Entanglement entropy for a Dirac field in a black shell

W. A. Rojas C.[1,*] and J. R. Arenas S.[2,†]

[1]*Departamento de Física, Universidad Nacional de Colombia, Bogotá UN.11001, Colombia*
[2]*Observatorio Astronómico Nacional, Universidad Nacional de Colombia, Bogotá UN.11001, Colombia*

(Received 18 May 2021; revised 13 August 2021; accepted 13 September 2021; published 9 November 2021)

A quantum field of $s = 1/2$ in the vicinity of a dust shell contracting at a distance $r(t_0)$ to near its gravitational radius r_s as seen by a FIDO observer is considered. Such an observer perceives a batch of particles around the event horizon. The origin of the particles around the spherical surface of radius $r = r_s + \epsilon$ lies in the thermal excitations in the Boulware vacuum state, $|0\rangle_B$ for an external observer. The foregoing is done based on thermo field dynamics, as it allows one to explain the origin of S_{BH} as a state of entanglement between the modes of the fermionic field spreading through the Kruskal variety $S_{\mathrm{Ent}} \propto S_{\mathrm{BH}}$ with respect to a FIDO observer. A location of the degrees of freedom responsible for S_{BH} entropy is given. The occupation number for particles of a half-integer spin $s = 1/2$ is estimated, and it is compared with the occupation number of particles of spin $s = 0$, finding that the occupation number of the Dirac field is slightly lower than the occupation number of scalar particles in the vicinity of the gravitational radius; this allows confirming the exclusion principle near event horizons. The other thermodynamic properties of the field are estimated.

DOI: 10.1103/PhysRevD.104.105006

I. INTRODUCTION

General relativity predicts that spacetime is a Riemannian manifold, and some of the solutions lead to the existence of black holes. Such astronomical bodies are currently the subject of intense study from an observational and theoretical point of view. A black hole (BH) possesses unique physical characteristics: it has an event horizon, it has an intense gravitational field such that nothing can escape from within it, and it is characterized by only three physical parameters: its charge Q, mass M, and angular momentum $\vec{J}$. Hence, two black holes look identical to an external observer.

Recent discoveries have been of gravitational waves for black holes and compact exotic objects (ECOs) [1,2], as well as the first direct photograph of the vicinity of a black hole [3]. They definitely pave the way for a better understanding of black holes and the quantum nature of gravity in the coming decades.

Early in the 1970s, it was possible to identify how the horizon of a black hole is affected when a bit of information is added to it. This allows associating entropy with a BH [4]. By 1972, Hawking introduces the derivative of the electromagnetic radiation temperature for a Schwarzschild black hole [5],

$$T_H = \frac{\hbar c^3}{8\pi k_B G M}. \tag{1}$$

Subsequently Gibbons-Hawking introduced a statistical derivative of the entropy of a black hole,

$$S_{\mathrm{BH}} = \frac{k_B c^3}{4\hbar G} A, \tag{2}$$

where an analytic extension to the Euclidean sector is used by imposing a Matsubara period $1/T_H$ [6,7]. This explains which microscopic degrees of freedom are responsible for (1), given that the Euclidean approach suggests some kind of origin in the topological structure of spacetime [7]. In the same vein, S_{BH} is believed to describe a true thermodynamic entropy that is given by a generalized second law of thermodynamics, which expresses the sum of S_{BH} and the entropy of the universe never decreases [8,9].

A promising candidate for the origin of S_{BH} corresponds to entanglement entropy S_{Ent}. It is associated with the modes and quantum correlations of the field that are hidden from an external observer in the presence of a horizon. Under the consideration that a black hole is in an unknown pure quantum state, there are correlations between the modes inside and outside the horizon, so it is possible to determine the entanglement entropy S_{Ent} by counting the modes outside the horizon, according to the pioneering work of Bombelli [10], Srednicki [11], Terashima [12], and others. Having

*warojasc@unal.edu.co
†jrarenass@unal.edu.co

$$S_{\mathrm{Ent}} \propto A, \tag{3}$$

where A is the partition area of the wall and is not only an inherent characteristic of black holes, but rather extends to other types of scenarios [11,12]. In this sense, Srednicki shows that the base state density matrix for a scalar field, which is plotted over the degrees of freedom inside an imaginary sphere, leads to an entropy proportional to the area

$$S = KM^2 A, \tag{4}$$

where K is a numerical constant that depends on M [11]. In the same way, Terashima seeks to explain S_{BH} in terms of the inner and outer horizon modes. Hence, the thickness of the inner and outer horizon region is around the Planck length l_P. This makes it possible to obtain an entropy of

$$S \approx C\frac{A}{a^2}, \tag{5}$$

where a corresponds to the horizon fluctuations and C is a constant [12].

Interpreting the entanglement state is related to the wall model introduced by 't Hooft [13]. This model is considered as a black shell, in other words, a spherical shell compressing from infinity to near its gravitational radius, where the thermal atmosphere is shown to arise as the excitations of the Boulware ground state, and a precise location of such excitations in the Hartle-Hawking state is also noted [14–17]. One of the explanations for S_{BH} entropy is related to the physical properties of the vacuum in strong gravitational fields, where there are always zero-point field fluctuations in a vacuum state. Thus, an observer at rest with respect to the horizon sees the vacuum excitations as an atmosphere around the horizon [18]. Within such a context, the free energy of the quantum scalar field around the horizon is

$$F(\beta) \approx -\frac{\pi^2}{90}\int \sqrt{-g}T^4 d^3x, \tag{6}$$

where g is the determinant of the metric (19) and the temperature is determined by Tolman's law as

$$T(r) = \frac{T_\infty}{\sqrt{f(r)}}, \tag{7}$$

where T_∞ is the temperature measured by an observer at infinity. Thus, for a Schwarzschild black hole, the entropy is found using standard statistical mechanics as

$$S = \beta^2\frac{\partial F(\beta)}{\partial\beta} \approx \frac{1}{360\pi\epsilon^2}A, \tag{8}$$

where $\beta = \frac{1}{T_\infty}$, ϵ is cutoff near the horizon, and A is the horizon area. Here, the field is considered to be in thermal equilibrium with the horizon, leading to the field temperature coinciding with the Hawking temperature, which allows the entropy of the field external to the horizon to be of around the same magnitude as the entropy of the horizon S_{BH}. Under this model, entropy was related to properties of the vacuum. Its explanation lies in the fact that a static observer near the horizon perceives the vacuum as a mixed state. That occurs because an observer cannot measure beyond the horizon, since there is a nontrivial density matrix $\hat{\rho}$, which occurs since the vacuum fluctuations of the field are correlated in an entangled state between what is observable and what is nonobservable at the horizon and the loss of information is quantified by means of the entanglement entropy:

$$S_{\mathrm{Ent}} = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}. \tag{9}$$

Hence, the entanglement entropy coincides with thermal atmosphere entropy, because $\hat{\rho}$ is a thermal density matrix. At this point, Fursaev indicates that if S_{Ent} explains the origin of S_{BH}, then the following questions arise:

(1) Entropy S in (8) depends on the cutoff ϵ; therefore there must be a natural reason that explains why ϵ must be adjusted so that $S = S_{\mathrm{BH}}$. The fact of introducing a cutoff means that the quantum field does not spread entirely throughout spacetime.
(2) In general, S receives contributions from all fields present in nature. This depends on the total number of fields and their spins. However, S_{BH} does not seem to show such dependence [7].

In this same line of research, Arenas and Tejeiro [14] propose a black shell model where the existence of thermal energy is strongly concentrated near the horizon with respect to a uniformly accelerated observer according to the equivalence principle. For such a model, an interpretation of S_{BH} entropy requires a consistent cross-interpretation of the entanglement state with thermal field dynamics, which allows one to assert the origin and location of such entropy [14,15,19,20]. In this sense we extend this research program to a Dirac field.

This research is distributed as follows: in Sec. II, a brief outline of the nature of thermo field dynamics is discussed. Section III contains the entanglement entropy for the Dirac field in a black shell. Section IV describes the thermo field dynamics for the Dirac field. Similarly, Sec. V provides the quantum formulation of the Dirac field close to a black shell. Section VI discusses the thermal energy of the Dirac field in a black shell. Section VII describes the entanglement entropy S_{Ent} for $s = 1/2$ fermions. It ends with the Discussion and Conclusions, Sec. VIII.

II. NATURE OF THERMO FIELD DYNAMICS

There is a system H, which can be subdivided into two subsystems H_1 and H_2, such that H_1 and H_2 are physically indistinguishable and the subsystems are coupled such that the state of H is described as $|\Psi\rangle = |\Psi_1, \Psi_2\rangle$. Consequently, H_i can be considered the thermal batch of H_j at a temperature T. Moreover, the subsystem H_i has a Hamiltonian $\mathcal{H}_i$, whose eigenvalue equation is

$$\mathcal{H}_i|\Psi_i\rangle = E_n|\Psi_i\rangle. \tag{10}$$

On the other hand, the pure state $|\Psi\rangle$ of H at a temperature T coupling H_1 and H_2 is of the form

$$|\Psi\rangle = \frac{1}{\sqrt{Z}}\sum_n e^{-E_n/2T}|\Psi_1, \Psi_2\rangle, \tag{11}$$

where Z is the partition function of system H,

$$Z = e^{-E_n/T} \tag{12}$$

with the condition that the pure states of H are normalized $\langle\Psi|\Psi\rangle = 1$.

Therefore, the entropy for the pure state Ψ of system H is zero given that

$$S = -\mathbf{Tr}[\rho \ln|\rho|] = 0 \tag{13}$$

when

$$\rho = |\Psi\rangle\langle\Psi| = |\Psi_1, \Psi_2\rangle\langle\Psi_1, \Psi_2|. \tag{14}$$

Under this same line of action, it is possible to estimate the reduced density matrix ρ_1 for H_1, when partially plotted over the quantum states $|\Psi_2\rangle$. This allows determining the eigenvalues proportional to the Boltzmann factor $e^{-E_n/T}$, such that the reduced density matrix takes the form

$$\begin{aligned}\rho_1 &= \mathbf{Tr}_2[\rho] = \mathbf{Tr}_2[|\Psi\rangle\langle\Psi|]\\ &= \mathbf{Tr}_2[|\Psi_1, \Psi_2\rangle\langle\Psi_1, \Psi_2|]\\ &= \frac{1}{\sqrt{Z}}e^{-E_n/T}|\Psi_1\rangle\langle\Psi_1|.\end{aligned} \tag{15}$$

Consequently, entropy for H_1 is

$$S(\rho_1) = -\mathbf{Tr}[\rho_1 \ln|\rho_1|] = \ln|Z| + \frac{\langle E\rangle_T}{T}. \tag{16}$$

The same result can be determined based on (15) and (16) for H_2, then obtaining

$$S_{\mathrm{Ent}} = S_1 = S_2 > 0. \tag{17}$$

With the condition that the total entropy of the system $H(H_1, H_2)$ is zero, $S(H) = S_1 + S_2 = 0$.

Based on the foregoing, there is a mixed state of the system H in thermal equilibrium, and it is purified when Fock space is doubled. This allows converting the statistical value of an operator $\langle A_i\rangle$ acting on H_i with the expected value of this same operator on the extended Fock space $\langle\Psi|A_i|\Psi\rangle$ [16,21–24],

$$\begin{aligned}\langle A_i\rangle &= \mathbf{Tr}[\rho_i A_i],\\ &= \sum_n e^{-E_n/T}\langle\Psi_i|A_i|\Psi_i\rangle = \langle\Psi|A_i|\Psi\rangle.\end{aligned} \tag{18}$$

The key spirit of the thermo field dynamics technique lies in the ability to encode the bilateral symmetry existing between H_1 and H_2, which is analogous to the bilateral symmetry existing between the modes of the Φ field spreading into the two regions R and L of the maximally extended Schwarzschild spacetime, as shown in Fig. 1.

III. ENTANGLEMENT ENTROPY FOR DIRAC FIELD IN A BLACK SHELL

An external observer in a flat region sees around the black shell (BS) spacetime as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\phi^2, \tag{19}$$

$$f(r) = 1 - \frac{2M}{r}. \tag{20}$$

Thus, for a spherical thin shell of dust collapsing from infinity to the Schwarzschild radius according to an external stationary observer, its motion equation for BS is [25]

$$a = \frac{M}{\mu}, \qquad \frac{dr}{d\tau} = \sqrt{\left[a + \frac{M}{2ar}\right]^2 - 1}, \tag{21}$$

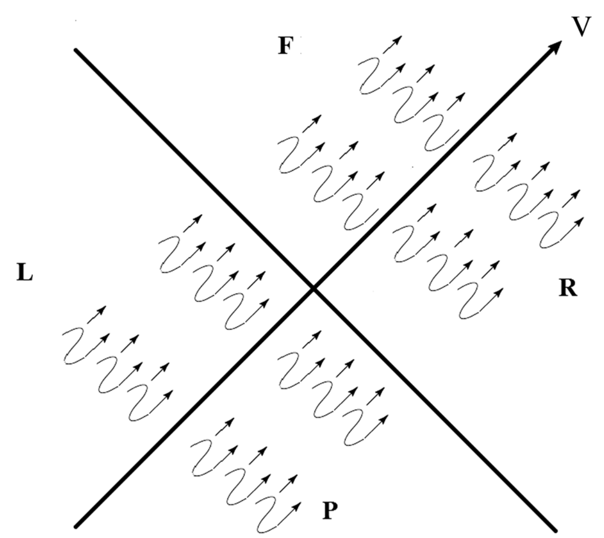

FIG. 1. Bilateral symmetry existing between the field modes spreading into the two regions R and L of the maximally extended Schwarzschild spacetime.

where a is the ratio between gravitational mass M and rest mass μ of BS. Also τ is the proper time for free fall observer (FFO). Arenas and Castro (AC) proposed an alternative solution to the motion equation of BS [26],

$$r(t) = r_s + \delta r e^{-t/\bar{\tau}}, \qquad \delta r = r_0 - r_s,$$
$$r_s = 2M, \qquad \bar{\tau} = \frac{4M}{3}, \tag{22}$$

where t is the coordinate time for a BS that is contracting and measured by a FIDO observer, r_0 is the initial position of BS in $t = t_0$, r_s is the Schwarzschild radius, and M is the mass of BS.

Consider a Dirac field in a curved spacetime with the metric (19) [27–63]. Based on the foregoing, the Dirac equation $[i\gamma^\mu \nabla_\mu + m]\Psi = 0$, $\gamma^\mu = \gamma^a e_a{}^\mu$ is rewritten as

$$[\gamma^a e_a{}^\mu(\partial_\mu + \Omega_\mu) + m]\Psi = 0, \tag{23}$$

where the tetrads are determined as

$$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b, \qquad \eta = \mathbf{diag}(-1, 1, 1, 1); \tag{24}$$

furthermore, the matrices γ^a are

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \tag{25}$$

where $i = 1, 2, 3$. And the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{26}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{27}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28}$$

Hence, according to (24), the tetrads for the Schwarzschild spacetime are

$$e_a^\mu = \left(\sqrt{f(r)}, \frac{1}{\sqrt{f(r)}}, r, r\sin\theta\right), \tag{29}$$

where Ω_μ is called the spin connection

$$\Omega_\mu = \frac{1}{8}[\gamma^a, \gamma^b] e_a{}^\nu e_{b\nu;\mu}. \tag{30}$$

The radial equation given in (34) describes particles whose positive and negative frequency modes, respectively, have the same energy spectrum in the vicinity of the horizon [28,64] (for specific details see Appendix A). Thus, for this research, we assume that the radial equations are identical for the positive and negative frequency particles $F(r) = G(r)$. Thus, in the high-frequency limit, obtaining

$$Q(r) = \frac{1}{4\kappa_0^2 \delta r}[\omega^2 - V(r, \kappa, m, \omega)], \tag{31}$$

where

$$W = \sqrt{\frac{2\kappa_0}{\delta r}\left(\frac{\kappa^2}{r^2} + m^2\right)}\delta r, \tag{32}$$

$$V(r, \kappa, m, \omega) = 2\kappa_0 \delta r \frac{dW}{dr} + W^2. \tag{33}$$

Hence,

$$\frac{d}{dr}\left(\frac{dF}{dr}\right) + Q(r)F = 0. \tag{34}$$

Explicitly, obtaining

$$Q(r) = \frac{2\delta r^2}{f(r)}\left[\frac{\omega^2}{2f(r)\delta r} - \frac{m^2}{2\delta r} + \frac{\kappa^2}{r^2}T(\kappa)\right], \tag{35}$$

where

$$T(\kappa) = \frac{1}{2\delta r}\left(\frac{1}{r\sqrt{\frac{1}{f(r)}[m^2 + \frac{\kappa^2}{r^2}]}} - 1\right). \tag{36}$$

Thus, under the Wentzel-Kramers-Brillouin approximation (WKB) approximation, the radial solution for (34) is [65–68]

$$F(r) = \sqrt[4]{\frac{\omega^2}{4\pi^2 Q(r)}} e^{-i\int \sqrt{Q(r)}dr}. \tag{37}$$

The condition of validity for (37) is given by

$$|\phi(r)''| \approx \frac{1}{2}\left|\frac{Q(r)'}{\sqrt{Q(r)}}\right| \ll |Q(r)|. \tag{38}$$

Solving Eq. (34) for the Dirac field in the Schwarzschild spacetime, (35) and (37), the modes of the Dirac field, according to

$$\Phi(t, r, \theta, \phi) = \frac{F(r)}{r}\begin{pmatrix} i\varphi^{\pm}_{j,m}(\theta, \phi) \\ \varphi^{\mp}_{j,m}(\theta, \phi) \end{pmatrix} e^{-i\omega t}, \tag{39}$$

with $F(r)$ defined in (37), $Q(r)$ and $T(\kappa)$ in (35) and (36) such that these are normalized under the inner product defined as [69]

$$(\Phi_\Omega(t,\underline{x}),\Phi_\Omega(t',\underline{x}')) = \int d^3x \bar{\Phi}_\Omega(t,\underline{x})\gamma_0\Phi_\Omega(t,\underline{x}), \tag{40}$$

the Dirac adjoint is defined as

$$\bar{\Phi}_\Omega(t,\underline{x}) = \Phi^*_\Omega(t,\underline{x})\gamma^0. \tag{41}$$

In addition, the harmonic spinors Y^s_{lm} are orthogonal when [70]

$$\int_{4\pi} Y^{*s}_{lm} Y^s_{lm} d\Omega = \delta_{ll'}\delta_{mm'}. \tag{42}$$

On the other hand, the field modes (39) are rewritten as

$$\Phi_\Omega(t,\underline{x}) = \Phi_\Omega(\underline{x})e^{i\omega t}, \Phi_\Omega(\underline{x}) = \frac{F(r)}{r}\begin{pmatrix} i\varphi^{\pm}_{j,m}(\theta,\phi) \\ \varphi^{\mp}_{j,m}(\theta,\phi)\end{pmatrix}, \tag{43}$$

where $\underline{x} = (r,\theta,\phi)$ and $\Omega = \omega, \kappa(l), l, m$.

The modes (43) are taken to the null coordinates U, V [14,71,72],

$$\begin{aligned}\Phi^{(\epsilon)}_\Omega(U,\underline{x}) &= \Theta(-\epsilon)\Phi^{\rm out}_\Omega(u,\underline{x}),\\ \Phi^{(\epsilon)}_\Omega(V,\underline{x}) &= \Theta(\epsilon)\Phi^{\rm in}_\Omega(v,\underline{x}),\end{aligned} \tag{44}$$

such that Θ defines the unit step function with $\epsilon = \pm 1$.

Consider two representations for the fermionic field given in terms of the Killing-Boulware modes (KB), $\Phi^{(\epsilon)}_\Omega(U,\underline{x})$ and $\Phi^{(\epsilon)}_\Omega(V,\underline{x})$, and the Hartle-Hawking modes (HH), $\Psi^{(\epsilon)}_\Omega(U,\underline{x})$ y $\Psi^{(\epsilon)}_\Omega(V,\underline{x})$. The modes are orthogonal under the inner product [44], leading to orthogonality relations for the modes of frequency $\omega > 0$ and $\omega < 0$ defined as [46]

$$(\Phi^{\pm}_\Omega(\underline{x}),\Phi^{\mp}_\Omega(\underline{x}')) = 0, \tag{45}$$

$$(\Phi^{+}_\Omega(\underline{x}),\Phi^{+}_{\Omega'}(\underline{x}')) = (\Phi^{-}_\Omega(\underline{x}),\Phi^{-}_{\Omega'}(\underline{x}')) = \delta_{\Omega\Omega'}. \tag{46}$$

The transformation between the HH and KB modes in the form [14,16,21,69,72]

$$\Psi^{(\epsilon)}_\Omega(t,\underline{x}) = \sqrt{\sin\chi\cos\chi} \times e^{-i\omega t_{\epsilon(\omega)\epsilon}}\Phi_\Omega(\underline{x}), \tag{47}$$

In the Fig. 2, we can see the modes of fermionic field for FIDO and FFO

$$\tan\chi = e^{-\beta\omega/2} = e^{-\pi\omega/\kappa_0}, \tag{48}$$

In the Fig. 3, the modes of fermionic field in the Carter-Penrose diagram can be expanded in terms of creation and annihilation operators

$$\beta = \frac{1}{T} \quad \text{and} \quad T = T_H = \frac{\kappa_0}{2\pi}. \tag{49}$$

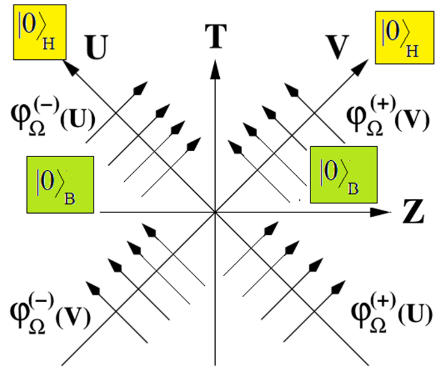

FIG. 2. Modes of the fermionic field in terms of FIDO and FFO observers.

IV. THERMO FIELD DYNAMICS FOR DIRAC FIELD IN A BLACK SHELL

Under the canonical quantization scheme, the fermionic field expands into [31,55–57,59,65,73,74]

$$\Phi(t,\underline{x}) = \sum_\Omega [b_\Omega F_\Omega + d^\dagger_\Omega G_\Omega], \tag{50}$$

$$\Phi(t,\underline{x}) = \sum_\Omega [b_\Omega u_\Omega e^{-i\omega t} + d^\dagger_\Omega v_\Omega e^{i\omega t}], \tag{51}$$

defining for (50)

$$F_\Omega = u_\Omega e^{-i\omega t}, \qquad G_\Omega = v_\Omega e^{i\omega t}, \tag{52}$$

and

$$u_\Omega = \frac{F(r)}{r}\begin{pmatrix} i\varphi^{+}_{j,m}(\theta,\phi) \\ \varphi^{-}_{j,m}(\theta,\phi)\end{pmatrix}, \quad v_\Omega = \frac{F(r)}{r}\begin{pmatrix} i\varphi^{-}_{j,m}(\theta,\phi) \\ \varphi^{+}_{j,m}(\theta,\phi)\end{pmatrix}. \tag{53}$$

Thus, the bispinors u_Ω and v_Ω contain the harmonic spinors defined in (A4). Moreover, they are orthogonal under the spinor product [50–53,60,62,70], then

$$\Phi^{(\epsilon)}_\Omega(t,\underline{x}) = \sum_\Omega \left[b^{(\epsilon)}_\Omega F_\Omega + d^{\dagger(\epsilon)}_\Omega G_\Omega\right], \quad \epsilon = \pm, \tag{54}$$

and they satisfy the anticommutator properties for fractional spin particles $s = 1/2$ [59,61–63],

$$\left\{\Phi^{(\epsilon)}_\Omega(t,\underline{x}),\Phi^{\dagger(\epsilon)}_{\Omega'}(t,\underline{x}')\right\} = \delta_{\Omega\Omega'}\delta^3(\underline{xx}'), \tag{55}$$

$$\left\{\Phi^{(\epsilon)}_\Omega(t,\underline{x}),\Phi^{(\epsilon)}_{\Omega'}(t,\underline{x}')\right\} = \left\{\Phi^{\dagger(\epsilon)}_\Omega(t,\underline{x}),\Phi^{\dagger(\epsilon)}_{\Omega'}(t,\underline{x}')\right\} = 0. \tag{56}$$

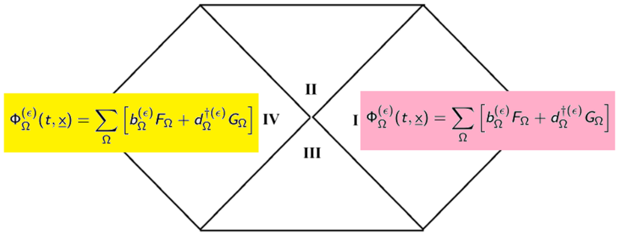

FIG. 3. The modes of the field $\Phi^{(\epsilon)}_{\Omega}(t,\underline{x})$ in each region R and L of the Penrose diagram can be extended in terms of operator creation and annihilation.

Once the field operator $\Phi^{(\epsilon)}_{\Omega}(t,\underline{x})$ under the KB scheme has been established, it is possible to apply the transformation (47) to determine the field operator under the HH scheme. This leads to

$$\Psi^{(\epsilon)}_{\Omega}(t,\underline{x}) = \sqrt{\sin\chi\cos\chi}\,e^{-i\omega t_{\epsilon(\omega)\epsilon}}\sum_{\Omega}[b_{\Omega}F_{\Omega} + d^{\dagger}_{\Omega}G_{\Omega}], \tag{57}$$

where $\epsilon = \pm$ and χ is defined by (48).

The Lagrangian density $\mathcal{L}$ for the Dirac field in the vicinity of a shell in the form [27,29–32,69]

$$\mathcal{L} = \frac{i}{2}[\bar{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - (\nabla_{\mu}\bar{\Psi})\gamma^{\mu}\Psi] - m\bar{\Psi}\Psi, \tag{58}$$

where the covariant derivatives of $\nabla_{\mu}\Psi$ and $\nabla_{\mu}\bar{\Psi}$ are in the form

$$\nabla_{\mu}\Psi = \partial_{\mu}\Psi + \Omega_{\mu}\Psi, \qquad \nabla_{\mu}\bar{\Psi} = \partial_{\mu}\bar{\Psi} - \Omega_{\mu}\bar{\Psi}. \tag{59}$$

In addition, the spin connection defined above allows expressing the Lagrangian density $\mathcal{L}$ as

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\Psi - m\bar{\Psi}\Psi + \bar{\Psi}\left[\frac{1}{4}\epsilon^{abcd}\omega_{abc}\gamma^{5}\gamma_{d}\right]\Psi,$$
$$\overleftrightarrow{\partial}_{\mu} = \left(\frac{\vec{\partial}_{\mu} - \overleftarrow{\partial}_{\mu}}{2}\right). \tag{60}$$

Thus, for (60), a Lagrangian density for a classical Dirac field is defined as [59,61–63]

$$\mathcal{L} = \mathcal{L}_{\text{classic}} + \bar{\Psi}\left[\frac{1}{4}\epsilon^{abcd}\omega_{abc}\gamma^{5}\gamma_{d}\right]\Psi, \tag{61}$$

where

$$\omega_{abc} = e^{\mu}_{c}[\eta_{ad}e^{d}_{\nu}e^{\nu}_{b;\mu}]. \tag{62}$$

If we estimate the antisymmetric part of ω_{abc}, we obtain

$$\omega_{[abc]} = 0. \tag{63}$$

Therefore, the Lagrangian density $\mathcal{L}$ in (61) is

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\Psi - m\bar{\Psi}\Psi. \tag{64}$$

At this point, the following comment regarding the term $\epsilon^{abcd}\omega_{abc}\gamma^{5}\gamma_{d}$ is noteworthy:

$$ie^{\mu}_{a}\gamma^{a}\Omega_{\mu} = \frac{1}{4}\epsilon^{abcd}\omega_{abc}\gamma^{5}\gamma_{d}. \tag{65}$$

Such a term is associated with a spin tensor current. In this particular case, we have $\omega_{[abc]} = 0$, which implies that the spin connection term is zero in the case of the Schwarzschild spacetime and also for the Minkowski spacetime. Thus, for the Schwarzschild spacetime there is no spin-gravitational field interaction [31,75].

Near the gravitational radius r_s, the Hamiltonian density can be rewritten as (for specific details see Appendix B)

$$\mathcal{H} = \frac{i}{2\kappa\delta r}\Psi^{\dagger}\overleftrightarrow{\partial}_{t}\Psi. \tag{66}$$

V. QUANTUM FORMULATION OF DIRAC FIELD CLOSE TO BLACK SHELL

Additionally, for the Ansatz considered above

$$\Psi_{\Omega}(t,\underline{x}) = \frac{1}{\sqrt[4]{f(r)}}\Phi(t,\underline{x}), \qquad \bar{\Psi}(t,\underline{x}) = \frac{1}{\sqrt[4]{f(r)}}\bar{\Phi}(t,\underline{x}) \tag{67}$$

allow rewriting the Hamiltonian density as

$$\mathcal{H} = \frac{i}{2f(r)}[\bar{\Phi}(t,\underline{x})\gamma^{0}\partial_{0}(\Phi_{\Omega}(t,\underline{x})) - \partial_{0}(\bar{\Phi}_{\Omega}(t,\underline{x}))\gamma^{0}\Phi_{\Omega}(t,\underline{x})]. \tag{68}$$

Moreover, it has previously been written that the field $\Phi_{\Omega}(t,\underline{x})$ has been promoted to a field operator in the form given in (54)

$$\Phi^{(\epsilon)}_{\Omega}(t,\underline{x}) = \sum_{\Omega}\left[b^{(\epsilon)}_{\Omega}F_{\Omega} + d^{\dagger(\epsilon)}_{\Omega}G_{\Omega}\right],$$
$$\bar{\Phi}^{(\epsilon)}_{\Omega}(t,\underline{x}) = \sum_{\Omega}\left[b^{\dagger(\epsilon)}_{\Omega}\bar{F}_{\Omega} + d^{(\epsilon)}_{\Omega}\bar{G}_{\Omega}\right], \qquad \epsilon = \pm. \tag{69}$$

Consequently, the field operator $H^{(\epsilon)}$ for the R and L regions of the maximally extended Schwarzschild spacetime is written as

$$H^{(\epsilon)} = \sum_{\Omega}\left[b^{\dagger(\epsilon)}_{\Omega}b^{(\epsilon)}_{\Omega}\omega + d^{\dagger(\epsilon)}_{\Omega}d^{(\epsilon)}_{\Omega}\omega - \frac{\omega}{2} - \frac{\omega}{2}\right]. \tag{70}$$

If, in addition, we define for (70) the occupation number operator as $N_\Omega^{(\epsilon)} = b_\Omega^{\dagger(\epsilon)} b_\Omega^{(\epsilon)}$ for particles with frequencies $\omega > 0$ and $\bar{N}_\Omega^{(\epsilon)} = d_\Omega^{\dagger(\epsilon)} d_\Omega^{(\epsilon)}$ with frequencies $\omega < 0$, additionally for (70) we identify the zero-point energy defined as Z.P.E. $= \frac{\omega}{2} + \frac{\omega}{2}$, which corresponds to each contribution for the considered particles. Thus, finally arriving at [76]

$$H^{(\epsilon)} = \sum_\Omega \left[N_\Omega^{(\epsilon)} \omega + \bar{N}_\Omega^{(\epsilon)} \omega - \mathrm{Z.P.E.} \right]. \qquad (71)$$

Given the thermo field dynamics (TFD) scheme, it is necessary to introduce a copy system (accent), which is identical to the original one, with a Hamiltonian denoted $\tilde{H}$ and state vectors $|\tilde{n}\rangle$ obey an eigenvalue equation. Consequently, under this scheme and by considering the Schwarzschild variety in Kruskal coordinates, it is necessary to consider $H^{(+)}$ as the real system and $H^{(-)}$ as the copy system. This is why their eigenvalue equations are [21–24,36,77–79]

$$H^{(+)}|n^{(+)}\rangle = E^{(+)}|n^{(+)}\rangle \in R, \qquad (72)$$

$$H^{(-)}|n^{(-)}\rangle = E^{(-)}|n^{(-)}\rangle \in L. \qquad (73)$$

Under such conditions, based on (72) and (73), the entire Hamiltonian of the Dirac field can be expressed in the form

$$H = H^{(+)} - H^{(-)} = \sum_\Omega \left[N_\Omega^{(+)} - N_\Omega^{(-)} + \bar{N}_\Omega^{(+)} - \bar{N}_\Omega^{(-)} \right] \omega. \qquad (74)$$

On the other hand, for the region R, the Dirac field has four incoming and four outgoing modes and similar modes for the region L. Hence, 16 modes are needed to describe the whole Dirac field in the Schwarzschild spacetime as shown in Fig. 4.

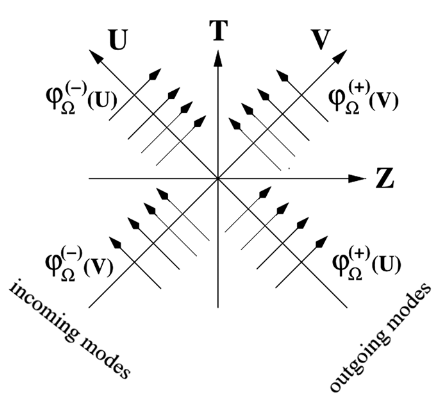

FIG. 4. Incoming and outgoing modes in the maximal representation of the Schwarzschild spacetime in Kruskal coordinates.

In addition, the anticommutation rules for the creation and annihilation operators are in the form $\{b^{(\epsilon)}, b^{\dagger(\epsilon)}\} = \{d^{(\epsilon)}, d^{\dagger(\epsilon)}\} = \mathbb{I}$, and the other combinations that may arise are $\{b^{(\epsilon)}, b^{(\epsilon)}\} = \{d^{(\epsilon)}, d^{(\epsilon)}\} = \{b^{\dagger(\epsilon)}, b^{\dagger(\epsilon)}\} = \{d^{\dagger(\epsilon)}, d^{\dagger(\epsilon)}\} = \cdots = 0$.

The Hilbert space for the Dirac field allows building the vacuum state [80,81]

$$|0\rangle_B = |0\rangle_B^{(+)+} \otimes |0\rangle_B^{(-)-}, \qquad (75)$$

where $|0\rangle^{(+)+}$ corresponds to the vacuum state for positive frequency particles in the region R and $|0\rangle^{(-)-}$ corresponds to the vacuum state for positive frequency particles in the region L. Also know that the state is normalized when ${}_B\langle 0|0\rangle_B = 1$ for regions R and L including the positive and negative frequency modes particles

$$b^{(+)}b^{(-)}|0^{(+)}, 0^{(-)}\rangle_B^+ = 0,$$
$$d^{(+)}d^{(-)}|0^{(+)}, 0^{(-)}\rangle_B^- = 0. \qquad (76)$$

Therefore, the thermal vacuum state $|0(\beta)\rangle$ for the Dirac field is defined as

$$|0(\beta)\rangle_B^+ = |0(\beta)^{(+)}\rangle_B^+ \otimes |0(\beta)^{(-)}\rangle_B^+,$$
$$|0(\beta)\rangle_B^- = |0(\beta)^{(+)}\rangle_B^- \otimes |0(\beta)^{(-)}\rangle_B^-, \qquad (77)$$

which corresponds to the entanglement state between the R and L regions for particles of positive frequency modes. Consequently, the most general possible full vacuum state including particles with positive and negative frequency modes corresponds with [22–24,28,64,78–80,82]

$$\begin{aligned} |0(\beta)\rangle_B = \frac{1}{Z(\beta)} & [1 + e^{-\beta E_n/2} b^{\dagger(+)} b^{\dagger(-)}] \\ & \times [1 + e^{-\beta E_n/2} d^{\dagger(+)} d^{\dagger(-)}] |0^{(+)}, 0^{(-)}\rangle_B^+ \\ & \otimes |0^{(+)}, 0^{(-)}\rangle_B^+. \end{aligned} \qquad (78)$$

Finally, once the Bogoliubov coefficients have been determined, it is possible to estimate the expected value of the occupation numbers $\langle N_\Omega^{(\epsilon)}\rangle$ and $\langle \bar{N}_\Omega^{(\epsilon)}\rangle$ (for specific details see Appendix C),

$$\begin{aligned} \langle N_\Omega^{(\epsilon)}\rangle = \langle \bar{N}_\Omega^{(\epsilon)}\rangle &= \langle 0(\beta)| b_\Omega^{(\epsilon)\dagger} b_\Omega^{(\epsilon)} |0(\beta)\rangle \\ &= \langle 0(\beta)| d_\Omega^{(\epsilon)\dagger} d_\Omega^{(\epsilon)} |0(\beta)\rangle = v^2(\beta) = \frac{1}{1 + e^{\beta\omega}}. \end{aligned} \qquad (79)$$

VI. THERMAL ENERGY OF DIRAC FIELD IN A BLACK SHELL

Therefore, the momentum-energy tensor of the Dirac field for a spacetime is [69,83,84]

$$T^a_\mu = \frac{1}{e}\frac{\delta(e\mathcal{L}_M)}{\delta e^\mu_a}, \tag{80}$$

which allows obtaining

$$T_{\mu\nu} = \frac{i}{2}[\bar{\Psi}\gamma_\mu \overleftrightarrow{\nabla}_\nu \Psi + \bar{\Psi}\gamma_\nu \overleftrightarrow{\nabla}_\mu \Psi]. \tag{81}$$

Estimating the value of the Wightman function for the Dirac field in a curved spacetime is [14,69,85–87]

$$W_{ab}(x,x')_H = {}_H\langle 0|\Psi_a(x)\bar{\Psi}_b(x')|0\rangle_H, \tag{82}$$

$$W_{ab}(x,x')_B = {}_B\langle 0|\Phi_a(x)\bar{\Phi}_b(x')|0\rangle_B, \tag{83}$$

where a and b are the indexes that run 1–4 for the spinor space. On the other hand, the field $\psi_a(x)$ is expressed as

$$\psi_a(x) = \sum_{\epsilon,\omega}\left[b^{(\epsilon)}_\Omega + d^{(\epsilon)}_\Omega\right]\Phi^{(\epsilon)}_{a,\Omega}(x) = \sum_{\epsilon,\omega}\left[f^{(\epsilon)}_\Omega + g^{(\epsilon)}_\Omega\right]\Psi^{(\epsilon)}_{a,\Omega}(x). \tag{84}$$

Based on the foregoing, we find that (for specific details see Appendix D)

$$\begin{aligned} -\langle T^0_0(x,x')\rangle &= \int_0^\infty \frac{E}{1+e^{E/T(r)}}\frac{4\pi p^2}{(2\pi)^3}dp \\ &= \int_0^\infty \frac{E}{1+e^{E/T(r)}}\frac{4\pi p^2}{h^3}dp. \end{aligned} \tag{85}$$

We know the T_{00} component of the energy-momentum tensor is associated with the matter-energy density of the field, $\rho = -\langle T^0_0(x,x')\rangle$. Hence, we have

$$\rho = -\int_0^\infty \frac{E}{1+e^{E/T(r)}}\frac{4\pi p^2}{h^3}dp, \tag{86}$$

which corresponds to a remarkable finding since the energy density of the fermionic field coincides with that reported by [7,14] for scalar fields. If, in addition, considering that the modeled Dirac field corresponds to Majorana fermions, which are weakly interacting, then the energy-momentum tensor $T_{\mu\nu}$ can be considered as an ideal fluid such that $\rho = -\langle T^0_0(x,x')\rangle$ and the pressures $p = \frac{1}{3}\langle T^a_a(x,x')\rangle = \frac{1}{3}\rho$. The foregoing necessarily leads to

$$P = \frac{1}{3}\int_0^\infty \frac{E}{1+e^{E/T(r)}}\frac{4\pi p^2}{h^3}dp. \tag{87}$$

Following Mukohyama [7] with the ultrarelativistic approximation for massive particles whose velocities are $v\sim c$ and $p \gg m$, we get $E \sim pv$; hence, Eq. (87) is simplified to

$$P = \frac{1}{3}\int_0^\infty \frac{pv}{1+e^{E/T(r)}}\frac{4\pi p^2}{h^3}dp. \tag{88}$$

Now, recalling (D29), which corresponds to the spatial components of the Wightman function [14],

$$\langle T_{ij}(x,x')\rangle|_{x=x'} = 0. \tag{89}$$

VII. ENTANGLEMENT ENTROPY OF DIRAC FIELD

Considering the partition function $Z_\Omega(\beta)$ per mode, given in (C1), for when $\Omega = \omega, \kappa(l), m$ [14,22,23,78,88,89],

$$Z_\Omega(\beta) = \sum_{n=0}^{1}[e^{-\beta\omega}]^n = (1+e^{-\beta E_n}), \tag{90}$$

where $\underline{n} = \{n_\Omega \ \forall \ \Omega, \omega > 0\}$. Moreover, under the high energy range, we have

$$E_{\underline{n}} = \sum_{\Omega_{\omega>0}} n_\Omega \omega. \tag{91}$$

Consequently, we find that the partition function for the Dirac field in the vicinity of the shell corresponds to the product of the partition functions for each mode with quantum numbers $\Omega = \omega, \kappa(l), m$ [88,89], which is equally valid for bosons and fermions. Therefore, the partition function becomes

$$Z = \prod_{\Omega_{\omega>0}} Z_\Omega(\beta) = \prod_{\Omega;\omega>0}\sum_{n=0}^{1}[e^{-\beta\omega}]^n = \prod_{\Omega;\omega>0}(1+e^{-\beta E_n}). \tag{92}$$

We require that (for specific details see Appendix E)

$$\sin\left[\int_{r_s+\epsilon}^{r_{\kappa n}}\sqrt{Q(r)}dr\right] = 0, \tag{93}$$

which is equivalent to

$$\int_{r_s+\epsilon}^{r_{\kappa n}}\sqrt{Q(r)}dr = \int_{r_s+\epsilon}^{r_{\kappa n}}\sqrt{Q(r_{\kappa n},\omega_{\kappa n},\kappa(l))}dr = n\pi, \tag{94}$$

where $Q(r)$ is defined by (35) and (36) for the fermionic field in the vicinity of the shell. Arenas and Tejeiro propose the case of a scalar field as [14]

$$k^2 \equiv \frac{1}{f}\left[\frac{\omega^2}{f} - \frac{l(l+1)}{r^2} - m^2 - \frac{(r^2 f)''}{2r^2}\right]. \tag{95}$$

Figure 5 shows the surface diagram for $Q(r,l)$ and $k(r,l)$ near the gravitational radius. It is possible to make the following comments regarding the model:

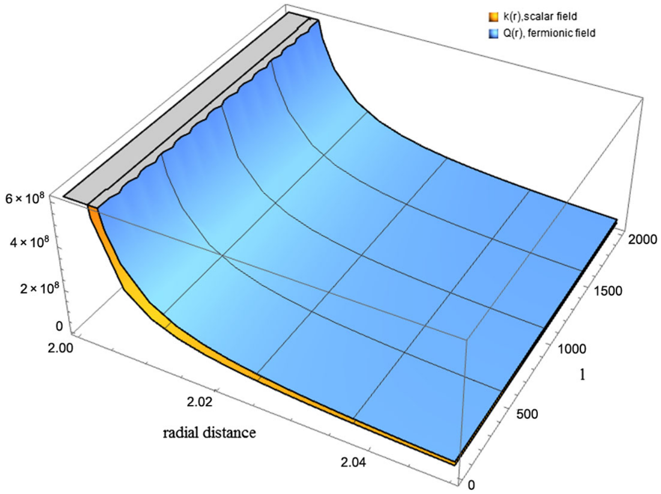

FIG. 5. Comparison of $Q(r)$ and $k(r)$ for fermionic and bosonic fields, respectively, in the vicinity of a shell that has contracted from infinity to near its gravitational radius r_s.

(1) The behavior of $Q(r)$ and $k(r)$ is similar in the high energy range, $\omega \to \infty$.
(2) $Q(r)$ and $k(r)$ are increasing when $r \to r_s$. This allows one to confirm that for the fields under study, the field modes are strongly concentrated very close to the horizon [14,21].
(3) $Q(r)$ and $k(r)$ decrease with distance, $r \to \infty$.
(4) $Q(r)$ and $k(r)$ exhibit similar behavior and are independent of spin [90].

On the other hand, taking (E12)

$$\sqrt{Q(r_{\kappa n}, \omega_{\kappa n}, \kappa(l))} = Q'(r'_{\kappa n}, \omega_{\kappa n}, \kappa(l)). \tag{96}$$

Therefore, Eq. (94) is rewritten as

$$\int_{r_s+\epsilon}^{r_{\kappa n}} Q'(r'_{\kappa n}, \omega_{\kappa n}, \kappa(l)) dr' = n\pi. \tag{97}$$

Next, we must change the independent variables (κ, n) to other independent variables (κ, ω) with $n = n(\omega, \kappa)$. Hence, Eq. (97) is rewritten as

$$n(\omega, \kappa) = \frac{1}{\pi} \int_{r_s+\epsilon}^{r_{\kappa n}} Q'(r'(\omega, \kappa), \omega, \kappa(l)) dr', \tag{98}$$

$$\begin{aligned} \frac{\partial n(\omega, \kappa)}{\partial \omega} = & \frac{1}{\pi} \int_\omega \int_\kappa \int_{r_s+\epsilon}^{r(\omega,\kappa)} d\omega d\kappa dr' (2l+1) \ln|1 + e^{\beta\omega}| \\ & \times \left[\frac{\partial Q'(r'(\omega, \kappa), \omega, \kappa(l))}{\partial \omega}\right], \end{aligned} \tag{99}$$

where $Q^2(r', \omega, \kappa(l)) = 0$ when $r' = r_s + \epsilon$ and $r' = R$, $\forall\, \omega, \kappa \geq 0$ (for specific details see Appendix F).

On the other hand, the Helmholtz free energy is related to the partition function as

$$\begin{aligned} F &= -\frac{1}{\beta} \ln|Z| \\ &= -\sqrt{\delta r} \int_0^\infty \frac{N(\omega)}{e^{\beta\omega} + 1} d\omega. \end{aligned} \tag{100}$$

It is also true that according to (D37), we find that (F9) is rewritten as

$$\begin{aligned} N(\omega) &= \frac{4}{3\pi} \int_{r_s+\epsilon}^R dr \frac{1}{r\sqrt{f(r)}} L_{\max}^3 \\ &= \int_{r_s+\epsilon}^R N_{\mathrm{F}}^*(\omega) dr. \end{aligned} \tag{101}$$

Based on (F8), we define the occupation number per unit frequency for fermions in the vicinity of a shell, $N^*(\omega)$, as

$$\begin{aligned} N_{\mathrm{F}}^*(\omega) &= \frac{4}{3\pi} \frac{1}{r\sqrt{f(r)}} \left[r^2 \left(\frac{\omega^2}{f(r)} - m^2\right)\right]^{3/2} \\ &= \frac{4}{3\pi} \frac{r^2}{f(r)^2} [\omega^2 - m^2 f(r)]^{3/2}. \end{aligned} \tag{102}$$

It is possible to compare the occupation number per unit frequency for bosons (103) and fermions (F9),

$$N(\omega)_{\mathrm{B}} = \int_{r_s+\epsilon}^R N_{\mathrm{B}}^*(\omega) dr, \tag{103}$$

where $N_{\mathrm{B}}^*(\omega)$ is

$$N_{\mathrm{B}}^*(\omega) = \frac{2}{3\pi} \frac{r^2}{f^2(r)} \left[\omega^2 - \left(m^2 + \frac{(r^2 f)''}{2r^2}\right) f\right]^{3/2}. \tag{104}$$

Figure 6 shows the comparison between the occupation numbers per unit frequency for bosons $N_{\mathrm{B}}^*(\omega)$ and fermions $N_{\mathrm{F}}^*(\omega)$. It shows that for the high frequency range $N_{\mathrm{B}}^*(\omega) > N_{\mathrm{F}}^*(\omega)$. This type of behavior is expected, since the fermions are subject to the exclusion principle in the vicinity of the shell, since only a few fermions can be located per energy level provided they do not have the same

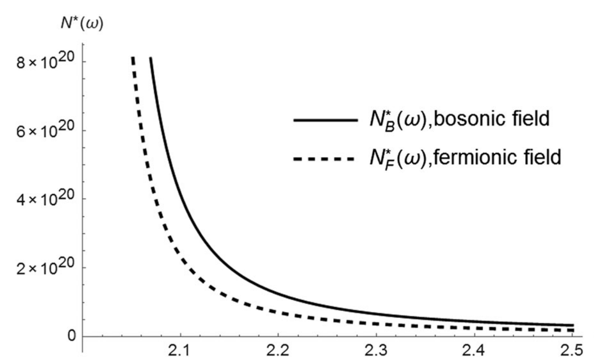

FIG. 6. Comparison between the occupation numbers per unit frequency for bosons $N_{\mathrm{B}}^*(\omega)$ and fermions $N_{\mathrm{F}}^*(\omega)$.

quantum numbers. In the case of bosons, the occupation number in the vicinity of the shell is higher. It is interesting to note that $N^*_{\mathrm{B}}(\omega)$ and $N^*_{\mathrm{F}}(\omega)$ decrease when $r \to \infty$.

Consider the internal energy U if the form

$$\begin{aligned} U &= -\frac{\partial}{\partial \beta} \ln |Z| \\ &= \int_0^\infty \frac{1}{e^{\beta\omega}+1} \omega \frac{\partial N(\omega)}{\partial \omega} d\omega, \end{aligned} \tag{105}$$

where the quantity $\frac{\partial N(\omega)}{\partial \omega}$ is obtained from (101)

$$\begin{aligned} N'(\omega) &= \frac{\partial N(\omega)}{\partial \omega} \\ &= \int_{r_s+\epsilon}^R dr \frac{4}{\pi} \frac{r^2}{f(r)^{3/2}} \omega \left[\frac{\omega^2}{f(r)} - m^2 \right]^{1/2}. \end{aligned} \tag{106}$$

Consequently, the internal energy U is simplified to

$$U = \int_0^\infty \frac{\omega N'(\omega)}{e^{\beta\omega}+1} d\omega. \tag{107}$$

Revisiting (D37)

$$p^2 = \frac{\omega^2}{2f(r)\delta r} - \frac{m^2}{2\delta r}, \tag{108}$$

such that

$$p dp = \frac{\omega}{2f(r)\delta r} d\omega. \tag{109}$$

Based on the foregoing, taking (107) we obtain

$$\begin{aligned} U &= \int_0^\infty \frac{\omega N'(\omega)}{e^{\beta\omega}+1} d\omega \\ &= 2\delta r \sqrt{2\delta} \int_{r_s+\epsilon}^R 4\pi r^2 dr \rho(r), \end{aligned} \tag{110}$$

where, for (110), the quantity $\rho(r)$ given in (86) is acknowledged; consequently we find

$$U = cte \int_{r_s+\epsilon}^R 4\pi r^2 dr \rho(r). \tag{111}$$

The foregoing makes it possible to calculate the Dirac field entropy as [67]

$$\begin{aligned} S &= \beta[U - F] \\ &= \beta^2 \int_0^\infty \omega N(\omega) \frac{e^{\beta\omega}}{(e^{\beta\omega}+1)^2} d\omega. \end{aligned} \tag{112}$$

To solve (112), the variables (r, ω) need to be substituted by (r, p). Therefore,

$$\begin{aligned} N(\omega)_{\mathrm{F}} &= \int_{r_s+\epsilon}^R N^*_{\mathrm{F}}(\omega) dr \\ &= \int_{r_s+\epsilon}^R dr \frac{4}{3\pi} \frac{r^2}{\sqrt{f(r)}} [2\delta r p^2]^{3/2}. \end{aligned} \tag{113}$$

Inserting (101) into (112) with $\omega d\omega = f(r) p dp$,

$$S = \int_{r_s+\epsilon}^R \frac{4\pi r^2}{\sqrt{f(r)}} dr s(r), \tag{114}$$

where $s(r)$ is identified as the entropy density of the fermionic field in the vicinity of the shell as

$$s(r) = \frac{\beta^2}{3} f(r) \int_0^\infty \frac{p^2 e^{\beta\omega}}{(e^{\beta\omega}+1)^2} \frac{4\pi p^2}{(h)^3} dp. \tag{115}$$

Moreover, the term

$$\beta^2 f(r) = \frac{1}{T^2(r)}, \tag{116}$$

and then

$$s(r) = \frac{1}{3T^2(r)} f(r) \int_0^\infty \frac{p^2 e^{\beta\omega}}{(e^{\beta\omega}+1)^2} \frac{4\pi p^2}{(h)^3} dp. \tag{117}$$

Considering again the energy density of the Dirac field given by (86)

$$\rho = \int_0^\infty \frac{E}{1+e^{E/T(r)}} \frac{4\pi p^2}{h^3} dp \tag{118}$$

with the condition that

$$\frac{E}{T(r)} \approx \frac{p}{T(r)} = x \tag{119}$$

leads to (119)

$$\begin{aligned} \rho(r) &= \frac{4\pi}{h^3} T^4(r) \int_0^\infty \frac{x^2}{e^x+1} dx \\ &= \frac{7\pi^2}{240\hbar^3} T^4(r). \end{aligned} \tag{120}$$

Similarly, for the entropy density defined in (117)

$$\begin{aligned} s(r) &= \frac{1}{3T^2(r)} f(r) \int_0^\infty \frac{p^2 e^{\beta\omega}}{(e^{\beta\omega}+1)^2} \frac{4\pi p^2}{(h)^3} dp \\ &= \frac{7\pi^2}{180\hbar^3} T^3(r). \end{aligned} \tag{121}$$

The combination of (120) and (121) leads to

$$s(r) = \frac{4}{3}\frac{\rho(r)}{T(r)}. \tag{122}$$

Now, at this point of the discussion, it must be assumed that the energy-momentum tensor ($T_{\mu\nu}$) of the Dirac field in the vicinity corresponds to a perfect fluid [7] such that the equation of state of the Dirac field corresponds to $\mathrm{Tr}[T_{\mu\nu}]$, which yields the relationship between the pressure P and the energy density ρ, and is in the form

$$P = \frac{1}{3}\rho. \tag{123}$$

Hence, the entropy density $s(r)$ is simplified to [7,14]

$$s(r) = \beta(r)[\rho + P], \tag{124}$$

$$s = \frac{16\pi k_B}{3h^3c^3k_BT}\int_0^\infty \frac{E^3}{e^{E/k_BT}+1}dE. \tag{125}$$

For an ideal gas fluid, the existing relationship between the energy density ρ and the pressure P is in the form

$$\rho = \frac{4\pi}{h^3c^3}\int_0^\infty \frac{E^3}{e^{E/k_BT}+1}dE, \tag{126}$$

$$P = \frac{1}{3}\rho = \frac{1}{3}\frac{4\pi}{h^3c^3}\int_0^\infty \frac{E^3}{e^{E/k_BT}+1}dE. \tag{127}$$

It is possible to determine the entropy of the quantum field once the proper distance α is known [7,14,15,19,20,91,92]

$$S_{\text{field}} = \frac{32k_B^4c^3}{h^3\alpha^2}\left(\frac{T_\infty^3}{\kappa_0^3}\right)\frac{A_H}{4}2\mathfrak{C}_1, \tag{128}$$

where

$$\mathfrak{C}_1 = \frac{1}{\bar{\tau}}\int_{t_1}^{t_2} e^{t/\bar{\tau}}dt. \tag{129}$$

Hence

$$S_{\text{field}} = S_{\text{BH}}. \tag{130}$$

VIII. DISCUSSION AND CONCLUSIONS

In the absence of a quantum gravity theory, it is useful to propose a classical approximation that allows one to explain the effects of quantum fields in high curvature scenarios. The model proposed herein corresponds to a statistical quantum description of a fermion field with spin $s = 1/2$ near the gravitational radius r_s.

The origin of the particles around the spherical surface of radius $r = r_s + \epsilon$ lies in the thermal excitations of the Boulware vacuum state $|0\rangle_B$ for an external observer (FIDO). The foregoing is done based on thermo field dynamics, as it allows one to explain the origin of S_{BH} as a state of entanglement between the modes of the fermionic field spreading by the Kruskal manifold (see Fig. 4). We show that for the case of a fermionic field $s = 1/2$, the entropy of the field is

$$S_{\text{Ent}}|_{N=1;s=1/2} \approx S_{\text{BH}}. \tag{131}$$

Arenas and Tejeiro [14] calculate that the case of a scalar field $s = 0$ has an entropy

$$S_{\text{Ent}}|_{N=1;s=0} \approx S_{\text{BH}}. \tag{132}$$

In the case of the superposition of two or more quantum fields near the gravitational radius r_s, we believe that

$$S_{\text{Ent}}|_{N=2,3,\ldots;s=0,1/2,\ldots} \approx S_{\text{BH}}, \tag{133}$$

Eq. (133), leads to the species problem, a question still not clearly understood. Furthermore, consider there is some kind of particle responsible for gravity, graviton. Such particles must be in a certain quantum state forming a Bose-Einstein condensate. The foregoing results in total entropy, as proposed by Chen *et al.* [93], in the form

$$S = S_{\text{bare}} + S_{\text{atm}}, \tag{134}$$

where S_{bare} is the entropy associated with spacetime and therefore to gravitons (if they exist) and S_{atm} in the entropy is associated with thermal atmosphere, such that if the Boltzmann principle is valid even in the gravitational context, which allows asserting

$$S_{\text{bare}} \propto \ln|\Omega_{\text{bare}}|, \tag{135}$$

where Ω_{bare} is the quantum state of the gravitons forming the Bose-Einstein condensate.

This study is possible since we can build the Hilbert space for the quantum field under consideration, which allows obtaining a self-consistent result with the techniques of quantum field theory on a curved spacetime. The use of typical methods of second quantization applied directly to the gravitational field leads to a nonrenormalizable theory [94]. However, recently Ulloa *et al.* calculate the energy-momentum tensor $T_{\mu\nu}$ for the gravitational field based on thermo field dynamics, such that they are able to directly estimate the thermal properties to the spacetime and intuit a corpuscular structure of spacetime [95].

This model allows providing a precise location of the field modes responsible for S_{BH}, since we found that such field modes are strongly concentrated in a thin layer $r = r_s + \epsilon$ around the horizon; see Fig. 6 [7,14,15,19,20].

The components of $T_{\mu\nu}$ for the fermionic field were calculated where it was possible to determine that

$$\langle T_{00}(x,x')\rangle|_{x=x'} \propto \rho, \tag{136}$$

$$\langle T_{ij}(x,x')\rangle|_{x=x'} = 0, \tag{137}$$

based on the quantum vacuum states $|0\rangle_B$ and $|0\rangle_H$. These vacuum states are associated with the quantum state $|0\rangle_{\mathrm{TUB}}$, which corresponds to the generalization of a Hartle-Hawking vacuum state,

$$(T_{\mu\nu})_{\mathrm{TUB}} = (T_{\mu\nu})_B - \Delta T_{\mu\nu}, \tag{138}$$

where $(T_{\mu\nu})_B$ is associated with the quantum field near the horizon and is associated with the source $\Delta T_{\mu\nu}$ [7,14,15,19,20]. According to the above mentioned, it is important to note that the role of FIDO is remarkable in this study.

Data Availability—Data analyzed in this study were a reanalysis of existing data, which are openly available at locations cited in the reference section. Further documentation about data processing is available at [96].

ACKNOWLEDGMENTS

This work was supported by the Departamento Administrativo de Ciencia, Tecnología e Innovación, Colciencias and Departamento de Física, Universidad Nacional de Colombia, Bogotá, Colombia, under convocatoria 6172 of Colciencias.

APPENDIX A: RADIAL EQUATION FOR DIRAC FIELD IN SCHWARZSCHILD SPACETIME

The Dirac equation for the Schwarzschild spacetime is obtained inserting (30) into (23) [28],

$$\begin{aligned}\Bigg[&\gamma^0\frac{1}{\sqrt{f(r)}}\frac{\partial}{\partial t}+\gamma^1\sqrt{f(r)}\left(\frac{\partial}{\partial r}+\frac{1}{r}+\frac{M}{2r(r-2M)}\right)\\&+\gamma^2\left(\frac{1}{r}\right)\left(\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta\right)+\gamma^3\left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial\phi}+m\Bigg]\Psi=0.\end{aligned} \tag{A1}$$

Equation (A1) is simplified when using the Ansatz $\Psi = \frac{1}{\sqrt[4]{f(r)}}\Phi$

$$\begin{aligned}\Bigg[&\gamma^0\frac{1}{\sqrt{f(r)}}\frac{\partial}{\partial t}+\gamma^1\sqrt{f(r)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\\&+\gamma^2\left(\frac{1}{r}\right)\left(\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta\right)+\gamma^3\left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial\phi}+m\Bigg]\Phi=0.\end{aligned} \tag{A2}$$

Consider as a solution [60]

$$\Phi(t,r,\theta,\phi) = \begin{pmatrix}\frac{iG^{(\pm)}(r)}{r}\varphi_{jm}^{(\pm)}(\theta,\phi)\\ \frac{F^{(\pm)}(r)}{r}\varphi_{jm}^{(\mp)}(\theta,\phi)\end{pmatrix}e^{-iEt}, \tag{A3}$$

where $j = l \pm 1/2$,

$$\varphi_{jm}^{(+)} = \begin{pmatrix}\sqrt{\frac{l+1/2+m}{2l+1}}Y_l{}^{m-1/2}\\ \sqrt{\frac{l+1/2-m}{2l+1}}Y_l{}^{m+1/2}\end{pmatrix}, \tag{A4}$$

$$\varphi_{jm}^{(-)} = \begin{pmatrix}\sqrt{\frac{l+1/2-m}{2l+1}}Y_l{}^{m-1/2}\\ -\sqrt{\frac{l+1/2+m}{2l+1}}Y_l{}^{m+1/2}\end{pmatrix}, \tag{A5}$$

which allows obtaining the radial equations

$$\frac{d}{dr_*}\begin{pmatrix}F^{(\pm)}\\G^{(\pm)}\end{pmatrix}-\sqrt{1-\frac{2M}{r}}\begin{pmatrix}\kappa_{(\pm)}/r & m\\ m & -\kappa_{(\pm)}/r\end{pmatrix}\begin{pmatrix}F^{(\pm)}\\G^{(\pm)}\end{pmatrix} = \begin{pmatrix}0 & -E\\ E & 0\end{pmatrix}\begin{pmatrix}F^{(\pm)}\\G^{(\pm)}\end{pmatrix}.$$

Consider the turtle coordinate

$$r_* = r + 2M\ln\left|\frac{r}{2M}-1\right|, \tag{A6}$$

and also, constant κ is [57,58,60,97]

$$\kappa_{(\pm)} = \begin{cases}-(j+1/2), & j = l+1/2,\\ (j+1/2), & j = l-1/2.\end{cases} \tag{A7}$$

Again, we consider a rotation for (A6) in the form

$$\begin{pmatrix}\hat{F}^{(+)}\\ \hat{G}^{(+)}\end{pmatrix} = \begin{pmatrix}\sin(\theta_{(+)}/2) & \cos(\theta_{(+)}/2)\\ \cos(\theta_{(+)}/2) & -\sin(\theta_{(+)}/2)\end{pmatrix}\begin{pmatrix}F^{(+)}\\G^{(+)}\end{pmatrix}, \tag{A8}$$

defining $\theta_{(+)} = \tan^{-1}\frac{mr}{|\kappa_{(+)}|}$. Therefore (A6) is simplified to

$$\frac{d}{dr_*}\begin{pmatrix}\hat{F}^{(+)}\\ \hat{G}^{(+)}\end{pmatrix} - \sqrt{1-\frac{2M}{r}}\sqrt{\left(\frac{\kappa_{(+)}}{r}\right)^2+m^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\hat{F}^{(+)}\\ \hat{G}^{(+)}\end{pmatrix}$$
$$= -E\left(1+\frac{1}{2E}\left(1-\frac{2M}{r}\right)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\hat{F}^{(+)}\\ \hat{G}^{(+)}\end{pmatrix}\frac{m|\kappa_{(+)}|}{\kappa_{(+)}^2+m^2r^2}\right). \tag{A9}$$

At this point, it is possible to further rewrite (A9) as

$$\hat{r}_* = r_* + \frac{1}{2E}\tan^{-1}\left(\frac{mr}{|\kappa_{(+)}|}\right). \tag{A10}$$

Hence, the radial equations are ultimately rewritten as

$$\frac{d}{d\hat{r}_*}\begin{pmatrix}\hat{F}^{(+)}\\ \hat{G}^{(+)}\end{pmatrix} + W_{(+)}\begin{pmatrix}-\hat{F}^{(+)}\\ \hat{G}^{(+)}\end{pmatrix} = E\begin{pmatrix}\hat{G}^{(+)}\\ -\hat{F}^{(+)}\end{pmatrix}, \tag{A11}$$

defining the superpotential $W_{(+)}$ as

$$W_{(+)} = \frac{\sqrt{1-\frac{2M}{r}}\sqrt{(\frac{\kappa_{(+)}}{r})^2+m^2}}{1+\frac{1}{2E}(1-\frac{2M}{r})\left(\frac{m|\kappa_{(+)}|}{\kappa_{(+)}^2+m^2r^2}\right)}. \tag{A12}$$

These equations for $\hat{F}^{(+)}$ and $\hat{G}^{(+)}$ are decoupled as

$$\left(-\frac{d^2}{d\hat{r}_*^2}+V_{(+)1}\right)\hat{F}^{(+)} = E^2\hat{F}^{(+)} \tag{A13}$$

and

$$\left(-\frac{d^2}{d\hat{r}_*^2}+V_{(+)2}\right)\hat{G}^{(+)} = E^2\hat{G}^{(+)}. \tag{A14}$$

Additionally,

$$V_{(+)1,2} = \pm\frac{dW_{(+)}}{d\hat{r}_*} + W_{(+)}^2. \tag{A15}$$

If $(-)$ yields

$$\left(-\frac{d^2}{d\hat{r}_*^2}+V_{(-)1}\right)\hat{F}^{(-)} = E^2\hat{F}^{(-)}, \tag{A16}$$

$$\left(-\frac{d^2}{d\hat{r}_*^2}+V_{(-)2}\right)\hat{G}^{(-)} = E^2\hat{G}^{(-)}, \tag{A17}$$

where

$$V_{(-)1,2} = \pm\frac{dW_{(-)}}{d\hat{r}_*} + W_{(-)}^2, \tag{A18}$$

then

$$W_{(-)} = \frac{\sqrt{1-\frac{2M}{r}}\sqrt{(\frac{\kappa_{(-)}}{r})^2+m^2}}{1-\frac{1}{2E}(1-\frac{2M}{r})\left(\frac{m\kappa_{(-)}}{\kappa_{(-)}^2+m^2r^2}\right)}. \tag{A19}$$

The integers are represented as $(+)$ and $(-)$ with

$$\kappa_{(+)} = j+\frac{1}{2}, \quad \kappa_{(+)} = -\left(j+\frac{1}{2}\right) \text{ and } j = l\pm\frac{1}{2}. \tag{A20}$$

APPENDIX B: HAMILTONIAN DENSITY FOR DIRAC FIELD IN SCHWARZSCHILD SPACETIME

Using the Lagrangian density given in (64), it is possible to estimate the Hamiltonian density $\mathcal{H}$ by a Legendre transformation [59] if the form $\mathcal{H} = \Pi_\Psi\dot{\Psi} + \dot{\bar{\Psi}}\Pi_{\bar{\Psi}} - \mathcal{L}$ where the canonically conjugate moments are $\Pi_\Psi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = \frac{i}{2}\dot{\bar{\Psi}}\gamma^0$ and $\Pi_{\bar{\Psi}} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{\Psi}}} = -\frac{i}{2}\gamma^0\Psi$. Moreover, having that $\partial_0\Psi = \dot{\Psi}$ and $\partial_0\bar{\Psi} = \dot{\bar{\Psi}}$,

$$\mathcal{H} = \frac{i}{2}[(\partial_i\bar{\Psi}\gamma^i\Psi) - \bar{\Psi}\gamma^i\partial_i\Psi] + m\bar{\Psi}\Psi. \tag{B1}$$

Considering the Dirac equation $[i\gamma^\mu\vec{\nabla}_\mu - m]\Psi = 0$ and the Dirac adjoint equation $\bar{\Psi}[i\gamma^\mu\overleftarrow{\nabla}_\mu + m] = 0$ are given by (59), they can also be broken down into their temporal and spatial components. Based on the foregoing, the Hamiltonian density is transformed as

$$\mathcal{H} = i\bar{\Psi}\gamma^0\overleftrightarrow{\partial}_0\Psi + \bar{\Psi}\left[\frac{1}{4}\epsilon^{abcd}\omega_{abc}\gamma^5\gamma_d\right]\Psi$$
$$= \mathcal{H}_{\text{classic}*} + \bar{\Psi}\left[\frac{1}{4}\epsilon^{abcd}\omega_{abc}\gamma^5\gamma_d\right]\Psi, \tag{B2}$$

where

$$\mathcal{H}_{\text{classic}*} = i\bar{\Psi}\gamma^0\overleftrightarrow{\partial}_0\Psi = i\bar{\Psi}\gamma^0 e_0^0\overleftrightarrow{\partial}_0\Psi. \tag{B3}$$

Moreover, it has been written for (B3) that the gamma matrices $\gamma^\mu = e_a^\mu\gamma^a$, $e_a^\mu = \frac{1}{\sqrt{f(r)}}$, $(\gamma^0)^2 = \mathbb{I}$, and the Dirac adjoint written as $\bar{\Psi} = \Psi^\dagger\gamma^0$. Consequently, Eq. (B3) is reduced to

$$\mathcal{H}_{\text{classic}*} = \frac{1}{\sqrt{f(r)}}[i\Psi^{\dagger}\overset{\leftrightarrow}{\partial}_0\Psi] = \frac{1}{\sqrt{f(r)}}\mathcal{H}_{\text{classic}}. \quad \text{(B4)}$$

Therefore, having that $\mathcal{H}_{\text{classic}} = i\Psi^{\dagger}\overset{\leftrightarrow}{\partial}_0\Psi$ corresponds to the Hamiltonian density of the Dirac field in the Minkowski spacetime [62], those field modes are defined by (35), (36), and (37). In this context, it is possible to conclude that the Hamiltonian density of the Dirac field in the vicinity of a shell of matter, which has contracted to near the Schwarzschild radius r_s, corresponds to the Hamiltonian density of the gravitationally adjusted classical fermionic field by a factor $\frac{1}{\sqrt{f(r)}}$, with $f(r) = 1 - \frac{2M}{r}$. Under the foregoing conditions, we obtain

$$\lim_{r\to\infty} \frac{1}{\sqrt{1-\frac{2M}{r}}}[i\Psi^{\dagger}\overset{\leftrightarrow}{\partial}_0\Psi] = \mathcal{H}_{\text{classic}}. \quad \text{(B5)}$$

This result is important since it is possible to perform the second quantization scheme of the fermionic field in the Schwarzschild spacetime analogously to the Minkowski spacetime [31]. The Hamiltonian H is obtained integrated over the three-volume as

$$H = \int d^3x \mathcal{H}_{\text{classic}*}. \quad \text{(B6)}$$

APPENDIX C: THE BOGOLIUBOV COEFFICIENTS FOR DIRAC FIELD IN SCHWARZSCHILD SPACETIME

The partition function $Z(\beta)$ is in the form [78,88]

$$Z(\beta) = (1 + e^{-\beta E_n}). \quad \text{(C1)}$$

The relation between the Boulware vacuum state $|0(\beta)\rangle_B$ given in (78) and the Hartle-Hawking vacuum state $|0(\beta)\rangle_{\text{HH}}$ is determined by a transformation in the form

$$|0(\beta)\rangle_{\text{HH}} = U(\beta)|0(\beta)\rangle_B,$$
$$U(\beta) = U(\beta)^+U(\beta)^- = e^{-iG(\theta)^+}e^{-iG(\theta)^-}, \quad \text{(C2)}$$

where

$$G(\theta)^+ = \theta(\beta)^+[b^{(-)}b^{(+)} - b^{(+)\dagger}b^{(-)\dagger}],$$
$$G(\theta)^- = \theta(\beta)^-[d^{(-)}d^{(+)} - d^{(+)\dagger}d^{(-)\dagger}]. \quad \text{(C3)}$$

Also, the Bogoliubov coefficients allow building the canonical transformation that leaves the Hamiltonian invariant defined as

$$\sin\theta^+ = \sin\theta^- = \frac{1}{\sqrt{1+e^{\beta E_D}}} = v(\beta),$$
$$\cos\theta^+ = \cos\theta^- = \frac{1}{\sqrt{1+e^{-\beta E_D}}} = u(\beta). \quad \text{(C4)}$$

For the thermal vacuum state $|0(\beta)\rangle_{\text{HH}} = U(\beta)|0(\beta)\rangle_B$, it is expressed as

$$|0(\beta)\rangle_{\text{HH}} = [\sin\theta + \cos\theta].|0(\beta)\rangle_B. \quad \text{(C5)}$$

Additionally, from (C5) we obtain

$$\tan\theta = \frac{v(\beta)}{u(\beta)} = e^{-\beta\omega/2}, \quad \text{(C6)}$$

such that it fulfills

$$\sin^2\theta + \cos^2\theta = v^2(\beta) + u^2(\beta) = 1. \quad \text{(C7)}$$

The temperature-dependent creation and annihilation operators are obtained as

$$b^{(\epsilon)}(\beta) = e^{-iG(\beta)}b^{(\epsilon)}e^{iG(\theta)}, \quad b^{(\epsilon)\dagger}(\beta) = e^{-iG(\beta)}b^{(\epsilon)\dagger}e^{iG(\theta)}, \quad \text{(C8)}$$

and

$$d^{(\epsilon)}(\beta) = e^{-iG(\beta)}d^{(\epsilon)}e^{iG(\theta)}, \quad d^{(\epsilon)\dagger}(\beta) = e^{-iG(\beta)}d^{(\epsilon)\dagger}e^{iG(\theta)}, \quad \text{(C9)}$$

which leads to

$$\begin{pmatrix} b^{(\epsilon)}(\beta) \\ b^{(-\epsilon)\dagger}(\beta) \end{pmatrix} = \begin{pmatrix} u(\beta) & -v(\beta) \\ v(\beta) & u(\beta) \end{pmatrix}\begin{pmatrix} b^{(\epsilon)} \\ b^{(-\epsilon)\dagger} \end{pmatrix}, \quad \text{(C10)}$$

$$\begin{pmatrix} d^{(\epsilon)}(\beta) \\ d^{(-\epsilon)\dagger}(\beta) \end{pmatrix} = \begin{pmatrix} u(\beta) & -v(\beta) \\ v(\beta) & u(\beta) \end{pmatrix}\begin{pmatrix} d^{(\epsilon)} \\ d^{(-\epsilon)\dagger} \end{pmatrix}. \quad \text{(C11)}$$

APPENDIX D: THE WIGHTMAN FUNCTION OF DIRAC FIELD IN SCHWARZSCHILD SPACETIME

The Dirac adjoint is defined as

$$\bar{\psi}_b(x') = \sum_{\epsilon',\omega'}\left[b_{\Omega'}^{\dagger(\epsilon')} + d_{\Omega'}^{\dagger(\epsilon')}\right]\bar{\Phi}_{b,\Omega'}^{(\epsilon')}(x'), \quad \text{(D1)}$$

$$\bar{\psi}_b(x') = \sum_{\epsilon',\omega'}\left[f_{\Omega'}^{\dagger(\epsilon')} + g_{\Omega'}^{\dagger(\epsilon')}\right]\bar{\Psi}_{b,\Omega'}^{(\epsilon')}(x'). \quad \text{(D2)}$$

Based on the foregoing, it is possible to determine the Wightman function in the modes of the Boulware scheme. Inserting (84) and (D1) in (83) allows finding

$$W_{ab}(x,x')_B = \sum_{\epsilon,\Omega,\epsilon',\Omega'} {}_B\langle 0|\left[b_\Omega^{(\epsilon)} + d_\Omega^{(\epsilon)}\right]\left[b_{\Omega'}^{\dagger(\epsilon')} + d_{\Omega'}^{\dagger(\epsilon')}\right]|0\rangle_B \times \Phi_{a,\Omega}^{(\epsilon)}(x)\bar{\Phi}_{b,\Omega'}^{(\epsilon')}(x'). \tag{D3}$$

Consider the anticommutation rules for fermions $\{b_\Omega^{(\epsilon)}, b_{\Omega'}^{\dagger(\epsilon')}\} = \{d_\Omega^{(\epsilon)}, d_{\Omega'}^{\dagger(\epsilon')}\} = \epsilon\epsilon(\omega)\delta_{\epsilon\epsilon'}\delta_{\Omega\Omega'}$. Once the anticommutator has been developed as

$$b_\Omega^{(\epsilon)} b_{\Omega'}^{\dagger(\epsilon')} + b_{\Omega'}^{\dagger(\epsilon')} b_\Omega^{(\epsilon)} = \epsilon\epsilon(\omega)\delta_{\epsilon\epsilon'}\delta_{\Omega\Omega'} \tag{D4}$$

and likewise for the anticommutator $\{d_\Omega^{(\epsilon)}, d_{\Omega'}^{\dagger(\epsilon')}\}$, we get that the Wightman function is rewritten as

$$\begin{aligned} W_{ab}(x,x')_B &= \sum_{\epsilon,\Omega}[2\epsilon\epsilon(\omega)\delta_{\epsilon\epsilon'}\delta_{\Omega\Omega'}]\Phi_{a,\Omega}^{(\epsilon)}(x)\bar{\Phi}_{b,\Omega'}^{(\epsilon')}(x') \\ &= \sum_{\epsilon,\Omega} 2\Theta(\epsilon\omega)\Phi_{a,\Omega}^{(\epsilon)}(x)\bar{\Phi}_{b,\Omega'}^{(\epsilon')}(x'), \end{aligned} \tag{D5}$$

with the condition that $\epsilon(\omega) = \mathrm{sing}(\omega) = \Theta(\epsilon\omega)$, $\delta_{\Omega\Omega'} = \delta_{(\omega-\omega')}\delta_{\kappa(l)\kappa'(l)}\delta_{ll'}, \delta_{mm'}$ for the possible values of $\kappa(\pm)$ defined in (A7) [28,60] and $\Omega = \omega, \kappa(l), m$. Moreover $\Omega = \Omega'$, $\epsilon = \epsilon'$, and $\epsilon = 1$.

With a procedure identical to the one described above, we determine the Wightman function under the HH scheme, which is

$$W_{ab}(x,x')_H = \sum_{\epsilon,\Omega,} 2\Theta(\epsilon\omega)\Psi_{a,\Omega}^{(\epsilon)}(x)\bar{\Psi}_{b,\Omega'}^{(\epsilon')}(x'). \tag{D6}$$

The relationship between the two modes $\Psi_\Omega^{(\epsilon)}(t,\underline{x})$ and $\Phi_\Omega^{(\epsilon)}(t,\underline{x})$ is

$$\Psi_{a,\Omega}^{(\epsilon)}(t,\underline{x}) = \Phi_{a,\Omega}^{(\epsilon)}(t,\underline{x})\cos\chi + \Phi_{a,\Omega}^{(-\epsilon)}(t,\underline{x})\sin\chi \tag{D7}$$

and the adjoint is

$$\bar{\Psi}_{b,\Omega}^{(\epsilon)}(t,\underline{x}) = \bar{\Phi}_{b,\Omega}^{(\epsilon)}(t,\underline{x})\cos\chi + \bar{\Phi}_{b,\Omega}^{(-\epsilon)}(t,\underline{x})\sin\chi. \tag{D8}$$

Substituting (D7) and (D8) in (D5), we obtain

$$\begin{aligned} W_{ab}(x,x')_H &= \sum_{\epsilon,\Omega,} 2\Theta(\epsilon\omega)\left[\Phi_{a,\Omega}^{(\epsilon)}(t,\underline{x})\cos\chi + \Phi_{a,\Omega}^{(-\epsilon)}(t,\underline{x})\sin\chi\right] \\ &\quad \times \left[\bar{\Phi}_{b,\Omega}^{(\epsilon)}(t,\underline{x})\cos\chi + \bar{\Phi}_{b,\Omega}^{(-\epsilon)}(t,\underline{x})\sin\chi\right], \end{aligned} \tag{D9}$$

with the following products for (D9)

$$\Phi_\Omega^{(\epsilon)}(\underline{x}) = \Phi_\Omega(t,\underline{x})\Theta_\epsilon(x), \qquad \bar{\Phi}_\Omega^{(\epsilon)}(\underline{x}) = \bar{\Phi}_\Omega(t,\underline{x})\Theta_\epsilon(x) \tag{D10}$$

and

$$\Phi_\Omega^{(-\epsilon)}(\underline{x}) = \Phi_\Omega(t,\underline{x})\Theta_{-\epsilon}(x), \quad \bar{\Phi}_\Omega^{(-\epsilon)}(\underline{x}) = \bar{\Phi}_\Omega(t,\underline{x})\Theta_{-\epsilon}(x). \tag{D11}$$

Thus, for (D9), we determine products, with the condition that $x \in R$, $x' \in L$, such that $\Theta_\epsilon(x)\Theta_{-\epsilon}(x') = \Theta_\epsilon(x')\Theta_{-\epsilon}(x) = 0$, as it is in different regions of the Kruskal manifold

$$\Phi_\Omega^{(\epsilon)}(\underline{x})\Phi_\Omega^{(-\epsilon)}(\underline{x}') = \Phi_\Omega(t,\underline{x})\Phi_\Omega(t',\underline{x}')\Theta_\epsilon(x)\Theta_{-\epsilon}(x') = 0 \tag{D12}$$

and

$$\Phi_\Omega^{(-\epsilon)}(\underline{x})\Phi_\Omega^{(\epsilon)}(\underline{x}) = \Phi_\Omega(t',\underline{x}')\Phi_\Omega(t,\underline{x})\Theta_\epsilon(x')\Theta_{-\epsilon}(x) = 0. \tag{D13}$$

Based on the foregoing, the Wightman function for the Dirac field under the HH scheme is simplified to

$$W_{ab}(x,x')_H = \sum_{\epsilon,\Omega} 2\Theta(\epsilon\omega) \times \left[\Phi_{a,\Omega}^{(\epsilon)}(x)\bar{\Phi}_{b,\Omega}^{(\epsilon)}(x)\cos^2\chi + \Phi_{a,\Omega}^{(-\epsilon)}(x)\bar{\Phi}_{b,\Omega}^{(-\epsilon)}(x)\sin^2\chi\right]. \tag{D14}$$

The difference between $W_{ab}(x,x')_H - W_{ab}(x,x')_B$ leads to

$$\begin{aligned} &W_{ab}(x,x')_H - W_{ab}(x,x')_B \\ &= -\sum_{\epsilon,\Omega} 2\Theta(\epsilon\omega)\sin^2\chi[\Phi_{a,\Omega}^{(\epsilon)}(x)\bar{\Phi}_{b,\Omega}^{(\epsilon)}(x') + \Phi_{a,\Omega}^{(-\epsilon)}(x)\bar{\Phi}_{b,\Omega}^{(-\epsilon)}(x')]. \end{aligned} \tag{D15}$$

Moreover, it holds that $W_{ab}(x,x')_H - W_{ab}(x,x')_B = (W_H - W_B)_{ab}(x,x')$, and it is also true that $\epsilon = +1 \in R$ and $\epsilon = -1 \in L$ can be restricted to one of the regions of the Kruskal manifold as

$$(W_H - W_B)_{ab}(x,x') = -\sum_{\epsilon,\Omega} 2\sin^2\chi\left[\Phi_{a,\Omega}^{(+)}(x)\bar{\Phi}_{b,\Omega}^{(+)}(x')\right] \tag{D16}$$

$$(W_H - W_B)_{ab}(x,x') = -\sum_{\epsilon,\Omega} 2\frac{1}{1+e^{\beta E}}\left[\Phi_{a,\Omega}^{(+)}(x)\bar{\Phi}_{b,\Omega}^{(+)}(x')\right]. \tag{D17}$$

On the other hand, we have already considered the explicit form of the energy-momentum tensor ($T_{\mu\nu}$) in (81) as [98]

$$T_{00} = \rho = \frac{i}{2}[\bar{\Psi}\gamma_0\overleftrightarrow{\nabla}_0\Psi + \bar{\Psi}\gamma_0\overleftrightarrow{\nabla}_0\Psi], \tag{D18}$$

where ρ is the energy density and for this study it is rigorously $\rho > 0$, since no foreign forms of matter energy are considered. Therefore, we get that $\langle T_{00}(x,x')\rangle|_{x=x'} \propto \rho$, which leads to $(W_H - W_B)_{ab}(x,x') > 0$ [99]. Consequently, (D16) is rewritten as

$$(W_H - W_B)_{ab}(x,x') = \sum_{\epsilon,\Omega,} 2\frac{1}{1+e^{\beta E}}\left[\Phi^{(+)}_{a,\Omega}(x)\bar{\Phi}^{(+)}_{b,\Omega}(x')\right]. \tag{D19}$$

At this point, Wightman's function for the Dirac field [14] must be expressed as

$$T_{\mu\nu} = \frac{i}{2}[\bar{\Psi}\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu \Psi + \bar{\Psi}\gamma_\nu \overset{\leftrightarrow}{\partial}_\mu \Psi], \tag{D20}$$

which is rewritten as

$$T_{\mu\nu}(x,x') = \bar{\Psi}(x')\frac{i}{2}[\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu + \gamma_\nu \overset{\leftrightarrow}{\partial}_\mu]\Psi(x), \tag{D21}$$

where

$$\overset{\leftrightarrow}{\partial}_\nu = \frac{1}{2}(\vec{\partial}_\nu - \overset{\leftarrow}{\partial}_{\nu'})$$

and

$$\overset{\leftrightarrow}{\partial}_\mu = \frac{1}{2}(\vec{\partial}_\mu - \overset{\leftarrow}{\partial}_{\mu'}),$$

which is why (D21) is rewritten as [7,14,69,85,86,100]

$$\langle T_{\mu\nu}(x,x')\rangle = \lim_{x\to x'}\mathrm{Tr}\left\{\left[\frac{i}{2}\gamma_{(\mu}\partial_{\nu)} - \frac{i}{2}\gamma_{(\mu'}\partial_{\nu')}\right]\right\}W_{ab}(x,x'). \tag{D22}$$

According to harmonic spinors, we get [60]

$$\Phi^{(+)}_\Omega(\underline{x}) = \Phi_{\omega,\kappa}(r)\begin{pmatrix} i\varphi^{(+)}_{j,m}(\theta,\phi) \\ \varphi^{(-)}_{j,m}(\theta,\phi)\end{pmatrix}, \tag{D23}$$

$$\Phi^{(-)}_\Omega(\underline{x}) = \Phi_{\omega,\kappa}(r)\begin{pmatrix} i\varphi^{(-)}_{j,m}(\theta,\phi) \\ \varphi^{(+)}_{j,m}(\theta,\phi)\end{pmatrix}. \tag{D24}$$

Therefore, the most general solution is in the form $\Phi_\Omega(\underline{x}) = \Phi^+_\Omega(\underline{x}) + \Phi^-_\Omega(\underline{x})$, and then

$$\Phi_\Omega(\underline{x}) = \Phi_{\omega,\kappa}(r)\begin{pmatrix} i\varphi^{(+)}_{j,m}(\theta,\phi) \\ \varphi^{(-)}_{j,m}(\theta,\phi)\end{pmatrix} + \Phi_{\omega,\kappa}(r)\begin{pmatrix} i\varphi^{(-)}_{j,m}(\theta,\phi) \\ \varphi^{(+)}_{j,m}(\theta,\phi)\end{pmatrix}. \tag{D25}$$

At this point, it is necessary to mention that in line with Bjorken [60], for the hydrogen atom, there is a relationship between the spherical harmonics in the form $Y^*_{l,m} = (-1)^m Y_{l,-m}$ and that the solution φ^- only exists for values of $l > 0$. Hence, the form of such an operator is $\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r}$, and that allows expressing the relationship between as [31,82]

$$\varphi^{(+)}_{jm} = \frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r}\varphi^{(-)}_{jm} = \boldsymbol{\sigma}\cdot\hat{r}\varphi^{(-)}_{jm}. \tag{D26}$$

They allow expressing (D22) as

$$\frac{i}{4}\left\{\mathrm{Tr}\left[\gamma_\mu\frac{\partial\Phi_\Omega(x)}{\partial x^\nu}\bar{\Phi}_\Omega(x') + \gamma_\nu\frac{\partial\Phi_\Omega(x)}{\partial x^\mu}\bar{\Phi}_\Omega(x')\right] - \mathrm{Tr}\left[\gamma_{\mu'}\Phi_\Omega(x)\frac{\partial\bar{\Phi}_\Omega(x')}{\partial x^{\nu'}} + \gamma_{\nu'}\Phi_\Omega(x)\frac{\partial\bar{\Phi}_\Omega(x')}{\partial x^{\mu'}}\right]\right\}. \tag{D27}$$

In turn, each term contained in (D27) is separated into its temporal and spatial components of

$$\langle T_{00}(x,x')\rangle|_{x=x'} = \lim_{x\to x'}\sum_{\epsilon,\Omega} 2\frac{1}{1+e^{\beta E}}\frac{i}{4}\times 2\mathrm{Tr}\left[\gamma_0\frac{\partial\Phi_\Omega(x)}{\partial x^0}\bar{\Phi}_\Omega(x') - \gamma_{0'}\Phi_\Omega(x)\frac{\partial\bar{\Phi}_\Omega(x')}{\partial x^{0'}}\right] \tag{D28}$$

and

$$\langle T_{ij}(x,x')\rangle|_{x=x'} = \lim_{x\to x'}\sum_{\epsilon,\Omega} 2\frac{1}{1+e^{\beta E}}\frac{i}{4} \times \mathrm{Tr}\left[\gamma_i\frac{\partial\Phi_\Omega(x)}{\partial x^j}\bar{\Phi}_\Omega(x') + \gamma_j\frac{\partial\Phi_\Omega(x)}{\partial x^i}\bar{\Phi}_\Omega(x') - \gamma_{i'}\Phi_\Omega(x)\frac{\partial\bar{\Phi}_\Omega(x')}{\partial x^{j'}} - \gamma_{j'}\Phi_\Omega(x)\frac{\partial\bar{\Phi}_\Omega(x')}{\partial x^{i'}}\right]. \tag{D29}$$

According to the properties of the spinor spherical harmonics we obtain

$$\langle T_{00}(x,x')\rangle|_{x=x'} = 4\sum_{\omega}\frac{\omega}{1+e^{\beta\omega}}\times\sum_{\kappa}\sqrt{f(r)}\,|\Phi_{\omega\kappa}(r)|^2\left(\frac{2l+1}{4\pi}\right). \tag{D30}$$

Now, the sum over the frequency ω in (D30) at the limit to the continuum can be expressed as an integral in the form

$$\sum_{\omega}\frac{\omega}{1+e^{\beta\omega}} \to \int_{-\infty}^{\infty}\frac{\omega}{1+e^{\beta\omega}}d\omega = 2\int_0^{\infty}\frac{\omega}{1+e^{\beta\omega}}d\omega. \tag{D31}$$

Inserting (D31) into (D30), we obtain

$$\langle T_{00}(x,x')\rangle|_{x=x'} = 4\left[2\int_0^{\infty}\frac{\omega}{1+e^{\beta\omega}}d\omega\right] \times\sum_{\kappa}\sqrt{f(r)}\,|\Phi_{\omega\kappa}(r)|^2\frac{2l+1}{4\pi}. \tag{D32}$$

On the other hand, under the WKB approximation, we obtain

$$|\Phi_{\omega\kappa}(r)|^2 = \frac{\omega}{2\pi r^2\sqrt{Q(r)}}. \tag{D33}$$

Substituting (D33) in (D32), we obtain

$$\langle T_{00}(x,x')\rangle = 4\left[2\int_0^{\infty}\frac{\omega^2}{1+e^{\beta\omega}}\frac{d\omega}{2}\right]\frac{1}{4\pi r^2}\sum_{\kappa}\sqrt{\frac{f(r)}{Q(r)}}(2l+1), \tag{D34}$$

where $f(r) = 1+\frac{2M}{r}$ and $Q(r)$ is defined by (35) and $T(\kappa)$ (36). Consequently, near the gravitational radius $r_s = 2M$, considering that is under the approach $f(r)\approx 2\kappa_0\delta r e^{-t/\bar{\tau}}$. We get that $\lim_{t\to t_u} f(r) = 2\kappa_0\delta r$, where $\kappa_0 = \frac{1}{4M}$. Based on the foregoing, we get that $T(\kappa)$ is rewritten as

$$\lim_{\kappa\to\infty} T(\kappa) = -\frac{1}{2\delta r}. \tag{D35}$$

It should be noted that the values of κ are bounded as $\kappa_{(+)} = -(l+1)$, $\kappa_{(-)} = l$, $\kappa_{(-)} > \kappa_{(+)}$. We want (35) to adopt the maximum possible values of $\kappa_{\max}\sim l_{\max}$,

$$Q(r) = \frac{2\delta r^2}{f(r)}\left[\frac{\omega^2}{2f(r)\delta r} - \frac{m^2}{2\delta r} - \frac{\kappa_{\max^2}}{2r^2\delta r}\right]. \tag{D36}$$

The fact that $\kappa\to\kappa_{\max}$ leads to $Q(r) = 0$,

$$\frac{2\delta r^2}{f(r)}\left[\frac{\omega^2}{2f(r)\delta r} - \frac{m^2}{2\delta r} - \frac{\kappa_{\max^2}}{2r^2\delta r}\right] = 0,$$
$$\frac{\kappa_{\max^2}}{2r^2\delta r} = \frac{\omega^2}{2f(r)\delta r} - \frac{m^2}{2\delta r} = p^2, \tag{D37}$$

where p^2 defines the centrifugal potential. Accordingly, inserting (D37) into (D36) we obtain

$$\sqrt{Q(r)} = \sqrt{\frac{2\delta r^2}{f(r)}\left[p^2 - \frac{\kappa_{\max}^2}{r^2}\frac{1}{2\delta r}\right]}. \tag{D38}$$

On the other hand, the sum over κ contained in (D34) is transformed into an integral when $\kappa\to\infty$ and $\kappa_{\max}\sim l$,

$$\sum_{\kappa}\sqrt{\frac{f(r)}{Q(r)}}(2l+1) \to \int_0^{\kappa_{\max}}\sqrt{\frac{f(r)}{Q(r)}}(2l+1)d\kappa, \tag{D39}$$

$$\int_0^{\kappa_{\max}}\sqrt{\frac{f(r)}{Q(r)}}(2l+1)d\kappa = \int_0^{l_{\max}}\sqrt{\frac{f(r)}{Q(r)}}(2l+1)dl. \tag{D40}$$

Inserting (D40) into (D35)

$$\langle T_{00}(x,x')\rangle = 4\left[2\int_0^{\infty}\frac{\omega^2}{1+e^{\beta\omega}}\frac{d\omega}{2}\right] \times\frac{1}{4\pi r^2}\int_0^{l_{\max}}\sqrt{\frac{f(r)}{Q(r)}}(2l+1)dl. \tag{D41}$$

Developing the integral over l in (D41) according to (D38)

$$\int_0^{l_{\max}}\sqrt{\frac{f(r)}{Q(r)}}(2l+1)dl = \frac{rf(r)}{\sqrt{\delta r}}\times\int_0^{l_{\max}}\frac{2l+1}{\sqrt{2p^2r^2\delta r - l^2}}dl$$
$$= \frac{rf(r)}{\sqrt{\delta r}}\left\{2\sqrt{2p^2r^2\delta r} - 2\sqrt{2p^2r^2\delta r - l_{\max}^2} + \tan^{-1}\left[\frac{l_{\max}}{\sqrt{2p^2r^2\delta r - l_{\max}^2}}\right]\right\}, \tag{D42}$$

where the term $\tan^{-1}[\frac{l_{\max}}{\sqrt{2p^2r^2\delta r - l_{\max}^2}}]$ can be written as

$$\tan^{-1}\left[\frac{l_{\max}}{\sqrt{2p^2r^2\delta r - l_{\max}^2}}\right] = \tan^{-1}\left[\frac{1}{\sqrt{x^2-1}}\right],$$
$$x^2 = \frac{2r^2p^2\delta r}{l_{\max}^2}, \tag{D43}$$

which can be transformed into a Taylor series at zero to [101]

$$\tan^{-1}\left[\frac{1}{\sqrt{x^2-1}}\right] = \cot^{-1}\left[\sqrt{x^2-1}\right] \approx \frac{\pi}{2}. \tag{D44}$$

Substituting (D44) in (D42)

$$\int_0^{l_{\max}} \sqrt{\frac{f(r)}{Q(r)}}(2l+1)dl = \frac{rf(r)}{\sqrt{\delta r}}\left\{2\sqrt{2p^2r^2\delta r} - 2\sqrt{2p^2r^2\delta r - l_{\max}^2} + \frac{\pi}{2}\right\}. \tag{D45}$$

For (D45), the term $\sqrt{p^2r^2 2\delta r - l_{\max}^2} = 0$, given that $p^2r^2 2\delta r = l_{\max}^2$, which was defined in (D37). Therefore, we have that the integral contained in (D42) is

$$\int_0^{l_{\max}} \sqrt{\frac{f(r)}{Q(r)}}(2l+1)dl \approx \frac{rf(r)}{\sqrt{\delta r}}\left\{2pr\sqrt{2\delta r} + \frac{\pi}{2}\right\}. \tag{D46}$$

Inserting (D47) in (D41)

$$\langle T_{00}(x,x')\rangle = 4\left[\int_0^\infty \frac{\omega^2}{1+e^{\beta\omega}}\frac{d\omega}{2}\right] \times \frac{1}{4\pi^2r^2}\frac{rf(r)}{\sqrt{\delta r}}\left[2\left(2pr\sqrt{2\delta r} + \frac{\pi}{2}\right)\right]. \tag{D47}$$

Consider the centrifugal potential defined earlier p, (D37). We obtain that (D47) is simplified to

$$\langle T_{00}(x,x')\rangle = \frac{1}{2\pi^2}2\left[\int_0^\infty \frac{\omega^2}{1+e^{\beta\omega}}d\omega\right] \times \frac{\sqrt{f(r)}}{f(r)}\left[\frac{2f(r)^{3/2}}{r\sqrt{\delta r}}\left(2l_{\max} + \frac{\pi}{2}\right)\right], \tag{D48}$$

where $f(r)$ can be recovered in (D48) in the following manner: $f(r) \approx 2\kappa_0\delta r$, $l_{\max}^2 = 2r^2p^2\delta r$, and also the surface gravity on the horizon $\kappa_0 = \frac{1}{4M}$. Therefore

$$\langle T_{00}(x,x')\rangle = \frac{1}{2\pi^2}2\left[\int_0^\infty \frac{\omega^2}{1+e^{\beta\omega}}d\omega\right]\frac{\sqrt{f(r)}}{f(r)}p\Delta. \tag{D49}$$

Defining in (D49) Δ as

$$\Delta = \frac{2}{r}\sqrt{8\kappa_0^3\delta r^2}\left(2r\sqrt{2\delta r} + \frac{\pi}{2p}\right) = cte. \tag{D50}$$

On (D50), two conditions are required:

(i) $\kappa_0|_{r_s+\epsilon} = cte$, which implies that the surface gravity on the horizon is constant.

(ii) $p|_{r_s+\epsilon} = cte$, the momenta distribution near the gravitational radius is uniform.

Based on the foregoing, the Wightman function for the Dirac field in the vicinity of a shell of mass M, which has contracted to near its gravitational radius r_s, is in the form

$$\langle T_{00}(x,x')\rangle = \frac{1}{2\pi^2}\left[\int_0^\infty \frac{\omega^2 p}{1+e^{\beta\omega}}d\omega\right]\frac{\sqrt{f(r)}}{f(r)}cte. \tag{D51}$$

Again, we have that

(i) The local energy per field mode is

$$E = \frac{\omega}{\sqrt{f(r)}}, \qquad \to \omega = E\sqrt{f(r)}.$$

(ii) The temperature is mediated by Tolman's law,

$$T(r) = \frac{T_H}{\sqrt{f(r)}}, \qquad T_H = \frac{1}{\beta} \to \beta = \frac{1}{T_H}.$$

Then

$$\beta\omega = \frac{E}{T(r)}. \tag{D52}$$

(iii) The relativistic energy $E^2 = m^2 + p^2$; therefore it can be written according to the foregoing

$$\frac{\omega d\omega}{f(r)} = pdp. \tag{D53}$$

Consequently (D51)

$$\langle T_{00}(x,x')\rangle = \frac{1}{2\pi^2}\int_0^\infty \frac{p^2}{1+e^{E/T(r)}}dpEf(r). \tag{D54}$$

In addition, we know that for the metric (19), $g^{00} = -\frac{1}{f(r)}$, so (D54) is rewritten as

$$-\langle T_0^0(x,x')\rangle = \frac{1}{2\pi^2}\int_0^\infty \frac{E}{1+e^{E/T(r)}}p^2dp, \tag{D55}$$

where the term

$$\frac{1}{2\pi^2} = \frac{4\pi}{(2\pi)^3}, \qquad \hbar = \frac{h}{2\pi} = 1.$$

APPENDIX E: THE ENTROPY OF THE DIRAC FIELD IN SCHWARZSCHILD SPACETIME

Once the partition function Z has been established, the entropy of the Dirac field is determined as

$$S = -\beta \frac{\partial}{\partial \beta} \ln |Z| + \ln |Z|, \tag{E1}$$

and the average value of internal energy as

$$\langle U \rangle = -\frac{d}{d\beta} \ln |Z|. \tag{E2}$$

Let $f(\omega)$ be a function such that

$$\lim_{\omega \to \infty} f(\omega) = 0. \tag{E3}$$

Consequently, the contribution of $f(\omega)$ for each state with Ω values allow stating

$$\sum_{\Omega} f(\omega) = \int_0^{\infty} N(\omega) f(\omega) d\omega. \tag{E4}$$

On the other hand, taking (92)

$$\begin{aligned} \ln |Z| &= \ln \left| \prod_{\Omega;\omega>0} Z_{\Omega}(\beta) \right| \\ &= \sum_{\Omega} f(\omega) = \int_0^{\infty} N(\omega) f(\omega) d\omega, \end{aligned} \tag{E5}$$

which leads to

$$\sum_{\Omega} \ln |Z_{\Omega}| = \sum_{\Omega} f(\omega), \qquad f(\omega) = 1 + e^{\beta\omega}. \tag{E6}$$

The foregoing is valid when the number of modes, $N(\omega)$ of the field $\Phi_{\Omega}(x^{\alpha})$, lie within the interval ω and $\omega + d\omega$. Accordingly, the partition function given in (E5) is simplified to [89]

$$\ln |Z| = \sum_{\omega} \ln |Z_{\omega}|. \tag{E7}$$

Also, under the condition that the fermions do not interact, the partition function explicitly is in the form [88]

$$\ln |Z| = \sum_{\eta=\pm} \sum_{\kappa} \sum_{n} (2l+1) \ln |1 + e^{\beta\omega}|, \tag{E8}$$

where $\eta = \pm$ has been included, which corresponds to the number of incoming and outgoing field modes in the Kruskal variety (see Fig. 4).

On the other hand, the field modes under the WKB approximation

$$F(r) = \sqrt[4]{\frac{\omega^2}{4\pi Q(r)}} e^{-i\left[\int_{r_s+\epsilon}^{r_{\kappa n}} \sqrt{Q(r)} dr\right]}, \tag{E9}$$

such that they become null on the shell with a Dirichlet condition [13]

$$F(r)|_{r_s+\epsilon} = 0, \tag{E10}$$

requires that

$$\int_{r_s+\epsilon}^{r_{\kappa n}} \sqrt{Q(r)} dr = n\pi, \tag{E11}$$

and also

$$Q^2(r_{\kappa n}, \omega_{\kappa n}, \kappa(l))|_{r=r_s+\epsilon} = 0 \tag{E12}$$

evaluated on the shell. And outside of the shell, we get

$$Q^2(r_{\kappa n}, \omega_{\kappa n}, \kappa(l))|_{r=r'} \geq 0, \qquad r' < r_{\kappa n}. \tag{E13}$$

According to the foregoing, the modes become null on the shell [13], and therefore we get

$$F(r) = \sqrt[4]{\frac{\omega^2}{4\pi^2 Q(r)}} \sin \left[\int_{r_s+\epsilon}^{r_{\kappa n}} \sqrt{Q(r)} dr \right]. \tag{E14}$$

APPENDIX F: THE SURFACE AND VOLUMETRIC CONTRIBUTIONS OF THE PARTITION FUNCTION OF DIRAC FIELD IN SCHWARZSCHILD SPACETIME

The integration by parts of (99) with respect to ω is

$$\begin{aligned} &\int d\omega \ln |1 + e^{\beta\omega}| \frac{\partial Q'}{\partial \omega} \\ &\quad = Q' \ln |1 + e^{\beta\omega}| + \int Q' \frac{\beta}{e^{\beta\omega} + 1} d\omega. \end{aligned} \tag{F1}$$

Inserting (F1) into (99), we find

$$\begin{aligned} \ln |Z| = &\frac{1}{\pi} \int_{\omega} \int_{\kappa} \int_{r_s+\epsilon}^{r(\omega,\kappa)} d\omega d\kappa dr' (2l+1) Q' \frac{\beta}{e^{\beta\omega} + 1} \\ &+ \frac{1}{\pi} \int_{\kappa} d\omega d\kappa dr' Q' \ln |1 + e^{\beta\omega}|. \end{aligned} \tag{F2}$$

For (F2), the first integral corresponds to the volumetric contribution in the space of $Q'(r'(\omega, \kappa), \omega, \kappa(l))$ and the second one to the surface enclosing said volume, under the condition that the partition function $\ln|Z|$ receives its main contribution from the volume of $Q'(r'(\omega,\kappa),\omega,\kappa(l))$ and not from the enclosed area. It is possible to approximate (F2) as

$$\ln|Z| \approx \frac{1}{\pi}\int_{\omega}\int_{\kappa}\int_{r_s+\epsilon}^{r(\omega,\kappa)} d\omega d\kappa dr'(2l+1)Q'\frac{\beta}{e^{\beta\omega}+1}. \tag{F3}$$

Again, the integral in (F3),

$$\int_{\kappa} d\kappa(2l+1)Q'(r'(\omega,\kappa),\omega,\kappa(l)), \tag{F4}$$

is developed according to condition (E12) and (D38). So,

$$\frac{\kappa_{\max}^2}{r^2\delta r} = \left[\frac{\omega^2}{f(r)\delta r} - \frac{m^2}{\delta r}\right]. \tag{F5}$$

The foregoing under the condition $\kappa \equiv L \leq L_{\max}(\omega', r')$ is [14]

$$\frac{\kappa_{\max}^2}{r^2\delta r} = \frac{L_{\max}^2}{r^2\delta r} = \left[\frac{\omega^2}{f(r)\delta r} - \frac{m^2}{\delta r}\right]. \tag{F6}$$

Therefore, integral (F4) is simplified to

$$\begin{aligned}\int_{\kappa} & d\kappa(2l+1)Q'(r'(\omega,\kappa),\omega,\kappa(l)) \\ &= \frac{1}{r}\sqrt{\frac{\delta r}{f(r)}}\int dL(2L+1)\sqrt{L_{\max}^2 - L^2} \\ &= \frac{\delta r}{r}\sqrt{\frac{\delta r}{f(r)}}\frac{4}{3}L_{\max}^3,\end{aligned} \tag{F7}$$

where $\delta r = r_0 - r_s$. Substituting (F7) in (F3), we find

$$\begin{aligned}\ln|Z| &= \frac{1}{\pi}\int_{\omega}\int_{r_s+\epsilon}^{R} d\omega dr \frac{\beta}{e^{\beta\omega}+1}\frac{1}{r}\sqrt{\frac{\delta r}{f(r)}}\frac{4}{3}L_{\max}^3. \\ &= \int_{\omega} d\omega \frac{\beta}{e^{\beta\omega}+1}\left[\sqrt{\delta r}\frac{4}{3\pi}\int_{r_s+\epsilon}^{R} dr\frac{1}{r\sqrt{f(r)}}L_{\max}^3\right].\end{aligned} \tag{F8}$$

Acknowledging in (F8) quantity $N(\omega)$,

$$N(\omega) = \frac{4}{3\pi}\int_{r_s+\epsilon}^{R} dr\frac{1}{r\sqrt{f(r)}}L_{\max}^3. \tag{F9}$$

Consequently, the partition function is written as

$$\ln|Z| = \int_0^{\infty}\frac{\beta N(\omega)}{e^{\beta\omega}+1}d\omega\sqrt{\delta r}. \tag{F10}$$

[1] V. Cardoso and P. Pani, Testing the nature of dark compact objects: A status report, Living Rev. Relativity **22**, 4 (2019).

[2] B. P. Abbott *et al.*, Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[3] D. Castelvecchi, Black hole pictured for first time—in spectacular detail, Nature (London) **568**, 7752 (2019).

[4] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).

[5] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).

[6] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752 (1977).

[7] S. Mukohyama and W. Israel, Black holes, brick walls and the Boulware state, Phys. Rev. D **58**, 104005 (1998).

[8] W. H. Zurek and K. S. Thorne, Statistical Mechanical Origin of the Entropy of a Rotating, charged Black Hole, Phys. Rev. Lett. **54**, 2171 (1985).

[9] J. D. Bekenstein, Generalized second law of thermodynamics in black-hole physics, Phys. Rev. D **9**, 3292 (1974).

[10] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Quantum source of entropy for black holes, Phys. Rev. D **34**, 373 (1986).

[11] M. Srednicki, Entropy and Area, Phys. Rev. Lett. **71**, 666 (1993).

[12] H. Terashima, Entanglement entropy of the black hole horizon, Phys. Rev. D **61**, 104016 (2000).

[13] G. 't Hooft, On the quantum structure of a black hole, Nucl. Phys. **B256**, 727 (1985).

[14] J. R. Arenas and J. M. Tejeiro, Entanglement entropy of black shells, Nuovo Cimento B **125**, 1223 (2010).

[15] F. Pretorius, D. Vollick, and W. Israel, An operational approach to black hole entropy, Phys. Rev. D **57**, 6311 (1998).

[16] W. Israel, Thermo field dynamics of black holes, Phys. Lett. **57A**, 107 (1976).

[17] S. Liberati, Quantum vacuum effects in gravitational fields: Theory and detectability, arXiv:gr-qc/0009050.

[18] D. V. Fursaev, Can one understand black hole entropy without knowing much about quantum gravity? Phys. Part. Nucl. **36**, 81 (2005), http://www1.jinr.ru/publish/Pepan/2005-v36/v-36-1/pdf/v-36-1_06.pdf.

[19] W. A. Rojas C, Tesis de Maestría: Termodinámica de un gas de fotones en la vecindad de una superficie de Schwarzschild. Observatorio Astronómico Nacional. Universidad Nacional de Colombia. Director: J. R. Arenas S. (2010).
[20] W. A. Rojas C, Tesis de Doctorado: Mecánica Estadística de la Termodinámica Black Shells. Departamento de Física. Universidad Nacional de Colombia. Director: J. Robel Arenas S. (2018), available at https://repositorio.unal.edu.co/handle/unal/63982.
[21] W. Israel, *Black Hole Thermodynamics* (Springer Berlin Heidelberg, Berlin, Heidelberg, 2003).
[22] Y. Takahashi and H. Umezawa, Thermo field dynamics, Int. J. Mod. Phys. B **10**, 1755 (1996).
[23] H. Umezawa, *Advanced Field Theory: Micro, Macro, and Thermal Physics* (Springer, 1995), https://www.springer.com/gp/book/9781563964565.
[24] S. N. Biswas and A. K. Das, Thermo field dynamics and para statistical mechanics, Mod. Phys. Lett. A **03**, 549 (1988).
[25] W. Israel, Gravitational collapse and causality, Phys. Rev. **153**, 1388 (1967).
[26] J. R. Arenas S and F. Castro, Euclidean approach for entropy of black shells, arXiv:1606.06786.
[27] D. R. Brill and J. A. Wheeler, Interaction of neutrinos and gravitational fields, Rev. Mod. Phys. **29**, 465 (1957).
[28] H. T. Cho, Dirac quasinormal modes in Schwarzschild black hole space-times, Phys. Rev. D **68**, 024003 (2003).
[29] L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2009), https://doi.org/10.1017/CBO9780511813924.
[30] D.-M. Sjøstrøm, *Bosons and Fermions in Curved Spacetime* (Department of Physics, Norwegian University of Science and Technology, Supervisor: Kåre Olaussen, IFY, 2013), http://hdl.handle.net/11250/246917.
[31] S. A. B. Ahmad, *Fermion Quantum Field Theory in Black Hole Spacetimes* (Faculty of the Virginia Polytechnic Institute and State University, Lay Nam Chang, Chair, 1997), https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.411.4072&rep=rep1&type=pdf.
[32] D. G. Boulware, Spin-$\frac{1}{2}$ quantum field theory in Schwarzschild space, Phys. Rev. D **12**, 350 (1975).
[33] S. Xu, X. k. Song, J. d. Shi, and L. Ye, Probing the quantum correlation and Bell non-locality for Dirac particles with Hawking effect in the background of Schwarzschild black hole, Phys. Lett. B **733**, 1 (2014).
[34] W. L. Bade and H. Jehle, An introduction to spinors, Rev. Mod. Phys. **25**, 714 (1953).
[35] Y. Lyu and Y.-X. Gui, Dirac equation and its solution around schwarzschildde sitter black hole, Int. J. Theor. Phys. **43**, 975 (2004).
[36] I. Shirai, M. Jimbo, and T. Kon, Dirac fields in non-equilibrium thermo field dynamics, J. Phys. Soc. Jpn. **58**, 2597 (1989).
[37] J. Li, M. Hong, and K. Lin, Dirac quasinormal modes in spherically symmetric regular black holes, Phys. Rev. D **88**, 064001 (2013).
[38] R. Sini, N. Varghese, and V. C. Kuriakose, Quasi normal modes of spherically symmetric black hole spacetimes with cosmic string in a Dirac field, arXiv:0802.0788.
[39] K. Jusufi, Dirac particles tunneling from black holes with topological defects, Gen. Relativ. Gravit. **48**, 105 (2016).
[40] Z. Y. Liu and J. R. Ren, Fermions tunnelling with quantum gravity correction, Commun. Theor. Phys. **62**, 819 (2014).
[41] S. Chen, B. Wang, and R. Su, Influence of Lorentz violation on Dirac quasinormal modes in the Schwarzschild black hole spacetime, Classical Quantum Gravity **23**, 7581 (2006).
[42] M. Saleh, B. T. Bouetou, and T. C. Kofane, Quasinormal modes of a quantum-corrected Schwarzschild black hole: gravitational and Dirac perturbations, Astrophys. Space Sci. **361**, 137 (2016).
[43] W. M. Jin, Scattering of massive Dirac fields on the Schwarzschild black hole space-time, Classical Quantum Gravity **15**, 3163 (1998).
[44] D. Ahn, Y. H. Moon, R. B. Mann, and I. Fuentes-Schuller, The black hole final state for the Dirac fields In Schwarzschild spacetime, J. High Energy Phys. 06 (2008) 062.
[45] M. Casals, S. R. Dolan, B. C. Nolan, A. C. Ottewill, and E. Winstanley, Quantization of fermions on Kerr space-time, Phys. Rev. D **87**, 064027 (2013).
[46] P. M. Alsing, I. Fuentes-Schuller, R. B. Mann, and T. E. Tessier, Entanglement of Dirac fields in non-inertial frames, Phys. Rev. A **74**, 032326 (2006).
[47] G. G. Nyambuya, New curved spacetime dirac equations, Found. Phys. **38**, 665 (2008).
[48] F. W. Hehl and B. K. Datta, Nonlinear spinor equation and asymmetric connection in general relativity, J. Math. Phys. (N.Y.) **12**, 1334 (1971).
[49] V. M. Villalba and U. Percoco, Separation of variables and exact solution to dirac and weyl equations in robertson-walker space-times, J. Math. Phys. (N.Y.) **31**, 715 (1990).
[50] G. F. T. del Castillo, Una introducción a los armónicos esféricos espinoriales, Rev. Mex. de Fis. **36**, 446 (1990), https://www.researchgate.net/profile/Gerardo-Torres-Del-Castillo/publication/268173198_An_introduction_to_spin-weighted_spherical_harmonics/links/5498be380cf2eeefc30f9cc3/An-introduction-to-spin-weighted-spherical-harmonics.pdf.
[51] G. T. del Castillo, Solución de la ecuación de dirac en términos de los armónicos esféricos espinoriales, Rev. Mex. de Fis. **38**, 162 (1992), https://rmf.smf.mx/ojs/index.php/rmf/article/view/2231.
[52] G. T. del Castillo, Espinores en espacios de dimensión arbitraria, Rev. Mex. de Fis. **35**, 123 (1989), https://rmf.smf.mx/ojs/index.php/rmf/article/view/2039.
[53] E. Ley-Koo and R. C. Wang, Dirac equation in ortogonal curvilinear coordinates, Rev. Mex. de Fis. **34**, 296 (1988), https://rmf.smf.mx/ojs/index.php/rmf/article/view/1988.
[54] M. D. Pollock, On the dirac equation in curved space-time, Acta Phys. Pol. B **41**, 1827 (2010), http://www.actaphys.uj.edu.pl/fulltext?series=Reg&vol=41&page=1827.
[55] C. J. Quimbay, R. A. Hernandez, and Y. F. Perez, Canonical quantization of the dirac oscillator field in $(1+1)$ and $(3+1)$ dimensions, Electron. J. Theor. Phys. **11**, 19 (2014), https://arxiv.org/abs/1201.3389.

[56] J. Vignatti and F. Villegas S, Dinámica del oscilador de Dirac, Rev. Invest. de Fís. **19**, 1 (2016), https://revistasinvestigacion.unmsm.edu.pe/index.php/fisica/article/view/13553.
[57] A. Davydov, *Quantum Mechanics*, International Series of Monographs in Natural Philosophy (Pergamon Press, New York, 1965), https://www.elsevier.com/books/quantum-mechanics/davydov/978-0-08-020438-3.
[58] W. Greiner, Relativistic quantum mechanics, *Theoretical Physics: Text and Exercise Books* (Springer, New York, 1990), https://www.springer.com/gp/book/9783642880827.
[59] B. D. Greiner and J. Reinhardt, *Field Quantization* (Springer, Berlin, Heidelberg, 2013), https://www.springer.com/gp/book/9783642614859.
[60] J. Bjorken and S. Drell, *Relativistic Quantum Fields*, International Series in Pure and Applied Physics (McGraw-Hill, New York, 1965), https://inspirehep.net/literature/873163.
[61] T. Lancaster and S. Blundell, *Quantum Field Theory for the Gifted Amateur* (Oxford University Press, Oxford, 2014), https://oxford.universitypressscholarship.com/view/10.1093/acprof:oso/9780199699322.001.0001/acprof-9780199699322.
[62] L. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1996), https://inspirehep.net/literature/220441.
[63] R. Klauber, *Student Friendly Quantum Field Theory: Basic Principles and Quantum Electrodynamics* (Sandtrove Press, 2013), https://books.google.com.co/books?id=FbpongEACAAJ&dq=Student+Friendly+Quantum+Field+Theory&hl=es-419&sa=X&redir=y.
[64] I. M. Ternov and A. B. Gaina, Energy spectrum of the dirac equation for the schwarzschild and kerr fields, Sov. Phys. J. **31**, 157 (1988).
[65] J. Wang, Q. Pan, and J. Jing, Projective measurements and generation of entangled Dirac particles in schwarzschild spacetime, Ann. Phys. (Amsterdam) **325**, 1190 (2010).
[66] J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Addison-Wesley Publishing Company, Inc, New York, 1970), https://www.worldcat.org/title/mathematical-methods-of-physics/oclc/90095.
[67] L. Susskind and J. Lindesay, *An introduction to black holes, information and the string*, Theory Revolution: The Holographic Universe (World Scientific, 2004), https://www.worldscientific.com/worldscibooks/10.1142/5689.
[68] S. Iyer and C. M. Will, Black-hole normal modes: A wkb approach. I. Foundations and application of a higher-order wkb analysis of potential-barrier scattering, Phys. Rev. D **35**, 3621 (1987).
[69] N. Birrell and P. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1984), https://inspirehep.net/literature/181166.
[70] W. B. Campbell, Tensor and spinor spherical harmonics and the spins harmonics, J. Math. Phys. (N.Y.) **12**, 1763 (1971).
[71] J. R. Arenas and S. Tesis, de Doctorado Termodinámica de entanglement del efecto Unruh, Departamento de Física. Universidad Nacional de Colombia. Director: J. M. Tejeiro S. (2005).
[72] J. R. Arenas and S. Agujeros Negros Cuánticos, Universidad Nacional de Colombia, Notas de Clase, Bogotá (2014).
[73] L. H. Ford, Quantum vacuum energy in general relativity, Phys. Rev. D **11**, 3370 (1975).
[74] L. H. Ford, Quantum vacuum energy in a closed universe, Phys. Rev. D **14**, 3304 (1976).
[75] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, General relativity with spin and torsion: Foundations and prospects, Rev. Mod. Phys. **48**, 393 (1976).
[76] I. J. R. Aitchison, Nothing's plenty. The vacuum in modern quantum field theory, Contemp. Phys. **50**, 261 (2009).
[77] W. A. Pulido and G. Tesis de Maestría, Dinámica de Campos Térmicos y Agujeros Negros, Observatorio Astronómico Nacional. Universidad Nacional de Colombia. Director: J. R. Arenas S. (2008).
[78] G. Ribeiro de Melo, Maestrado em Física: Dinâmica de Campos Térmicos e Estados Superposição., Universidade Federal Da Bahia. Instituto de Física (2004), https://repositorio.ufba.br/ri/handle/ri/10685.
[79] W. Paniago de Souza, Bosonic string at finite temperature, Ph.D thesis, Sao Paulo, IFT, 2002, https://arxiv.org/abs/hep-th/0208134.
[80] K. Soutome, Note on fermion propagator in thermo field dynamics, Z. Phys. C Part. Fields **40**, 479 (1988).
[81] E. Martin-Martinez, Relativistic quantum information: Developments in quantum information in general relativistic scenarios, Ph.D thesis, Waterloo University, 2011, https://arxiv.org/abs/1106.0280.
[82] A. Lasenby, C. Doran, J. Pritchard, A. Caceres, and S. Dolan, Bound states and decay times of fermions in a schwarzschild black hole background, Phys. Rev. D **72**, 105014 (2005).
[83] T. Ortín, *Gravity and Strings*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2004), https://www.cambridge.org/core/books/gravity-and-strings/03D4FD92AC7349B1C9161F53C8922AC3.
[84] D. N. Blaschke, F. Gieres, M. Reboud, and M. Schweda, The energy-momentum tensor(s) in classical gauge theories, Nucl. Phys. **B912**, 192 (2016).
[85] R. M. Wald, Trace anomaly of a conformally invariant quantum field in curved space-time, Phys. Rev. D **17**, 1477 (1978).
[86] C. Dappiaggi, T.-P. Hack, and N. Pinamonti, The extended algebra of observables for dirac fields and the trace anomaly of their stress-energy tensor, Rev. Math. Phys. **21**, 1241 (2009).
[87] J. Louko and V. Toussaint, Unruh-dewitt detector's response to fermions in flat spacetimes, Phys. Rev. D **94**, 064027 (2016).
[88] W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics*, Classical Theoretical Physics (Springer-Verlag, New York, 1995), https://www.springer.com/gp/book/9780387942995.
[89] J. Kapusta and C. Gale, *Finite-Temperature Field Theory: Principles and Applications*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2006), https://inspirehep.net/literature/738588.

[90] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics* (University of Chicago Press, 1994).
[91] W. A. Rojas C and J. Robel Arenas S, A conceptual model for the origin of the cutoff parameter in exotic compact objects, Symmetry **12**, 2072 (2020).
[92] W. A. Rojas C and J. Robel Arenas S, Relativistic origin of the cutoff parameter in exotic compact objects, Gravitation Cosmol. **27**, 136 (2021).
[93] Y. Z. Chen, W. D. Li, and W. S. Dai, Why the entropy of spacetime is independent of species of particles—the species problem, Eur. Phys. J. C **78**, 635 (2018).
[94] C. Rovelli, Notes for a brief history of quantum gravity, arXiv:gr-qc/0006061.
[95] S. C. Ulhoa, E. P. Spaniol, R. Gomes, A. F. Santos, and A. E. Santana, On regular black holes at finite temperature, Adv. High Energy Phys. **2020**, 1 (2020).
[96] https://repositorio.unal.edu.co/handle/unal/63982.
[97] H. Arslan, Dirac equation and hydrogen atom, M.Sc thesis, Departament of Physics, Institute of Natural and Applied Sciences, University of Cukurova, 1998.
[98] C. J. Smith, An Absolute quantum energy inequality for the Dirac field in curved spacetime, Classical Quantum Gravity **24**, 4733 (2007).
[99] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, England, 2004), https://www.cambridge.org/core/books/relativists-toolkit/DA7ED68B971708A0F782257F948981E7.
[100] V. E. Ambruş and E. Winstanley, Fermions on adS, Springer Proc. Phys. **170**, 331 (2016).
[101] M. Spiegel and J. Liu, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline of Theory and Problems (McGraw-Hill, New York, 1999).