No-short scalar hair theorem for spinning acoustic black holes in a photon-fluid model

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It has recently been revealed that spinning black holes of the photon-fluid model can support acoustic 'clouds'—stationary density fluctuations whose spatially regular radial eigenfunctions are determined by the $(2+1)$-dimensional Klein-Gordon equation of an effective massive scalar field. Motivated by this intriguing observation, we use analytical techniques in order to prove a no-short hair theorem for the composed acoustic black hole scalar-clouds configurations. In particular, it is proved that the effective lengths of the stationary bound-state corotating acoustic scalar clouds are bounded from below by the series of inequalities $r_{\text{hair}} > \frac{1+\sqrt{5}}{2}\cdot r_{\text{H}} > r_{\text{null}}$, where r_{H} and r_{null} are respectively the horizon radius of the supporting black hole and the radius of the corotating null circular geodesic that characterizes the acoustic spinning black hole spacetime.

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I. INTRODUCTION

Early mathematical studies of the Einstein-scalar field equations [1–4], which were motivated by the influential no-hair conjecture [5,6], have revealed the interesting fact that asymptotically flat black holes with regular horizons cannot support, in their exterior regions, static matter configurations which are made of minimally coupled scalar fields.

However, subsequent analyses (see [7–18] and references therein) of the Einstein-matter field equations have explicitly demonstrated that black hole spacetimes may not be as simple as suggested by the original no-hair conjecture [5,6]. In particular, it is by now well established in the physics literature [7–18] that spherically symmetric asymptotically flat black holes can support various types of hairy matter configurations; static fields which are well behaved on and outside the black hole horizon.

In addition, it has been proven analytically [19] that the superradiant scattering phenomenon of bosonic fields in spinning black hole spacetimes [20,21] allows nonstatic Kerr black holes to support stationary bound-state matter configurations which are made of minimally coupled linearized massive scalar fields. These externally supported scalar field configurations, which corotate with the central spinning black hole, have received the nickname 'scalar clouds' in the linearized regime [19,22]. Using sophisticated numerical techniques, the existence of genuine hairy (scalarized) spinning black hole solutions of the nonlinearly coupled Einstein-scalar field equations has been explicitly demonstrated in [22].

Interestingly, the stationary corotating externally supported bosonic field configurations are characterized by proper frequencies which are in resonance with the horizon angular velocity of the central supporting black hole [19,22–24],

$$\omega = m\Omega_{\text{H}}. \tag{1}$$

In addition, the proper frequencies of the supported bound-state field configurations are bounded from above by the proper mass of the supported scalar field [25]

$$\omega^2 < \mu^2. \tag{2}$$

Given the intriguing fact that hairy black hole solutions of the Einstein-matter field equations do exist, one may raise the following interesting question: How short can a black hole hair be?

For static spherically-symmetric hairy black hole spacetimes, the answer to this question has been provided in [26], where it was proven that the effective lengths of spatially regular hairy matter configurations whose energy-momentum trace is nonpositive must extend beyond the innermost null circular geodesics of the corresponding curved black-hole spacetimes:

$$r_{\text{hair}} > r_{\text{null}}. \tag{3}$$

As explicitly proven in [27], the effective lengths of the corotating nonspherically-symmetric scalar cloudy configurations of the spinning Kerr spacetime [19,22] also conform to the lower bound (3).

Interestingly, it is well established that fluid systems share many features with curved black hole spacetimes (see [28–43] and references therein). In particular, it has recently been proven in the important work [28] that acoustic black holes of the $(2+1)$-dimensional rotating photon-fluid system can support stationary bound-state density fluctuations (acoustic scalar 'clouds') whose spatiotemporal behavior in the black hole spacetime is governed by the linearized Klein-Gordon equation of an effective massive scalar field.

As nicely emphasized in [28], the corotating acoustic scalar clouds of the photon-fluid model, like the more familiar scalar hairy configurations of the Kerr black hole spacetime [19,22], owe their existence to the intriguing phenomenon of superradiant scattering of corotating bosonic field modes in the spinning physical system. In particular, the $(2+1)$-dimensional stationary acoustic clouds revealed in [28] are characterized by the same resonance condition [see Eq. (1)] as the Kerr scalar clouds [44].

The main goal of the present paper is to analyze the spatial functional behavior of the stationary bound-state acoustic scalar field configurations (linearized scalar clouds) that are supported by the effective spinning black-hole spacetime of the photon-fluid model [28]. In particular, motivated by the existence of the lower bound (3) on the effective lengths of hairy matter configurations in the black-hole spacetime solutions of the Einstein field equations, we shall use analytical techniques in order to derive an analogous generic lower bound on the effective lengths of the composed acoustic black hole scalar field cloudy configurations of the interesting photon-fluid model.

II. DESCRIPTION OF THE SYSTEM

The spinning $(2+1)$-dimensional acoustic black hole spacetime of the photon-fluid model is characterized by the curved line element [28]

$$ds^2 = -\left(1-\frac{r_{\rm H}}{r}-\frac{\Omega^2_{\rm H}r^4_{\rm H}}{r^2}\right)dt^2+\left(1-\frac{r_{\rm H}}{r}\right)^{-1}dr^2 - 2\Omega_{\rm H}r^2_{\rm H}d\theta dt + r^2d\theta^2, \tag{4}$$

where $\{r,\theta\}$ are the familiar polar coordinates in a two-dimensional plane and the physical parameters $\{r_{\rm H},\Omega_{\rm H}\}$ are respectively the horizon radius [45] and the angular velocity of the spinning acoustic horizon. The acoustic black hole spacetime (4), like the spinning Kerr black hole spacetime, possesses an ergoregion whose outer radial location [28]

$$r_{\rm E}=\frac{1}{2}r_{\rm H}\left(1+\sqrt{1+4\Omega^2_{\rm H}r^2_{\rm H}}\right), \tag{5}$$

is determined by the root of the metric function g_{tt}.

As explicitly shown in [28,32], long-wavelength excitations (phonons) of the photon-fluid system behave as effective massive scalar fields that propagate in the acoustic curved spacetime (4). In particular, given a linearized acoustic density fluctuation [46]

$$\rho(t,r,\theta)=\frac{\psi(r)}{\sqrt{r}}e^{im\theta-i\Omega t} \tag{6}$$

of the photon-fluid model, it has been proven that its spatiotemporal behavior is determined by the $(2+1)$-dimensional Klein-Gordon differential equation [28,32]

$$\left[\Delta\frac{d}{dr}\left(\Delta\frac{d}{dr}\right)-V(r;\Omega)\right]\psi(r)=0;\qquad \Delta\equiv 1-\frac{r_{\rm H}}{r}. \tag{7}$$

The radial potential [28]

$$V(r;\Omega)=-\left(\Omega-\frac{m\Omega_{\rm H}r^2_{\rm H}}{r^2}\right)^2+\Delta\left(\Omega^2_0+\frac{m^2}{r^2}+\frac{r_{\rm H}}{2r^3}-\frac{\Delta}{4r^2}\right), \tag{8}$$

which determines the spatial behavior of the density fluctuations (6) in the acoustic curved spacetime (4), corresponds to an effective scalar field ψ of mass Ω_0 [47,48].

In the present paper we shall use analytical techniques in order to analyze the spatial behavior of the composed acoustic black hole stationary linearized massive scalar field configurations of the photon-fluid system. The stationary bound-state scalar clouds of the spinning acoustic spacetime (4) are characterized by the resonant frequency [49]

$$\Omega = m\Omega_{\rm H}. \tag{9}$$

In addition, the scalar eigenfunctions of the supported acoustic clouds are assumed to be regular at the acoustic black-hole horizon [28]

$$\psi(r=r_{\rm H})<\infty. \tag{10}$$

The bound-state acoustic scalar eigenfunctions are also assumed to be normalizable (decay exponentially) at spatial infinity [28]

$$\psi(r\to\infty)\sim e^{-\sqrt{\Omega^2_0-\Omega^2}r}, \tag{11}$$

for [50]

$$\Omega^2 < \Omega_0^2. \tag{12}$$

As demonstrated numerically in [28] and proved analytically in [43], the stationary bound-state acoustic black hole massive scalar field cloudy configurations of the photon-fluid model, which respect the boundary conditions (10) and (11), are characterized by the dimensionless regime of existence

$$\frac{\Omega_0}{m\Omega_H} \in \left(1, \sqrt{\frac{32}{27}}\right). \tag{13}$$

In the next section we shall reveal, using analytical techniques, the existence of a generic lower bound on the effective radial lengths of the supported corotating acoustic scalar clouds of the photon-fluid model.

III. LOWER BOUND ON THE EFFECTIVE RADIAL LENGTHS OF THE STATIONARY BOUND-STATE ACOUSTIC SCALAR CLOUDS OF THE PHOTON-FLUID MODEL

In the present section we shall explore the spatial functional behavior of the scalar eigenfunctions $\psi(r; r_H, \Omega_H, \Omega_0, m)$ which characterize the linearized massive scalar field configurations (stationary scalar clouds) that are supported by the $(2+1)$-dimensional acoustic black hole spacetime (4) of the photon-fluid model [28]. In particular, we shall explicitly prove that the stationary bound-state acoustic scalar clouds cannot be arbitrarily compact.

To this end, we shall first prove that the bound-state scalar clouds of the photon-fluid model are characterized by a nonmonotonic radial eigenfunction $\psi(r)$. We shall then derive, using the explicit functional behavior of the effective radial potential (8), a generic (parameter-independent) lower bound [see Eq. (24) below] on the peak location r_{max} of the radial scalar eigenfunctions that characterize the supported corotating acoustic scalar clouds of the photon-fluid model.

Before proceeding, we would like to emphasize that the interesting lower bound

$$\frac{r_{min}}{r_H} > \frac{3}{2(\Omega_H r_H)^2}, \tag{14}$$

on the radial location of the minimum $r = r_{min}$ of the effective potential (8) [51] has been derived in the important work [28]. The lower bound (14) of [28] nicely demonstrates the important fact that, in the slow rotation $\Omega_H \to 0$ limit of the central supporting acoustic black holes, the scalar clouds are effectively located far away from the central black hole. However, it should be realized that the interesting bound (14), which is based on the asymptotic large-r expansion of the effective radial potential (8) [see [28] for details], is unable to describe the genuine near-horizon radial behavior of the stationary scalar clouds in the regime $\Omega_H r_H \gg 1$ of rapidly-spinning central supporting acoustic black holes [52]. In particular, one finds that the right-hand side of (14) is less than 1 for $\Omega_H r_H \gtrsim 1$, thus suggesting that the bound (14) works well for slowly-rotating black holes (for which $r_{min} \gg r_H$) but breaks down for rapidly-spinning acoustic black holes.

In the present section we shall use the exact functional form of the composed acoustic black hole massive scalar field binding potential (8) in order to derive an alternative lower bound on the peak location r_{max} of the radial eigenfunctions $\psi(r; r_H, \Omega_H, \Omega_0, m)$ that characterize the bound-state linearized scalar clouds of the photon-fluid model. In particular, we shall explicitly prove below that, for all values of the dimensionless rotation parameter $\Omega_H r_H$ of the central supporting acoustic black hole, the peak location $r = r_{max}$ of the acoustic scalar configurations cannot be located arbitrarily close to the black hole horizon.

The radial functional behavior of the stationary bound-state cloudy field configurations of the photon-fluid system is determined by the ordinary differential equation (7) with the effective binding potential [see Eqs. (8) and (9)]

$$V(r) = -(m\Omega_H)^2 \cdot \left(1 - \frac{r_H^2}{r^2}\right)^2 + \Delta\left(\Omega_0^2 + \frac{m^2}{r^2} + \frac{r_H}{2r^3} - \frac{\Delta}{4r^2}\right). \tag{15}$$

We first point out that in the near-horizon region,

$$x \equiv \frac{r - r_H}{r_H} \ll 1, \tag{16}$$

the effective radial potential (15) of the composed acoustic black hole stationary bound state massive scalar field configurations is characterized by the functional behavior

$$V(x \ll 1) = \left(\Omega_0^2 + \frac{m^2 + \frac{1}{2}}{r_H^2}\right) \cdot x + O(x^2/r_H^2), \tag{17}$$

which implies

$$V(x) > 0 \quad \text{for } 0 < x \ll 1. \tag{18}$$

We shall now consider two mathematically distinct cases for the possible near-horizon functional behaviors of the scalar eigenfunction $\psi(r)$.

Case (i): If $\psi(r = r_H) = 0$, then the asymptotic boundary condition (11), which characterizes the radial behavior of the bound-state cloudy scalar configurations at spatial infinity, implies that the scalar eigenfunction $\psi(r)$ must have an extremum point $r = r_{max}$ [53] in the exterior region of the effective black hole spacetime.

Case (ii): If $\psi(r \to r_H) \neq 0$ and $[d\psi(r)/dr]_{r=r_H} \neq 0$ [54], then one deduces from the radial differential equation (7) and the near-horizon functional behavior (17) of the effective radial potential that $\psi(r) \cdot d\psi(r)/dr > 0$ for

$r \to r_H$. This observation together with the asymptotic boundary condition (11) imply again that the scalar eigenfunction $\psi(r)$, which characterizes the spatial behavior of the acoustic scalar clouds, must have an extremum point $r = r_{max}$ in the exterior region of the spinning acoustic black hole spacetime.

We therefore conclude that the stationary bound-state scalar clouds of the photon-fluid model are characterized by nonmonotonic radial eigenfunctions. In particular, the acoustic scalar eigenfunction $\psi(r)$ is characterized by the presence of an extremum radial point $r = r_{max}$ in the exterior region of the acoustic black hole spacetime with the properties

$$\left\{\psi \neq 0; \frac{d\psi}{dr} = 0; \psi\frac{d^2\psi}{dr^2} < 0\right\} \quad \text{for } r = r_{max}. \tag{19}$$

Substituting the characteristic functional relations (19) into the radial differential equation (7), one finds the simple relation

$$V(r = r_{max}) < 0. \tag{20}$$

Taking cognizance of Eqs. (15) and (20), one finds the characteristic series of inequalities [55]

$$(m\Omega_H)^2 \cdot \left(1 - \frac{r_H^2}{r^2}\right)^2 > \Delta\left(\Omega_0^2 + \frac{m^2}{r^2} + \frac{r_H}{2r^3} - \frac{\Delta}{4r^2}\right) > \Delta \cdot \Omega_0^2 \quad \text{for } r = r_{max}, \tag{21}$$

which implies [see Eq. (7)]

$$\left(1 - \frac{r_H}{r_{max}}\right)\left(1 + \frac{r_H}{r_{max}}\right)^2 > \left(\frac{\Omega_0}{m\Omega_H}\right)^2. \tag{22}$$

From the analytically derived cubic inequality (22) one obtains the dimensionless lower bound

$$\frac{r_{max}}{r_H} > F\left(\frac{\Omega_0}{m\Omega_H}\right) \tag{23}$$

on the location $r = r_{max}$ of the radial peak of the acoustic scalar eigenfunctions, where the (mathematically cumbersome) dimensionless function $F = F(\frac{\Omega_0}{m\Omega_H})$ is a monotonically increasing function in the regime of existence $\Omega_0/m\Omega_H \in (1, \sqrt{32/27})$ [see Eq. (13)] of the composed acoustic black hole stationary bound state massive scalar field configurations. In particular, from (22) one directly finds that the function $F(\frac{\Omega_0}{m\Omega_H})$ in the lower bound (23) increases from $F(\frac{\Omega_0}{m\Omega_H} \to 1^+) \to [(1+\sqrt{5})/2]^+$ to $F(\frac{\Omega_0}{m\Omega_H} \to \sqrt{32/27^-}) \to 3^-$. We therefore find the generic (that is, rotation-independent) lower bound

$$\frac{r_{max}}{r_H} > \frac{1+\sqrt{5}}{2} \tag{24}$$

on the effective radial lengths of the stationary bound-state acoustic scalar clouds which are supported by the spinning black hole spacetime (4).

It is interesting to emphasize the fact that the lower bound (24) on the effective lengths of the corotating acoustic scalar clouds is universal in the sense that it does not depend on the physical parameters (proper mass Ω_0 and azimuthal harmonic index m) of the supported acoustic scalar field.

IV. COROTATING ACOUSTIC SCALAR CLOUDS AND NULL CIRCULAR GEODESICS

In the present section we shall explicitly prove that the composed acoustic black hole stationary bound-state linearized massive scalar field configurations of the photon-fluid model, like the scalarized spinning black hole solutions of the Einstein field equations, conform to the no-short hair relation (3). In particular, as we shall now show, the radial peak location r_{max}, which characterizes the nonmonotonic eigenfunctions $\psi(r)$ of the bound-state acoustic scalar clouds, is located beyond the corotating null circular geodesic of the effective spinning black hole spacetime (4).

A remarkably economic way to determine the radial location of the corotating null circular geodesic of a curved black hole spacetime has been revealed in [56]. In particular, it has been proven in [56,57] that the corotating null circular geodesic provides the fastest way, as measured by asymptotic observers, to circle the central black hole. Substituting $ds = dr = 0$ and $d\theta = 2\pi$ into the curved line element (4), one finds the functional expression

$$\frac{T(r)}{r_H} = 2\pi\Omega_H r_H \cdot \frac{\sqrt{1 + \frac{r^2}{\Omega_H^2 r_H^4} \cdot \left(1 - \frac{r_H}{r} - \frac{\Omega_H^2 r_H^4}{r^2}\right)} - 1}{1 - \frac{r_H}{r} - \frac{\Omega_H^2 r_H^4}{r^2}} \tag{25}$$

for the (radius-dependent) dimensionless orbital period of light-like test particles around the central black hole [58].

As explicitly shown in [56,57], the corotating null circular geodesics of curved black hole spacetimes are characterized by the relation

$$\frac{dT(r)}{dr} = 0 \quad \text{for } r = r_{null}. \tag{26}$$

Substituting (25) into Eq. (26), one obtains the characteristic equation

$$\frac{r^2(r-r_{\rm H})(2r-3r_{\rm H})+r(5r_{\rm H}-6r)\Omega_{\rm H}^2r_{\rm H}^4+2\Omega_{\rm H}r_{\rm H}^3\sqrt{r(r-r_{\rm H})}(r+2\Omega_{\rm H}^2r_{\rm H}^3)}{\left(1-\frac{r_{\rm H}}{r}-\frac{\Omega_{\rm H}^2r_{\rm H}^4}{r^2}\right)^2}=0 \quad \text{for } r=r_{\rm null} \tag{27}$$

for the radial location $r = r_{\rm null}$ of the corotating null circular geodesic of the acoustic spinning black hole spacetime (4).

From Eq. (27) one finds that the $\Omega_{\rm H}r_{\rm H}$-dependent radial location $r_{\rm null} = r_{\rm null}(\Omega_{\rm H}r_{\rm H})$ of the corotating null circular geodesic is restricted to the interval

$$\frac{r_{\rm null}}{r_{\rm H}} \in \left(1, \frac{3}{2}\right]. \tag{28}$$

In particular, from (27) one finds that $r_{\rm null}(\Omega_{\rm H}r_{\rm H})$ is a monotonically decreasing function of the dimensionless black hole rotation parameter $\Omega_{\rm H}r_{\rm H}$ with the simple asymptotic behaviors

$$\frac{r_{\rm null}}{r_{\rm H}} = \frac{3}{2} - \frac{2}{\sqrt{3}} \cdot \Omega_{\rm H}r_{\rm H} + O[(\Omega_{\rm H}r_{\rm H})^2] \quad \text{for } \Omega_{\rm H}r_{\rm H} \ll 1, \tag{29}$$

and

$$\frac{r_{\rm null}}{r_{\rm H}} = 1 + \frac{1}{16(\Omega_{\rm H}r_{\rm H})^2} + O[(\Omega_{\rm H}r_{\rm H})^{-3}] \quad \text{for } \Omega_{\rm H}r_{\rm H} \gg 1. \tag{30}$$

Taking cognizance of Eqs. (24) and (28), one concludes that the stationary bound-state scalar clouds of the spinning acoustic black hole spacetime (4) are characterized by the lower bound

$$r_{\rm max} > r_{\rm null}. \tag{31}$$

V. SUMMARY

A decade ago it was proven that spinning Kerr black holes can support corotating scalar clouds, stationary bound-state linearized configurations of spatially regular massive scalar fields whose orbital frequencies are in resonance with the angular velocity $\Omega_{\rm H}$ of the black-hole horizon [19,22]. The bound-state scalar configurations are known to be characterized by the no-short hair property [26,27], according to which their effective lengths extend beyond the null circular geodesics of the supporting black-hole spacetimes.

Intriguingly, it has recently been revealed in the important work [28] that an analogous physical phenomenon occurs in a rotating photon-fluid model [28]. In particular, it has been demonstrated [28] that in the presence of vortex flows, the photon-fluid system may be described by an effective rotating acoustic black hole spacetime [see Eq. (4)] which, like the spinning Kerr black hole spacetime, may support stationary linearized density fluctuations (acoustic scalar clouds) whose spatial behavior is governed by the Klein-Gordon equation of a $(2+1)$-dimensional scalar field with an effective proper mass Ω_0.

The main goal of the present paper was to analyze the spatial behavior of the corotating acoustic scalar clouds that are supported by the $(2+1)$-dimensional acoustic black hole (4) of the photon-fluid model. Interestingly, we have established the fact that the supported acoustic scalar configurations cannot be made arbitrarily compact. In particular, using analytical techniques, we have derived a generic lower bound on the effective lengths of the bound-state acoustic black hole scalar field cloudy configurations of the photon-fluid model. This parameter-independent bound can be expressed in a remarkably compact way by the dimensionless series of inequalities [see Eqs. (24) and (31)]

$$r_{\rm max} > \frac{1+\sqrt{5}}{2} \cdot r_{\rm H} > r_{\rm null}, \tag{32}$$

where $\{r_{\rm H}, r_{\rm null}\}$ are respectively the horizon radius and the radius of the corotating null circular geodesic that characterize the supporting acoustic black hole spacetime.

Finally, it is worth emphasizing the interesting fact that the analytically derived lower bound (32) on the effective lengths of the bound-state acoustic scalar clouds of the photon-fluid model is universal in the sense that it is valid for all possible sets $\{r_{\rm H}, \Omega_{\rm H}, \Omega_0, m\}$ of the physical parameters that characterize the supporting spinning acoustic black hole and the effective massive scalar fields.

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[44] For the co-rotating acoustic scalar clouds discussed in [28], the physical parameter $\Omega_{\rm H}$ in the resonance condition is the horizon angular velocity of the central supporting $(2+1)$-dimensional acoustic black hole.
[45] The radius $r = r_{\rm H}$ of the acoustic black-hole horizon is determined by the physical requirement $v_{\rm r} = c_{\rm s}$, where $\{v_{\rm r}, c_{\rm s}\}$ are respectively the inward radial velocity of the fluid flow and the speed of sound in the fluid [28]. We shall use natural units in which $c_{\rm s} \equiv 1$.
[46] Note that the circular periodicity of the azimuthal eigenfunction $e^{im\theta}$ implies that the discrete field parameter m is an integer.
[47] As discussed in [28], the effective scalar mass Ω_0 in the composed acoustic-black-hole-scalar-field curvature

potential (8) is the rest energy of the collective excitations (phonons) [28].
[48] Note that the flat-space $r \to \infty$ limit of the effective potential (8) is given by the simple functional expression $V(r \to \infty) = -\Omega^2 + \Omega_0^2$, and it therefore describes an effective scalar field of proper frequency Ω and an effective proper mass Ω_0.
[49] See (1) for the analogous resonance relation in spinning black-hole spacetimes.
[50] See the analogous upper bound (2) for the stationary supported field configurations in asymptotically flat black-hole spacetimes.
[51] As emphasized in [28], the radial eigenfunction of the fundamental scalar mode attains its peak location in correspondence with the minimum of the effective binding potential (8) of the photon-fluid model.
[52] It is well known that the angular velocity of spinning Kerr black holes is characterized by the dimensionless upper bound $\Omega_H r_H \le 1/2$. On the other hand, as discussed in [28], there is no fundamental upper bound on the angular velocity of the $(2+1)$-dimensional acoustic black hole (4) of the photon-fluid model.
[53] Note that one can assume $\psi(r = r_H) \ge 0$ for the radial scalar eigenfuntion without loss of generality.
[54] Note that if $d\psi(r)/dr \to 0$ and $d^2\psi(r)/dr^2 < \infty$ for $r \to r_H$, then one finds from Eqs. (7) and (17) the near-horizon relation $\psi(r \to r_H) \to 0$. This observation together with the asymptotic boundary condition (11) imply again that the scalar eigenfunction must have an extremum point $r = r_{max}$ in the exterior region of the acoustic black-hole spacetime.
[55] Note that in the last inequality of (21) we have used the relations $m^2 \ge 1 > 1/4 > \Delta/4$ for $m \ge 1$.
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[58] Our goal here is to identify the unique circular trajectory that minimizes the orbital period of test particles around the central $(2+1)$-dimensional acoustic black hole (4) as measured by asymptotic observers. We therefore assume the relation $v/c \to 1^-$ for the tangential velocity of the orbiting test particle.