),\alpha}(x_1) J_{\beta(4)}(x_2) Q_{\gamma(4),\gamma}(x_3)\rangle = \frac{1}{(x_{13}^2)^{7/2}(x_{23}^2)^3} \hat{x}_{13\alpha}{}^{\alpha'} \hat{x}_{13\alpha_1}{}^{\alpha'_1} \hat{x}_{13\alpha_2}{}^{\alpha'_2} \hat{x}_{13\alpha_3}{}^{\alpha'_3} \hat{x}_{13\alpha_4}{}^{\alpha'_4} \times \hat{x}_{23\beta_1}{}^{\beta'_1} \hat{x}_{23\beta_2}{}^{\beta'_2} \hat{x}_{23\beta_3}{}^{\beta'_3} \hat{x}_{23\beta_4}{}^{\beta'_4} \mathcal{T}_{\alpha',\alpha'(4)\beta'(4)\gamma,\gamma(4)}(X_{12}), \tag{B15}$$

with

$$\mathcal{T}_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) = \mathcal{T}^A_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X) + \mathcal{T}^B_{\alpha,\alpha(4)\beta(4)\gamma,\gamma(4)}(X). \tag{B16}$$

APPENDIX C: CONSISTENCY CHECKS

1. Correlator $\langle LL\mathcal{O}\rangle$

In this sub-appendix we derive the general form of the correlation function $\langle LL\mathcal{O}\rangle$. We also demonstrate that our solution is consistent with the results of [20] in terms of the number of independent tensor structures. The ansatz for $\langle LL\mathcal{O}\rangle$ is

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2)\mathcal{O}(z_3)\rangle = \delta^{\bar{a}\bar{b}} \frac{\hat{\boldsymbol{x}}_{13\alpha}{}^{\alpha'} \hat{\boldsymbol{x}}_{23\beta}{}^{\beta'}}{(\boldsymbol{x}_{13}^2)^{3/2}(\boldsymbol{x}_{23}^2)^{3/2}} \mathcal{H}_{\alpha'\beta'}(\boldsymbol{X}_3, \Theta_3). \tag{C1}$$

The constraints on this three-point function are summarized below:

(i) *Homogeneity constraint*

Covariance under scale transformations of superspace results in the following constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha\beta}(\lambda^2 \boldsymbol{X}, \lambda\Theta) = (\lambda^2)^{-\tau} \mathcal{H}_{\alpha\beta}(\boldsymbol{X}, \Theta), \tag{C2}$$

which implies that $\mathcal{H}$ is homogeneous degree $\tau = 3 - \Delta$.

(ii) *Differential constraints*

The conservation equations (3.1) imply the following constraints:

$$D^\alpha_{(1)} \langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2)\mathcal{O}(z_3)\rangle = 0. \tag{C3}$$

Application of the identities (2.22a) to (C3) gives

$$\mathcal{D}^\alpha \mathcal{H}_{\alpha\beta}(\boldsymbol{X}, \Theta) = 0. \tag{C4}$$

(iii) *Point-switch identity*

Invariance under permutation of the superspace points z_1 and z_2 is equivalent to the condition

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2)\mathcal{O}(z_3)\rangle = -\langle L^{\bar{b}}_\beta(z_2) L^{\bar{a}}_\alpha(z_1)\mathcal{O}(z_3)\rangle, \tag{C5}$$

which results in the following constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X}, \Theta) = -\mathcal{H}_{\alpha\beta}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta). \tag{C6}$$

An irreducible expansion for $\mathcal{H}$ is

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X}, \Theta) = \varepsilon_{\alpha\beta} A(\boldsymbol{X}, \Theta) + (\gamma^a)_{\alpha\beta} S_a(\boldsymbol{X}, \Theta), \tag{C7}$$

with

$$A(\boldsymbol{X}, \Theta) = A^1(X) + \Theta^2 A^2(X), \tag{C8a}$$

$$S_a(\boldsymbol{X}, \Theta) = S^1_a(X) + \Theta^2 S^2_a(X). \tag{C8b}$$

The point-switch identity (C6) implies

$$A^1(X) = A^1(-X), \qquad A^2(X) = A^2(-X), \tag{C9a}$$

$$S^1_a(X) = -S^1_a(-X), \qquad S^2_a(X) = -S^2_a(-X). \tag{C9b}$$

Imposing the differential relation (C4) results in the following constraints on the tensors A^i and S^i:

$$A^2(X) = \frac{\mathrm{i}}{2}\partial^m S^1_m(X), \tag{C10a}$$

$$S^2_a(X) = -\frac{\mathrm{i}}{2}\{\partial_a A^1(X) + \epsilon_a{}^{mn}\partial_m S^1_n(X)\}. \tag{C10b}$$

Hence, we may treat A^1 and S^1 as independent. Explicit solutions for the tensors A^1 and S^1 are

$$A^1(X) = \frac{a}{X^\tau}, \qquad S^1_a(X) = b\frac{X_a}{X^{\tau+1}}. \tag{C11}$$

These solutions are trivially compatible with (C9a) and (C9b). Using (C10a) and (C10b) we obtain expressions for A^2 and S^2

$$A^2(X) = \frac{\mathrm{i}b}{2}(2-\tau)\frac{1}{X^{\tau+1}}, \qquad S^2_a(X) = \frac{\mathrm{i}a}{2}\tau\frac{X_a}{X^{\tau+2}}. \tag{C12}$$

Following [20] we set $\Delta = 1$ ($\tau = 2$) and obtain the set of solutions

$$A^1(X) = \frac{a}{X^2}, \qquad A^2(X) = 0, \tag{C13a}$$

$$S^1_a(X) = b\frac{X_a}{X^3}, \qquad S^2_a(X) = \mathrm{i}a\frac{X_a}{X^4}. \tag{C13b}$$

The solution for $\mathcal{H}$ in spinor notation is then

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta) = a\left\{\frac{\varepsilon_{\alpha\beta}}{\boldsymbol{X}^2} + \frac{\mathrm{i}\boldsymbol{X}_{\alpha\beta}\Theta^2}{\boldsymbol{X}^4}\right\} + b\left\{\frac{\boldsymbol{X}_{\alpha\beta}}{\boldsymbol{X}^3} + \frac{\mathrm{i}}{2}\frac{\varepsilon_{\alpha\beta}\Theta^2}{\boldsymbol{X}^3}\right\}, \tag{C14}$$

which clearly contains both parity-even and parity-odd contributions. Our notation is quite different so it is difficult to make a direct comparison; however, we agree on the number of independent tensor structures.

2. Correlator $\langle\mathcal{F}LL\rangle$—alternative ansatz

In this subsection we investigate an alternative formulation of the correlation function $\langle\mathcal{F}LL\rangle$, it serves as a consistency check of our result in Sec. V B. The starting point is the alternative ansatz $\langle LL\mathcal{O}\rangle$:

$$\langle L^{\bar a}_\beta(z_1)L^{\bar b}_\gamma(z_2)\mathcal{F}_{\alpha(4)}(z_3)\rangle = \delta^{\bar a\bar b}\frac{\hat{\boldsymbol{x}}_{13\beta}{}^{\beta'}\hat{\boldsymbol{x}}_{23\gamma}{}^{\gamma'}}{(\boldsymbol{x}^2_{13})^{3/2}(\boldsymbol{x}^2_{23})^{3/2}}\mathcal{H}_{\beta'\gamma'\alpha(4)}(\boldsymbol{X}_3,\Theta_3). \tag{C15}$$

The constraints on this three-point function are summarized below:

(i) *Homogeneity constraint*
Covariance under scale transformations of superspace results in the following constraint on $\mathcal{H}$:

$$\mathcal{H}_{\beta\gamma\alpha(4)}(\lambda^2\boldsymbol{X},\lambda\Theta) = \mathcal{H}_{\beta\gamma\alpha(4)}(\boldsymbol{X},\Theta), \tag{C16}$$

which implies that $\mathcal{H}$ is homogeneous degree 0.

(ii) *Differential constraints*
The conservation equations (3.1) imply the following constraints:

$$D^\beta_{(1)}\langle L^{\bar a}_\beta(z_1)L^{\bar b}_\gamma(z_2)\mathcal{F}_{\alpha(4)}(z_3)\rangle = 0, \tag{C17a}$$

$$D^\sigma_{(3)}\langle L^{\bar a}_\beta(z_1)L^{\bar b}_\gamma(z_2)\mathcal{F}_{\sigma\alpha(3)}(z_3)\rangle = 0. \tag{C17b}$$

Application of the identities (2.22a) to (C17a) gives

$$\mathcal{D}^\beta\mathcal{H}_{\beta\gamma\alpha(4)}(\boldsymbol{X},\Theta) = 0. \tag{C18}$$

Imposing (5.15b) is rather nontrivial, and it will be handled later in this section.

(iii) *Point-switch identity*
Invariance under permutation of the superspace points z_1 and z_2 is equivalent to the condition

$$\langle L^{\bar a}_\beta(z_1)L^{\bar b}_\gamma(z_2)\mathcal{F}_{\alpha(4)}(z_3)\rangle = -\langle L^{\bar b}_\gamma(z_2)L^{\bar a}_\beta(z_1)\mathcal{F}_{\alpha(4)}(z_3)\rangle, \tag{C19}$$

which results in the following constraint on $\mathcal{H}$

$$\mathcal{H}_{\beta\gamma\alpha(4)}(\boldsymbol{X},\Theta) = -\mathcal{H}_{\gamma\beta\alpha(4)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta). \tag{C20}$$

First we combine symmetric pairs of spinor indices into vector ones as follows:

$$\mathcal{H}_{\beta\gamma\alpha_1\alpha_2\alpha_3\alpha_4}(\boldsymbol{X},\Theta) = (\gamma^{a_1})_{\alpha_1\alpha_2}(\gamma^{a_2})_{\alpha_3\alpha_4}\mathcal{H}_{\beta\gamma,a_1a_2}(\boldsymbol{X},\Theta), \tag{C21}$$

where it is required that $\mathcal{H}_{a_1a_2,\beta\gamma}$ is symmetric and traceless in a_1, a_2. We then expand this in irreducible components as follows:

$$\mathcal{H}_{\beta\gamma,a_1a_2}(\boldsymbol{X},\Theta) = \varepsilon_{\beta\gamma}A_{a_1a_2}(\boldsymbol{X},\Theta) + (\gamma^c)_{\beta\gamma}S_{a_1a_2,c}(\boldsymbol{X},\Theta), \tag{C22}$$

with

$$A_{a_1a_2}(\boldsymbol{X},\Theta) = A^1_{a_1a_2}(X) + \Theta^2A^2_{a_1a_2}(X), \tag{C23a}$$

$$S_{a_1a_2,c}(\boldsymbol{X},\Theta) = S^1_{a_1a_2,c}(X) + \Theta^2S^2_{a_1a_2,c}(X). \tag{C23b}$$

Here the A^i and S^i are both symmetric and traceless in a_1, a_2. The point-switch identity (C20) implies

$$A^1_{a_1a_2}(X) = A^1_{a_1a_2}(-X), \qquad A^2_{a_1a_2}(X) = A^2_{a_1a_2}(-X), \tag{C24a}$$

$$S^1_{a_1a_2,c}(X) = -S^1_{a_1a_2,c}(-X), \qquad S^2_{a_1a_2,c}(X) = -S^2_{a_1a_2,c}(-X). \tag{C24b}$$

Imposing the differential relation (C18) results in the following constraints on the tensors A^i and S^i:

$$A^2_{a_1a_2}(X) = \frac{\mathrm{i}}{2}\partial^m S^1_{a_1a_2,m}(X), \tag{C25a}$$

$$S^2_{a_1a_2,c}(X) = -\frac{\mathrm{i}}{2}\{\partial_c A^1_{a_1a_2}(X) + \epsilon_c{}^{mn}\partial_m S^1_{a_1a_2,n}(X)\}. \tag{C25b}$$

Hence, we may treat A^1 and S^1 as independent. The only solution for the tensor A^1 compatible with the symmetries is

$$A^1_{a_1a_2}(X) = c\left\{\eta_{a_1a_2} - \frac{3X_{a_1}X_{a_2}}{X^2}\right\}, \tag{C26}$$

while for S we have the general ansatz

$$S^1_{a_1a_2,c}(X) = k_1\frac{X_{a_1}X_{a_2}X_c}{X^3} + k_2\left\{\frac{\epsilon_{a_1c}{}^m X_m X_{a_2}}{X^2} + \frac{\epsilon_{a_2c}{}^m X_m X_{a_1}}{X^2}\right\} + k_3\left\{\frac{\eta_{ca_1}X_{a_2}}{X} + \frac{\eta_{ca_2}X_{a_1}}{X}\right\} + k_4\frac{\eta_{a_1a_2}X_c}{X}. \tag{C27}$$

Imposing tracelessness on a_1, a_2 on this expansion, along with the conditions (C24a) and (C24b) results in the constraints

$$k_3 = -\frac{3}{2}k_4 - \frac{1}{2}k_1, \qquad k_2 = 0. \tag{C28}$$

The solution then becomes

$$S^1_{a_1a_2,c}(X) = k_1\left\{\frac{X_{a_1}X_{a_2}X_c}{X^3} - \frac{1}{2}\frac{\eta_{ca_2}X_{a_1}}{X} - \frac{1}{2}\frac{\eta_{ca_1}X_{a_2}}{X}\right\} + k_4\left\{\frac{\eta_{a_1a_2}X_c}{X} - \frac{3}{2}\frac{\eta_{ca_2}X_{a_1}}{X} - \frac{3}{2}\frac{\eta_{ca_1}X_{a_2}}{X}\right\}. \tag{C29}$$

The expressions for A^2 and S^2 are

$$A^2_{a_1a_2}(X) = \frac{\mathrm{i}(k_1+k_4)}{2}\left\{\frac{\eta_{a_1a_2}}{X} - \frac{3X_{a_1}X_{a_2}}{X^3}\right\}, \tag{C30a}$$

$$S^2_{a_1a_2,c}(X) = -3\mathrm{i}c\left\{\frac{X_{a_1}X_{a_2}X_c}{X^4} - \frac{1}{2}\frac{\eta_{ca_2}X_{a_1}}{X^2} - \frac{1}{2}\frac{\eta_{ca_1}X_{a_2}}{X^2}\right\} + \frac{\mathrm{i}k_1}{4}\left\{\frac{\epsilon_{a_1c}{}^m X_m X_{a_2}}{X^3} + \frac{\epsilon_{a_2c}{}^m X_m X_{a_1}}{X^3}\right\} - \frac{3\mathrm{i}k_4}{4}\left\{\frac{\epsilon_{a_1c}{}^m X_m X_{a_2}}{X^3} + \frac{\epsilon_{a_2c}{}^m X_m X_{a_1}}{X^3}\right\}. \tag{C30b}$$

Imposing the symmetry conditions (C24a) and (C24b) on these solutions results in the constraint $k_1 = 3k_4$. After making the replacement $k_4 \to \tilde{c}$, the solutions for the tensors A^i and S^i become

$$A^1_{a_1a_2}(X) = c\left\{\eta_{a_1a_2} - \frac{3X_{a_1}X_{a_2}}{X^2}\right\}, \tag{C31a}$$

$$A^2_{a_1a_2}(X) = 2\mathrm{i}\tilde{c}\left\{\frac{\eta_{a_1a_2}}{X} - \frac{3X_{a_1}X_{a_2}}{X^3}\right\}, \tag{C31b}$$

$$S^1_{a_1a_2,c}(X) = \tilde{c}\left\{\frac{3X_{a_1}X_{a_2}X_c}{X^3} + \frac{\eta_{a_1a_2}X_c}{X} - \frac{3\eta_{ca_2}X_{a_1}}{X} - \frac{3\eta_{ca_1}X_{a_2}}{X}\right\}, \tag{C31c}$$

$$S^2_{a_1a_2,c}(X) = -3\mathrm{i}c\left\{\frac{X_{a_1}X_{a_2}X_c}{X^4} - \frac{1}{2}\frac{\eta_{ca_2}X_{a_1}}{X^2} - \frac{1}{2}\frac{\eta_{ca_1}X_{a_2}}{X^2}\right\}, \tag{C31d}$$

where we note that the correlation function is presently determined up to two coefficients. However, it remains to impose the final relation (C17b). To provide a comparison with our results in Sec. V B it is sufficient to analyze conservation on one of the component correlators to see whether this reduces the number of tensor structures. First, let us define the component fields by bar projection[13]

$$\psi^{\bar{a}}_\alpha(x) = L^{\bar{a}}_\alpha(z)|,$$
$$J_{a_1a_2}(x) = \frac{1}{4}(\gamma_{a_1})^{\alpha_1\alpha_2}(\gamma_{a_2})^{\alpha_3\alpha_4}\mathcal{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}(z)|. \tag{C32}$$

The three-point function $\langle\psi\psi J\rangle$ is then defined as follows:

$$\langle\psi^{\bar{a}}_\beta(x_1)\psi^{\bar{b}}_\gamma(x_2)J_{a_1a_2}(x_3)\rangle = \langle L^{\bar{a}}_\beta(z_1)L^{\bar{b}}_\gamma(z_2)\mathcal{F}_{a_1a_2}(z_3)\rangle|, \tag{C33}$$

where bar projection denotes switching off the fermionic variables at each superspace point. Using (C15), this three-point function has the general form

$$\langle\psi^{\bar{a}}_\beta(x_1)\psi^{\bar{b}}_\gamma(x_2)J_{a_1a_2}(x_3)\rangle = \delta^{\bar{a}\bar{b}}\frac{\hat{x}_{13\beta}{}^{\beta'}\hat{x}_{23\gamma}{}^{\gamma'}}{(x^2_{13})^{3/2}(x^2_{23})^{3/2}}H_{\beta'\gamma',a_1a_2}(X_{12}), \tag{C34a}$$

$$H_{\beta\gamma,a_1a_2}(X) = \varepsilon_{\beta\gamma}A^1_{a_1a_2}(X) + (\gamma^c)_{\beta\gamma}S^1_{a_1a_2,c}(X), \tag{C34b}$$

[13]There are three component fields contained within the flavor current multiplet, and they are $\{\psi_\alpha, V_{\alpha\beta}, \chi_\alpha\}$. The superfield conservation equation (5.12) then implies that V satisfies $\partial^a V_a = 0$, while χ is auxiliary. The calculations are similar to those in Sec. III.

where A^1 and S^1 are the solutions given above. We will then impose conservation by transforming the three-point function such that it is represented in the following way:

$$\langle \psi^{\bar{a}}_{\beta}(x_1)\psi^{\bar{b}}_{\gamma}(x_2)J_{a_1a_2}(x_3)\rangle \Rightarrow \langle J_{a_1a_2}(x_3)\psi^{\bar{b}}_{\gamma}(x_2)\psi^{\bar{a}}_{\beta}(x_1)\rangle. \tag{C35}$$

Using the explicit expressions for A^1 and S^1, after some calculation it may be shown that

$$\langle J_{a_1a_2}(x_3)\psi^{\bar{b}}_{\gamma}(x_2)\psi^{\bar{a}}_{\beta}(x_1)\rangle = \delta^{\bar{a}\bar{b}}\frac{\mathcal{I}_{a_1a_2,a_1'a_2'}(\hat{x}_{31})\hat{x}_{21\gamma}{}^{\gamma'}}{(x_{31}^2)^3(x_{21}^2)^{3/2}} \times \tilde{H}_{a_1'a_2',\gamma'\beta}(X_{32}), \tag{C36}$$

$$\tilde{H}_{a_1a_2,\gamma\beta}(X) = \varepsilon_{\beta\gamma}\tilde{A}^1_{a_1a_2}(X) + (\gamma^c)_{\beta\gamma}\tilde{S}^1_{a_1a_2,c}(X), \tag{C37}$$

where $\tilde{H}$ is homogeneous degree -3 and the tensors $\tilde{A}^1$ and $\tilde{S}^1$ are found to be

$$\tilde{A}^1_{a_1a_2}(X) = \frac{\tilde{c}}{X^3}\left\{\eta_{a_1a_2} - \frac{3X_{a_1}X_{a_2}}{X^2}\right\}, \tag{C38}$$

$$\tilde{S}^1_{a_1a_2,c}(X) = c\left\{\frac{3X_{a_1}X_{a_2}X_c}{X^6} - \frac{\eta_{a_1a_2}X_c}{X^4}\right\} - 3\tilde{c}\left\{\frac{\epsilon_{a_1cm}X^mX_{a_2}}{X^5} + \frac{\epsilon_{a_2cm}X^mX_{a_1}}{X^5}\right\}. \tag{C39}$$

At this stage we note that the $\tilde{c}$ terms exactly match the solutions (5.29a) and (5.29b); however, we have picked up an extra tensor structure (the c terms). If we now relabel the points in the $\langle J\psi\psi\rangle$ ansatz such that $x_3 \to x_1, x_1 \to x_3$, then the tensors $\tilde{A}^1$ and $\tilde{S}^1$ must satisfy the constraints (5.22a) and (5.22b). The solutions above are compatible with these constraints provided that $c = 0$. Hence, we have found that this correlator is fixed up to a single tensor structure and fully agrees with what we found in Sec. V B.

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