Four-dimensional spinfoam quantum gravity with a cosmological constant: Finiteness and semiclassical limit

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We present an improved formulation of 4-dimensional Lorentzian spinfoam quantum gravity with a cosmological constant. The construction of spinfoam amplitudes uses the state-integral model of $\mathrm{PSL}(2,\mathbb{C})$ Chern-Simons theory and the implementation of a simplicity constraint. The formulation has two key features: (1) spinfoam amplitudes are all finite, and (2) with suitable boundary data, the semiclassical asymptotics of the vertex amplitude has two oscillatory terms, with phase plus or minus the 4-dimensional Lorentzian Regge action with a cosmological constant for the constant curvature 4-simplex.

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I. INTRODUCTION

Spinfoam quantum gravity is the covariant formulation of loop quantum gravity (LQG) in 4 spacetime dimensions [1,2]. There are two motivations to include the cosmological constant Λ in the spinfoam quantum gravity: First, spinfoam models without Λ are well known to have the infrared divergence (see e.g., [3–5]); then, Λ is expected to provide a natural infrared cutoff to make spinfoam amplitudes finite. Second, the simplest consistent explanation for the cosmological accelerating expansion is a positive Λ, so quantum gravity should reproduce Λ in the semiclassical regime. Based on these motivations, a satisfactory spinfoam quantum gravity with Λ is expected to (1) define finite spinfoam amplitudes and (2) consistently recover classical gravity with Λ in the semiclassical limit. This work covers both positive and negative Λ.

The semiclassical limit of LQG scales the Planck length $\ell_P \to 0$ while keeping the geometrical area $\mathfrak{a}$ fixed. Using the LQG area spectrum $\mathfrak{a} = \gamma \ell_P^2 \sqrt{j(j+1)}$, the semiclassical limit implies the SU(2) spin $j \to \infty$. We do not scale the Barbero-Immirzi parameter γ. In the presence of Λ, we require, in addition, that Λ should not scale in the semiclassical limit; then in 4d, the dimensionless quantity $k \propto (|\Lambda|\ell_P^2)^{-1}$ scales as $k \to \infty$ in addition to $j \to \infty$, whereas $j/k \propto |\Lambda|\mathfrak{a}$ is fixed. This suggests that the semiclassical limit of spinfoam quantum gravity with Λ should be a double-scaling limit, i.e., $j, k \to \infty$ while fixing j/k. In our following discussion, k becomes the integer Chern-Simons (CS) level.

In 3 dimensions, the Turaev-Viro (TV) model [6] with quantum group $\mathrm{SU}(2)_\mathfrak{q}$ ($\mathfrak{q} = e^{\pi i/k}, k \in \mathbb{Z}$) is the spinfoam quantum gravity with Λ that satisfies both expectations (1) and (2): It gives finite amplitudes due to the cutoff of spins given by $\mathrm{SU}(2)_\mathfrak{q}$; the vertex amplitude, the 6j symbol of $\mathrm{SU}(2)_\mathfrak{q}$, recovers the Regge action of 3d gravity with $\Lambda > 0$ in the semiclassical limit [7].[1]

In contrast, a 4d spinfoam quantum gravity with Λ has not yet been achieved that satisfies both expectations (1) and (2) in the literature. There are 4d spinfoam models based on the quantum Lorentz group, as generalizations from the 3d quantum group TV model [8–10] (see also e.g., [11,12] for the LQG kinematics with a quantum group). These models produce finite spinfoam amplitudes due to the spin cutoff from the quantum group. But it is difficult to examine the semiclassical limits of these models due to the complexity of their vertex amplitudes in terms of quantum group symbols. More recently, a more promising spinfoam model was found based on the $\mathrm{SL}(2,\mathbb{C})$ CS theory instead of the quantum group [13]. The vertex amplitude A_v^0 of this model is defined to be the CS evaluation of the projective $\mathrm{SL}(2,\mathbb{C})$ spin-network function Ψ_{Γ_5} based on the Γ_5 graph embedded in S^3 (see Fig. 1):

$$A_v^0 := \int D\mathcal{A}D\bar{\mathcal{A}} e^{-iS_{\mathrm{CS}}(\mathcal{A},\bar{\mathcal{A}})} \Psi_{\Gamma_5}(\mathcal{A},\bar{\mathcal{A}}), \tag{1}$$

where S_{CS} is the unitary $\mathrm{SL}(2,\mathbb{C})$ CS action with the complex level $t = k + \sigma$ ($k \in \mathbb{Z}_+$, $\sigma \in i\mathbb{R}$) that unifies Λ and γ by $k = \mathrm{Re}(t) = \frac{12\pi}{|\Lambda|\ell_P^2\gamma}$, $\sigma = i\mathrm{Im}(t) = ik\gamma$,

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[1]The semiclassical limit in 3d is the same double-scaling limit since $\mathfrak{a} \propto \ell_P \sqrt{j(j+1)}$ becomes the length and $k^2 \propto (\Lambda \ell_P^2)^{-1}$.

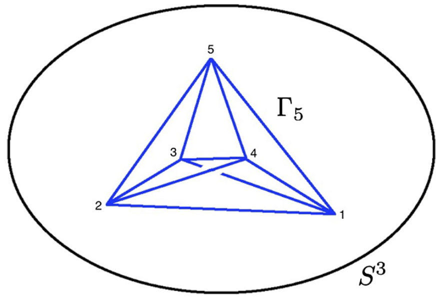

FIG. 1. Γ_5 graph embedded in S^3.

$$S_{\rm CS} = \frac{t}{8\pi}\int_{S^3}\mathrm{Tr}\left(\mathcal{A}\wedge \mathrm{d}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{\bar{t}}{8\pi}\int_{S^3}\mathrm{Tr}\left(\bar{\mathcal{A}}\wedge \mathrm{d}\bar{\mathcal{A}}+\frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\right). \quad (2)$$

Here, Ψ_{Γ_5} reduces to the EPRL vertex amplitude [14] when $\mathcal{A},\bar{\mathcal{A}}\to 0$. The derivation of the model (1) from the BF_Λ theory is given in [13] and is reviewed briefly around Eq. (3).

In the semiclassical limit ($j,k\to\infty$, $\sigma = ik\gamma\to i\infty$, keeping j/k fixed), and with a suitable boundary condition, A_v^0 reproduces the constant curvature 4-simplex geometry and gives the asymptotics as two oscillatory terms, with phase plus or minus the Regge action of 4d Lorentzian gravity with Λ. The sign of Λ is not fixed *a priori*, but rather it emerges semiclassically and dynamically from equations of motion and boundary data, as shown in the asymptotic analysis in [13].[2] However, the drawback of A_v^0 is that the formal path integral in (1) is not mathematically well defined, which makes the finiteness of the spinfoam amplitude obscure.

In this work, we present an improved formulation of 4d spinfoam quantum gravity with cosmological constant Λ, which gives *both* finite spinfoam amplitudes *and* the correct semiclassical behavior. We construct a new vertex amplitude A_v, which replaces the formal CS path integral in A_v^0 by a finite sum and finite-dimensional integral, based on the recent state-integral model of complex CS theory [15–17]. The resulting A_v is a bounded function of boundary data. The spinfoam amplitude made by A_v is finite on any triangulation. On the other hand, we are able to apply the stationary phase analysis to the finite-dimensional integral to show that A_v indeed reproduces the constant curvature 4-simplex and the 4d Lorentzian Regge action with Λ (positive or negative) in the semiclassical limit.

[2]First, the sign of Λ of boundary tetrahedra is determined by the boundary data, and then the critical equations from the stationary phase analysis cause the sign of Λ to propagate between tetrahedra and 4-simplices. The critical equations have no solution if the boundary tetrahedra fail to have a common sign of Λ; then, the spinfoam amplitude is suppressed in the semiclassical regime.

The new vertex amplitude A_v is closely related to the partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$ of the $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ CS theory on $S^3\backslash\Gamma_5$, which is the complement of an open tubular neighborhood of the Γ_5 graph in S^3. Here, $\Gamma_5\subset S^3$ is dual to the triangulation of S^3 given by the 4-simplex's boundary. This duality delivers flat connections of the CS theory to decorate the 4-simplex. We adopt the method proposed in [16] to explicitly construct $\mathcal{Z}_{S^3\backslash\Gamma_5}$ as a state-integral model with a finite sum and finite-dimensional integral (see Sec. II). Here, $\mathcal{Z}_{S^3\backslash\Gamma_5}$ quantizes the moduli space $\mathcal{L}_{S^3\backslash\Gamma_5}$ of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $S^3\backslash\Gamma_5$ and is a wave function of flat connection data on the boundary of $S^3\backslash\Gamma_5$. Given a manifold M, the moduli space of the flat connection with structure group G is the space of G connections modulo gauge transformations with vanishing curvature, equivalent to the character variety of representations of $\pi_1(M)$ in G modulo conjugation [18].

The new vertex amplitude A_v contains only finite sums and finite-dimensional integrals and thus improves the earlier formulation (1). It is also different from the state-integral model obtained in [19], which mainly focuses on the holomorphic block of CS and does not specify the integration cycle.[3] Note that A_v has both holomorphic and antiholomorphic parts of the CS theory. As a key to prove the finiteness, the integration cycle is specified in A_v.

By the construction in [16], the state-integral model converges absolutely if the underlying 3-manifold admits a "positive angle structure." Our construction of $\mathcal{Z}_{S^3\backslash\Gamma_5}$ manifests that $S^3\backslash\Gamma_5$ indeed admits a positive angle structure $(\vec{\alpha},\vec{\beta})\in\mathfrak{P}_{\rm new}$, where $\mathfrak{P}_{\rm new}$ is a 30-dimensional open convex polytope. The finiteness of $\mathcal{Z}_{S^3\backslash\Gamma_5}$ is a prerequisite for the finiteness of A_v and spinfoam amplitudes on triangulations.

The simplicity constraint needs to be imposed in order to define A_v: The derivation of (1) in [13] starts from the Holst-BF_Λ theory on a 4-ball $\mathcal{B}_4$, which is topologically identical to a 4-simplex,

$$S_{\text{H-BF}_\Lambda} = -\frac{1}{2}\int_{\mathcal{B}_4}\mathrm{Tr}\left[\left(\star+\frac{1}{\gamma}\right)B\wedge\mathcal{F}(\mathcal{A})\right] - \frac{|\Lambda|}{12}\int_{\mathcal{B}_4}\mathrm{Tr}\left[\left(\star+\frac{1}{\gamma}\right)B\wedge B\right]. \quad (3)$$

Considering the formal path integral of $S_{\text{H-BF}_\Lambda}$, integrating out the $so(1,3)$-valued 2-form B gives the action $\frac{3i}{4|\Lambda|}\int_{B_4}\mathrm{Tr}[(\star+\frac{1}{\gamma})\mathcal{F}\wedge\mathcal{F}]$, which is a total derivative and gives the CS action (2) on the boundary $S^3\simeq\partial\mathcal{B}_4$. Using the feature of the Gaussian integral, integrating out B constraints $|\Lambda|B/3 = \mathcal{F}(\mathcal{A})$, which encodes B in the

[3]In addition, the construction here uses different symplectic coordinates from [19].

$so(1,3)$ curvature $\mathcal{F}(\mathcal{A})$. On the boundary S^3, $\mathcal{F}(\mathcal{A})$ is the $so(1,3)$ curvature of the CS connection $\mathcal{A}$. Classically, $S_{\mathrm{H\Lambda BF}}$ reduces to the Holst action of gravity with $\pm|\Lambda|$ when the simplicity constraint $B = \pm e \wedge e$ is imposed, where e is the cotetrad 1-form. At the quantum level, the simplicity constraint must be imposed on the CS partition function in order to obtain the spinfoam vertex amplitude.

Using the relation $\mathcal{F}(\mathcal{A}) = |\Lambda|B/3$, the simplicity constraint of B can be translated to constraining $\mathcal{A}$. By the CS symplectic structure, the resulting simplicity constraint can be divided into the first-class and second-class components. The first-class components are imposed *strongly* on $\mathcal{Z}_{S^3\setminus\Gamma_5}$ and restrict certain boundary data to a discrete set $\{2j_{ab}\}_{a<b}$, $a,b = 1,\ldots,5$, where $j_{ab} \in \mathbb{N}_0/2$ and $j_{ab} \le (k-1)/2$. Note that $\{j_{ab}\}_{a<b}$ are analogs of SU (2) spins associated to 10 boundary faces of the 4-simplex. Interestingly, a consistency condition "4d area = 3d area" (similar to [20]) gives restrictions to the positive angle structure $(\vec{\alpha},\vec{\beta})$. The second-class components of the simplicity constraint have to be imposed *weakly*. We propose coherent states Ψ_ρ peaked at points ρ in the (subspace of) phase space of $\mathcal{A}$ and apply the simplicity constraint to restrict ρ. The restricted ρ is equivalent to the set of 20 spinors $\xi_{ab} \in \mathbb{C}^2$ normalized by the Hermitian inner product, such that for each $a = 1,\ldots,5$, $\{j_{ab},\xi_{ab}\}_{b\neq a}$ are subject to the generalized closure condition of a constant curvature tetrahedra [21]. In our model, all tetrahedra and triangles are spacelike. We denote the ρ restricted by the simplicity constraint by $\rho_{\vec{j},\vec{\xi}}$. As a result, the vertex amplitude is defined by the inner product

$$A_v(\vec{j},\vec{\xi}) = \langle \bar{\Psi}_{\rho_{\vec{j},\vec{\xi}}} | \mathcal{Z}_{S^3\setminus\Gamma_5}\rangle, \tag{4}$$

where the complex conjugate of Ψ_ι is conventional. This inner product is a finite-dimensional integral of L^2 type. We show that the integral converges absolutely and A_v is a bounded function of $\vec{j},\vec{\xi}$. Here, A_v as an inner product (4) resembles the idea of A_v^0, but now A_v is well defined.

Given a simplicial complex $\mathcal{K}$ made by 4-simplices v, tetrahedra e, and faces f, following the general scheme of spinfoam state-sum models, the spinfoam amplitude associated to $\mathcal{K}$ is defined by

$$A = \sum_{\{j_f\}}^{(k-1)/2}{}' \prod_f A_f(j_f) \int [\mathrm{d}\xi \mathrm{d}\xi'] \prod_e A_e(\vec{j},\vec{\xi}_e,\vec{\xi}'_e) \prod_v A_v(\vec{j},\vec{\xi})$$

where j_f is associated to a face f and $\vec{\xi}_e = (\xi_1,\ldots,\xi_4)_e$ is associated to a tetrahedron e. The CS level $k = \frac{12\pi}{|\Lambda|\ell_P^2\gamma} \in \mathbb{Z}$ provides the cutoff to the sum over half-integer $0 \le j_f \le (k-1)/2$. The face and edge amplitudes A_f, A_e are not specified here except for requiring that A_e is a Gaussian-like continuous function approaching $\delta(\vec{\xi}_e,\vec{\xi}'_e)$ as $j \to \infty$. Given the boundedness of A_v, the amplitude A is finite because the sum over j_f's is finite and the integral over $\vec{\xi}$'s is compact. Here $\sum'$ indicates that some special spins are excluded in the sum.

When $\mathcal{K}$ has a boundary, the boundary data of A are j_f, $\vec{\xi}_e$ for boundary faces f and boundary tetrahedra e. These data are deformations of the data of coherent intertwiners in spin-network states. We conjecture that the boundary states of A are $\mathfrak{q}$-deformed spin-network states of quantum group $\mathrm{SU}(2)_\mathfrak{q}$ with $\mathfrak{q}$ the root of unity.

After accomplishing the finiteness of the spinfoam amplitude with Λ, we demonstrate the correct semiclassical behavior for the new vertex amplitude A_v in Sec. IV. Note that A_v in (4) as a finite-dimensional integral can be expressed in the form $\int e^{k\mathcal{I}}$ where $\mathcal{I}$ depends on j's only by j/k. Therefore, we use the stationary phase analysis to study the semiclassical behavior of A_v as $j,k \to \infty$ keeping j/k fixed: The dominant contribution of A_v comes from critical points, i.e., solutions of the critical equation $\delta\mathcal{I} = 0$. Given any boundary data $\{j_{ab},\xi_{ab}\}$ corresponding to the geometrical boundary of a nondegenerate convex constant curvature 4-simplex, there are exactly two critical points, which are two flat connections $\mathfrak{A}$, $\tilde{\mathfrak{A}} \in \mathcal{L}_{S^3\setminus\Gamma_5}$ having geometrical interpretations as the constant curvature 4-simplex. Note that $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ give the same 4-simplex geometry but opposite 4d orientations and that $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ are analogous to the two critical points related by parity in the EPRL vertex amplitude [22]. As a result, the asymptotic behavior of A_v is given up to an overall phase by

$$A_v = (\mathcal{N}_+ e^{iS_{\mathrm{Regge}}+C} + \mathcal{N}_- e^{-iS_{\mathrm{Regge}}-C})[1 + O(1/k)], \tag{5}$$

where $\mathcal{N}_\pm$ are nonoscillatory and relate to the Hessian matrix of $\mathcal{I}$. In the exponents,

$$S_{\mathrm{Regge}} = \frac{\Lambda k\gamma}{12\pi}\left(\sum_{a<b}\mathfrak{a}_{ab}\Theta_{ab} - \Lambda|V_4|\right) \tag{6}$$

is the 4d Lorentzian Regge action with Λ of the constant curvature 4-simplex reconstructed by $\mathfrak{A}$ or $\tilde{\mathfrak{A}}$. The gravitational coupling is effectively given by $\ell_P^2 = \frac{12\pi}{|\Lambda|k\gamma}$. Note that C is an undetermined geometry-independent integration constant. This semiclassical result of A_v is similar to the one related to A_v^0 in [13,23,24].

Lastly, it is known that the formalism of state-integral models that we use to construct $\mathcal{Z}_{S^3\setminus\Gamma_5}$ excludes the contributions from Abelian flat connections [15,16,25]. This does not cause trouble for us since Abelian flat connections only relate to degenerate tetrahedron geometries, which we exclude in the model.

The paper is organized as follows. In Sec. II, we construct the state-integral model of $\mathcal{Z}_{S^3\setminus\Gamma_5}$, and include the discussion of ideal triangulation of $S^3\setminus\Gamma_5$, and a brief review of $\mathrm{PSL}(2,\mathbb{C})$ CS theory on an ideal tetrahedron; we define convenient phase space coordinates, construct octahedron partition functions and then the partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$, and discuss coherent states. In Sec. III, we impose a simplicity constraint and construct A_v, and then we construct the spinfoam amplitude A on a simplicial complex and prove the finiteness; we also discuss the relation between boundary data of A and LQG spin networks, as well as various choices that we make in the definition of A. In Sec. IV, we derive the asymptotic behavior of A_v in the semiclassical limit.

II. COMPLEX CHERN-SIMONS THEORY ON $S^3\backslash\Gamma_5$

The purpose of this section is to construct the complex CS theory on the 3-manifold $S^3\backslash\Gamma_5$. In Sec. II A, we first review the ideal triangulation of $S^3\backslash\Gamma_5$ (see also [19]). As the building block, the CS theory on the ideal tetrahedron is reviewed in Sec. II B. Then, as an intermediate step, we construct the CS partition function on the ideal octahedron in Sec. II C since the ideal triangulation of $S^3\backslash\Gamma_5$ is made by five ideal octahedra. Section II D defines the phase space coordinates of the CS theory on $S^3\backslash\Gamma_5$ and the symplectic transformation from the phase space coordinates of the CS theory on the octahedra. The symplectic transformation defines the Weil-like transformations which relate the octahedron partition functions to the CS partition function on $S^3\backslash\Gamma_5$, as discussed in Sec. II E. In Sec. II F, we discuss the coherent state of the CS theory, which will be useful for the spinfoam model.

A. Ideal triangulation of $S^3\backslash\Gamma_5$

The 3-manifold $M_3 = S^3\backslash\Gamma_5$ is the complement in S^3 of an open tubular neighborhood of the Γ_5 graph (see Fig. 3). Here, M_3 can be triangulated by a set of (topological) ideal tetrahedra. An ideal tetrahedron Δ is a tetrahedron whose vertices are located at infinities. It is convenient to truncate the vertices to define the ideal tetrahedron as the "truncated tetrahedron" as in Fig. 2. There are two types of boundary components for the ideal tetrahedron: (a) the original boundary of the tetrahedron and (b) the boundaries created by truncating tetrahedron vertices. Following e.g., [15,26,27], the type-(a) boundary is called the geodesic boundary, and the type-(b) boundary is called the cusp boundary.

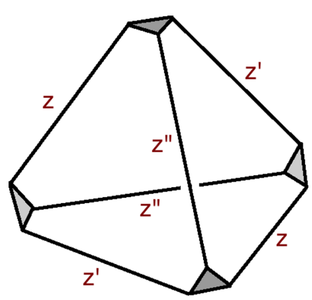

FIG. 2. Ideal tetrahedron.

Note that M_3 also has two types of boundary components: (A) the boundaries created by removing the open ball containing vertices of the graph, and (B) the boundaries created by removing tubular neighborhoods of edges. Here each type-(A) boundary component is a 4-holed sphere. Each type-(B) boundary component is an annulus which begins and ends at a pair of holes of two type-(A) boundaries. The type-(A) boundary is called the geodesic boundary of M_3, and the type-(B) boundary is called the cusp boundary. An ideal triangulation decomposes M_3 into a set of ideal tetrahedra, such that the geodesic boundary of M_3 is triangulated by geodesic boundaries of the ideal tetrahedra, while the cusp boundary of M_3 is triangulated by cusp boundaries of the ideal tetrahedra. This ideal triangulation of $S^3\backslash\Gamma_5$ is *not* the triangulation of S^3 dual to Γ_5 (the latter is given by the boundary of the 4-simplex). It is important to distinguish these two triangulations.

Here the geodesic boundary of $S^3\backslash\Gamma_5$ consists of five 4-holed spheres $\{\mathcal{S}_a\}_{a=1}^5$, while the cusp boundary consists of 10 annuli $\{\ell_{ab}\}_{a<b}$. The Γ_5 graph in Fig. 3 motivates us to subdivide $S^3\backslash\Gamma_5$ into five tetrahedron-like regions (five gray tetrahedra in Fig. 3, whose vertices coincide with the vertices of the graph). Every tetrahedron-like region should actually be understood as an ideal octahedron (with vertices truncated). The octahedron faces triangulate the 4-holed spheres, and the octahedron cusp boundaries (created by truncating vertices) triangulate the annuli. The way of gluing five ideal octahedra to form $S^3\backslash\Gamma_5$ is shown in Fig. 3. Each ideal octahedron can be subdivided into four ideal tetrahedra as shown in Fig. 4. A specific way of subdividing the octahedron is specified by a choice of octahedron equator. As a result, $S^3\backslash\Gamma_5$ is triangulated by 20 ideal tetrahedra.

Given M_3 with both geodesic and cusp boundaries, a framed $\mathrm{PSL}(2,\mathbb{C})$ flat connection on M_3 is a $\mathrm{PSL}(2,\mathbb{C})$ flat connection A on M_3 with a choice of flat section s (called the framing flag) in an associated $\mathbb{CP}_1$ bundle over every cusp boundary (see e.g., [27–29]). The flat section s can be viewed as a $\mathbb{C}^2$ vector field on a cusp boundary, defined up a complex rescaling and satisfying the flatness equation $(\mathrm{d}-A)s=0$ (d is the exterior derivative). Consequently, the vector $s(\mathfrak{p})$ at a point $\mathfrak{p}$ of the cusp boundary is an eigenvector of the holonomy of A around the cusp based at $\mathfrak{p}$. The eigenvector fixes the Weyl symmetry. Similarly, a framed flat connection on ∂M_3 is a flat connection $\mathfrak{A}$ on ∂M_3 with the same choice of framing flag on every cusp boundary. In addition, if a cusp boundary component of a certain 3-manifold is a small disc, such as the boundaries created by truncating of tetrahedron vertices, the holonomy of any framed flat connection $\mathfrak{A}$ around the disc is unipotent. The moduli space of framed $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\partial(S^3\backslash\Gamma_5)$ is denoted by $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$, which is a symplectic manifold with the Atiyah-Bott symplectic form.

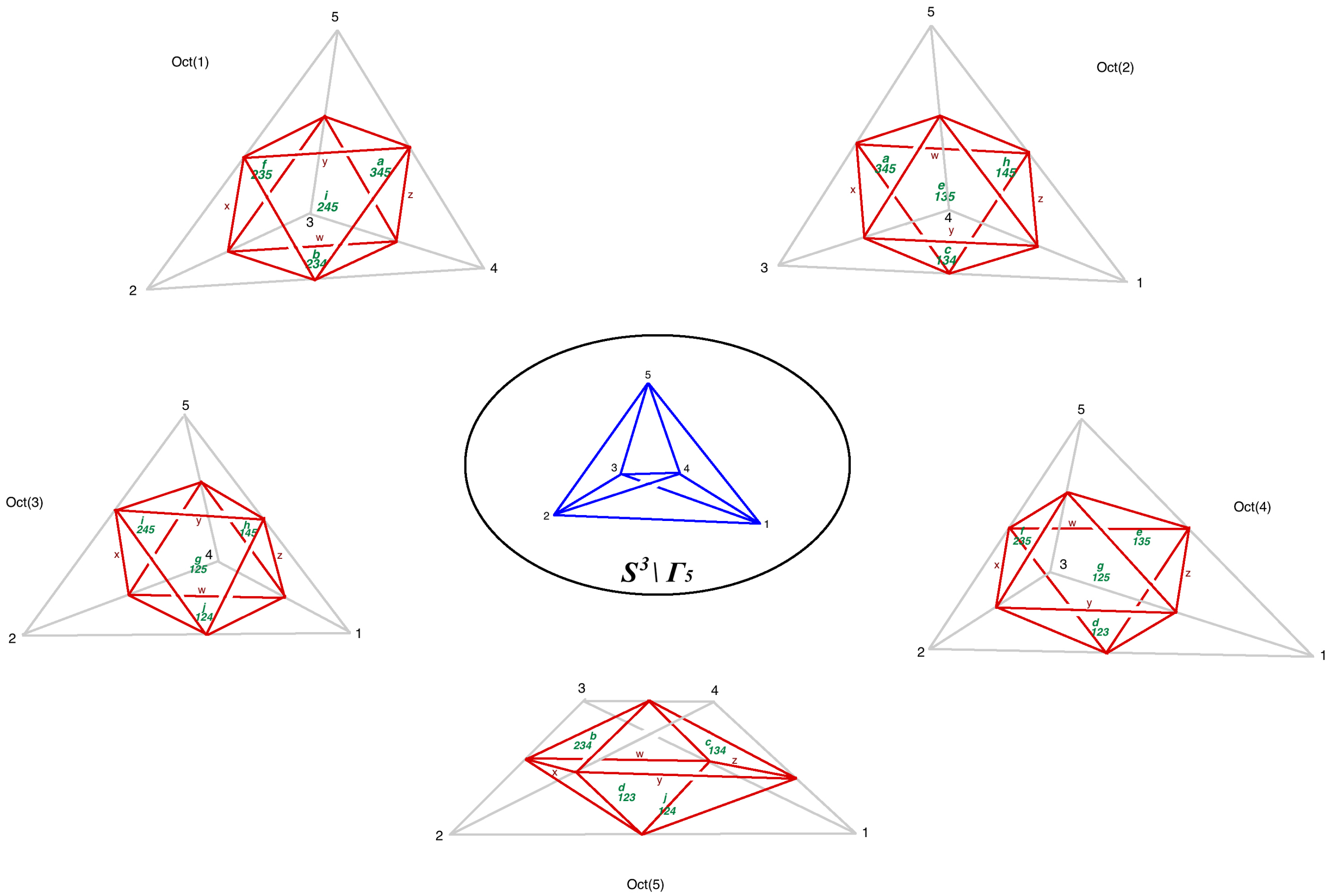

FIG. 3. Decomposition of $S^3\backslash\Gamma_5$ with five ideal octahedra (red), each of which can be decomposed into four ideal tetrahedra. The truncations of octahedron vertices are not drawn in the figure. The faces with green labels $a, b, c, d, e, f, g, h, i, j$ are the faces where a pair of octahedra are glued. Two ideal octahedra are glued through a pair of faces having the same label. In each ideal octahedron, we have chosen the edges with red label x, y, z, w to form the equator of the octahedron. This ideal triangulation first appeared in [19].

The moduli space of framed $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $S^3\backslash\Gamma_5$ is denoted by $\mathcal{L}_{S^3\backslash\Gamma_5}$, which is a Lagrangian submanifold in $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$.

B. Complex Chern-Simons theory on ideal tetrahedron

Given the ideal triangulation, the building block of the CS theory on $S^3\backslash\Gamma_5$ is the theory on an ideal tetrahedron Δ. In this subsection, we review the main results of the CS theory on Δ and refer to e.g., [15,16,27] for details. The boundary $\partial\Delta$ of the ideal tetrahedron is a sphere with four cusp discs. We denote by $\mathcal{P}_{\partial\Delta}$ the phase space of $\mathrm{PSL}(2,\mathbb{C})$ CS theory on Δ. Note that $\mathcal{P}_{\partial\Delta}$ is the moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on a 4-holed sphere, where the holonomy around each hole is unipotent.

The moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on an n-holed sphere can be described as follows: A 2-sphere in which n discs are removed is an n-holed sphere. We make a 2d ideal triangulation of the n-holed sphere such that edges in the triangulation end at the boundary of the holes. For example, the boundary of the ideal tetrahedron is an ideal triangulation of the 4-holed sphere. The 2d ideal triangulation has $3(n-2)$ edges. Each edge E is associated to a coordinate x_E of the moduli space. Given a framed flat connection, x_E is a cross-ratio of four framing flags s_1, s_2, s_3, s_4 associated to the vertices of the quadrilateral containing E as the diagonal (see Fig. 5),

$$x_E = \frac{\langle s_1 \wedge s_2\rangle\langle s_3 \wedge s_4\rangle}{\langle s_1 \wedge s_3\rangle\langle s_2 \wedge s_4\rangle} \tag{7}$$

where $\langle s_i \wedge s_j\rangle$ is an $\mathrm{SL}(2,\mathbb{C})$ invariant volume on $\mathbb{C}^2$ computed by parallel transporting $s_1, \ldots, s_4$ to a common point inside the quadrilateral by the flat connection. The set of $\{x_E\}_E$ are the Fock-Goncharov (FG) edge coordinates of the moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on the n-holed sphere. The correspondence between $\{x_E\}_E$'s and framed $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ is 1-to-1 [29]. By the "snake rule" [27,28], $\mathrm{PSL}(2,\mathbb{C})$ holonomies on the n-holed sphere can be expressed as 2×2 matrices whose entries are functions of $\{x_E\}$. In particular, the eigenvalue λ of the counterclockwise holonomy (of the flat connection) around a single hole relates to x_E by

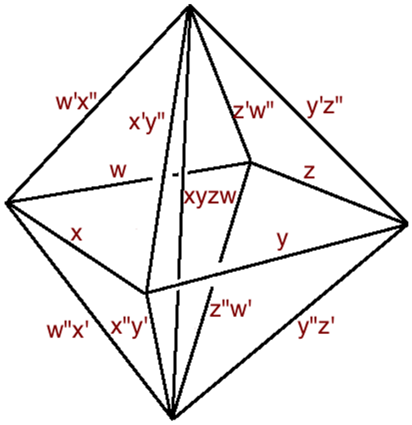

FIG. 4. By choosing the equator edges with labels x, y, z, w, an ideal octahedron can be subdivided into four ideal tetrahedra by drawing a vertical line connecting the remaining two vertices which does not belong to the equator. Vertices are truncated, although truncations are not shown in the figure.

$$\prod_{E\,\text{around hole}} (-x_E) = \lambda^2. \tag{8}$$

It is convenient to lift it to a logarithmic relation

$$\sum_{E\,\text{around hole}} (\chi_E - i\pi) = 2L, \tag{9}$$

where $x_E = e^{\chi_E}$, $\lambda = e^L$. The moduli space has a natural Poisson structure with

$$\{\chi_E, \chi_{E'}\} = \epsilon_{E,E'}, \tag{10}$$

where $\epsilon_{E,E'} \in 0, \pm 1, \pm 2$ counts the number of oriented triangles shared by E, E', and $\epsilon_{E,E'} = +1$ if E' occurs to the left of E in a triangle. Note that the moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on any n-holed sphere is not a symplectic manifold unless λ of all holes are fixed.

Applying this to the boundary of the ideal tetrahedron, we denote the FG coordinates at the edges around a given hole (cusp disc) by z, z', z'' (see Fig. 2). The trivial holonomy around each hole gives

$$zz'z'' = -1. \tag{11}$$

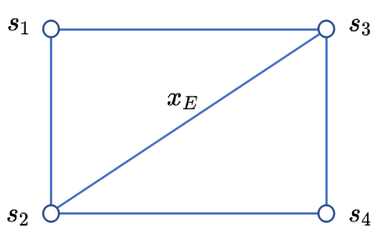

FIG. 5. Quadrilateral in the 2d ideal triangulation for defining x_E.

Similar conditions for all four cusps identify the FG coordinates at opposite edges. As a result, we find

$$\mathcal{P}_{\partial\Delta} = \{z, z', z'' \in \mathbb{C}^* | zz'z'' = -1\} \simeq (\mathbb{C}^*)^2. \tag{12}$$

Here, $\mathcal{P}_{\partial\Delta}$ is a symplectic manifold since the holonomy eigenvalues at all holes are fixed. The Atiyah-Bott symplectic form is $\Omega = \frac{\mathrm{d}z''}{z''} \wedge \frac{\mathrm{d}z}{z}$. We also define the logarithmic phase space coordinates $Z = \log(z)$, $Z' = \log(z')$, $Z'' = \log(z'')$ with canonical lifts that satisfy

$$Z + Z' + Z'' = i\pi, \tag{13}$$

$$\{Z, Z''\}_\Omega = \{Z'', Z'\}_\Omega = \{Z', Z\}_\Omega = 1. \tag{14}$$

The $\mathrm{PSL}(2,\mathbb{C})$ CS theory at levels $k \in \mathbb{Z}$, $\sigma \in i\mathbb{R}$ endows the following symplectic form $\omega_{k,\sigma}$ on $\mathcal{P}_{\partial\Delta}$:

$$\omega_{k,\sigma} \coloneqq \frac{1}{4\pi}(t\Omega + \bar{t}\bar{\Omega}), \quad t \coloneqq k + \sigma, \quad \bar{t} \coloneqq k - \sigma, \tag{15}$$

where k, σ relates to the cosmological constant Λ by

$$k = \frac{12\pi}{|\Lambda|\ell_P^2 \gamma}, \qquad \sigma = ik\gamma \tag{16}$$

where γ is the Barbero-Immirzi parameter [13]. We use the following parametrization to change from γ to b [16]:

$$i\gamma = \frac{1-b^2}{1+b^2}, \qquad b^2 = \frac{1-i\gamma}{1+i\gamma}, \tag{17}$$

$$\frac{4\pi i}{t} = \frac{2\pi i}{k}(1+b^2), \qquad \frac{4\pi i}{\bar{t}} = \frac{2\pi i}{k}(1+b^{-2}), \tag{18}$$

with complex b satisfying

$$\mathrm{Re}(b) > 0, \qquad \mathrm{Im}(b) \neq 0, \qquad |b| = 1. \tag{19}$$

We reparametrize z, z' and define $\tilde{z}, \tilde{z}''$ by

$$z = \exp\left[\frac{2\pi i}{k}(-ib\mu - m)\right], \tag{20}$$

$$\tilde{z} = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\mu + m)\right], \tag{21}$$

$$z'' = \exp\left[\frac{2\pi i}{k}(-ib\nu - n)\right], \tag{22}$$

$$\tilde{z}'' = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\nu + n)\right], \tag{23}$$

where (m, n) are real and periodic ($m \sim m + k, n \sim n + k$). When (μ, ν) are real, $\tilde{z}, \tilde{z}''$ are complex conjugates of z, z'.

But in the following, (μ,ν) will be analytically continued away from the real axis. Here, $\omega_{k,\sigma}$ written in terms of μ, ν, m, n gives

$$\omega_{k,\sigma} = \frac{2\pi}{k}(\mathrm{d}\nu\wedge\mathrm{d}\mu - \mathrm{d}n\wedge\mathrm{d}m). \tag{24}$$

The quantization of $(\mathcal{P}_{\partial\Delta},\omega_{k,\sigma})$ promotes μ, ν, m, n to operators $\boldsymbol{\mu}$, $\mathbf{m}$, $\boldsymbol{\nu}$, $\mathbf{n}$ satisfying the commutation relations

$$[\boldsymbol{\mu},\boldsymbol{\nu}] = [\mathbf{n},\mathbf{m}] = -\frac{k}{2\pi i}, \qquad [\boldsymbol{\nu},\mathbf{m}] = [\boldsymbol{\mu},\mathbf{n}] = 0. \tag{25}$$

The variables m, n are both canonical conjugate and periodic, so the spectra of $\mathbf{m}$, $\mathbf{n}$ are discrete and bounded: $m,n\in\mathbb{Z}/k\mathbb{Z}$. A representation of (25) is the kinematical Hilbert space

$$\mathcal{H}^{(k,\sigma)}_{\mathrm{kin}} = L^2(\mathbb{R})\otimes\mathbb{C}^k. \tag{26}$$

For any wave function $f(\mu|m)\in\mathcal{H}^{(k,\sigma)}_{\mathrm{kin}}$ where $\mu\in\mathbb{R}$ and $m\in\mathbb{Z}/k\mathbb{Z}$, the actions of $\boldsymbol{\mu}$, $\mathbf{m}$, $\boldsymbol{\nu}$, $\mathbf{n}$ are given by

$$\boldsymbol{\mu} f(\mu|m) = \mu f(\mu|m), \qquad e^{-\frac{2\pi i}{k}\mathbf{m}} f(\mu|m) = e^{-\frac{2\pi i}{k}m} f(\mu|m),$$
$$\boldsymbol{\nu} f(\mu|m) = \frac{k}{2\pi i}\partial_\mu f(\mu|m), \qquad e^{-\frac{2\pi i}{k}\mathbf{n}} f(\mu|m) = f(\mu|m+1). \tag{27}$$

We also define the operators corresponding to $z, z'', \tilde{z}, \tilde{z}''$,

$$z = \exp\left[\frac{2\pi i}{k}(-ib\boldsymbol{\mu} - \mathbf{m})\right], \tag{28}$$

$$\tilde{z} = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\boldsymbol{\mu} + \mathbf{m})\right], \tag{29}$$

$$z'' = \exp\left[\frac{2\pi i}{k}(-ib\boldsymbol{\nu} - \mathbf{n})\right], \tag{30}$$

$$\tilde{z}'' = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\boldsymbol{\nu} + \mathbf{n})\right]. \tag{31}$$

They satisfy q- and $\tilde{q}$-Weyl algebras

$$zz'' = qz''z, \qquad \tilde{z}\tilde{z}'' = \tilde{q}\tilde{z}''\tilde{z},$$
$$z\tilde{z}'' = \tilde{z}''z, \qquad \tilde{z}z'' = z''\tilde{z},$$
$$q = \exp\left(\frac{4\pi i}{t}\right) = \exp\left[\frac{2\pi i}{k}(1+b^2)\right], \tag{32}$$

$$\tilde{q} = \exp\left(\frac{4\pi i}{\bar{t}}\right) = \exp\left[\frac{2\pi i}{k}(1+b^{-2})\right]. \tag{33}$$

The above discussion focuses on flat connections on the boundary $\partial\Delta$. Only a subset of the flat connections on the boundary can be extended into the bulk. The moduli space of the $\mathrm{PSL}(2,\mathbb{C})$ flat connection on the ideal tetrahedron Δ, denoted by $\mathcal{L}_\Delta$, is a holomorphic Lagrangian submanifold in $\mathcal{P}_{\partial\Delta}$. Note that $\mathcal{L}_\Delta$ can be expressed as the holomorphic algebraic curve in terms of z, z' (see e.g., [15,27]):

$$\mathcal{L}_\Delta = \{z^{-1} + z'' - 1 = 0\}\subset\mathcal{P}_{\partial\Delta}, \tag{34}$$

and similarly for the antiholomorphic variables $\tilde{z}, \tilde{z}''$. In the quantum theory, we promote the algebraic curve to the quantum constraints imposed on the wave functions,

$$(z^{-1} + z'' - 1)\Psi_\Delta(\mu|m) = (\tilde{z}^{-1} + \tilde{z}'' - 1)\Psi_\Delta(\mu|m) = 0.$$

The solution is the quantum dilogarithm function (see e.g., [16,30–32])

$$\Psi_\Delta(\mu|m) = \begin{cases} \prod_{j=0}^{\infty}\frac{1-q^{j+1}z^{-1}}{1-\tilde{q}^{-j}\tilde{z}^{-1}} & |q|<1, \\ \prod_{j=0}^{\infty}\frac{1-\tilde{q}^{j+1}\tilde{z}^{-1}}{1-q^{-j}z^{-1}} & |q|>1. \end{cases} \tag{35}$$

Here, $\Psi_\Delta(\mu|m)$ is the CS partition function on the ideal tetrahedron Δ, and $\Psi_\Delta(\mu|m)$ defines a meromorphic function of $\mu\in\mathbb{C}$ for each $m\in\mathbb{Z}/k\mathbb{Z}$ and is analytic in b in each regime $\mathrm{Im}(b)>0$ and $\mathrm{Im}(b)<0$ (correspondingly $|q|<1$ and $|q|>1$). The poles and zeros of $\Psi_\Delta(\mu|m)$ are

$$\mu_{\mathrm{pole/zero}} = ibu + ib^{-1}v, \quad \text{with} \quad u,v\in\mathbb{Z},$$
$$u - v = -m + k\mathbb{Z}\begin{cases}\text{zeroes:} & u,v\geq 1,\\ \text{poles:} & u,v\leq 0.\end{cases} \tag{36}$$

Poles of Ψ_Δ are in the lower-half plane,

$$\mathrm{Im}(\mu_{\mathrm{pole}}) = \mathrm{Re}(b)(u+v)\leq 0. \tag{37}$$

Note that $\Psi_\Delta(\mu|m)$ is holomorphic in μ when $\mathrm{Im}(\mu)>0$.

The asymptotic behavior of $\Psi_\Delta(\mu|m)$ as $\mathrm{Re}(\mu)\to\infty$ with fixed $\mathrm{Im}(\mu)$ is

$$\Psi_\Delta(\mu|m) = \begin{cases} O(1) & \mathrm{Re}(\mu)\to+\infty \\ \exp\left[\frac{i\pi}{k}(\mu-\frac{i}{2}Q)^2 + O(1)\right] & \mathrm{Re}(\mu)\to-\infty, \end{cases}$$
$$Q = b + b^{-1} > 0. \tag{38}$$

The asymptotic behavior indicates that $\Psi_\Delta(\mu|m)$ does not belong to the Hilbert space $\mathcal{H}^{(k,\sigma)}_{\mathrm{kin}}$ but is a tempered distribution. Note that $\Psi_\Delta(\mu|m)$ is analytic in the upper-half plane $\mathrm{Im}(\mu)>0$. We have the following useful observation from the asymptotic behavior: Let $\alpha>0$; then

$$|e^{-\frac{2\pi}{k}\beta\mu}\Psi_\Delta(\mu+i\alpha|m)| \sim \begin{cases} \exp[-\frac{2\pi}{k}\beta\mu] & \mu\to\infty \\ \exp[-\frac{2\pi}{k}\mu(\alpha+\beta-Q/2)] & \mu\to-\infty. \end{cases} \quad (39)$$

Therefore, $e^{-\frac{2\pi}{k}\beta\mu}\Psi_\Delta(\mu+i\alpha|m)$ is a Schwartz function of μ if α, β is inside the open triangle $\mathfrak{P}(\Delta)$:

$$\mathfrak{P}(\Delta)=\{(\alpha,\beta)\in\mathbb{R}^2|\alpha,\beta>0,\alpha+\beta<Q/2\}. \quad (40)$$

The Fourier transform $\int d\mu e^{\frac{2\pi i}{k}\nu\mu}\Psi_\Delta(\mu|m)$ is convergent if the integration contour is shifted away from the real axis while $\alpha=\mathrm{Im}(\mu)$, $\beta=\mathrm{Im}(\nu)$ belong to $\mathfrak{P}(\Delta)$. Here, α, β can be understood as angles associated with coordinates z, z' in the context of hyperbolic geometry. Note that $(\alpha,\beta)\in\mathfrak{P}(\Delta)$ is called a "positive angle structure" of Δ [16,17].

C. Octahedron partition function

Four ideal tetrahedra are glued to form an ideal octahedron as shown in Fig. 4. The phase space $\mathcal{P}_{\partial\mathrm{oct}}$ is a symplectic reduction from four copies of $\mathcal{P}_{\partial\Delta}$: The FG edge coordinates $\{x_E\}$ of $\mathcal{P}_{\partial\mathrm{oct}}$ are a product of the tetrahedron edge coordinates. In general, for any edge on the boundary or in the bulk, it associates [27]

$$x_E=\prod(z,z',z''\,\text{incident at}\,E)\quad\text{or}\quad \chi_E=\sum(Z,Z',Z''\,\text{incident at}\,E) \quad (41)$$

as a product or sum over all the tetrahedron edge coordinates incident at the edge E. For boundary edges, x_E are the FG coordinates of $\mathcal{P}_{\partial\mathrm{oct}}$. The lift of $\chi_E=\log(x_E)$ is determined by the lifts of Z, Z', Z'' of ideal tetrahedra. For the bulk edge, x_E or χ_E is rather a constraint which is denoted by $c_E=\exp(C_E)$, satisfying

$$c_E=1\quad\text{or}\quad C_E=2\pi i, \quad (42)$$

because the flat connection holonomy around a bulk edge is trivial. We denote the edge coordinates in four copies of $\mathcal{P}_{\partial\Delta}$ by X, Y, Z, W and their double primes. All the edge coordinates of $\mathcal{P}_{\partial\mathrm{oct}}$ are expressed in Fig. 4, where we have a single constraint at the bulk edge,

$$C=X+Y+Z+W=2\pi i. \quad (43)$$

We make a symplectic transformation in $\mathcal{P}_{\partial\Delta}\times\mathcal{P}_{\partial\Delta}\times\mathcal{P}_{\partial\Delta}\times\mathcal{P}_{\partial\Delta}$ from the tetrahedron coordinates (X,X''), (Y,Y''), (Z,Z''), (W,W'') to a set of new symplectic coordinates (X,P_X), (Y,P_Y), (Z,P_Z), (C,Γ), where

$$P_X=X''-W'',\qquad P_Y=Y''-W'',\qquad P_Z=Z''-W'',\qquad \Gamma=W'' \quad (44)$$

and each pair are canonical conjugate variables, Poisson commutative with other pairs. The octahedron phase space $\mathcal{P}_{\partial\mathrm{oct}}$ is a symplectic reduction by imposing the constraint $C=2\pi i$ and removing the "gauge orbit" variable Γ. A set of symplectic coordinates of $\mathcal{P}_{\partial\mathrm{oct}}$ are given by $\vec{\phi}=(X,Y,Z)$, $\vec{\pi}=(P_X,P_Y,P_Z)$. The Atiyah-Bott symplectic form Ω implies

$$\{\phi_i,\pi_j\}_\Omega=\delta_{ij},\qquad \{\phi_i,\phi_j\}_\Omega=\{\pi_i,\pi_j\}_\Omega=0. \quad (45)$$

The CS partition function on the ideal octahedron, Z_{oct}, is a product of four tetrahedron partition functions followed by the restriction on the quantum deformed constraint surface $e^C=q$, $e^{\tilde{C}}=\tilde{q}$[4]:

$$\begin{aligned} &Z_{\mathrm{oct}}(\mu_X,\mu_Y,\mu_Z|m_X,m_Y,m_Z)\\ &=\Psi_\Delta(\mu_X|m_X)\Psi_\Delta(\mu_Y|m_Y)\Psi_\Delta(\mu_Z|m_Z)\\ &\quad\times\Psi_\Delta(iQ-\mu_X-\mu_Y-\mu_Z|-m_X-m_Y-m_Z). \end{aligned}$$

The octahedron partition function gives the following asymptotics behavior

$$|e^{-\frac{2\pi}{k}\sum_i\beta_i\mu_i}Z_{\mathrm{oct}}(\{\mu_i+i\alpha_i\}|\{m_i\})| \sim \begin{cases} e^{-\frac{2\pi}{k}\mu_X(\alpha_X+\beta_X+\alpha_Y+\alpha_Z-Q/2)} & \mu_X\to\infty \\ e^{-\frac{2\pi}{k}\mu_X(\alpha_X+\beta_X-Q/2)} & \mu_X\to-\infty \end{cases}$$

where $i=X$, Y, Z. Similar behaviors are satisfied for $\mu_Y\to\pm\infty$ or $\mu_Z\to\pm\infty$. Therefore, $e^{-\frac{2\pi}{k}\sum_i\beta_i\mu_i}Z_{\mathrm{oct}}\times(\{\mu_i+i\alpha_i\}|\{m_i\})$ is a Schwartz function of μ_X, μ_Y, μ_Z, if $(\alpha_X,\beta_X,\alpha_Y,\beta_Y,\alpha_Z,\beta_Z)\in\mathbb{R}^6$ is contained by the open polytope $\mathfrak{P}(\mathrm{oct})$ defined by the following inequalities:

$$\begin{aligned} &\alpha_X,\alpha_Y,\alpha_Z>0,\qquad \alpha_X+\alpha_Y+\alpha_Z<Q,\\ &\alpha_X+\beta_X<\frac{Q}{2},\qquad \alpha_Y+\beta_Y<\frac{Q}{2},\qquad \alpha_Z+\beta_Z<\frac{Q}{2},\\ &\alpha_X+\alpha_Y+\alpha_Z+\beta_X>\frac{Q}{2},\qquad \alpha_X+\alpha_Y+\alpha_Z+\beta_Y>\frac{Q}{2},\\ &\alpha_X+\alpha_Y+\alpha_Z+\beta_Z>\frac{Q}{2}. \end{aligned} \quad (46)$$

To see that $\mathfrak{P}(\mathrm{oct})$ is not empty, Appendix A shows a plot, Fig. 9, of the intersection between $\mathfrak{P}(\mathrm{oct})$ and the plane of $\alpha_X=\alpha_Y=\alpha_Z$, $\beta_X=\beta_Y=\beta_Z$. Here, $(\vec{\alpha},\vec{\beta})\in\mathfrak{P}(\mathrm{oct})$ is a positive angle structure of the ideal octahedron.

Following [16], we consider any $2N$-dimensional phase space $(\mathcal{P},\omega)$ with Darboux coordinates (μ_i,m_i) and (ν_i,m_i) such that $\omega=\frac{2\pi}{k}\sum_{i=1}^n(d\nu_i\wedge d\mu_i-dn_i\wedge dm_i)$. The phase space is associated with an "angle space"

[4]The quantum deformation is necessary to make the partition function invariant under the 3d Pachner move (see e.g., [15]).

$(\mathcal{P}_{\rm angle}, \omega_{\rm angle})$ whose universal cover is $T^*\mathbb{R}^N \simeq \mathbb{R}^{2N}$; the Darboux coordinates of $\mathcal{P}_{\rm angle}$ are

$$\alpha_i = \mathrm{Im}(\mu_i), \qquad \beta_i = \mathrm{Im}(\nu_i), \tag{47}$$

so $\omega_{\rm angle} = \sum_{i=1}^N d\beta_i \wedge d\alpha_i$. Given a $2N$-dimensional open convex symplectic polytope $\mathfrak{P} \in \mathcal{P}_{\rm angle}$, we define $\pi(\mathfrak{P})$ to be the projection of $\mathfrak{P}$ to the base of $T^*\mathbb{R}^N$, with coordinates $\vec{\alpha}$; we then define

$$\mathrm{strip}(\mathfrak{P}) \coloneqq \{\vec{\mu} \in \mathbb{C}^N | \mathrm{Im}(\vec{\mu}) \in \pi(\mathfrak{P})\}. \tag{48}$$

We define the functional space as

$$\begin{aligned}\mathcal{F}_{\mathfrak{P}} \coloneqq \{&\text{holomorphic functions } f\colon \mathrm{strip}(\mathfrak{P}) \to \mathbb{C} \text{ s.t.}\\ \forall\, (\vec{\alpha}, \vec{\beta}) \in \mathfrak{P},\ &\text{the function } e^{-\frac{2\pi}{k}\vec{\mu}\cdot\vec{\beta}} f(\vec{\mu} + i\vec{\alpha}) \in \mathcal{S}(\mathbb{R}^N)\\ &\text{is the Schwartz class}\}.\end{aligned}$$

We have the convergence for any $f \in \mathcal{F}_{\mathfrak{P}}$,

$$\int \mathrm{d}^N\mu\, e^{\frac{2\pi i}{k}\vec{\mu}\cdot\vec{\nu}} f(\vec{\mu}) < \infty \tag{49}$$

when the integration contour is shifted away from the real axis while $\vec{\alpha} = \mathrm{Im}(\vec{\mu})$, $\vec{\beta} = \mathrm{Im}(\vec{\nu})$ belong to $\mathfrak{P}$. Note that $f \in \mathcal{F}_{\mathfrak{P}}$ implies the Fourier transform of f also belongs to $\mathcal{F}_{\mathfrak{P}}$.

To accommodate partition functions of complex Chern-Simons theory at level k, we define

$$\mathcal{F}^{(k)}_{\mathfrak{P}} = \mathcal{F}_{\mathfrak{P}} \otimes_{\mathbb{C}} (V_k)^{\otimes N}, \qquad V_k \simeq \mathbb{C}^k. \tag{50}$$

As examples, the tetrahedron partition function Ψ_Δ belongs to $\mathcal{F}^{(k)}_{\mathfrak{P}(\Delta)}$ with $N = 1$, and the octahedron partition function $Z_{\rm oct}$ belongs to $\mathcal{F}^{(k)}_{\mathfrak{P}(\rm oct)}$ with $N = 3$.

D. Phase space coordinates of $\mathcal{P}_{\partial(S^3\setminus\Gamma_5)}$

The geodesic boundary of $S^3\setminus\Gamma_5$ consists of five 4-holed spheres, denoted by $\mathcal{S}_{a=1,\ldots,5}$. In Fig. 3, each $\mathcal{S}_a$ is made by the triangles from the geodesic boundaries of the octahedra. We compute all FG edge coordinates $\chi^{(a)}_{mn}$ (a labels the 4-holed sphere and mn labels the edge E) of flat connections on $\mathcal{S}_{a=1,\ldots,5}$ using Eq. (41) and list them in Table I in Appendix B.

The phase space $\mathcal{P}_{\partial(S^3\setminus\Gamma_5)}$ is the moduli space of framed $\mathrm{PSL}(2,\mathbb{C})$ flat connections on the 2d boundary $\partial(S^3\setminus\Gamma_5)$. We choose the Darboux coordinates of $\mathcal{P}_{\partial(S^3\setminus\Gamma_5)}$ as follows: First, the complex Fenchel-Nielsen (FN) length variables $\lambda^2_{ab} = e^{2L_{ab}}$ are squared eigenvalues of $\mathrm{PSL}(2,\mathbb{C})$ holonomies meridian to the 10 annuli ℓ_{ab} connecting 4-holed spheres $\mathcal{S}_a$ and $\mathcal{S}_b$. They relate edge coordinates $\chi^{(a)}_{mn}$ using Eq. (9). Ten $2L_{ab}$ are linear combinations of $(X_a, P_{X_a}), (Y_a, P_{Y_a}), (Z_a, P_{Z_a})$ from five $\mathrm{Oct}(a)$ with integer coefficients. Their expressions are given in Appendix B. The resulting L_{ab} are mutually Poisson commutative and commute with all edge coordinates $\chi^{(a)}_{mn}$.

All L_{ab} commute with 4-holed sphere edge coordinates $\chi^{(a)}_{mn}$, and $\mathcal{P}_{\partial(S^3\setminus\Gamma_5)}$ is complex 30-dimensional. Among the Darboux coordinates, the position variables include ten $2L_{ab}$ and five variables $\mathcal{X}_a$ ($a = 1, \ldots, 5$), one for each 4-holed sphere. We choose $\mathcal{X}_a$ to be one of $\chi^{(a)}_{mn}$:

$$\begin{aligned}&\mathcal{X}_1 = \chi^{(1)}_{25}, \qquad \mathcal{X}_2 = \chi^{(2)}_{15}, \qquad \mathcal{X}_3 = \chi^{(3)}_{15},\\ &\mathcal{X}_4 = \chi^{(4)}_{15}, \qquad \mathcal{X}_5 = \chi^{(5)}_{14}.\end{aligned} \tag{51}$$

We denote the conjugate momentum variables by $\mathcal{T}_{ab}$ and $\mathcal{Y}_a$, and denote

$$\mathcal{Q}_I = (2L_{ab}, \mathcal{X}_a), \qquad \mathcal{P}_I = (\mathcal{T}_{ab}, \mathcal{Y}_a), \qquad I = 1, \ldots, 15,$$

where I labels the boundary components $(\ell_{ab}, \mathcal{S}_a)$. The momentum variables $\mathcal{T}_{ab}$ conjugate to $2L_{ab}$ are called the twist variables. On each $\mathcal{S}_a$, the momentum variable $\mathcal{Y}_a$ conjugate to $\mathcal{X}_a$ also turns out to be the FG edge coordinates up to sign and $2\pi i$.

$$\begin{aligned}&\mathcal{Y}_1 = \chi^{(1)}_{23}, \qquad \mathcal{Y}_2 = \chi^{(2)}_{14}, \qquad \mathcal{Y}_3 = \chi^{(3)}_{45} - 2\pi i,\\ &\mathcal{Y}_4 = -\chi^{(4)}_{35} + 2\pi i, \qquad \mathcal{Y}_5 = \chi^{(5)}_{34} - 2\pi i.\end{aligned} \tag{52}$$

Explicit expressions of $2L_{ab}$, $\mathcal{T}_{ab}$, $\mathcal{X}_a$, $\mathcal{Y}_a$ in terms of $(X_a, P_{X_a}), (Y_a, P_{Y_a}), (Z_a, P_{Z_a})$ are given in Appendix B.

There exists a linear symplectic transformation from $\vec{\Phi} \equiv (X_a, Y_a, Z_a)^5_{a=1}$ and $\vec{\Pi} \equiv (P_{X_a}, P_{Y_a}, P_{Z_a})^5_{a=1}$ to $\vec{\mathcal{Q}}, \vec{\mathcal{P}}$,

$$\begin{pmatrix}\vec{\mathcal{Q}}\\ \vec{\mathcal{P}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ -(\mathbf{B}^T)^{-1} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\vec{\Phi}\\ \vec{\Pi}\end{pmatrix} + i\pi\begin{pmatrix}\vec{t}\\ \vec{0}\end{pmatrix}, \tag{53}$$

such that all entries in $\mathbf{A}, \mathbf{B}, \vec{t}$ are integers. Here, $\vec{t}$ is a 15-dimensional vector, and $\mathbf{A}, \mathbf{B}$ are 15×15 blocks satisfying the fact that $\mathbf{A}\mathbf{B}^T$ is a symmetric matrix. Matrices $\mathbf{A}, \mathbf{B}, \vec{t}$ are given explicitly in Appendix C.

Following from (45), the Atiyah-Bott symplectic form Ω on $\mathcal{P}_{\partial(S^3\setminus\Gamma_5)}$ is expressed as

$$\begin{aligned}\Omega &= \sum_{I=1}^{15} \mathrm{d}\mathcal{P}_I \wedge \mathrm{d}\mathcal{Q}_I\\ &= 2\sum_{a<b} \mathrm{d}\mathcal{T}_{ab} \wedge \mathrm{d}L_{ab} + \sum_{a=1}^{5} \mathrm{d}\mathcal{Y}_a \wedge \mathrm{d}\mathcal{X}_a.\end{aligned} \tag{54}$$

The coordinates $\vec{\mathcal{Q}}, \vec{\mathcal{P}}$ are used below for constructing the CS partition function of $S^3\setminus\Gamma_5$. We sometimes use the notations $\mathcal{Q}_{ab} = 2L_{ab}$, $\mathcal{Q}_a = \mathcal{X}_a$, $\mathcal{P}_{ab} = \mathcal{T}_{ab}$, $\mathcal{P}_a = \mathcal{Y}_a$ in our following discussion.

It is remarkable that there is no additional constraint for gluing octahedra to form $S^3\backslash\Gamma_5$ since gluing octahedra does not produce an additional bulk edge. Therefore, $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)} \simeq \times_{a=1}^5 \mathcal{P}_{\partial\mathrm{oct}(a)}$. It is simply a symplectic transformation from the octahedra Darboux coordinates $\vec{\Phi}$, $\vec{\Pi}$ to $\mathcal{P}_I$, $\mathcal{Q}_I$ of $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$. The moduli space of framed flat connections on each octahedron is a Lagrangian submanifold $\mathcal{L}_{\mathrm{oct}(a)} \subset \mathcal{P}_{\partial\mathrm{oct}(a)}$. Then, $\times_{a=1}^5 \mathcal{L}_{\mathrm{oct}(a)} \simeq \mathcal{L}_{S^3\backslash\Gamma_5}$ is a Lagrangian submanifold in $\times_{a=1}^5 \mathcal{P}_{\partial\mathrm{oct}(a)} \simeq \mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$. Given any five framed flat connections on five octahedra, respectively, they define a flat connection on $S^3\backslash\Gamma_5$.

E. $S^3\backslash\Gamma_5$ partition function

The symplectic matrix in (53) can be decomposed into generators

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -(\mathbf{B}^T)^{-1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}\mathbf{B}^T & \mathbf{I} \end{pmatrix} \begin{pmatrix} -(\mathbf{B}^{-1})^T & \mathbf{0} \\ \mathbf{0} & -\mathbf{B} \end{pmatrix}. \tag{55}$$

We start with a product of five octahedron partition functions, each of which is associated to an octahedron in the decomposition of $S^3\backslash\Gamma_5$,

$$Z_\times(\vec{\mu}|\vec{m}) = \prod_{a=1}^5 Z_{\mathrm{oct}}(\mu_{X_a}, \mu_{Y_a}, \mu_{Z_a} | m_{X_a}, m_{Y_a}, m_{Z_a}) \in \mathcal{F}^{(k)}_{\mathfrak{P}(\mathrm{oct})^{\times 5}}. \tag{56}$$

The generators of the symplectic transformation are represented as a Weil-like action on $Z_\times$ according to the order in (55) [15,16].

1. U-type transformation:

$$U = \begin{pmatrix} -(\mathbf{B}^{-1})^T & \mathbf{0} \\ \mathbf{0} & -\mathbf{B} \end{pmatrix}, \tag{57}$$

$$\begin{aligned} Z_1(\vec{\mu}|\vec{m}) &= (U \triangleright Z_\times)(\vec{\mu}|\vec{m}) \\ &= \sqrt{\det(-\mathbf{B})} Z_\times(-\mathbf{B}^T\vec{\mu}| - \mathbf{B}^T\vec{m}), \end{aligned} \tag{58}$$

where $\sqrt{\det(-B)} = 4i$. The fact that all entries of $\mathbf{B}$ are integers guarantees that Z_1 is well defined for $\vec{m} \in \mathbb{Z}/k\mathbb{Z}$. In addition, $Z_\times \in \mathcal{F}^{(k)}_{\mathfrak{P}(\mathrm{oct})^{\times 5}}$ indicates that the following function is of Schwartz class when $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}(\mathrm{oct})^{\times 5}$,

$$\begin{aligned} &e^{-\frac{2\pi}{k}(-\mathbf{B}^T\vec{\mu})\cdot\vec{\beta}} Z_\times(-\mathbf{B}^T\vec{\mu} + i\vec{\alpha}|\vec{m}) \\ &= e^{-\frac{2\pi}{k}\vec{\mu}\cdot(-\mathbf{B}\vec{\beta})} Z_\times(-\mathbf{B}^T(\vec{\mu} - i(\mathbf{B}^{-1})^T\vec{\alpha})|\vec{m}), \end{aligned} \tag{59}$$

where $\mu_i \in \mathbb{R}$. Therefore, Z_1 belongs to $\mathcal{F}^{(k)}_{\mathfrak{P}_1}$, where $\mathfrak{P}_1 = U \circ \mathfrak{P}(\mathrm{oct})^{\times 5}$, with U acting on the angle space $\mathcal{P}_{\mathrm{angle}}$ as a symplectic transformation.

2. T-type transformation:

$$T = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}\mathbf{B}^T & \mathbf{I} \end{pmatrix}, \tag{60}$$

$$\begin{aligned} Z_2(\vec{\mu}|\vec{m}) &= (T \triangleright Z_1)(\vec{\mu}|\vec{m}) \\ &= (-1)^{\vec{m}\cdot\mathbf{A}\mathbf{B}^T\cdot\vec{m}} e^{\frac{i\pi}{k}(-\vec{\mu}\cdot\mathbf{A}\mathbf{B}^T\cdot\vec{\mu} + \vec{m}\cdot\mathbf{A}\mathbf{B}^T\cdot\vec{m})} Z_1(\vec{\mu}|\vec{m}). \end{aligned} \tag{61}$$

All entries of $\mathbf{A}\mathbf{B}^T$ are integers so that Z_2 is well defined for $\vec{m} \in (\mathbb{Z}/k\mathbb{Z})^{15}$. Note that $Z_1 \in \mathcal{F}^{(k)}_{\mathfrak{P}_1}$ implies that the following function is of Schwartz class when $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_1$,

$$e^{-\frac{2\pi}{k}\vec{\mu}\cdot\vec{\beta}} Z_1(\vec{\mu} + i\vec{\alpha}|\vec{m}) = \mathrm{phase} \cdot e^{-\frac{2\pi}{k}\vec{\mu}\cdot(\vec{\beta} + \mathbf{A}\mathbf{B}^T\cdot\vec{\alpha})} Z_2(\vec{\mu} + i\vec{\alpha}|\vec{m}). \tag{62}$$

Therefore, Z_2 belongs to $\mathcal{F}^{(k)}_{\mathfrak{P}_2}$, where $\mathfrak{P}_2 = T \circ \mathfrak{P}_1$.

3. S-type transformation:

$$S = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{63}$$

$$\begin{aligned} Z_3(\vec{\mu}|\vec{m}) &= (S \triangleright Z_2)(\vec{\mu}|\vec{m}) \\ &= \frac{1}{k^{15}} \sum_{\vec{n} \in (\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathcal{C}} \mathrm{d}^{15}\nu\, e^{\frac{2\pi i}{k}(-\vec{\mu}\cdot\vec{\nu} + \vec{m}\cdot\vec{n})} Z_2(\vec{\nu}|\vec{n}). \end{aligned} \tag{64}$$

If we set $\alpha_i = \mathrm{Im}(\mu_i)$ and $\beta_i = \mathrm{Im}(\nu_i)$ $(i = 1, \ldots, 15)$,

$$\begin{aligned} e^{\frac{2\pi i}{k}(-\vec{\mu}\cdot\vec{\nu})} Z_2(\vec{\nu}|\vec{n}) &= [e^{\frac{2\pi}{k}\vec{\alpha}\cdot\mathrm{Re}(\vec{\nu})} Z_2(\mathrm{Re}(\vec{\nu}) + i\vec{\beta}|\vec{n})] \\ &\quad \times e^{\frac{2\pi i}{k}[-\mathrm{Re}(\vec{\mu})\cdot\mathrm{Re}(\vec{\nu}) + \vec{\alpha}\cdot\vec{\beta}] + \frac{2\pi}{k}\mathrm{Re}(\vec{\mu})\cdot\vec{\beta}} \end{aligned}$$

is a Schwartz function in $\mathrm{Re}(\vec{\nu})$, when $(\vec{\beta}, -\vec{\alpha}) \in \mathfrak{P}_2$ (the function in the square brackets is a Schwartz function, and $e^{\frac{2\pi i}{k}[-\mathrm{Re}(\vec{\mu})\cdot\mathrm{Re}(\vec{\nu})]}$ is a phase), or equivalently,

$$(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_3 = S \circ \mathfrak{P}_2 = S \circ T \circ U \circ \mathfrak{P}(\mathrm{oct})^{\times 5}. \tag{65}$$

Given any $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_3$, let $\mathrm{Im}(\mu_i) = \alpha_i$ and the integration contour $\mathcal{C}$ be defined such that $\mathrm{Im}(\nu_i) = \beta_i$; then $Z_3(\vec{\mu}|\vec{m})$ converges absolutely, and $Z_3 \in \mathcal{F}^{(k)}_{\mathfrak{P}_3}$. As long as the contour $\mathcal{C}$ satisfies the condition $\mathrm{Im}(\nu_i) = \beta_i$, $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_3$, $Z_3(\vec{\mu}|\vec{m})$ is independent of the choices of $\mathcal{C}$, i.e., choices of β_i, due to the analyticity of Z_2 and the fast decay of the integrand at infinity.

4. Affine shift[5]:

$$\sigma_{\vec{t}}: \begin{pmatrix} \vec{X} \\ \vec{P} \end{pmatrix} \mapsto \begin{pmatrix} \vec{X} \\ \vec{P} \end{pmatrix} + i\pi \begin{pmatrix} \vec{t} \\ \vec{0} \end{pmatrix}, \tag{66}$$

$$\begin{aligned} \mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m}) &= (\sigma_{\vec{t}} \rhd Z_3)(\vec{\mu}|\vec{m}) \\ &= Z_3\left(\vec{\mu} - \frac{iQ}{2}\vec{t}|\vec{m}\right). \end{aligned} \tag{67}$$

We have $\mathcal{Z}_{S^3\backslash\Gamma_5} \in \mathcal{F}^{(k)}_{\mathfrak{P}_{\rm new}}$, where

$$\begin{aligned} \mathfrak{P}_{\rm new} &= \sigma'_{\vec{t}} \circ \mathfrak{P}_3 = \sigma'_{\vec{t}} \circ S \circ T \circ U \circ \mathfrak{P}(\mathrm{oct})^{\times 5}, \\ \sigma'_{\vec{t}}&: \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix} \mapsto \begin{pmatrix} \vec{\alpha}' \\ \vec{\beta}' \end{pmatrix} \coloneqq \begin{pmatrix} \vec{\alpha} + \frac{Q}{2}\vec{t} \\ \vec{\beta} \end{pmatrix}. \end{aligned} \tag{68}$$

The resulting $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$ is the CS partition function on $S^3\backslash\Gamma_5$. Here, $\mathfrak{P}_{\rm new}$ is obviously nonempty since $\mathfrak{P}(\mathrm{oct})$ is nonempty. Every $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_{\rm new}$ is a positive angle structure of $S^3\backslash\Gamma_5$, and it leads to the absolute convergence of $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$.

Note that $\vec{\mu}, \vec{m}$ relate to $\{\mathcal{Q}_I, \tilde{\mathcal{Q}}_I\}_{I=1,\ldots,15}$ by

$$\mu_I = \frac{k(\tilde{\mathcal{Q}}'_I + \mathcal{Q}'_I)}{2\pi(b + b^{-1})}, \qquad m_I = \frac{ik(\mathcal{Q}'_I - b^2\tilde{\mathcal{Q}}'_I)}{2\pi(b^2+1)}, \tag{69}$$

$$\mathcal{Q}'_I = \mathcal{Q}_I - i\pi t_I, \qquad \tilde{\mathcal{Q}}'_I = \tilde{\mathcal{Q}}_I - i\pi t_I \tag{70}$$

or in terms of exponentials,

$$(-1)^{t_I} e^{\mathcal{Q}_I} = \exp\left[\frac{2\pi i}{k}(-ib\mu_I - m_I)\right], \tag{71}$$

$$(-1)^{t_I} e^{\tilde{\mathcal{Q}}_I} = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\mu_I + m_I)\right]. \tag{72}$$

Consider the shifts $\mathcal{Q}_I \to \mathcal{Q}_I + 2\pi i p_I$, $\tilde{\mathcal{Q}}_I \to \tilde{\mathcal{Q}}_I - 2\pi i \tilde{p}_I$ ($p_I, \tilde{p}_I \in \mathbb{Z}$), which leave $e^{\mathcal{Q}_I}, e^{\tilde{\mathcal{Q}}_I}$ invariant. Fixing $\mathrm{Im}(\mu_I) = \alpha_I$ implies $\tilde{p}_I = p_I$, and then the shifts reduce to the gauge freedom $m_I \to m_I + kp_I$ in $\mathbb{Z}/k\mathbb{Z}$.

F. Coherent states

Given the 4-holed sphere $\mathcal{S}_a$, we transform the corresponding phase space coordinates from $\mathcal{X}_a, \mathcal{Y}_a, \tilde{\mathcal{X}}_a, \tilde{\mathcal{Y}}_a$ to μ_a, ν_a, m_a, n_a by

$$\mathcal{X}_a - i\pi t_a = \frac{2\pi i}{k}(-ib\mu_a - m_a), \tag{73}$$

$$\tilde{\mathcal{X}}_a - i\pi t_a = \frac{2\pi i}{k}(-ib^{-1}\mu_a + m_a), \tag{74}$$

$$\mathcal{Y}_a = \frac{2\pi i}{k}(-ib\nu_a - n_a), \tag{75}$$

$$\tilde{\mathcal{Y}}_a = \frac{2\pi i}{k}(-ib^{-1}\nu_a + n_a), \tag{76}$$

where μ_a is the component in $\vec{\mu} \in \mathrm{strip}(\mathfrak{P}_{\rm new})$. These coordinates parametrize $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ with fixed $e^{2L_{ab}}$ at the holes. The moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ is locally $\mathbb{C}^6$, but fixing $e^{2L_{ab}}$ reduces the space to $\mathbb{C}^2$ locally. Let us fix $\mathrm{Im}(\mu_a) = \alpha_a$ and focus on degrees of freedom $\mathrm{Re}(\mu_a), m_a$. In the following discussions of this section, we use $\mu_a \in \mathbb{R}$ to represent $\mathrm{Re}(\mu_a)$. We define the Hilbert space

$$\mathcal{H}_{\mathcal{S}_a} = L^2(\mathbb{R}) \otimes_{\mathbb{C}} V_k, \qquad V_k \simeq \mathbb{C}^k, \tag{77}$$

containing functions of $\mu_a \in \mathbb{R}, m_a \in \mathbb{Z}/k\mathbb{Z}$. Operators $\boldsymbol{\mu}_a$, $\boldsymbol{\nu}_a$, $\mathbf{m}_a$, $\mathbf{n}_a$ on $\mathcal{H}_{\mathcal{S}_a}$ are defined in the same way as in (27). We suppress the a index in the following discussions.

We first focus on $L^2(\mathbb{R})$ and define the "annihilation operator" and coherent state $\psi_z(\mu)$ labeled by $z \in \mathbb{C}$. Here, $\psi_z(\mu)$ satisfies

$$\frac{1}{\sqrt{2}}\left(\sqrt{\frac{2\pi}{k}}\boldsymbol{\mu} + i\sqrt{\frac{2\pi}{k}}\boldsymbol{\nu}\right)\psi^0_z(\mu) = \sqrt{\frac{k}{2\pi}} z\psi^0_z(\mu).$$

The solution is

$$\psi^0_z(\mu) = \left(\frac{2}{k}\right)^{1/4} e^{-\frac{\pi}{k}(\mu - \frac{k}{\pi\sqrt{2}}\mathrm{Re}(z))^2} e^{i\sqrt{2}\mu\mathrm{Im}(z)}, \tag{78}$$

where $\psi^0_z(\mu)$ is normalized by the standard L^2-norm. The coherent state label z relates to the classical phase space coordinates μ_0, ν_0 as

$$z = \frac{1}{\sqrt{2}}\frac{2\pi}{k}(\mu_0 + i\nu_0). \tag{79}$$

We can multiply ψ^0_z by a prefactor that relates to the polytope $\mathfrak{P}_{\rm new}$; namely, for each $\mathcal{S}_a$ we define

[5]The affine shifted classical coordinate $X + i\pi t$ ($t \in \mathbb{Z}$) has the quantum deformation $X + (i\pi + \frac{\hbar}{2})t$ when entering the partition function [15]. In terms of the exponential variables, the affine shift is given by $(-q^{\frac{1}{2}})^t e^X = (-q^{\frac{1}{2}})^t x$. Here, we define $q^{\frac{1}{2}} = e^{\frac{\hbar}{2}}$, where $\hbar = \frac{2\pi i}{k}(1 + b^2)$. If we parametrize $e^X = \exp[\frac{2\pi i}{k}(-ib\mu - m)]$, the affine shift $X \to X + (i\pi + \frac{\hbar}{2})t$ corresponds to $\mu \to \mu + \frac{1}{2}i(b + b^{-1})t$, $m \to m$, and adding an overall $(-1)^t$ to e^X.

$$\psi_{z_a}(\mu_a) = e^{-\sqrt{2}\beta_a \mathrm{Re}(z_a)} \psi^0_{z_a}(\mu_a), \tag{80}$$

where β_a is the component in $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_{\mathrm{new}}$. The prefactor does not affect the semiclassical behavior of ψ_z, but it relates to the finiteness of the amplitude. Note that $\{\beta_a\}_{a=1}^5$ cannot be zero because e.g., $\beta_1 = \alpha_{Z_2} + \alpha_{Z_3} > 0$ according to (46). It is still a viable choice to work with the normalized coherent state $\psi^0_{z_a}$; then, certain requirements should be implemented to the spinfoam edge amplitude. We come back to this point in Sec. III E.

We denote the coherent state in V_k by $\xi_{(x,y)}(m)$, where $(x, y) \in [0, 2\pi) \times [0, 2\pi)$ and $m \in \mathbb{Z}/k\mathbb{Z}$ [33],

$$\xi_{(x,y)}(m) = \left(\frac{2}{k}\right)^{\frac{1}{4}} e^{-\frac{ikxy}{4\pi}} \sum_{n\in\mathbb{Z}} e^{-\frac{k}{4\pi}(\frac{2\pi m}{k} - 2\pi n - x)^2} e^{-\frac{ik}{2\pi} y (\frac{2\pi m}{k} - 2\pi n - x)}. \tag{81}$$

Note that (x, y) relates to the classical phase space coordinates m_0, n_0 by

$$y = \frac{2\pi}{k} n_0, \qquad x = \frac{2\pi}{k} m_0, \quad \mathrm{mod}\ 2\pi \tag{82}$$

and $\xi_{(x,y)}(m)$ satisfy the overcompleteness relation in V_k,

$$\frac{k}{4\pi^2} \int_{\mathbb{T}^2} \mathrm{d}x\mathrm{d}y \xi_{(x,y)}(m) \bar{\xi}_{(x,y)}(m') = \delta_{m,m'}. \tag{83}$$

We define coherent states in $\mathcal{H}_{\mathcal{S}_a}$ by tensor products

$$\psi_{z_a} \otimes \xi_{(p_a,q_a)} \in \mathcal{H}_{\mathcal{S}_a}. \tag{84}$$

Note that $z_a, \bar{z}_a, x_a, y_a$ coordinatize the part of the phase space associated to $\mathcal{S}_a$, and they form a coordinate system on the moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ (with fixed $e^{2L_{ab}}$). We get the following relation:

$$\bar{\psi}_{z_a} \otimes \bar{\xi}_{(x_a,y_a)} = \psi_{\bar{z}_a} \otimes \xi_{(x_a,-y_a)}. \tag{85}$$

We multiply the coherent states over five $\mathcal{S}_a$,

$$\Psi_\rho(\{\mu_a\}|\{m_a\}) = \prod_{a=1}^5 \psi_{z_a}(\mu_a) \xi_{(x_a,y_a)}(m_a) \in \otimes_a \mathcal{H}_{\mathcal{S}_a}$$
$$\rho = \{z_a, x_a, y_a\}_{a=1}^5, \tag{86}$$

where $\mu_a \in \mathbb{R}$. The partition function $\mathcal{Z}_{S^3\setminus\Gamma_5}$ is a function of $\vec{\mu}, \vec{m}$, including μ_a, m_a. We consider the (partial) L^2 inner product between $\mathcal{Z}_{S^3\setminus\Gamma_5}$ and $\bar{\Psi}_\rho$ (this may be understood as $\mathcal{Z}_{S^3\setminus\Gamma_5}$ acting on $\bar{\Psi}_\rho$ since $\mathcal{Z}_{S^3\setminus\Gamma_5}$ is a tempered distribution),

$$\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota) = \langle \bar{\Psi}_\rho | \mathcal{Z}_{S^3\setminus\Gamma_5} \rangle_{\otimes_a \mathcal{H}_{\mathcal{S}_a}} = \sum_{\{m_a\}\in(\mathbb{Z}/k\mathbb{Z})^5} \int_{\mathbb{R}^5} \prod_{a=1}^5 \mathrm{d}\mu_a \mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{\mu} + i\vec{\alpha}|\vec{m}) \Psi_\rho(\{\mu_a\}|\{\vec{m}_a\}), \tag{87}$$

where $\vec{\mu} + i\vec{\alpha} \in \mathrm{strip}(\mathfrak{P}_{\mathrm{new}})$. Here, $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$ is a function of

$$\iota = (\{\mu_{ab} + i\alpha_{ab}, m_{ab}\}_{a<b}, \{z_a, x_a, y_a\}_{a=1}^5, \{\alpha_a, \beta_a\}_{a=1}^5), \quad \mu_{ab} \in \mathbb{R}, \quad m_{ab} \in \mathbb{Z}/k\mathbb{Z}, \quad z_a \in \mathbb{C}, \quad (x_a, y_a) \in \mathbb{T}^2, \tag{88}$$

which includes the position variables of annuli and both the position and momentum variables of 4-holed spheres. Note that ι determines a unique $\mathrm{PSL}(2,\mathbb{C})$ flat connection on each $\mathcal{S}_a$: Given an ι and using (79) and (82), z_a, x_a, y_a determine phase space coordinates that relate to FG coordinates by (73)–(76). The resulting FG coordinates and $e^{2L_{ab}}$ given by μ_{ab}, m_{ab} of the same ι determine a unique $\mathrm{PSL}(2,\mathbb{C})$ flat connection on $\mathcal{S}_a$.

Theorem II.1. Fixing the annulus data $\{\mu_{ab}, m_{ab}\}_{a<b}$, $|\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)|$ is bounded for all $\{z_a, x_a, y_a\}_{a=1}^5$.

Proof: In $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$, the sum over $\vec{m}'$ is finite, and for any m,

$$\xi_{(x,y)}(m) = \frac{\sqrt{4}2 e^{-\frac{ky(y+ix)}{4\pi}} \vartheta_3(\frac{1}{2}(-\frac{2\pi m}{k} + x - iy), e^{-\frac{\pi}{k}})}{k^{3/4}}$$

is smooth in $(x, y) \in [0, 2\pi) \times [0, 2\pi) \simeq \mathbb{T}^2$; thus, $|\xi_{(x,y)}(m)|$ is bounded on $\mathbb{T}^2$ for any m. Therefore, the boundedness of $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$ is implied by the boundedness of the following integral for all $\vec{m}$:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^5}\prod_{a=1}^{5}\mathrm{d}\mu'_a\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}'+i\vec{\alpha}|\vec{m}')\prod_{a=1}^{5}\psi_{z_a}(\vec{\mu}'_a)\right|\\
&=\left|e^{-\sqrt{2}\sum_a\beta_a\mathrm{Re}(z_a)}\int_{\mathbb{R}^5}\prod_{a=1}^{5}\mathrm{d}\mu'_a\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}'+i\vec{\alpha}|\vec{m}')\prod_{a=1}^{5}\psi^0_{z_a}(\vec{\mu}'_a)\right|\\
&\leq\left(\frac{1}{k}\right)^{\frac{5}{4}}e^{-\sqrt{2}\sum_a\beta_a\mathrm{Re}(z_a)}\int_{\mathbb{R}^5}\prod_a\mathrm{d}\mu'_a|\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}'+i\vec{\alpha}|\vec{m}')e^{-\frac{2\pi}{k}\sum_a\beta_a\mu'_a}|\prod_{a=1}^{5}|\psi^0_{z_a}(\vec{\mu}'_a)e^{\frac{2\pi}{k}\beta_a\mu'_a}|\\
&\leq C\left(\frac{1}{k}\right)^{\frac{5}{4}}e^{-\sqrt{2}\sum_a\beta_a\mathrm{Re}(z_a)}\int_{\mathbb{R}^5}\prod_a\mathrm{d}\mu'_a e^{-\frac{\pi}{k}\sum_a(\mu'_a-\frac{k}{\pi\sqrt{2}}\mathrm{Re}(z_a))^2}e^{\frac{2\pi}{k}\sum_a\beta_a\mu'_a}\\
&=Ck^{\frac{5}{4}}e^{\sum_a\frac{\pi\beta_a^2}{k}}.
\end{aligned}\tag{89}$$

In the third step we use $\mathcal{Z}_{S^3\backslash\Gamma_5}\in\mathcal{F}^{(k)}_{\mathfrak{P}_{\mathrm{new}}}$; thus, as a function of μ'_a $(a=1,\ldots,5)$, $\forall\,(\vec{\alpha},\vec{\beta})\in\mathfrak{P}_{\mathrm{new}}$,

$$e^{-\frac{2\pi}{k}\sum_a\mu'_a\beta_a}\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}'+i\vec{\alpha}|\vec{m})\in\mathcal{S}(\mathbb{R}^5),\tag{90}$$

where C is the upper bound of $|e^{-\frac{2\pi}{k}\sum_a\mu'_a\beta_a}\mathcal{Z}_{S^3\backslash\Gamma_5}\times(\vec{\mu}'+i\vec{\alpha}|\vec{m})|$. ■

III. SPINFOAM AMPLITUDE WITH A COSMOLOGICAL CONSTANT

The purpose of this section is to impose the simplicity constraint on $\mathcal{Z}_{S^3\backslash\Gamma_5}(\iota)$ in order to relate the CS partition function to the spinfoam vertex amplitude in 4d. The simplicity constraint reduces the $\mathrm{PSL}(2,\mathbb{C})$ flat connection to PSU(2) on five $\mathcal{S}_a$'s. Based on the resulting vertex amplitude, we define the spinfoam amplitude with Λ on any simplicial complex and prove its finiteness, as well as discuss several related perspectives.

A. Simplicity constraint and vertex amplitude

In the simplicial context with $\Lambda=0$, the simplicity constraint (in the EPRL/FK model) imposes that for any spacelike tetrahedron e, there exists a timelike unit vector N^I in 4d Minkowski space such that $B_f^{IJ}N_J=0$, where B_f^{IJ} $(f=1,\ldots,4)$ are bivectors associated to four faces f. The simplicity constraint and closure condition endow every e with a convex geometrical tetrahedron in flat space. Indeed, the B_f^{IJ} that satisfy the constraint are equivalent to 3d vectors $\mathfrak{a}_f\mathfrak{n}_f^I=\frac{1}{2}\epsilon^{IJKL}N_JB_{KL}$ $(\mathfrak{n}^I\mathfrak{n}_I=1)$ in the plane normal to N^I. Then, the BF closure condition $\sum_{f=1}^4B^{IJ}=0$ implies $\sum_{f=1}^4\mathfrak{a}_f\mathfrak{n}_f^I=0$, which endows e with a convex geometrical tetrahedron (whose face areas and normals are $\mathfrak{a}_f$ and $\mathfrak{n}_f^I$) using Minkowski's theorem [34]. At the quantum level, the first-class part of the simplicity constraint, i.e., the diagonal simplicity constraint $\epsilon_{IJKL}B_f^{IJ}B_f^{KL}=0$, is imposed strongly on the states, whereas the second-class part of the simplicity constraint is weakly imposed [14,20,35].

In the presence of nonvanishing Λ, $\Gamma_5\subset S^3$ is the dual graph of the triangulation of S^3 given by the 4-simplex's boundary. Each node of Γ_5, or each $\mathcal{S}_a\subset\partial(S^3\backslash\Gamma_5)$, is dual to a boundary tetrahedron e_a of the 4-simplex. Each link of Γ_5, or each annulus $\ell_{ab}\subset\partial(S^3\backslash\Gamma_5)$, is dual to a boundary triangle f_{ab} of the 4-simplex. All tetrahedra and triangles are spacelike, similar to the EPRL/FK model. Given any e_a, the generalization of the closure condition is the defining equation of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on the 4-holed sphere $\mathcal{S}_a$: $O_4O_3O_2O_1=1$, where $O_{f=1,\ldots,4}\in\mathrm{PSL}(2,\mathbb{C})$ are holonomies around four holes based at a common point $\mathfrak{p}_a\in\mathcal{S}_a$. Using the non-Abelian Stokes theorem, we identify $O_f=e^{|\Lambda|B_f/3}\in\mathrm{SO}(1,3)^+$ due to the relation $\mathcal{F}(\mathcal{A})=|\Lambda|B/3$ from integrating out B in (3). Here $\mathcal{F}(\mathcal{A})$, as the curvature of the CS connection $\mathcal{A}$ on S^3, is proportional to the delta function supported on Γ_5 (equivalent to the fact that $\mathcal{A}$ is flat on $S^3\backslash\Gamma_5$). Namely, $\mathcal{F}(\mathcal{A})=\frac{|\Lambda|}{3}B_f\delta^2(x)\mathrm{d}x^1\wedge\mathrm{d}x^2$ on face f coordinated by (x^1,x^2) transverse to an edge of Γ_5 at $\vec{x}=0$. $O_4O_3O_2O_1=1$ with $O_f=e^{|\Lambda|B_f/3}$ reduces to the linear closure condition $\sum_{f=1}^4B_f=0$ as $\Lambda\to0$. Moreover, the simplicity constraint $B_f^{IJ}N_J=0$ for all $f=1,\ldots,4$ restricts $O_{f=1,\ldots,4}$ to a common PSU(2) subgroup stabilizing the timelike vector N^I. The result in [21] shows that restricting all O_f to the subgroup PSU(2) endows e with a convex geometrical tetrahedron with constant curvature. The effect of restricting O_f to PSU(2) is analogous to the simplicity constraint reviewed above. This motivates us to define this restriction to be the simplicity constraint in the presence of nonvanishing Λ [36]:

Definition III.1. Semiclassically, in the presence of a nonvanishing cosmological constant, the simplicity constraint restricts the moduli spaces of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on 4-holed spheres to the ones that can be gauge transformed to $\mathrm{PSU}(2)\simeq\mathrm{SO}(3)$ flat connections.

1. First-class constraints

Our proposal is to quantize and impose the simplicity constraint on $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$. First, flat connections on all $\mathcal{S}_a$ are PSU(2), which implies $e^{2L_{ab}} \in \mathrm{U}(1)$, or equivalently $\mu_{ab} = 0$ for all annuli ℓ_{ab}. However, due to the presence of $\alpha_{ab} = \mathrm{Im}(\mu_{ab})$, at the quantum level we may have to decide whether we impose

$$\Re(\boldsymbol{\mu}_{ab})\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota) = 0 \quad \text{or} \quad \boldsymbol{\mu}_{ab}\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota) = 0. \tag{91}$$

In either case, these ten constraints are first class since $\{\boldsymbol{\mu}_{ab}\}_{a<b}$ are commutative; thus, they can be imposed strongly on $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$. Note that $\{\boldsymbol{\mu}_{ab}\}_{a<b}$ are multiplication operators acting on $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$. The former choice restricts

$$\mathrm{Re}(\mu_{ab}) = 0, \quad \forall\ \ell_{ab} \tag{92}$$

in ι. The latter choice restricts both $\mathrm{Re}(\mu_{ab})$ and the positive angle structure,

$$\mathrm{Re}(\mu_{ab}) = 0 \quad \text{and} \quad \alpha_{ab} = 0, \quad \forall\ \ell_{ab}, \tag{93}$$

and thus, it is much stronger than the former choice. However, the semiclassical limit of the theory is insensitive to the choices: Consider the former (weaker) choice; $e^{2L_{ab}}$ determined by ι is given by

$$\begin{aligned} e^{2L_{ab}} &= (-1)^{t_{ab}} \exp\left[\frac{2\pi i}{k}(b\alpha_{ab} - m_{ab})\right] \\ &= \exp\left[\frac{2\pi i}{k}\left(b\alpha_{ab} - \left(m_{ab} + t_{ab}\frac{k}{2}\right)\right)\right] \\ &= \exp\left[\frac{2\pi i}{k}\left(b\alpha_{ab} + \left(2j_{ab} + \frac{\epsilon_{ab}}{2}\right)\right)\right] \end{aligned} \tag{94}$$

where $\alpha_{ab} = \mathrm{Im}(\mu_{ab})$. In the last step, since $-(m_{ab} + t_{ab}\frac{k}{2}) \in \mathbb{Z}/k\mathbb{Z}$ (or $\mathbb{Z}/k\mathbb{Z} + 1/2$) if k is even (or odd), we introduce the half-integer "spin" j_{ab} such that $-(m_{ab} + t_{ab}\frac{k}{2}) = 2j_{ab} + \frac{\epsilon_{ab}}{2} \bmod k\mathbb{Z}$ where

$$\epsilon_{ab} = \begin{cases} \frac{1-(-1)^{t_{ab}}}{2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \tag{95}$$

$$j_{ab} = 0, \frac{1}{2}, \ldots, \frac{k-1}{2}. \tag{96}$$

The double-scaling limit $j_{ab}, k \to \infty$ with j_{ab}/k fixed is the semiclassical limit for the spinfoam amplitude with a cosmological constant (see Sec. IV for a discussion). In this limit, $e^{2L_{ab}}$ is insensitive to α_{ab}, ϵ_{ab} since they do not scale with k,

$$e^{2L_{ab}} \to \exp\left[\frac{4\pi i}{k} j_{ab}\right] \in \mathrm{U}(1). \tag{97}$$

Both choices in (91) lead to the same semiclassical result. At least semiclassically, each holonomy around holes on $\mathcal{S}_a$ can be individually conjugated to PSU(2), while j_{ab}/k determines the conjugacy class of the holonomy.

The stronger choice (93) is indeed viable. We can have $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_{\text{new}}$ with ten $\alpha_{ab} = 0$ because, for instance, all ten $\alpha_{ab} = 0$ can be given by $\alpha_{X_a} = \alpha_{Y_a} = \alpha_{Z_a} = Q/4$ and $\beta_{X_a} = \beta_{Y_a} = \beta_{Z_a} = 0$ $(a = 1, \ldots, 5)$, which satisfy (46). The simplicity constraint results in, restrictively, $e^{2L_{ab}} \in \mathrm{U}(1)$ when $\alpha_{ab} = 0$, whereas $e^{2L_{ab}} \notin \mathrm{U}(1)$ for other $\alpha_{ab} \neq 0$. Note that $\alpha_{ab} = 0$ is a preferred choice because $e^{2L_{ab}} \in \mathrm{U}(1)$ implies that after imposing the simplicity constraint, the area from the 4d bivector B_f coincides with the face area of the 3d tetrahedron at the quantum level: Recall the discussion above Definition III.1. We diagonalize an $O_f \in \mathrm{PSL}(2,\mathbb{C})$ by a gauge transformation

$$\begin{aligned} O_f &= \pm\mathrm{diag}(e^{L_{ab}}, e^{-L_{ab}}) = \pm e^{\mathrm{Re}(L_{ab})\boldsymbol{\sigma}^3 + i\mathrm{Im}(L_{ab})\boldsymbol{\sigma}^3} \\ &\leftrightarrow e^{2\mathrm{Re}(L_{ab})\mathbf{K}^3 - 2\mathrm{Im}(L_{ab})\mathbf{L}^3} = e^{\frac{|\Lambda|}{3}B_f} \in \mathrm{SO}(1,3)^+ \end{aligned}$$

where $\mathrm{Im}(L_{ab}) \in [0, \pi)$ and $\mathbf{K}^3, \mathbf{L}^3$ are $so(1,3)$ generators. We obtain $\frac{|\Lambda|}{3}B_f = 2\mathrm{Re}(L_{ab})\mathbf{K}^3 - 2\mathrm{Im}(L_{ab})\mathbf{L}^3$ for the preferred lift of B_f. Then, L_{ab} relates to the area from the 4d bivector, $|B_f| = |\frac{1}{2}\mathrm{Tr}(B_f^2)|^{1/2}$, by $\frac{|\Lambda|}{3}|B_f| = 2|\mathrm{Re}(L_{ab})^2 - \mathrm{Im}(L_{ab})^2|^{1/2}$. Restricting $\alpha_{ab} = 0$ and the simplicity constraint $\mathrm{Re}(\mu_{ab}) = 0$, we get

$$\frac{|\Lambda|}{3}|B_f| = 2\mathrm{Im}(L_{ab}) = \frac{4\pi}{k}(j_{ab} + \epsilon_{ab}/4) \equiv \frac{|\Lambda|}{3}\mathfrak{a}_{ab}, \tag{98}$$

where $\mathfrak{a}_{ab}$ is the face area of the 3d tetrahedron (this is implied by the generalized closure condition, see [21] or the discussion below). Both $\mathcal{Z}_{S^3\setminus\Gamma_5}$ and Ψ_ι are functions of L_{ab}; thus, both the 4d and 3d area operators, $\frac{|\Lambda|}{3}|B_f| = 2|\mathrm{Re}(L_{ab})^2 - \mathrm{Im}(L_{ab})^2|^{1/2}$ and $\frac{|\Lambda|}{3}\mathfrak{a}_{ab} = 2\mathrm{Im}(L_{ab})$, act as multiplications. The above shows that these two operators coincide when $\alpha_{ab} = 0$. A similar consistency constraint "4d area = 3d area" has also been imposed on the EPRL model [20].

However, to keep discussions general, we still use the weaker version (92) and keep α_{ab} general in the following discussion. But we prefer $\alpha_{ab} = 0$ using the above argument.

2. Second-class constraints

The first-class part of the simplicity constraint and j_{ab} fix $e^{2L_{ab}}$ on ten annuli. Classically, fixing $e^{2L_{ab}}$ reduces the moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ to two complex dimensions whose Darboux coordinates $\vartheta, \varphi \in \mathbb{C}$

are studied in [37], with $\{\vartheta, \varphi\} = 1$ (they are the complexification of θ, ϕ in Sec. III B). Constraining flat connections to PSU(2) restricts $\mathrm{Im}(\vartheta) = \mathrm{Im}(\varphi) = 0$. The restriction gives second-class constraints due to the noncommutativity of ϑ, φ. Using the lessons from the EPRL/FK model, the constraints have to be imposed weakly at the quantum level. Our strategy is to impose the constraints on the label (z_a, x_a, y_a) where the coherent state Ψ_ρ is peaked. Here, (z_a, x_a, y_a) is a point in the moduli space of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ with fixed $e^{2L_{ab}}$'s. We restrict (z_a, x_a, y_a) to the subspace of flat connections that can be gauge transformed to PSU(2).

Classically, our simplicity constraint is an analog of the linear simplicity constraint in the EPRL/FK model, as discussed at the beginning of this subsection. At the quantum level, although all spinfoam models weakly impose the second-class simplicity constraint, here the constraint is imposed on the coherent state labels, similar to the FK model [35] but different from the EPRL model where the constraint is imposed by a master constraint operator.

Although the following discussion does not assume large j_{ab}, before Eq. (108), we ignore α_{ab} so that $e^{2L_{ab}} \in \mathrm{U}(1)$ is assumed since we are only concerned with the semiclassical simplicity constraint here. After Eq. (108) we take into account, in general, $\alpha_{ab} \neq 0$ and $e^{2L_{ab}} \notin \mathrm{U}(1)$ at the quantum level.

On the 4-holed sphere $\mathcal{S}_a$, flat connections that can be gauge transformed to PSU(2) are described by four $\mathrm{PSL}(2,\mathbb{C})$ holonomies O_1, O_2, O_3, O_4 that can be simultaneously conjugated to PSU(2). Here, O_1, O_2, O_3, O_4 are based at a common point $\mathfrak{p}$, and each of them travels around a hole of $\mathcal{S}_a$. As holonomies of flat connections, they satisfy the generalized closure condition

$$O_4 O_3 O_2 O_1 = 1. \tag{99}$$

This equation is invariant under the $\mathrm{PSL}(2,\mathbb{C})$ gauge transformation. We apply the gauge transformation to make all $O_i \in \mathrm{PSU}(2)$, and we treat (99) as an equation of PSU(2) holonomies. The conjugacy class of each O_i has been fixed by (97), which specifies the squared eigenvalue of O_i. There exists a lift from O_i to $H_i \in \mathrm{SU}(2)$ such that

$$H_i = M(\xi_i) \begin{pmatrix} \pm e^{\frac{2\pi i}{k} j_i} & 0 \\ 0 & \pm e^{-\frac{2\pi i}{k} j_i} \end{pmatrix} M(\xi_i)^{-1}, \tag{100}$$

$$M(\xi) = \begin{pmatrix} \xi^1 & -\bar{\xi}^2 \\ \xi^2 & \bar{\xi}^1 \end{pmatrix}, \tag{101}$$

satisfying

$$H_4 H_3 H_2 H_1 = 1. \tag{102}$$

In each H_i, we neglect ϵ_{ab} when discussing the parametrization of PSU(2) flat connections,

$$j_i = j_{ab},$$

as ℓ_{ab} ends at the hole labeled by i, and similarly for t_i. Note that $\xi_i = (\xi_i^1, \xi_i^2)^T$ is defined up to a complex scaling by the above formula of H_i. If we fix $\det(M(\xi_i)) = 1$,

$$\vec{n}_i = \xi_i^\dagger \vec{\boldsymbol{\sigma}} \xi_i, \quad i = 1, \dots, 4, \quad \text{where } \vec{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3) \text{ are Pauli matrices} \tag{103}$$

giving four unit 3-vectors in $\mathbb{R}^3$. The geometrical interpretation of (99) relates the holonomies to a geometrical 3d tetrahedron with constant curvature (see [13,21] or Theorem IV.2), in which $\frac{4\pi}{k} j_i = \frac{|\Lambda|}{3} \mathfrak{a}_i$ is the face area and $\vec{n}_i$ are face normals parallel transported to a common vertex of the tetrahedron.[6] Note that $\{\vec{n}_i\}_{i=1}^4$ relates to the outward pointing normals $\{\mathfrak{n}_i\}_{i=1}^4$ of the tetrahedron by $\mathfrak{n}_i = \mathrm{sgn}(\Lambda) \vec{n}_i$. Equation (102) with $H_i = e^{\Lambda \vec{v}_i \cdot \vec{\boldsymbol{\sigma}}}$ reduces to the flat closure condition $\sum_i \vec{v}_i = 0$ for small Λ.

To clarify our convention, consider ℓ_{ab} connecting the ith hole of $\mathcal{S}_a$ to the jth hole of $\mathcal{S}_b$. We choose the framing flag $s_{\ell_{ab}}$ of ℓ_{ab} such that on $\mathcal{S}_a$, the eigenvector of the holonomy $O_i \equiv O_{ab}$, $\xi_i \equiv \xi_{ab}$ coincides with $s_{\ell_{ab}}$ parallel transported to the common base point $\mathfrak{p}_a \in \mathcal{S}_a$ of $\{O_i\}_{i=1}^4$. If our convention is (99) on both $\mathcal{S}_a$ and $\mathcal{S}_b$, the parallel transport of $O_i \equiv O_{ab}$ of $\mathcal{S}_a$ gives $O_j^{-1} \equiv O_{ba}$ of $\mathcal{S}_b$, i.e., $G_{ab}^{-1} O_{ab} G_{ab} = O_{ba}$ with a holonomy G_{ab} along ℓ_{ab}. Here, $s_{\ell_{ab}}$ evaluated at a point $\mathfrak{p}_b \in \mathcal{S}_b$ gives ξ_{ba} as the eigenvector of O_{ba} with upper eigenvalue $\pm e^{2\pi i j_i/k}$. But ξ_{ba} does not equal $\xi_j = (\xi_j^1, \xi_j^2)^T$ on $\mathcal{S}_b$, but it does equal $(-\bar{\xi}_j^2, \bar{\xi}_j^1)^T$ in the convention of (100).[7]

If a minus sign is present in (100), we write $-e^{\frac{2\pi i}{k} j} = e^{-\frac{2\pi i}{k} j'}$, where $j' = k/2 - j$; then, Eq. (100) can be rewritten as

$$H_i = M'(\xi_i) \begin{pmatrix} e^{\frac{2\pi i}{k} j_i'} & 0 \\ 0 & e^{-\frac{2\pi i}{k} j_i'} \end{pmatrix} M'(\xi_i)^{-1}, \tag{104}$$

$$M'(\xi) = \begin{pmatrix} -\bar{\xi}^2 & -\xi^1 \\ \bar{\xi}^1 & -\xi^2 \end{pmatrix}. \tag{105}$$

If there is a plus sign in (100), we set $j' = j$. Flipping $+ \to -$ in (100) corresponds to $j \to k/2 - j$ and $M(\xi) \to M'(\xi)$.

[6]Note that $\frac{4\pi}{k} j_i = \frac{|\Lambda|}{3} \mathfrak{a}_i$ mismatches (98) if $\epsilon_{ab} \neq 0$, but this is not a problem since here we discuss coherent state labels, whereas (98) is about operator eigenvalues.

[7]The inverse of H_i in (100) can be written as $H_i^{-1} = \pm M'(\xi_i) \mathrm{diag}(e^{\frac{2\pi i}{k} j_i}, e^{-\frac{2\pi i}{k} j_i}) M'(\xi_i)^{-1}$ where $M'(\xi)$ is given by (105).

Lemma III.1. The lifts $H_{i=1,...,4} \in \mathrm{SU}(2)$ satisfy $H_4H_3H_2H_1 = 1$, which exist if and only if $j'_{i=1,...,4}$ satisfy the triangle inequality, i.e., there exists J such that

$$|j'_1 - j'_2| \le J \le \min(j'_1 + j'_2, k - j'_1 - j'_2), \qquad (106)$$

$$|j'_3 - j'_4| \le J \le \min(j'_3 + j'_4, k - j'_3 - j'_4). \qquad (107)$$

The proof of this Lemma is given in Appendix D. Equations (106) and (107) agree with the spin-coupling rule of $\mathrm{SU}(2)_\mathfrak{q}$ with $\mathfrak{q} = e^{\pi i/(k+2)}$.

Lemma III.2. $O_4O_3O_2O_1 = 1$ has the solution $O_i \in \mathrm{PSU}(2)$ if j_i given by (97) equals either j'_i or $k/2 - j'_i$, where $\{j'_i\}$ satisfies the triangle inequality (106) and (107).

Proof: Given a solution $H_i \in \mathrm{SU}(2)$ to $H_4H_3H_2H_1 = 1$, both $\pm H_i$ project to $O_i \in \mathrm{PSU}(2)$, solving $O_4O_3O_2O_1 = 1$. If H_i is given by (104) with $j' = k/2 - j$,

$$-H_i = M(\xi_i)\begin{pmatrix} e^{\frac{2\pi i}{k}(k/2-j'_i)} & 0 \\ 0 & e^{-\frac{2\pi i}{k}(k/2-j'_i)} \end{pmatrix} M(\xi_i)^{-1}.$$

Since both $\pm H_i$ are allowed for the PSU(2) equation, j_i is given by the squared eigenvalue (97) of either H_i or $-H_i$, and thus can be either j'_i or $k/2 - j'_i$. ■

We restrict j_{ab} to satisfy the condition in Lemma III.2 so that $O_4O_3O_2O_1 = 1$ has a solution at every $\mathcal{S}_a$. The triangle inequality in Lemma III.1 is the analog of the triangle inequality for SU(2) intertwiners in spinfoam models without a cosmological constant.

The eigenvector of the holonomy O_i, $\xi'_i = (\xi^1_i, \xi^2_i)^T$ or $(-\bar{\xi}^2_j, \bar{\xi}^1_j)^T$ is the framing flag s_ℓ (of ℓ connecting the hole i) parallel transported to the base point $\mathfrak{p}$ of O_i, i.e.,

$$\xi'_i = s_\ell(\mathfrak{p}), \qquad \mathfrak{p} \in \mathcal{S}_a. \qquad (108)$$

The FG coordinates on $\mathcal{S}_a$ can be expressed in terms of ξ'_i: Without loss of generality, we assume that $\mathfrak{p}$ is inside the quadrilateral shown in Fig. 6, and each O_i travels around the hole i counterclockwise. We have

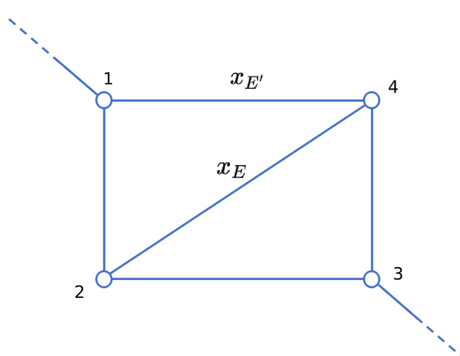

FIG. 6. Ideal triangulation of a 4-holed sphere.

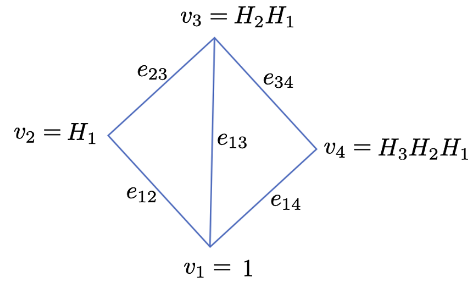

FIG. 7. The 4-gon in SU(2) determined by $H_4H_3H_2H_1 = 1$.

$$x_E(\vec{j},\vec{\xi}) = \frac{\langle \xi'_1 \wedge \xi'_2\rangle\langle \xi'_4 \wedge \xi'_3\rangle}{\langle \xi'_1 \wedge \xi'_4\rangle\langle \xi'_2 \wedge \xi'_3\rangle},$$
$$x_{E'}(\vec{j},\vec{\xi}) = \frac{\langle O_4\xi'_3 \wedge \xi'_1\rangle\langle \xi'_4 \wedge \xi'_2\rangle}{\langle O_4\xi'_3 \wedge \xi'_4\rangle\langle \xi'_1 \wedge \xi'_2\rangle}. \qquad (109)$$

Here, O_4 is given by

$$O_4 = M(\xi'_4)\begin{pmatrix} \pm e^{L_{ab}} & 0 \\ 0 & \pm e^{-L_{ab}} \end{pmatrix} M(\xi'_4)^{-1}, \qquad (110)$$

where $\pm e^{L_{ab}} = \pm\exp[\frac{\pi i}{k}(b\alpha_{ab} + (2j_{ab} + \frac{\epsilon_{ab}}{2}))]$ for ℓ_{ab} attached to the fourth hole. Here, $x_{E'}$ is independent of the $\pm$ sign. Both $x_E(\vec{j},\vec{\xi})$, $x_{E'}(\vec{j},\vec{\xi})$ are invariant under the $\mathrm{PSL}(2,\mathbb{C})$ gauge transformation of (99): $O_i \to hO_ih^{-1}$, $\xi'_i \to h\xi'_i$.

The correspondence between $\{x_E\}_E$'s and framed $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ is 1-to-1 [29], so $x_E, x_{E'}$ given by (109) and four $e^{2L_{ab}}$ at the holes uniquely determine a $\mathrm{PSL}(2,\mathbb{C})$ flat connection labeled by $\vec{j},\vec{\xi}$. This connection reduces to PSU(2) when $\alpha_{ab} = 0$. We choose E, E' to be such that x_E, $x_{E'}$ equals $e^{\mathcal{X}_a}$, $e^{\mathcal{Y}_a}$ in $(e^{\mathcal{Q}_I}, e^{\mathcal{P}_I})$. We lift x_E, $x_{E'}$ to logarithmic coordinates $\chi_E = \log(x_E), \chi_{E'} = \log(x_{E'})$ [the lift is uniquely given by (41) and the lifts of ideal-tetrahedra coordinates] and obtain $\mathcal{X}_a$, $\mathcal{Y}_a$ as functions of $\vec{j},\vec{\xi}$. Using (73)–(76), we have $\mu_a, \nu_a, m_a, n_a \in \mathbb{R}$ as functions of $\vec{j},\vec{\xi}$. Furthermore, using (79) and (82), we uniquely obtain the functions $z_a(\vec{j},\vec{\xi})$, $x_a(\vec{j},\vec{\xi})$, and $y_a(\vec{j},\vec{\xi})$.

Recalling (88), the implementation of the simplicity constraint restricts the label ι to the subspace

$$\iota_{\vec{j},\vec{\xi}} = (\{0, m_{ab}\}_{a<b}, \{\rho^{(a)}_{\vec{j},\vec{\xi}}\}^5_{a=1}),$$
$$\rho^{(a)}_{\vec{j},\vec{\xi}} = (z_a(\vec{j},\vec{\xi}), x_a(\vec{j},\vec{\xi}), y_a(\vec{j},\vec{\xi})),$$

where $\vec{j} = \{j_{ab} + \epsilon_{ab}/4\}_{a<b}$ and $\vec{\xi} = \{\xi_{ab}\}_{a,b=1,...,5}$. Note that m_{ab} relates to j_{ab} according to (94). Here, $\vec{j}$ has to satisfy the condition in Lemma III.2 so that the solution $O_{i=1,...,4} \in \mathrm{PSU}(2)$ to Eq. (99) exists, and $\vec{\xi}$ are eigenvectors of the solution $O_{i=1,...,4}$.

Therefore, the simplicity constraint restricts the partition function $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$ in (87) to

$$\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota_{\vec{j},\vec{\xi}}) \equiv A_v(\vec{j},\vec{\xi}), \tag{111}$$

which is defined to be the spinfoam vertex amplitude with a cosmological constant.

Note that only two FG coordinates x_E, $x_{E'}$ out of six are used in z_a, x_a, y_a. Only these two coordinates are restricted to be (109). The other four FG coordinates $x_{E''} \neq x_E, x_{E'}$ may not be simultaneously expressed in terms of $\vec{j}$, $\vec{\xi}'$ as (109) when $\alpha_{ab} \neq 0$, since otherwise $\lambda^2 = \prod_{E\,\text{around hole}} x_E$ would belong to U(1), whereas generally $e^{2L_{ab}} \notin \mathrm{U}(1)$ for $\alpha_{ab} \neq 0$. However, the other four $x_{E''} \neq x_E, x_{E'}$ are absent in the coherent label. Note that $\rho^{(a)}_{\vec{j},\vec{\xi}}$ is generally a $\mathrm{PSL}(2,\mathbb{C})$ flat connection, but it reduces to PSU(2) when $\alpha_{ab} = 0$ or in the semiclassical limit.

B. SU(2) flat connections on $\mathcal{S}_a$ and 4-gon

A simple counting of degrees of freedom shows that $\vec{\xi}$'s solving $O_4O_3O_2O_1 = 1$ modulo PSU(2) gauge transformations generically span real 2-dimensional space. This 2-dimensional space is denoted by $\mathcal{M}_{\vec{j}}$. Note that x_E, $x_{E'}$ in (109) are densely defined functions on $\mathcal{M}_{\vec{j}}$.

A description of $\mathcal{M}_{\vec{j}}$ [37] generalizes the Kapovich-Millson phase space description [38,39]: We lift to the cover space $\tilde{\mathcal{M}}_{\vec{j}}$ the moduli space of an SU(2) flat connection with fixed $\vec{j}$. Here, $\tilde{\mathcal{M}}_{\vec{j}}$ is the moduli space of solutions to $H_4H_3H_2H_1 = 1$ with

$$H_i = M(\xi_i)\begin{pmatrix} e^{\frac{2\pi i}{k}j_i} & 0 \\ 0 & e^{-\frac{2\pi i}{k}j_i}\end{pmatrix} M(\xi_i)^{-1},$$

where $j_i = j_{ab}$ of annuli ℓ_{ab} connecting to the holes.

Given the 4-dimensional complex vector space $V = \mathrm{Mat}_{2\times 2}(\mathbb{C}) \simeq \mathbb{C}^4$ of complex 2×2 matrices, we endow V with the complex metric $\langle X, Y\rangle = -\frac{1}{2}[\mathrm{Tr}(XY) - \mathrm{Tr}X\mathrm{Tr}Y]$. If we write $X = x^0 I + \sum_{a=1}^3 x^a \boldsymbol{\sigma}_a$ and $Y = y^0 I + \sum_{a=1}^3 y^a \boldsymbol{\sigma}_a$, $\langle X, Y\rangle$ is the complexified Minkowski metric on $\mathbb{C}^4$: $\langle X, Y\rangle = x^0y^0 - \sum_{a=1}^3 x^a y^a$. Here, SU(2) is the unit 3-sphere in $V_{\mathbb{R}} \simeq \mathbb{R}^4 \subset V$ defined by

$$H = h^0 + i\sum_{a=1}^3 h^a \boldsymbol{\sigma}_a, h_0, \qquad h_a \in \mathbb{R},$$
$$\langle H, H\rangle = (h^0)^2 + \sum_{a=1}^3 (h^a)^2 = 1.$$

When restricting $h^0 + i\sum_{a=1}^3 h^a\boldsymbol{\sigma}_a$ with h_0, $h_a \in \mathbb{R}$, $\langle\cdot,\cdot\rangle$ becomes the Euclidean metric on $\mathbb{R}^4$ and induces the spherical metric of S^3 on SU(2).

Given $H_{1,\ldots,4} \in \mathrm{SU}(2)$ satisfying $H_4H_3H_2H_1 = 1$, the set of H_i determines four points $v_1, \ldots, v_4$ in SU(2) Fig. 7, where

$$v_1 = 1, \quad v_2 = H_1, \quad v_3 = H_2H_1, \quad v_4 = H_3H_2H_1.$$

We first assume the generic situation that $v_1, \ldots, v_4$ are linearly independent in $\mathbb{R}^4$. Any pair (v_i, v_j) viewed as two vectors in $\mathbb{R}^4$ determines a 2-plane $E_{ij} = \mathrm{Span}_{\mathbb{R}}(v_i, v_j) \subset \mathbb{R}^4$. The intersection between E_{ij} and SU(2) is the geodesic e_{ij} connecting v_i, v_j [SU(2) is the unit 3-sphere in $\mathbb{R}^4$],

$$e_{ij} = E_{ij} \cap \mathrm{SU}(2) = \{t_1v_i + t_2v_j | \\ t_1^2 + t_2^2 + 2t_1t_2\langle v_i, v_j\rangle = 1, t_1, t_2 \geq 0\}.$$

The vertices v_i and edges e_{12}, e_{23}, e_{34}, e_{14} make a 4-gon in SU(2). The geodesic distance θ_{ij} between v_i and v_j is given by

$$\cos(\theta_{ij}) = \langle v_i, v_j\rangle \equiv c_{ij}, \qquad \theta_{ij} \in (0, \pi).$$

The lengths of e_{12}, e_{23}, e_{34}, e_{14} are $a_i = \theta_{i,i+1}$ such that

$$\cos(a_i) = \mathrm{Tr}(H_i)/2.$$

We draw the diagonal geodesic connecting v_1, v_3. Here, θ_{13} is the length of the diagonal.

The face f_{ijk} with the vertices v_i, v_j, v_k is the intersection of $F_{ijk} = \mathrm{Span}_{\mathbb{R}}(v_i, v_j, v_k)$ and SU(2),

$$f_{ijk} = F_{ijk} \cap \mathrm{SU}(2) = \{t_1v_i + t_2v_j + t_3v_k | t_1, t_2, t_3 \geq 0, \\ t_1^2 + t_2^2 + t_3^2 + 2t_1t_2c_{ij} + 2t_1t_3c_{ik} + 2t_2t_3c_{jk} = 1\}.$$

The unit normal n_{ijk} of F_{ijk} is defined by $\langle f, n\rangle = 0$, $\forall f \in F_{ijk}$, and $\langle n, n\rangle = 1$. A choice of orientation of F_{ijk} corresponds to the sign of n. We define the bending angle $\phi_{ij} \in (0, \pi)$ by

$$\cos(\phi_{ij}) = \langle n_{ikl}, n_{jkl}\rangle. \tag{112}$$

Note that $\theta = \theta_{13}$, $\phi = \phi_{24}$ are symplectic coordinates of $\tilde{\mathcal{M}}_{\vec{j}}$ [37]. Up to isometries of S^3, (θ, ϕ) determines a unique 4-gon in $S^3 \simeq \mathrm{SU}(2)$ whose geodesic edge lengths relate to the conjugacy classes of H_i. Indeed, geodesic edge lengths a_i, $\theta \in (0, \pi)$ uniquely determine two triangles sharing the diagonal e_{13}, up to isometries of S^3. We break the translational symmetry by fixing $v_1 = 1$. The remaining symmetry is the rotation leaving $v_1 = (1, 0, 0, 0) \in \mathbb{R}^4$ invariant. We use the freedom of the rotation to fix the position of v_2, v_3 of the triangle (v_1, v_2, v_3). Fixing the position of the triangle (v_1, v_2, v_3) breaks the continuous rotational symmetry, and v_1, v_2, v_3 determine the hyperplane $F_{123} \subset \mathbb{R}^4$. The freedom of v_4 is equivalent to choosing the hyperplane F_{134}, which is determined by the bending angle ϕ up to a parity symmetry with respect to F_{123}. This parity symmetry can be fixed by also choosing the orientation of the bending flow, i.e., fixing the orientation of $n_{123} \wedge n_{134}$ (see Appendix E). As a result, $v_1, \ldots, v_4 \in \mathrm{SU}(2)$ are uniquely determined by (θ, ϕ) once we fix $v_1 = 1$ and the rotation symmetry. Here, $v_2 = H_1$, $v_3 = H_2 H_1$, $v_4 = H_3 H_2 H_1$ determine $H_{1,\ldots,4}$ with $H_4 = (H_3 H_2 H_1)^{-1}$. Using (100) and the given $\{j_i\}_{i=1}^4$, we obtain all ξ_i as the eigenvector of H_i whose squared eigenvalue is $e^{4\pi i j_i/k}$. We normalize ξ_i's by $\det(M(\xi_i)) = 1$ up to individual phases. As a result, all ξ_i's are functions of j_i and θ, ϕ. Appendix E provides an algorithm to determine ξ_i's from θ, ϕ in practice.

For any function f on $\mathcal{M}_{\vec{j}}$, f can be lifted to a function on $\tilde{\mathcal{M}}_{\vec{j}}$ and is invariant under $H_i \to -H_i$. We define the following integral on $\mathcal{M}_{\vec{j}}$:

$$\int_{\mathcal{M}_{\vec{j}}} \mathrm{d}\xi f = \int \mathrm{d}\theta \wedge \mathrm{d}\phi f. \tag{113}$$

The integral on the right-hand side is over the compact domain; thus, it is finite, provided that $|f|$ is bounded. The degenerate 4-gons with θ, $\phi = 0$ are included as boundaries of the integral. This integral is needed for gluing vertex amplitudes to construct spinfoam amplitudes on complexes.

It may happen that for certain $\vec{j}$, $\tilde{\mathcal{M}}_{\vec{j}}$ only contain a degenerate 4-gon (i.e., becoming an n-gon with $n < 4$) where a vector v_i is a linear combination of another two vectors v_j, v_k in $\mathbb{R}^4$. In this case the dimension of $\tilde{\mathcal{M}}_{\vec{j}}$ is less than 2; thus, the above integral is ill-defined. The degenerate 4-gon leads to at least two H_i's belonging to a U(1) subgroup in SU(2). It sometimes gives a pair of collinear ξ_i''s that result in ill-defined x_E, $x_{E'}$ on the entire $\tilde{\mathcal{M}}_{\vec{j}}$ [see (109)]. We set the contribution from $\vec{j}$ such that $\dim(\tilde{\mathcal{M}}_{\vec{j}}) < 2$ so that it vanishes in the spinfoam amplitude. In particular, we set the contribution of $j_i = 0$ to vanish.

C. Finite spinfoam amplitude on a simplicial complex

Given a simplicial complex $\mathcal{K}$ made by a finite number of 4-simplices, we associate each 4-simplex with a vertex amplitude as a function on $\times_{a=1}^5 \mathcal{M}_{\vec{j}_a}$ when fixing $\vec{j}$,

$$A_v(\vec{j}, \vec{\xi}) = \mathcal{Z}_{S^3\setminus\Gamma_5}(\iota_{\vec{j},\vec{\xi}}) \tag{114}$$

where $\iota_{\vec{j},\vec{\xi}} = (j_{ab}, \rho^{(a)}_{\vec{j},\vec{\xi}})$. When gluing a pair of 4-simplices by identifying a pair of tetrahedra, we identify four spins j_f (of tetrahedron face areas) for the pair of tetrahedra; we associate $\rho_{\vec{j},\vec{\xi}} = (z(\vec{j}, \vec{\xi}), x(\vec{j}, \vec{\xi}), y(\vec{j}, \vec{\xi}))$ (of the tetrahedron shape) to one tetrahedron and associate

$$J\rho_{\vec{j},\vec{\xi}} = (\overline{z(\vec{j}, \vec{\xi})}, x(\vec{j}, \vec{\xi}), -y(\vec{j}, \vec{\xi})) \tag{115}$$

to the other tetrahedron [recall (85)]. We may define the gluing of the pair of vertex amplitudes by

$$\int_{\mathcal{M}_{\vec{j}}} \mathrm{d}\xi \mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{j}, \rho_{\vec{j},\vec{\xi}}) \mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{j}, J\rho_{\vec{j},\vec{\xi}}), \tag{116}$$

where we only focus on variables associated to the pair of tetrahedra identified by gluing. Here, $\int_{\mathcal{M}_{\vec{j}}} \mathrm{d}\xi$ is an analog of integrating SU(2) coherent intertwiners in the EPRL model. The gluing defined by (116) identifies $\vec{\xi}$ at the quantum level between the pair of tetrahedra. Generally speaking, it may only be necessary to identify $\vec{\xi}$ semiclassically, i.e., gluing 4-simplices by identifying two tetrahedra with shape matching only semiclassically. Thus, we define the more general gluing by

$$\int_{\mathcal{M}_{\vec{j}}} \mathrm{d}\xi \mathrm{d}\xi' \mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{j}, \rho_{\vec{j},\vec{\xi}}) A_e(\vec{j}, \vec{\xi}, \vec{\xi}') \mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{j}, J\rho'_{\vec{j},\vec{\xi}'}), \tag{117}$$

where A_e is called the edge amplitude. Note that A_e is a function of $\vec{j}, \vec{\xi}, \vec{\xi}'$ relating to the tetrahedron e (A_e may depend on k, γ, which is implicit in the formula). The precise form A_e is not determined in this work, but we require that A_e is a Gaussian-like continuous function peaked at $\vec{\xi} = \vec{\xi}'$ and suppressed elsewhere. Here, A_e approaches $\delta(\vec{\xi}, \vec{\xi}')$ when $j \to \infty$. Choices of the integration measures of $\vec{\xi}, \vec{\xi}'$ are included in choices of A_e.

Given any simplicial complex $\mathcal{K}$, we associate a "spin" $j_f = 0, \frac{1}{2}, \ldots, \frac{k-1}{2}$ to each (internal or boundary) face f and associate to each (internal or boundary) tetrahedron e a PSU(2) flat connection labeled by $\vec{j}, \vec{\xi}$ on the 4-holed sphere. These data enter vertex amplitudes $A_v = \mathcal{Z}_{S^3\setminus\Gamma_5}(\iota_{\vec{j},\vec{\xi}})$, edge amplitudes $A_e(\vec{j}, \vec{\xi}, \vec{\xi}')$, and face amplitudes $A_f(j_f)$. We construct the full spinfoam amplitude A on $\mathcal{K}$ by integrating over $\rho_{\vec{j},\vec{\xi}}$ of all internal tetrahedra e and summing over j_f of all internal faces,

$$A = \sum_{\{j_f\}}^{(k-1)/2}{}' \prod_f A_f(j_f) \int [\mathrm{d}\xi \mathrm{d}\xi'] \prod_e A_e(\vec{j}, \vec{\xi}_e, \vec{\xi}'_e) \prod_v A_v(\vec{j}, \vec{\xi}). \tag{118}$$

We use the subscript e to manifest that A_e only depends on variables relating to e. Here, $\int [\mathrm{d}\xi]$ is a product of integrals (117) over all internal tetrahedra e, and $A_f(j_f)$ is an undetermined face amplitude. Note that $\prod_v$ is a product of over all 4-simplices, and $\sum'_{\{j_f\}}$ sums j_f at all internal faces in $\mathcal{K}$. The sum of each j_f is *finite* according to (96). The cosmological constant relating to k provides a cutoff to the sum over spins. Here, $\sum'$ indicates that we exclude j_f's that do not satisfy the triangle inequality or lead to $\tilde{\mathcal{M}}_{\vec{j}}$ of dimension less than 2.

Theorem III.3. The amplitude A is finite for any choice of simplicial complex.

Proof: $|A_v|$ is bounded because of Theorem II.1 since it is continuous on the compact space of $\vec{\xi}_e, \vec{\xi}'_e$. The integral in A integrates a function whose absolute value is bounded on a compact domain and thus is absolutely convergent. Then, the finite sum over j_f implies the finiteness of A. ∎

D. Boundary data

The boundary data of the spinfoam amplitude A relate to the kinematical states of LQG up to a deformation. The boundary of the 4d simplicial complex $\mathcal{K}$ is a 3d simplicial complex $\partial\mathcal{K}$. The dual complex $\partial\mathcal{K}^* = \Gamma$ is an (oriented) graph with links $\mathfrak{l} \subset \Gamma$ dual to faces $f \subset \partial\mathcal{K}$ and nodes $\mathfrak{v} \in \Gamma$ dual to tetrahedra $e \subset \partial\mathcal{K}$. The boundary data of A color every link by a spin $j_\mathfrak{l} = 0, \frac{1}{2}, \ldots, \frac{k-1}{2}$, and color every node $\mathfrak{v}$ by an element $\rho_\mathfrak{v} = \mathcal{M}_{\vec{j}}$. There is a 1-to-1 correspondence between $\rho_\mathfrak{v}$ and a convex constant curvature tetrahedron (up to degenerate tetrahedra) whose face areas are determined by $j_\mathfrak{l}$ of $\mathfrak{l}$ adjacent to $\mathfrak{v}$ (see [21] or Theorem IV.1). These data are perfect analogs of LQG spin-network data on Γ: spins $j_\mathfrak{l}$ on links and coherent intertwiners $||\vec{j}, \vec{\xi}\rangle_\mathfrak{v}$ at nodes. The coherent intertwiners 1-to-1 correspond to convex flat tetrahedra whose face areas are proportional to $j_\mathfrak{l}$ [40–42]. The boundary data of A are a deformation of the spin-network data due to the cutoff $\frac{k-1}{2}$ of $j_\mathfrak{l}$ and $\rho_\mathfrak{v}$ for constant curvature tetrahedra versus $||\vec{j}, \vec{\xi}\rangle_\mathfrak{v}$ for flat tetrahedra. When $k \to \infty$ while fixing $j_\mathfrak{l}$ (different from the semiclassical limit j, $k \to \infty$ fixing j/k), the cutoff is removed and the constant curvature Λ given by (16) reduces so that it is flat; then, the boundary data of A reduce to the spin-network data.

We expect that A defines transition amplitudes of boundary states that are the eigenstates of area operators at links and coherent with respect to quantum tetrahedra at nodes, similar to spin-network states with coherent intertwiners. The coherent states at nodes are expected to quantize the phase space $\tilde{\mathcal{M}}_{\vec{j}}$: the moduli space of SU(2) flat connections on a 4-holed sphere with fixed conjugacy classes. The quantization of $(\tilde{\mathcal{M}}_{\vec{j}}, \frac{k}{2\pi}\Omega)$ is known to give the Hilbert space of quantum group $\mathrm{SU}(2)_\mathfrak{q}$ intertwiners with $\mathfrak{q} = e^{\pi i/(k+2)}$ (see e.g., [43,44]). Using these arguments, we conjecture that the boundary Hilbert space of A is spanned by $\mathfrak{q}$-deformed spin-network states $|\Gamma, j_\mathfrak{l}, i_\mathfrak{v}\rangle$ where $j_\mathfrak{l}$, $i_\mathfrak{v}$ are unitary irreps and intertwiners of $\mathrm{SU}(2)_\mathfrak{q}$, respectively. The proof of this conjecture is a work in progress. It involves the coherent intertwiner of $\mathrm{SU}(2)_\mathfrak{q}$ and shows the relation to the curved tetrahedron labeled by the SU(2) flat connection. Some earlier studies of the quantum group coherent intertwiner are given in [45]. Research related to constructing geometrical operators for the boundary Hilbert space is also in progress (see [46] for the first step).

E. Ambiguities

The construction of the spinfoam amplitude with a cosmological constant depends on several choices, which may relate to ambiguities of the model. In the following we classify and discuss these choices:

(1) The spinfoam amplitude depends on choices of coherent states in Sec. II F. This dependence is a result of the proposal of imposing the simplicity constraint on coherent state labels. In this work we choose the coherent states (80) and (81). But a different set of coherent states may be chosen, as long as they are peaked semiclassically at points in the phase space.

(2) There is freedom in choosing edge and face amplitudes A_e, A_f in (118). See e.g., [47,48] for some existing discussion about preferred choices of A_e, A_f in the absence of Λ. The freedom of A_e contains the freedom of the integration measure for $\vec{\xi}$. Moreover, the freedom of A_e has an overlap with the freedom of coherent states discussed in (1). Namely, if we make a change of coherent state $\Psi_{\rho_{\vec{j},\vec{\xi}}} \mapsto \Psi'_{\rho_{\vec{j},\vec{\xi}}} = \int \mathrm{d}\xi_e K(\vec{\xi}_e, \vec{\xi}'_e) \Psi_{\rho_{\vec{j},\vec{\xi}'}}$ with a certain function K of $\vec{\xi}_e$, $\vec{\xi}'_e$ of the tetrahedron e, the spinfoam amplitude constructed with the new state $\Psi'_{\rho_{\vec{j},\vec{\xi}}}$ can be written in the same form as (118) with A_v of the old state $\Psi_{\rho_{\vec{j},\vec{\xi}}}$, while A_e transforms as $A_e(\vec{\xi}_e, \vec{\xi}'_e) \mapsto \int \mathrm{d}\zeta_e \mathrm{d}\zeta'_e K(\vec{\zeta}_e, \vec{\xi}_e) A_e(\vec{\zeta}_e, \vec{\zeta}'_e) \overline{K(\vec{\zeta}'_e, \vec{\xi}'_e)}$.

(3) The vertex amplitude depends on the positive angle structure $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_{\mathrm{new}}$ since $\mathcal{Z}_{S^3\setminus\Gamma_5}$ depends on $(\vec{\alpha}, \vec{\beta})$. More precisely, $\mathcal{Z}_{S^3\setminus\Gamma_5}$ only depends on $\vec{\alpha}$ but is independent of specific $\vec{\beta}$ as long as $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}_{\mathrm{new}}$, according to the discussion below (65). The dependence on angles $\vec{\alpha} = (\{\alpha_{ab}\}_{a<b}, \{\alpha_a\}_{a=1}^5)$ in $\mathcal{Z}_{S^3\setminus\Gamma_5}$ may be analogous to the framing anomaly of CS theory with a compact group [49,50]. For the consistency "4d area = 3d area" at the quantum level, it is preferred to restrict all α_{ab} in A_v to vanish and still be inside $\mathfrak{P}_{\mathrm{new}}$, whereas there still exists some freedom of $\{\alpha_a\}_{a=1}^5$.

The spinfoam amplitude depends on $\{\beta_a\}_{a=1}^5$ because they enter the vertex amplitude A_v via the prefactor $e^{-\sqrt{2}\beta_a \mathrm{Re}(z_a)}$ of the coherent state ψ_{z_a} in (80). But this prefactor can be absorbed in A_e (or the definition of the integration measure of $\vec{\xi}$). Thus, this dependence on $\{\beta_a\}_{a=1}^5$ is part of the freedom of (1) and (2). In more detail, using the freedom of coherent states, we choose $\psi^0_{z_a}$

instead of ψ_{z_a} in the definition of $\mathcal{Z}_{S^3\backslash\Gamma_5}(\iota)$. Then (89) for the bound of $|\mathcal{Z}_{S^3\backslash\Gamma_5}(\iota)|$ is modified by

$$\left| \int_{\mathbb{R}^5} \prod_{a=1}^{5} \mathrm{d}\mu'_a \mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}' + i\vec{\alpha}|\vec{m}') \prod_{a=1}^{5} \psi^0_{z_a}(\vec{\mu}'_a) \right| \leq Ck^{5/4} \prod_a e^{\beta_a(\frac{\pi\beta_a}{k} + \sqrt{2}\Re(z_a))}. \tag{119}$$

The bound diverges if $\mathrm{Re}(z_a)$ approaches ∞ or $-\infty$ depending on $\mathrm{sgn}(\beta_a)$. This can happen even after imposing the simplicity constraint since x_E, $x_{E'}$ can approach infinity when a pair of ξ'_i becomes collinear in (109), particularly when the constant curvature tetrahedron becomes degenerate. In addition, we need to require the following behavior of A_e as $\mathrm{Re}(z)$ approaches ∞ or $-\infty$ correspondingly,

$$|A_e(\vec{j}, \vec{\xi}, \vec{\xi}')| \leq C' e^{-\sqrt{2}\beta_e \mathrm{Re}(z_e(\vec{j},\vec{\xi}))} e^{-\sqrt{2}\beta'_e \mathrm{Re}(z_e(\vec{j},\vec{\xi}'))}$$

where the exponential decay factors should cancel the exponential growth in (119) of two vertex amplitudes sharing the tetrahedron e. The freedom of β_e becomes part of the freedom of A_e. The integrand of $\int[\mathrm{d}\xi]$ in (118) still has a bounded absolute value; then, A is finite.

(4) The amplitude A generally depends on the choice of the simplicial complex $\mathcal{K}$, similar to spinfoam models in the absence of a cosmological constant.

IV. SEMICLASSICAL ANALYSIS

In this section, we examine the semiclassical behavior of the vertex amplitude A_v and show that the semiclassical limit of A_v reproduces the 4d Regge action with Λ.

The semiclassical limit of quantum gravity is $\ell_P \to 0$ while keeping geometrical quantities—e.g., areas, shapes, curvature, etc.—fixed. Note that A_v is the LQG transition amplitude associated to a 4-simplex whose boundary is made by five tetrahedra labeled by $a, b = 1, \ldots, 5$, and A_v depends on k, γ, j_{ab}, and ξ_{ab}. Using the result of [21] (to be reviewed in Sec. IV B), the ξ_{ab}'s parametrize geometrical shapes of five boundary constant curvature tetrahedra as boundary data of A_v, while j_{ab}/k (up to ε_{ab}/k) is proportional to $|\Lambda|\mathfrak{a}_{ab}$. Here, $\mathfrak{a}_{ab}$ is the area of the face f_{ab} shared by tetrahedra a and b. The cosmological constant Λ equals the constant curvature of tetrahedra. Therefore, the semiclassical limit in our context is $\ell_P \to 0$ while keeping ξ_{ab}'s, $\mathfrak{a}_{ab}$'s, and Λ fixed. The Barbero-Immirzi parameter γ is also fixed. The relation between k and Λ in (16) indicates that $k \to \infty$ in the semiclassical limit. These features motivate the following definition:

Definition IV.1 The semiclassical limit of A_v is the asymptotic behavior of A_v when we uniformly scale all $j_{ab} \to \infty$ and $k \to \infty$ (so $\sigma = ik\gamma \to i\infty$) while keeping j_{ab}/k fixed.

This limit generalizes the semiclassical limit of the Turaev-Viro model in 3d gravity and is studied in [13] for the 4d spinfoam vertex amplitude.

The semiclassical limit of the spinfoam amplitude is the same as the semiclassical limit of CS theory. Indeed, the flat connection position variables $\mathcal{Q}_I$ depend on j_{ab} only through the ratio j_{ab}/k [see (94)]. The above semiclassical limit sends $k \to \infty$ but leaves $\mathcal{Q}_I$ finite. The limit effectively removes the dependence of α_{ab}, ϵ_{ab} in $e^{2L_{ab}}$. The limit $k \to \infty$, keeping $\mathcal{Q}_I$ finite, is the same as the semiclassical limit of the CS partition function. Therefore, it is useful to first study the semiclassical limit of the CS partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$ in Sec. IV A; then, the result can be applied straightforwardly to the semiclassical limit of A_v in Secs. IV B and IV C.

A. Semiclassical analysis of Chern-Simons partition function

Recall the construction of $\mathcal{Z}_{S^3\backslash\Gamma_5}$ in Sec. II E. Equations (58), (61), (64), and (67) lead to

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m}) = \frac{4i}{k^{15}} \sum_{\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathcal{C}} \mathrm{d}^{15}\nu e^{S_0} Z_\times(-\mathbf{B}^T\vec{\nu}| - \mathbf{B}^T\vec{n}), \tag{120}$$

$$S_0 = \frac{\pi i}{k}\left[-2\left(\vec{\mu} - \frac{iQ}{2}\vec{t}\right)\cdot\vec{\nu} + 2\vec{m}\cdot\vec{n} - \vec{\nu}\cdot\mathbf{A}\mathbf{B}^T\cdot\vec{\nu} + (k+1)\vec{n}\cdot\mathbf{A}\mathbf{B}^T\cdot\vec{n}\right], \tag{121}$$

$$Z_\times(\vec{\mu}|\vec{m}) = \prod_{a=1}^{5} \Psi_\Delta(\mu_{X_a}|m_{X_a})\Psi_\Delta(\mu_{Y_a}|m_{Y_a})\Psi_\Delta(\mu_{Z_a}|m_{Z_a})\Psi_\Delta(\mu_{W_a}|m_{W_a}), \tag{122}$$

$$\mu_{W_a} = iQ - \mu_{X_a} - \mu_{Y_a} - \mu_{Z_a}, \qquad m_{W_a} = -m_{X_a} - m_{Y_a} - m_{Z_a}, \tag{123}$$

and Ψ_Δ is given by (35).

We use (69) to change variables from μ_I, m_I to $\mathcal{Q}'_I = \mathcal{Q}_I - i\pi t_I$ and $\tilde{\mathcal{Q}}_I' = \tilde{\mathcal{Q}}_I - i\pi t_I$. It is intuitive to make a similar change of variables from ν_I, n_I to $\mathcal{P}_I$,$\tilde{\mathcal{P}}_I$ for studying the semiclassical limit,

$$\nu_I = \frac{bk(\tilde{\mathcal{P}}_I + \mathcal{P}_I)}{2\pi(b^2+1)}, \qquad n_I = \frac{ik(\mathcal{P}_I - b^2\tilde{\mathcal{P}}_I)}{2\pi(b^2+1)}. \tag{124}$$

Semiclassically, $\vec{\mathcal{P}}$ here is identical to the classical momenta conjugate to $\vec{\mathcal{Q}}$ [recall (55) and the discussion there]. Using the change of variables,

$$\begin{aligned} S_0 = & -\frac{1}{2}\vec{t}\cdot(\vec{\mathcal{P}}+\vec{\tilde{\mathcal{P}}}) - \frac{ik}{4\pi(1+b^2)}\left[\vec{\mathcal{P}}\cdot(\mathbf{A}\mathbf{B}^T\cdot\vec{\mathcal{P}}+2\vec{\mathcal{Q}}') + b^2\vec{\tilde{\mathcal{P}}}\cdot(\mathbf{A}\mathbf{B}^T\cdot\vec{\tilde{\mathcal{P}}}+2\vec{\tilde{\mathcal{Q}}}')\right] \\ & - \frac{ik^2}{4\pi(1+b^2)^2}(\vec{\mathcal{P}}-b^2\vec{\tilde{\mathcal{P}}})\cdot\mathbf{A}\mathbf{B}^T\cdot(\vec{\mathcal{P}}-b^2\vec{\tilde{\mathcal{P}}}). \end{aligned} \tag{125}$$

We treat the sum $\sum_{n_I\in\mathbb{Z}/k\mathbb{Z}}$ by the Poisson resummation

$$\begin{aligned} \sum_{\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}} f(\vec{n}) = \sum_{n_I=0}^{k-1} f(\vec{n}) &= \sum_{\vec{p}\in\mathbb{Z}^{15}} \int_{-\delta}^{k-\delta} \mathrm{d}^{15}n f(\vec{n}) e^{2\pi i\vec{p}\cdot\vec{n}} \\ &= \left(\frac{k}{2\pi}\right)^{15} \sum_{\vec{p}\in\mathbb{Z}^{15}} \int_{-\delta'}^{2\pi-\delta'} \mathrm{d}^{15}\mathcal{J} f'(\vec{\mathcal{J}}) e^{ik\vec{p}\cdot\vec{\mathcal{J}}_I}, \end{aligned} \tag{126}$$

$$\mathcal{J}_I = \frac{2\pi n_I}{k} = \frac{i(\mathcal{P}_I - b^2\tilde{\mathcal{P}}_I)}{b^2+1}, \qquad f'(\vec{\mathcal{J}}) = f(\vec{n}). \tag{127}$$

Here, $f(\vec{n}) = \chi(\vec{n})g(\vec{n})$, where $g(\vec{n})$ is the summand in (120) extended from $\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}$ to $\vec{n}\in\mathbb{R}^{15}$. Note that $\chi(\vec{n})$ is a compact support function satisfying $\chi(\vec{n})=1$ for $\vec{n}\in\mathbb{Z}^{15}$, and $\chi(\vec{n})$ vanishes outside $[-\delta, k-\delta]^{15}\backslash\mathcal{U}$ (with arbitrarily small $\delta>0$) where $\mathcal{U}$ is an open neighborhood of singularities of $g(\vec{n})$ and $\mathcal{U}\cap\mathbb{Z}^{15}=\varnothing$.[8] The result does not depend on details of χ at $\vec{n}\notin\mathbb{Z}^{15}$ because $\sum_{p_I\in\mathbb{Z}} e^{2\pi i p_I n_I} = \sum_{n_I'\in\mathbb{Z}}\delta(n_I - n_I')$. By changing integration variables,

$$\mathrm{d}\nu_I \mathrm{d}\mathcal{J}_I = \frac{k}{2\pi i Q}\mathrm{d}\mathcal{P}_I \mathrm{d}\tilde{\mathcal{P}}_I. \tag{128}$$

The following large-k asymptotic formula of the quantum dilogarithm is useful [15,51]:

$$\Psi_\Delta = e^{-\frac{ik}{2\pi(1+b^2)}\mathrm{Li}_2(e^{-Z}) - \frac{ik}{2\pi(1+b^{-2})}\mathrm{Li}_2(e^{-\tilde{Z}})}[1+O(1/k)]. \tag{129}$$

The large-k asymptotic behavior of $Z_\times$ is given by

$$Z_\times(\vec{\mu}|\vec{m}) = e^{S_1+\tilde{S}_1}[1+O(1/k)], \tag{130}$$

$$S_1 = -\frac{ik}{2\pi(1+b^2)}\sum_{a=1}^{5}[\mathrm{Li}_2(e^{-X_a}) + \mathrm{Li}_2(e^{-Y_a}) + \mathrm{Li}_2(e^{-Z_a}) + \mathrm{Li}_2(e^{-W_a})], \tag{131}$$

$$\tilde{S}_1 = -\frac{ik}{2\pi(1+b^{-2})}\sum_{a=1}^{5}[\mathrm{Li}_2(e^{-\tilde{X}_a}) + \mathrm{Li}_2(e^{-\tilde{Y}_a}) + \mathrm{Li}_2(e^{-\tilde{Z}_a}) + \mathrm{Li}_2(e^{-\tilde{W}_a})]. \tag{132}$$

Here $(X_a, Y_a, Z_a)_{a=1}^5 \equiv -\mathbf{B}^T\vec{\mathcal{P}}$ and $(\tilde{X}_a, \tilde{Y}_a, \tilde{Z}_a)_{a=1}^5 \equiv -\mathbf{B}^T\vec{\tilde{\mathcal{P}}}$, and W_a, $\tilde{W}_a$ are given by

$$\begin{aligned} X_a + Y_a + Z_a + W_a &= 2\pi i + \frac{2\pi i}{k}(1+b^2), \\ \tilde{X}_a + \tilde{Y}_a + \tilde{Z}_a + \tilde{W}_a &= 2\pi i + \frac{2\pi i}{k}(1+b^{-2}) \end{aligned} \tag{133}$$

coinciding with the classical octahedron constraint (43) up to $O(1/k)$.

[8] When extending $\Psi_\Delta(\mu|m)$ to $m\in\mathbb{R}$, poles of $\Psi_\Delta(\mu|m)$ are given by e.g., $\mu_{\text{pole}} = ibu + ib^{-1}v$ with $v=-j$ and $u=-j-m+k\mathbb{Z}$ $(j\in\mathbb{N}_0)$ when $\mathrm{Im}(b)>0$. Poles with $u\geq 1$ cancel with zeros when $m\in\mathbb{Z}/k\mathbb{Z}$, but this cancellation does not apply for noninteger m. At poles, $\mathrm{Im}(\mu_{\text{pole}}) = \mathrm{Re}(b)(u+v) = \mathrm{Re}(b)(-2j-m+k\mathbb{Z})$. There exists m's such that $\mathrm{Im}(\mu_{\text{pole}})=\alpha$; i.e., the pole lies on the integration contour $\mathcal{C}$ and may cause the integral to diverge. Therefore, open neighborhoods of these m's should be removed.

Therefore, we rewrite $\mathcal{Z}_{S^3\backslash\Gamma_5}$ for large k by

$$\mathcal{Z}_{S^3\backslash\Gamma_5} = \mathcal{N}_0 \sum_{\vec{p}\in\mathbb{Z}^{15}} \int_{\mathcal{C}_\mathcal{P}} \mathrm{d}^{15}\mathcal{P}\mathrm{d}^{15}\tilde{\mathcal{P}} e^{S_{\vec{p}}}\chi[1+O(1/k)], \quad (134)$$

$$S_{\vec{p}} = S_0(\mathcal{P},\tilde{\mathcal{P}},\mathcal{Q},\tilde{\mathcal{Q}}) + S_1(-\mathbf{B}^T\mathcal{P}) + \tilde{S}_1(-\mathbf{B}^T\tilde{\mathcal{P}}) - \frac{k}{b^2+1}\vec{p}\cdot(\vec{\mathcal{P}} - b^2\vec{\tilde{\mathcal{P}}}). \quad (135)$$

where $\mathcal{N}_0 = -\frac{4k^{15}}{(2\pi)^{30}Q^{15}}$. The integration domain $\mathcal{C}_\mathcal{P}$ is the 30 (real)-dimensional submanifold of $(\vec{\mathcal{P}},\vec{\tilde{\mathcal{P}}})\in\mathbb{C}^{30}$ satisfying $\vec{\nu}\in\mathcal{C}$ and $\vec{\mathcal{J}}\in[-\delta', 2\pi-\delta']^{15}$.

The large-k asymptotics of $\mathcal{Z}_{S^3\backslash\Gamma_5}$ can be analyzed by the stationary phase approximation. The dominant contributions of integrals in (134) come from critical points that are solutions of the critical equations $\partial_{\mathcal{P}_I}S_{\vec{p}} = \partial_{\tilde{\mathcal{P}}_I}S_{\vec{p}} = 0$ (see Appendix F for details).

We make the linear transformation from $\vec{\mathcal{Q}}'$, $\vec{\mathcal{P}}$ to $\vec{\Phi}\equiv(X_a, Y_a, Z_a)_{a=1}^5$ and $\vec{\Pi}\equiv(P_{X_a}, P_{Y_a}, P_{Z_a})_{a=1}^5$, and similarly for tilded variables

$$\vec{\mathcal{Q}}' - 2\pi i(\vec{n}+\vec{p}) = \mathbf{A}\cdot\vec{\Phi} + \mathbf{B}\cdot\vec{\Pi}, \quad (136)$$

$$\vec{\tilde{\mathcal{Q}}}' + 2\pi i(\vec{n}+\vec{p}) = \mathbf{A}\cdot\vec{\tilde{\Phi}} + \mathbf{B}\cdot\vec{\tilde{\Pi}}, \quad (137)$$

$$\vec{\mathcal{P}} = -(\mathbf{B}^T)^{-1}\vec{\Phi}, \qquad \vec{\tilde{\mathcal{P}}} = -(\mathbf{B}^T)^{-1}\vec{\tilde{\Phi}}. \quad (138)$$

In terms of $\vec{\Phi}$, $\vec{\Pi}$, the critical equations reduce to

$$P_{X_a} = X''_a - W''_a, \qquad P_{Y_a} = Y''_a - W''_a, \quad (139)$$

$$P_{Z_a} = Z''_a - W''_a, \qquad \tilde{P}_{X_a} = \tilde{X}''_a - \tilde{W}''_a, \quad (140)$$

$$\tilde{P}_{Y_a} = \tilde{Y}''_a - \tilde{W}''_a, \qquad \tilde{P}_{Z_a} = \tilde{Z}''_a - \tilde{W}''_a, \quad (141)$$

where

$$X''_a = \log(1-e^{-X_a}), \qquad Y''_a = \log(1-e^{-Y_a}),$$
$$Z''_a = \log(1-e^{-Z_a}), \qquad W''_a = \log(1-e^{-W_a}), \quad (142)$$

$$\tilde{X}''_a = \log(1-e^{-\tilde{X}_a}), \qquad \tilde{Y}''_a = \log(1-e^{-\tilde{Y}_a}),$$
$$\tilde{Z}''_a = \log(1-e^{-\tilde{Z}_a}), \qquad \tilde{W}_a{}'' = \log(1-e^{-\tilde{W}_a}). \quad (143)$$

Equations (142) and (143) reproduce e.g., $z^{-1}+z''-1=0$ with $z=e^Z$ and $z''=e^{Z''}$, i.e., the Lagrangian submanifold $\mathcal{L}_\Delta\subset\mathcal{P}_{\partial\Delta}$ of framed flat $\mathrm{PSL}(2,\mathbb{C})$ connections on the ideal tetrahedron Δ. Here, W_a, $\tilde{W}_a$ are given by (133). The above logarithms are defined with the same canonical lifts as in (13). Moreover, $X_a, Y_a, Z_a, P_{X_a}, P_{Y_a}, P_{Z_a}$ satisfying Eqs. (139)–(141) parametrize the moduli space of framed flat $\mathrm{PSL}(2,\mathbb{C})$ connections on the ideal octahedron oct(a) made by gluing four ideal tetrahedra. Therefore, any solution of critical equations gives five flat connections, respectively, on five ideal octahedra and vice versa. As a result, all possible critical points are in $\mathcal{L}_{S^3\backslash\Gamma_5}$ since the set of five flat connections on five ideal octahedra, respectively, is equivalent to a flat connection on $S^3\backslash\Gamma_5$ [see the discussion below (54)]. Given a $\mathrm{PSL}(2,\mathbb{C})$ flat connection on $S^3\backslash\Gamma_5$, $\vec{\mathcal{Q}}'$, $\vec{\mathcal{P}}$ at the critical point are determined by (136)–(138), the same as in (53) up to $2\pi i(\vec{n}+\vec{p})$.

We set $n_I\in\mathbb{Z}$ in (136) and (137) as an approximation up to $O(1/k)$ because for large k any $\mathcal{J}_I\in\mathbb{R}$ in (127) can be approximated by $n_I\in\mathbb{Z}$ up to $O(1/k)$.[9] Semiclassically, critical equations are insensitive to $O(1/k)$. Then (136)–(138) are the same as (53) (only up to gauge shifts $m_I\to m_I+k\mathbb{Z}$ of $m_I\in\mathbb{Z}/k\mathbb{Z}$).

Fixing the range of m_I (e.g., fixing $m_I=0,\ldots,k-1$) in $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$ fixes the lifts of $\mathcal{Q}_I$, $\tilde{\mathcal{Q}}_I$ from $e^{\mathcal{Q}_I}$, $e^{\tilde{\mathcal{Q}}_I}$ and then uniquely fixes $\vec{p}=\vec{p}_0\in\mathbb{Z}$, given the lifts of logarithms in (142) and (143), since different $p_I\in\mathbb{Z}$ change $\mathcal{Q}_I$, $\tilde{\mathcal{Q}}_I$ by $\mp 2\pi i p_I$ (n_I is determined by $\mathcal{P}_I$). Therefore, only one term with $\vec{p}=\vec{p}_0$ in (134) has a critical point and contributes to the leading order, whereas other terms with $\vec{p}\neq\vec{p}_0$ have no critical point and thus are suppressed faster than $O(k^{-N})$ for all $N>0$.

Given $\vec{\mu}$, $\vec{m}$ or $\vec{\mathcal{Q}}$, $\vec{\tilde{\mathcal{Q}}}$ such that there exists a $\mathrm{PSL}(2,\mathbb{C})$ flat connection on $S^3\backslash\Gamma_5$ satisfying (136) and (137), $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$ has a critical point and thus is not suppressed fast, or in physics terms, $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$ has a semiclassical approximation. In this case, the critical point is generally nonunique; namely, there exists multiple critical points corresponding to the same $\vec{\mathcal{Q}}$, $\vec{\tilde{\mathcal{Q}}}$. Indeed, different $\vec{\mathcal{P}}$, and thus different $\vec{\Phi}$, $\vec{\Pi}$, satisfying (138)–(143) can give the same $\vec{\mathcal{Q}}$ via (136) (the critical equations expressed in terms of $e^{\mathcal{Q}_I}$, $e^{\mathcal{P}_I}$ are polynomial equations of degree higher than 1) and similarly for tilded variables. The critical points 1-to-1 correspond to the solutions of $(\vec{\mathcal{P}},\vec{\tilde{\mathcal{P}}})$ with given $\vec{\mathcal{Q}}$, $\vec{\tilde{\mathcal{Q}}}$. The solutions are denoted by $(\mathcal{P}^{(\alpha)}(\mathcal{P}),\tilde{\mathcal{P}}^{(\alpha)}(\tilde{\mathcal{Q}}))$, $\alpha\in\mathcal{I}$, where $\mathcal{I}$ is a set of indices labeling the solutions. Here, α labels the branches of $\mathcal{L}_{S^3\backslash\Gamma_5}$. Given any α, the coordinates $\vec{\mathcal{Q}}$ provide a local parametrization of $\mathcal{L}_{S^3\backslash\Gamma_5}$.

The asymptotic behavior of $\mathcal{Z}_{S^3\backslash\Gamma_5}$ relates to the action $S_{\vec{p}=\vec{p}_0}$ evaluated at critical points

$$S^{(\alpha)}_{\vec{p}_0}(\mathcal{Q},\tilde{\mathcal{Q}}) = S_{\vec{p}_0}(\mathcal{Q},\tilde{\mathcal{Q}},\mathcal{P}^{(\alpha)}(\mathcal{Q}),\tilde{\mathcal{P}}^{(\alpha)}(\tilde{\mathcal{Q}})). \quad (144)$$

The derivatives of $S^{(\alpha)}_{\vec{p}_0}$ with respect to $\vec{\mathcal{Q}}$, $\vec{\tilde{\mathcal{Q}}}$ are

[9] When $k=10000$, $\mathcal{J}_I/2\pi = 0.5624587\cdots$ can be approximated by $n_I=5625$, and the error bound is $|\mathcal{J}_I/2\pi - n_I/k| < 1/k$.

$$\partial_{\vec{\mathcal{Q}}} S_{\vec{p}_0}^{(\alpha)} = -\frac{ik}{2\pi(1+b^2)}\vec{\mathcal{P}}^{(\alpha)}(\mathcal{Q}), \tag{145}$$

$$\partial_{\vec{\tilde{\mathcal{Q}}}} S_{\vec{p}_0}^{(\alpha)} = -\frac{ik}{2\pi(1+b^{-2})}\vec{\tilde{\mathcal{P}}}^{(\alpha)}(\tilde{\mathcal{Q}}), \tag{146}$$

where we have used $\partial_{\mathcal{P}} S_{\vec{p}_0} = \partial_{\tilde{\mathcal{P}}} S_{\vec{p}_0} = 0$ since $\{\mathcal{P}^{(\alpha)}(\mathcal{Q}), \tilde{\mathcal{P}}^{(\alpha)}(\tilde{\mathcal{Q}})\}_{\alpha\in\mathcal{I}}$ satisfy the critical equations. It implies that[10]

$$S_{\vec{p}_0}^{(\alpha)}(\mathcal{Q},\tilde{\mathcal{Q}}) = -\frac{ik}{2\pi(1+b^2)}\int^{\vec{\mathcal{Q}}}\vec{\mathcal{P}}^{(\alpha)}(\mathcal{Q}')\cdot \mathrm{d}\vec{\mathcal{Q}}' - \frac{ik}{2\pi(1+b^{-2})}\int^{\vec{\tilde{\mathcal{Q}}}}\vec{\tilde{\mathcal{P}}}^{(\alpha)}(\tilde{\mathcal{Q}}')\cdot \mathrm{d}\vec{\tilde{\mathcal{Q}}}' + C^\alpha, \tag{147}$$

where C^α is an integration constant. The integrals are along certain curves embedded in $\mathcal{L}_{S^3\setminus\Gamma_5}$. The result is independent of smooth deformations of the integration contour in $\mathcal{L}_{S^3\setminus\Gamma_5}$ since $\Omega = 0$ on the Lagrangian submanifold $\mathcal{L}_{S^3\setminus\Gamma_5}$. From this result, $\exp(S_{\vec{p}_0}^{(\alpha)})$ is expressed as an analog of the WKB wave function. The large-k asymptotics of $\mathcal{Z}_{S^3\setminus\Gamma_5}$ is given by a finite sum over critical points,

$$\mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{\mu}|\vec{m}) = \sum_\alpha \mathcal{N}_0^{(\alpha)} e^{S_{\vec{p}_0}^{(\alpha)}(\mathcal{Q},\tilde{\mathcal{Q}})}[1+O(1/k)], \tag{148}$$

$$\mathcal{N}_0^{(\alpha)} = \frac{\mathcal{N}_0}{\sqrt{\det(-H_\alpha/2\pi)}} \tag{149}$$

where H_α is the Hessian matrix $\partial^2 S_{\vec{p}_0}$ evaluated at the critical point. Note that H_α is generically nondegenerate as supported by a large number of numerical experiments.

B. Critical points of the vertex amplitude and constant curvature 4-simplex

Let us recall $\mathcal{Z}_{S^3\setminus\Gamma_5}(\iota)$ and the coherent states ψ_{z_a}, $\xi_{(x_a,y_a)}$ defined in (80) and (81). Restricting $\iota = \iota_{\vec{j},\vec{\xi}}$ to satisfy the simplicity constraint, $A_v = \mathcal{Z}_{S^3\setminus\Gamma_5}(\iota_{\vec{j},\vec{\xi}})$ is the vertex amplitude with a cosmological constant.

The simplicity constraint restricts $\mathrm{Re}(\mu_{ab}) = 0$ (the semiclassical behavior is insensitive to α_{ab}); thus,

$$e^{2L_{ab}} = \exp\left[\frac{2\pi i}{k}\left(b\alpha_{ab} + 2j_{ab} + \frac{\epsilon_{ab}}{2}\right)\right] \simeq e^{\frac{4\pi i}{k}j_{ab}},$$

$$e^{2\tilde{L}_{ab}} = \exp\left[\frac{2\pi i}{k}\left(b^{-1}\alpha_{ab} - 2j_{ab} - \frac{\epsilon_{ab}}{2}\right)\right] \simeq e^{-\frac{4\pi i}{k}j_{ab}}.$$

Here, $\simeq$ stands for the semiclassical approximation.

We make the change of variable (69) in ψ_{z_a} (recall $\mathcal{Q}_I' = \mathcal{Q}_I - i\pi t_I$, $\tilde{\mathcal{Q}}_I' = \tilde{\mathcal{Q}}_I - i\pi t_I$)

$$\psi_{z_a} = \left(\frac{2}{k}\right)^{1/4} e^{S_{z_a}}, \qquad S_{z_a} \simeq \frac{bk(\tilde{\mathcal{Q}}_a' + \mathcal{Q}_a')}{2\pi(b^2+1)}\left[\sqrt{2}z_a - \frac{b(\tilde{\mathcal{Q}}_a' + \mathcal{Q}_a')}{2(b^2+1)}\right] - \frac{k(\bar{z}_a + z_a)^2}{8\pi}, \tag{150}$$

where we neglect the term $-\sqrt{2}\beta_a \mathrm{Re}(z_a)$ since it is subleading as $k\to\infty$. Note that $\xi_{(x_a,y_a)}$ is simplified by $k\to\infty$ and by restricting $m_a = 0,\ldots,k-1$ and $x_a, y_a \in (0,2\pi)$. After neglecting exponentially small contributions,

$$\xi_{(x_a,y_a)} \simeq \left(\frac{2}{k}\right)^{\frac{1}{4}} e^{\frac{ikx_a y_a}{4\pi}} e^{-\frac{k}{4\pi}\left(\frac{2\pi m_a}{k} - x_a\right)^2} e^{-iy_a m_a} = \left(\frac{2}{k}\right)^{\frac{1}{4}} e^{S_{(x_a,y_a)}}, \tag{151}$$

$$S_{(x_a,y_a)} = \frac{ikx_a y_a}{4\pi} - \frac{k}{4\pi}\left[\frac{i(\mathcal{Q}_a' - b^2\tilde{\mathcal{Q}}_a')}{b^2+1} - x_a\right]^2 + \frac{k(\mathcal{Q}_a' - b^2\tilde{\mathcal{Q}}_a')}{2\pi(b^2+1)}y_a. \tag{152}$$

[10]Given the $S(\vec{x})$ function on $\mathbb{R}^n$ and $\vec{\nabla}S(\vec{x}) = \vec{f}(\vec{x})$, we choose a curve $c\subset\mathbb{R}^n$ parametrized by $t\in[0,1]$ ending at x_0. We denote by $\vec{t}$ the tangent vector of c. Then, $\frac{\mathrm{d}}{\mathrm{d}t}S(\vec{x}(t)) = \vec{t}\cdot\vec{\nabla}S(\vec{x}(t)) = \vec{t}\cdot\vec{f}(\vec{x}(t))$. Therefore, $S(\vec{x}_0) = \int_c^{x_0}\vec{f}(\vec{x})\cdot \mathrm{d}\vec{x} + C$.

The vertex amplitude A_v is expressed as

$$A_v = \mathcal{N} \sum_{\vec{m}\in(\mathbb{Z}/k\mathbb{Z})^5} \sum_{\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathbb{R}^5\times\mathcal{C}} \mathrm{d}^5\mu \mathrm{d}^{15}\nu e^{\mathcal{I}(\mathcal{P},\tilde{\mathcal{P}},\mathcal{Q},\tilde{\mathcal{Q}})}, \qquad \mathcal{N} = \frac{4i}{k^{15}}\left(\frac{2}{k}\right)^{5/2},$$

$$\mathcal{I} = S_0(\mathcal{P},\tilde{\mathcal{P}},\mathcal{Q},\tilde{\mathcal{Q}}) + S_1(-\mathbf{B}^T\mathcal{P}) + \tilde{S}_1(-\mathbf{B}^T\tilde{\mathcal{P}}) + \sum_{a=1}^{5}[S_{z_a}(\mathcal{Q}_a,\tilde{\mathcal{Q}}_a) + S_{(x_a,y_a)}(\mathcal{Q}_a,\tilde{\mathcal{Q}}_a)]. \tag{153}$$

For finite z_a, the integrand is a Schwartz function of both $\vec{\mu}$ and $\vec{\nu}$ along the integration cycle [ψ_{z_a} is a Gaussian function; see the discussion below (64)], so interchanging the $\vec{\mu}$-integral with the $\vec{\nu}$-integral does not affect the result. We apply the Poisson resummation similarly to (126),

$$A_v = \mathcal{N}' \sum_{(\vec{p},\vec{s})\in\mathbb{Z}^{20}} \int_{\mathcal{C}_\mathcal{Q}\times\mathcal{C}_\mathcal{P}} \mathrm{d}^5\mathcal{Q}\mathrm{d}^5\tilde{\mathcal{Q}}\mathrm{d}^{15}\mathcal{P}\mathrm{d}^{15}\tilde{\mathcal{P}} e^{\mathcal{I}_{\vec{p},\vec{s}}(\mathcal{P},\tilde{\mathcal{P}},\mathcal{Q},\tilde{\mathcal{Q}})}, \qquad \mathcal{N}' = \frac{i(k/2)^{45/2}}{8192\pi^{40}Q^{20}} \tag{154}$$

$$\mathcal{I}_{\vec{p},\vec{s}} = \mathcal{I}(\mathcal{P},\tilde{\mathcal{P}},\mathcal{Q},\tilde{\mathcal{Q}}) - \frac{k}{b^2+1}\vec{p}\cdot(\vec{\mathcal{P}} - b^2\vec{\tilde{\mathcal{P}}}) - \frac{k}{b^2+1}\sum_{a=1}^{5} s_a(\mathcal{Q}_a - b^2\tilde{\mathcal{Q}}_a), \tag{155}$$

where $\mathcal{C}_\mathcal{Q}$ is a 10-dimensional real manifold satisfying $\mu_a \in \mathbb{R}$ and $m_a \in [0,k)$ [here μ_a, m_a are understood as continuous variables relating $\mathcal{Q}_a$, $\tilde{\mathcal{Q}}_a$ by (69)].

We again apply the stationary phase analysis to the integral as $k\to\infty$. The critical equations $\partial_\mathcal{P}\mathcal{I}_{\vec{p},\vec{s}} = \partial_{\tilde{\mathcal{P}}}\mathcal{I}_{\vec{p},\vec{s}} = 0$ give the same results as (136)–(143) whose solutions are flat connections on $S^3\backslash\Gamma_5$. The other set of critical equations $\partial_\mathcal{Q}\mathcal{I}_{\vec{p},\vec{s}} = \partial_{\tilde{\mathcal{Q}}}\mathcal{I}_{\vec{p},\vec{s}} = 0$ implies

$$\frac{2\pi}{k}\mathrm{Re}(\mu_a) = \sqrt{2}\mathrm{Re}(z_a), \qquad \frac{2\pi}{k}\mathrm{Re}(\nu_a) = \sqrt{2}\mathrm{Im}(z_a),$$
$$\frac{2\pi}{k}m_a = x_a, \qquad \frac{2\pi}{k}n_a = y_a, \qquad s_a = 0. \tag{156}$$

See Appendix F for derivations. At the critical point, the 4-holed sphere data $\mathcal{Q}_a$, $\tilde{\mathcal{Q}}_a$, $\mathcal{P}_a$, $\tilde{\mathcal{P}}_a$ are determined by the coherent state labels z_a, x_a, y_a. The determined 4-holed sphere data, together with $2L_{ab}$, $2\tilde{L}_{ab}$ determined by j_{ab}, provide the boundary condition to the flat connection solving (136)–(143).

The simplicity constraint requires that z_a, x_a, y_a are determined by the data $\vec{j}$, $\vec{\xi}$ via (109). Then (156) determines the 4-holed sphere FG coordinates $\mathcal{X}_a$, $\mathcal{Y}_a$. Because of the 1-to-1 correspondence between values of FG coordinates $\{x_E\}_E$ and framed $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $\mathcal{S}_a$ [29], the resulting $\mathcal{X}_a$, $\mathcal{Y}_a$, together with $e^{\mathcal{Q}_{ab}} = e^{2L_{ab}}$ [belonging to U(1) as $k\to\infty$], uniquely determine a $\mathrm{PSU}(2)\simeq\mathrm{SO}(3)$ flat connection on $\mathcal{S}_a$. We denote by $\mathcal{M}_{\mathrm{flat}}(\mathcal{S}_a,\mathrm{PSU}(2))$ the moduli space of PSU(2) flat connections on the 4-holed sphere $\mathcal{S}_a$. Flat connections in this moduli space have the following geometrical interpretations as constant curvature tetrahedra.

Theorem IV.1. There is a bijection between flat connections in $\mathcal{M}_{\mathrm{flat}}(\mathcal{S}_a,\mathrm{PSU}(2))$ and convex constant curvature tetrahedron geometries in $3d$, except for degenerate geometries. Nondegenerate tetrahedral geometries are dense in $\mathcal{M}_{\mathrm{flat}}(\mathcal{S}_a,\mathrm{PSU}(2))$.

The proof of this theorem is given in [21]. Both positive and negative constant curvature tetrahedra are included in $\mathcal{M}_{\mathrm{flat}}(\mathcal{S}_a, PSU(2))$.

Given the boundary condition leading to PSU(2) flat connections on $\{\mathcal{S}_a\}_{a=1}^5$, if there exists a $\mathrm{PSL}(2,\mathbb{C})$ flat connection on $S^3\backslash\Gamma_5$ satisfying the boundary condition, it is a critical point of $A_v = \mathcal{Z}_{S^3\backslash\Gamma_5}(\iota_{\vec{j},\vec{\xi}})$ and has the geometrical interpretation as a constant curvature 4-simplex.

Theorem IV.2. There is a bijection between $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $S^3\backslash\Gamma_5$ satisfying the boundary condition and the nondegenerate, convex, oriented, geometrical 4-simplex with constant curvature in the Lorentzian signature.

The proof of this theorem is given in [13]. Note that not every flat connection on $\times_{a=1}^5\mathcal{S}_a$ can extend to a flat connection $S^3\backslash\Gamma_5$. It is shown in [13] that there is a subset of PSU(2) flat connections on $\times_{a=1}^5\mathcal{S}_a$ that can serve as the boundary of $\mathrm{PSL}(2,\mathbb{C})$ flat connections on $S^3\backslash\Gamma_5$, and these boundary PSU(2) flat connections correspond to five constant curvature tetrahedra that can be glued[11] to form the close boundary of a nondegenerate 4-simplex with the same constant curvature Λ. Here, the A_v with these boundary data has critical points. However, any boundary PSU(2) flat connection corresponding to five tetrahedra that cannot be glued to form a 4-simplex boundary cannot extend to a

[11]Namely, they have the same constant curvature Λ and satisfy triangle shape matching and orientation matching when they are glued.

$\mathrm{PSL}(2,\mathbb{C})$ flat connection on $S^3\backslash\Gamma_5$; the result is that A_v has no critical point and thus is suppressed faster than $O(k^{-N})$ for all $N>0$.

We do not discuss the possible flat connections corresponding to the degenerate 4-simplex or tetrahedron. We also do not consider the boundary condition with $z_a\to\infty$ which leads to critical points located at infinity of the integration cycle.[12]

In this geometrical correspondence between the flat connection and 4-simplex geometry, the holonomy's squared eigenvalue $e^{2L_{ab}}$ relates to the area $\mathfrak{a}_{ab}$ of the 4-simplex boundary triangle f_{ab} shared by the pair of tetrahedra a, b (corresponding to $\mathcal{S}_a$, $\mathcal{S}_b$); i.e., semiclassically,

$$e^{2L_{ab}}\simeq e^{i\frac{|\Lambda|}{3}\mathfrak{a}_{ab}},\qquad \mathfrak{a}_{ab}\in[0,6\pi/|\Lambda|]. \tag{157}$$

The framing flag $s_{\ell_{ab}}$ evaluated at $\mathfrak{p}_a\in\mathcal{S}_a$, $s_{\ell_{ab}}(\mathfrak{p}_a)=\xi_{ab}$ relates to the unit normal $\vec{n}_{ab}$ (located at a vertex of the curved tetrahedron) of the face f_{ab} viewed in the frame of tetrahedra a by $\vec{n}_{ab}=\xi_{ab}^\dagger\vec{\boldsymbol{\sigma}}\xi_{ab}$. Note that ξ_{ab} is not always the same as ξ_i in (100); see the discussion in the paragraph above (104). Given the tetrahedra a, if we denote by $\vec{\mathfrak{n}}_i$ the geometrical outward pointing face normal of the tetrahedron, we have $\vec{n}_{ab}=\mathrm{sgn}(\Lambda)\vec{\mathfrak{n}}_i$ if $\xi_{ab}=\xi_i=(\xi_i^1,\xi_i^2)^T$, and $\vec{n}_{ab}=-\mathrm{sgn}(\Lambda)\vec{\mathfrak{n}}_i$ if $\xi_{ab}=(-\bar{\xi}_i^2,\bar{\xi}_i^1)^T$ [24].

In order to obtain the geometrical interpretation of the conjugate $\mathcal{T}_{ab}$, we review the definition of the complex FN twist variable: Let us consider the annulus cusps ℓ connecting a pair of 4-holed spheres $\mathcal{S}_0$, $\mathcal{S}_n$. Let s be the framing flag for ℓ and $s_{0,n}$, $s'_{0,n}$ be the framing flags for a pair of other cusps connecting $\mathcal{S}_{0,n}$. Then, the complex FN twist is defined by (see e.g., [27])

$$\tau_\ell=-\frac{\langle s_0\wedge s_0'\rangle}{\langle s_0\wedge s\rangle\langle s_0'\wedge s\rangle}\frac{\langle s_n\wedge s\rangle\langle s_n'\wedge s\rangle}{\langle s_n\wedge s_n'\rangle}, \tag{158}$$

where $\langle s\wedge s'\rangle$ are evaluated at a common point after parallel transportation. Without loss of generality, we evaluate the first ratio with factors $\langle s_0\wedge s_0'\rangle,\langle s_0\wedge s\rangle,\langle s_0'\wedge s\rangle$ at a point $\mathfrak{p}_0\in\mathcal{S}_0$, and we evaluate the second ratio with factors $\langle s_n\wedge s\rangle,\langle s_n'\wedge s\rangle,\langle s_n\wedge s_n'\rangle$ at a point $\mathfrak{p}_n\in\mathcal{S}_n$. The evaluation involves both $s(\mathfrak{p}_0)$ and $s(\mathfrak{p}_n)$ at two ends of ℓ, while the parallel transportation between $s(\mathfrak{p}_0)$ and $s(\mathfrak{p}_n)$ depends on a choice of contour γ_τ connecting $\mathfrak{p}_0$, $\mathfrak{p}_n$ (Fig. 8). Different γ_τ may transform $s(\mathfrak{p}_n)\to\lambda_\ell s(\mathfrak{p}_n)$ but keep $s(\mathfrak{p}_0)$ invariant. Moreover, by definition, τ_ℓ also depends on the choice of two other auxiliary cusps for each of $\mathcal{S}_0,\mathcal{S}_n$. The choices of γ_τ and the auxiliary cusps are part of the definition for τ_ℓ. The choices in defining τ_ℓ do not affect our later result. The Atiyah-Bott symplectic form implies that $\log(\tau_\ell)$ is the conjugate variable of the FN length variable $L_\ell=\log(\lambda_\ell)$ associated to the same annulus ℓ:

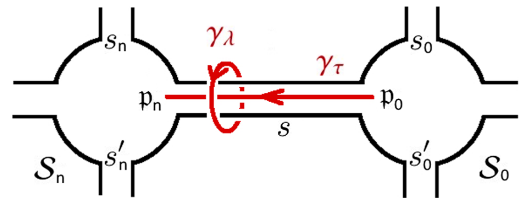

FIG. 8. Contour γ_τ used to define the complex FN twist τ_ℓ, and the meridian cycle γ_λ used to define the complex FN length λ_ℓ.

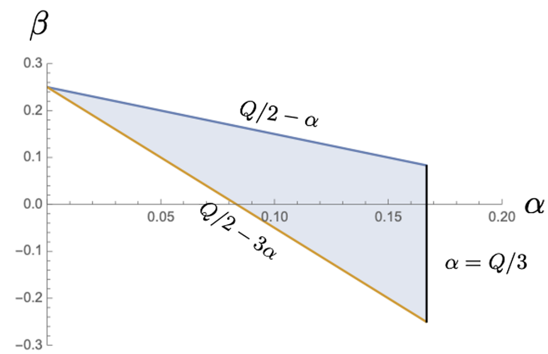

FIG. 9. Setting $\alpha_X=\alpha_Y=\alpha_Z=\alpha$, $\beta_X=\beta_Y=\beta_Z=\beta$, and $Q=1/2$, $\mathfrak{P}(\mathrm{oct})$ is restricted to the gray open triangle in the plot.

$$\{L_\ell,\log(\tau_{\ell'})\}_\Omega=\delta_{\ell,\ell'}. \tag{159}$$

Applying the above definition to $S^3\backslash\Gamma_5$, we set $\mathcal{S}_0=\mathcal{S}_b$, $\mathfrak{p}_0\equiv\mathfrak{p}_b$ and $\mathcal{S}_n=\mathcal{S}_a$, $\mathfrak{p}_n\equiv\mathfrak{p}_a$. Framing flags associated to holes in $\mathcal{S}_a$ (or $\mathcal{S}_b$) evaluated at $\mathfrak{p}_a$ (or $\mathfrak{p}_b$) are $\{\xi_{ac}\}_{c\neq a}$ (or $\{\xi_{bc}\}_{c\neq b}$). In particular, $s(\mathfrak{p}_a)=\xi_{ab}$ and $s(\mathfrak{p}_b)=\xi_{ba}$. We denote by G_{ab} the flat connection holonomy along γ_τ starting at $\mathfrak{p}_a$ and ending at $\mathfrak{p}_b$. Note that G_{ab} satisfies [13,19,24]

$$G_{ab}\xi_{ab}=e^{-\frac{1}{2}\nu\,\mathrm{sgn}(V_4)\Theta_{ab}+i\theta_{ab}}\xi_{ba},\qquad \nu=\mathrm{sgn}(\Lambda). \tag{160}$$

By the geometrical correspondence of the flat connection, Θ_{ab} is the hyperdihedral (boost) angle hinged by the face f_{ab} shared by the tetrahedra a, b on the boundary of the 4-simplex. Note that $\mathrm{sgn}(V_4)=\pm1$ is the orientation of the 4-simplex, and $\theta_{ab}\in[0,2\pi)$ is an angle relating to the phase convention of the ξ's. Inserting (160) in the definition of τ_ℓ, we obtain

$$\tau_{\ell_{ab}}\equiv\tau_{ab}=e^{-\nu\,\mathrm{sgn}(V_4)\Theta_{ab}+2i\theta_{ab}}\chi_{ab}(\xi),$$
$$\chi_{ab}(\xi)=\frac{\langle\xi_{bd}\wedge\xi_{bh}\rangle}{\langle\xi_{bd}\wedge\xi_{ba}\rangle\langle\xi_{bh}\wedge\xi_{ba}\rangle}\frac{\langle\xi_{ac}\wedge\xi_{ab}\rangle\langle\xi_{ae}\wedge\xi_{ab}\rangle}{\langle\xi_{ac}\wedge\xi_{ae}\rangle} \tag{161}$$

[12]Critical points at infinity give z, z' or $z''\to\infty$ of certain $\Delta\subset S^3\backslash\Gamma_5$. They either correspond to degenerate 4-simplex or to special 4-simplices which become close to degenerate if $|\Lambda\mathfrak{a}|\ll1$; i.e., scales of 4-simplices are small (see [19] and Appendix E therein).

where we have set $s_0(\mathfrak{p}_b) = \xi_{bd}$, $s'_0(\mathfrak{p}_b) = \xi_{bh}$ and $s_n(\mathfrak{p}_a) = \xi_{ac}$, $s'_n(\mathfrak{p}_a) = \xi_{ae}$. Here, $\chi(\xi)$ is a function only depending on the boundary condition on $\{\mathcal{S}_a\}_{a=1}^5$.

Theorem IV.3. Given a $\mathrm{PSL}(2,\mathbb{C})$ flat connection $\mathfrak{A}$ on $S^3\backslash\Gamma_5$ corresponding to a nondegenerate convex constant curvature 4-simplex, there exists a unique flat connection $\tilde{\mathfrak{A}} \neq \mathfrak{A}$ sharing the same boundary condition. Here, $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ correspond to the same constant curvature 4-simplex geometry but opposite orientations: $\mathrm{sgn}(V_4)|_{\mathfrak{A}} = -\mathrm{sgn}(V_4)|_{\tilde{\mathfrak{A}}}$.

The detailed proof is again given in [13]. The boundary condition corresponding to the boundary tetrahedra of the nondegenerate 4-simplex gives exactly two critical points $\mathfrak{A}$, $\tilde{\mathfrak{A}}$, which are called the parity pair, as an analog of a similar situation in the EPRL amplitude [22]. That $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ correspond to the same geometry means that they endow the same edge lengths, areas, angles, etc. to the 4-simplex. Implied by this result, $e^{2L_{ab}}$, $e^{\mathcal{X}_a}$, $e^{\mathcal{Y}_a}$ have the same value at $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ since they are determined by the geometry, whereas τ_{ab} are different,

$$\tau_{ab}|_{\mathfrak{A}} = e^{-\nu\Theta_{ab}+2i\theta_{ab}}\chi_{ab}(\xi), \qquad \tau_{ab}|_{\tilde{\mathfrak{A}}} = e^{\nu\Theta_{ab}+2i\theta_{ab}}\chi_{ab}(\xi),$$

since τ_{ab} relates to the orientation. Here $\theta_{ab}, \chi_{ab}(\xi)$ are the same at $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ since they are determined only by the boundary condition.

Lemma IV.4. At each annulus ℓ_{ab}, $\tau_{ab} = \tau_{\ell_{ab}}$ is related to $\mathcal{T}_{ab}$ by $\mathcal{T}_{ab} = \frac{1}{2}\log(\tau_{ab}) + f(\{L_{ab}\},\{\mathcal{X}_a,\mathcal{Y}_a\})$, where f is a linear function of $\{L_{ab}\},\{\mathcal{X}_a,\mathcal{Y}_a\}$.

Proof: Each τ_{ab} is a product of $z^{\pm 1}$, $z'^{\pm 1}$, $z''^{\pm 1}$ of some ideal tetrahedra in the triangulation of $S^3\backslash\Gamma_5$ (see Appendix A.3.3 in [27]). When expressing this in terms of octahedron phase space coordinates, each $\log(\tau_{ab})$ is a linear function of X_a, P_{X_a}, Y_a, P_{Y_a}, Z_a, P_{Z_a} $(a = 1,\ldots,5)$ when we impose $C_a = 2\pi i$; see [19] for explicit examples of $\log(\tau_{ab})$. Using the symplectic transformation (53), we express $\log(\tau_{ab}) = \sum_{c<d}(\alpha_{(ab),(cd)}\mathcal{T}_{cd} + \beta_{(ab),(cd)}L_{cd}) + \sum_{c=1}^5(\rho_c\mathcal{X}_c + \sigma_c\mathcal{Y}_c) + i\pi\mathbb{Z}$. Using $\{L_\ell, \log(\tau_{\ell'})\}_\Omega = \delta_{\ell,\ell'}$, we determine $\alpha_{(ab),(cd)} = 2\delta_{(ab),(cd)}$ and define $f = -\frac{1}{2}[\sum_{c<d}\beta_{(ab),(cd)}L_{cd} + \sum_{c=1}^5(\rho_c\mathcal{X}_c + \sigma_c\mathcal{Y}_c) + i\pi\mathbb{Z}]$. ∎

As a result, $\mathcal{T}_{ab}$ are given by

$$\mathcal{T}_{ab}|_{\mathfrak{A}} = -\frac{1}{2}\nu\Theta_{ab} + i\theta_{ab} + \frac{1}{2}\log\chi_{ab}(\xi) + f(\{L_{ab}\},\{\mathcal{X}_a,\mathcal{Y}_a\}) + \pi i N_{ab}^{(A)}, \tag{162}$$

$$\mathcal{T}_{ab}|_{\tilde{\mathfrak{A}}} = \frac{1}{2}\nu\Theta_{ab} + i\theta_{ab} + \frac{1}{2}\log\chi_{ab}(\xi) + f(\{L_{ab}\},\{\mathcal{X}_a,\mathcal{Y}_a\}) + \pi i N_{ab}^{(\tilde{A})}, \tag{163}$$

where $N_{ab}^{(A)}$, $N_{ab}^{(\tilde{A})} \in \mathbb{Z}$ label the lifts of logarithms.

C. Asymptotics of the vertex amplitude

The vertex amplitude A_v has precisely two critical points $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ when the boundary condition corresponds to five tetrahedra that can be glued to form the close boundary of a nondegenerate constant curvature 4-simplex. Using (148), the vertex amplitude has the following large-k asymptotics:

$$A_v(\vec{j},\vec{\xi}) = [\mathcal{N}_\alpha e^{S_{\vec{p}_0}^{(\alpha)}(\mathcal{Q},\tilde{\mathcal{Q}})} + \mathcal{N}_{\tilde{\alpha}} e^{S_{\vec{p}_0}^{(\tilde{\alpha})}(\mathcal{Q},\tilde{\mathcal{Q}})}], [1 + O(1/k)] \tag{164}$$

$$\mathcal{N}_\alpha = \frac{\mathcal{N}' e^{\frac{ik}{4\pi}\sum_{a=1}^5[4\mathrm{Re}(z_a)\mathrm{Im}(z_a) - x_a y_a]}}{\sqrt{\det(-\mathcal{H}_\alpha/2\pi)}}, \tag{165}$$

where $S_{\vec{p}_0}^{(\alpha)}$ is given in (147). The nondegeneracy of the Hessian matrix $\mathcal{H}_\alpha = \partial^2\mathcal{I}_{\vec{p}_0,\vec{0}}$ is supported by many numerical experiments. Note that $\mathcal{Q}_I$, $\tilde{\mathcal{Q}}_I$ are the same at the critical points $\mathfrak{A}$, $\tilde{\mathfrak{A}}$, and α, $\tilde{\alpha}$ are branches of the Lagrangian submanifold $\mathcal{L}_{S^3\backslash\Gamma_5}$ containing $\mathfrak{A}$, $\tilde{\mathfrak{A}}$, respectively. The asymptotics (164) of A_v reduces to the same form as the one studied in [23,24]. In the following we sketch the computation of (164) and refer to [23,24] for the details.

We rewrite (164) in $A_v \simeq e^{i\eta}(\mathcal{N}_+ e^S + \mathcal{N}_- e^{-S})$ where we factor out the overall phase $e^{i\eta}$, and we are interested in the phase difference e^{2S} between two exponentials in (164). To extract the phase difference, we consider a small variation $\delta\mathcal{Q}_I$, $\delta\tilde{\mathcal{Q}}_I$. The consequent variation of δS is given by

$$\begin{aligned}
2\delta S &= -\frac{ik}{2\pi(1+b^2)}(\vec{\mathcal{P}}^{(\alpha)} - \vec{\mathcal{P}}^{(\tilde{\alpha})})\cdot\delta\vec{\mathcal{Q}} - \text{c.c.} \\
&= -\frac{k\Lambda}{6\pi(1+b^2)}\sum_{a<b}(\Theta_{ab} + 2\pi i N_{ab})\delta\mathfrak{a}_{ab} - \text{c.c.} \\
&= -\frac{i\Lambda}{6\pi}\mathrm{Im}(t)\sum_{a<b}\Theta_{ab}\delta\mathfrak{a}_{ab} - \frac{i\Lambda}{3}\mathrm{Re}(t)\sum_{a<b}N_{ab}\delta\mathfrak{a}_{ab},
\end{aligned}$$

where $N_{ab} = \mathrm{sgn}(\Lambda)(N_{ab}^{(A)} - N_{ab}^{(\tilde{A})}) \in \mathbb{Z}$. Only Θ_{ab} and $N_{ab}^{(A)}$, $N_{ab}^{(\tilde{A})}$ in $\mathcal{T}_{ab}$ give nonvanishing contributions to $2\delta S$ because each of $\{L_{ab},\mathcal{X}_a,\mathcal{Y}_a,\chi_{ab}(\xi),\theta_{ab}\}$ gives the same value at $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ (see [23,24] for details). By the Schläfli identity $\sum_{a<b}\delta\Theta_{ab}\mathfrak{a}_{ab} = \Lambda|V_4|$ of the constant curvature 4-simplex [52], δS can be integrated as

$$\begin{aligned}
2S &= -\frac{i\Lambda k\gamma}{6\pi}\left(\sum_{a<b}\mathfrak{a}_{ab}\Theta_{ab} - \Lambda|V_4|\right) \\
&\quad - \frac{i\Lambda k}{3}\sum_{a<b}N_{ab}\mathfrak{a}_{ab} + 2C,
\end{aligned} \tag{166}$$

where $|V_4|$ is the 4-simplex volume. $2C$ is a geometry-independent integration constant. Equations (157) and (97) imply $\frac{|\Lambda|}{3}\mathfrak{a}_{ab}=\frac{4\pi}{k}j_{ab}$; thus, $\frac{i\Lambda k}{3}\sum_{a<b}N_{ab}\mathfrak{a}_{ab}\in 2\pi i\mathbb{Z}$ is negligible in e^{2S}. As a result, we obtain the leading asymptotics of A_v as

$$A_v=e^{i\eta}(\mathcal{N}_+e^{iS_{\mathrm{Regge}}+C}+\mathcal{N}_-e^{-iS_{\mathrm{Regge}}-C})[1+O(1/k)], \tag{167}$$

$$\mathcal{N}_{+,-}=\frac{\mathcal{N}'}{\sqrt{\det(-\mathcal{H}_{\alpha,\tilde{\alpha}}/2\pi)}} \tag{168}$$

where in the exponents

$$S_{\mathrm{Regge}}=\frac{\Lambda k\gamma}{12\pi}\left(\sum_{a<b}\mathfrak{a}_{ab}\Theta_{ab}-\Lambda|V_4|\right) \tag{169}$$

is the Regge action of the constant curvature 4-simplex. The coefficient $\frac{|\Lambda|k\gamma}{12\pi}$ is identified to be the inverse gravitational coupling $1/\ell_P^2$. This identification is consistent with (16).

V. CONCLUSION AND OUTLOOK

In this work, we propose an improved formulation of 4d spinfoam quantum gravity with a cosmological constant Λ. This formulation is featured with the finite spinfoam amplitudes on simplicial complexes and the correct semiclassical behavior of the vertex amplitude.

Despite the above promising aspects, this formulation still has several open issues, which are expected to be addressed in future work: First, it is conjectured in Sec. III D that the boundary Hilbert space of the spinfoam amplitude A is the Hilbert space of $\mathfrak{q}$-deformed spin-network states with $\mathfrak{q}$ root of unity. To prove this conjecture, we need to define and study coherent intertwiners of $\mathfrak{q}$-deformed spin networks and clarify if there is a canonical bijection between these coherent intertwiners and the boundary data of A. The expected coherent intertwiner should be a $\mathfrak{q}$ deformation of the Livine-Speciale coherent intertwiner [40].

We need to construct a geometrical operator on the boundary Hilbert space to understand the quantum geometrical interpretation of boundary states. The construction may be based on the combinatorial quantization of SU(2) CS theory [53,54]. It is interesting to define coherent states that are coherent in both spins (areas) and intertwiners (shapes of curved tetrahedra). The coherent state may be a $\mathfrak{q}$ deformation of the complexifier coherent states in [55]. In addition, we need to direct the sum over all graphs to define the entire $\mathfrak{q}$-deformed LQG kinematical Hilbert space and check the cylindrical consistency of operators. This should generalize the work [12] from the real $\mathfrak{q}$ to $\mathfrak{q}$ root of unity.

In Sec. III E, we discuss that the spinfoam amplitude A has ambiguities in which the freedom of choosing coherent states is due to imposing a semiclassical simplicity constraint on coherent state labels. It may be useful to develop an operator formalism or other ways to impose the simplicity constraint (such as the master constraint, Gupta-Bleuler, etc.) at the quantum level for reducing the freedom of the amplitude. Another possible drawback of our implementation of the simplicity constraint is that spins such that $\dim(\tilde{\mathcal{M}}_{\vec{j}})<2$ ($\tilde{\mathcal{M}}_{\vec{j}}$ only contains degenerate 4-gons) have to be excluded from our formalism.

In the present work, we only study the semiclassical behavior of the vertex amplitude. The semiclassical analysis should generalize to the spinfoam amplitude with Λ on an arbitrary simplicial complex, as well as taking into account the sum over j.

Note that Λ in this spinfoam model should be understood as the ultraviolet value of the cosmological constant. It would be interesting to apply the Wilson renormalization to the spinfoam model with Λ (see e.g., [56] for some earlier results). The spinfoam renormalization is expected to result in a flow of Λ from the ultraviolet to infrared, where the infrared value of Λ should relate to the observation.

It would also be interesting to develop a group field theory (GFT) based on the spinfoam formulation with Λ. The notion of group fields might be suitably generalized to include Λ. The "group fields" might actually be fields on the moduli space of flat connections. The GFT is expected to reproduce spinfoam amplitudes A, which are finite order by order in the perturbative expansion.

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APPENDIX A: A PLOT FOR THE POLYTOPE $\mathfrak{P}(\mathrm{oct})$

The open polytope $\mathfrak{P}(\mathrm{oct})$ is defined by the following inequalities:

$$\alpha_X,\alpha_Y,,\alpha_Z>0,\qquad \alpha_X+\alpha_Y+\alpha_Z<Q,$$
$$\alpha_X+\beta_X<\frac{Q}{2},\qquad \alpha_Y+\beta_Y<\frac{Q}{2},\qquad \alpha_Z+\beta_Z<\frac{Q}{2},$$
$$\alpha_X+\alpha_Y+\alpha_Z+\beta_X>\frac{Q}{2},\qquad \alpha_X+\alpha_Y+\alpha_Z+\beta_Y>\frac{Q}{2},$$
$$\alpha_X+\alpha_Y+\alpha_Z+\beta_Z>\frac{Q}{2}.$$

Figure 9 plots the intersection between $\mathfrak{P}(\mathrm{oct})$ and the plane of $\alpha_X=\alpha_Y=\alpha_Z$, $\beta_X=\beta_Y=\beta_Z$.

APPENDIX B: DARBOUX COORDINATES OF $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$

Darboux coordinates $\mathcal{Q}_I = (2L_{ab}, \mathcal{X}_a)$, $\mathcal{P}_I = (\mathcal{T}_{ab}, \mathcal{Y}_a)$ expressed in terms of (X_a, P_{X_a}),(Y_a, P_{Y_a}),(Z_a, P_{Z_a}),(C_a, Γ_a) are listed below:

$$2L_{12} = -C_3 - C_4 - C_5 + P_{Y_3} + P_{Y_4} + P_{Y_5} + X_3 + X_4 + X_5 + Y_3 + Y_4 + Y_5 + 3i\pi, \tag{B1}$$

$$2L_{13} = -C_2 - C_5 + P_{Y_2} + P_{Y_4} - P_{Z_4} + P_{Z_5} + X_2 + X_5 + Y_2 + Y_5 + 2Z_5 + i\pi, \tag{B2}$$

$$2L_{14} = -C_3 + P_{Y_2} + P_{Y_5} - P_{Z_2} + P_{Z_3} - P_{Z_5} + X_3 + Y_3 + 2Z_3, \tag{B3}$$

$$2L_{15} = -C_2 - C_4 + P_{Y_3} + P_{Z_2} - P_{Z_3} + P_{Z_4} + X_2 + X_4 + Y_2 + Y_4 + 2Z_2 + 2Z_4, \tag{B4}$$

$$2L_{23} = -P_{X_1} + P_{X_4} - P_{X_5} - P_{Y_4} + X_4 - Y_4, \tag{B5}$$

$$2L_{24} = -P_{X_3} + P_{X_5} - P_{Y_1} - P_{Y_5} - X_1 + X_5 - Y_1 - Y_5 + i\pi, \tag{B6}$$

$$2L_{25} = P_{X_1} + P_{X_3} - P_{X_4} - P_{Y_1} - P_{Y_3} + X_1 + X_3 - Y_1 - Y_3, \tag{B7}$$

$$2L_{34} = C_1 - C_5 + P_{X_2} + P_{X_5} - P_{Y_2} - P_{Z_1} - P_{Z_5} - X_1 + X_2 + X_5 - Y_1 - Y_2 + Y_5 - 2Z_1 + i\pi, \tag{B8}$$

$$2L_{35} = -C_1 + P_{X_1} - P_{X_2} - P_{X_4} - P_{Z_1} + P_{Z_4} + X_1 - X_4 + Y_1 - Y_4 + 2i\pi, \tag{B9}$$

$$2L_{45} = -C_3 - P_{X_2} + P_{X3} + P_{Y_1} - P_{Z_1} + P_{Z_2} - P_{Z_3} - X_2 + X_3 - Y_2 + Y_3 + 2i\pi. \tag{B10}$$

$$\mathcal{X}_1 = \chi_{25}^{(1)} = P_{Y_2} - P_{Z_2} - Z_2 + Z_5 + i\pi, \tag{B11}$$

TABLE I. Edge coordinates $\chi_{mn}^{(a)}$ of 4-holed spheres. Recall in Fig. 3 that the octahedra are glued through the triangles labeled by $a, b, c, d, e, f, g, h, i, j$. For example, a_2' labels the triangles symmetric to the triangle a with respect to the equator of Oct(2). The "primed triangles" with primed labels triangulate the geodesic boundary of $S^3\backslash\Gamma_5$. Here X_a, Y_a, Z_a, W_a $(a = 1, \ldots, 5)$ are the tetrahedron edge coordinates from the four tetrahedra triangulating Oct(a).

$\mathcal{S}_1$:	$h_2' \cap h_3'$: $\chi_{23}^{(1)} = Z_2 + Z_3$	$h_3' \cap e_4'$: $\chi_{34}^{(1)} = Y_3'' + Z_3' + Z_4'' + W_4'$
	$h_2' \cap e_4'$: $\chi_{24}^{(1)} = Z_2'' + W_2' + Z_4$	$h_3' \cap c_5'$: $\chi_{35}^{(1)} = Z_3'' + W_3' + Y_5'' + Z_5'$
	$h_2' \cap c_5'$: $\chi_{25}^{(1)} = Y_2'' + Z_2' + Z_5$	$e_4' \cap c_5'$: $\chi_{45}^{(1)} = Y_4'' + Z_4' + Z_5'' + W_5'$
$\mathcal{S}_2$:	$f_1' \cap i_3'$: $\chi_{13}^{(2)} = X_1'' + Y_1' + X_3$	$i_3' \cap f_4'$: $\chi_{34}^{(2)} = X_3'' + Y_3' + W_4'' + X_4'$
	$f_1' \cap f_4'$: $\chi_{14}^{(2)} = X_1 + X_4$	$i_3' \cap b_5'$: $\chi_{35}^{(2)} = W_3'' + X_3' + X_5'' + Y_5'$
	$f_1' \cap b_5'$: $\chi_{15}^{(2)} = W_1'' + X_1' + X_5$	$f_4' \cap b_5'$: $\chi_{45}^{(2)} = X_4'' + Y_4' + W_5'' + X_5'$
$\mathcal{S}_3$:	$b_1' \cap a_2'$: $\chi_{12}^{(3)} = Z_1' + W_1'' + X_2$	$a_2' \cap d_4'$: $\chi_{24}^{(3)} = W_2'' + X_2' + Y_4' + Z_4''$
	$b_1' \cap d_4'$: $\chi_{14}^{(3)} = W_1' + X_1'' + X_4' + Y_4''$	$a_2' \cap d_5'$: $\chi_{25}^{(3)} = X_2'' + Y_2' + Z_5' + W_5''$
	$b_1' \cap d_5'$: $\chi_{15}^{(3)} = W_1 + W_5' + X_5''$	$d_4' \cap d_5'$: $\chi_{45}^{(3)} = Y_4 + W_5$
$\mathcal{S}_4$:	$a_1' \cap c_2'$: $\chi_{12}^{(4)} = Z_1 + X_2' + Y_2''$	$c_2' \cap j_3'$: $\chi_{23}^{(4)} = Y_2' + Z_2'' + Z_3' + W_3''$
	$a_1' \cap j_3'$: $\chi_{13}^{(4)} = Y_1'' + Z_1' + W_3' + X_3''$	$c_2' \cap j_5'$: $\chi_{25}^{(4)} = Y_2 + Y_5' + Z_5''$
	$a_1' \cap j_5'$: $\chi_{15}^{(4)} = Z_1'' + W_1' + X_5' + Y_5''$	$j_3' \cap j_5'$: $\chi_{35}^{(4)} = W_3 + Y_5$
$\mathcal{S}_5$:	$i_1' \cap e_2'$: $\chi_{12}^{(5)} = Y_1' + Z_1'' + W_2' + X_2''$	$e_2' \cap g_3'$: $\chi_{23}^{(5)} = Z_2' + W_2'' + Y_3' + Z_3''$
	$i_1' \cap g_3'$: $\chi_{13}^{(5)} = Y_1 + X_3' + Y_3''$	$e_2' \cap g_4'$: $\chi_{24}^{(5)} = W_2 + Z_4' + W_4''$
	$i_1' \cap g_4'$: $\chi_{14}^{(5)} = X_1' + Y_1'' + W_4' + X_4''$	$g_3' \cap g_4'$: $\chi_{34}^{(5)} = Y_3 + W_4$

$$\mathcal{X}_2 = \chi_{15}^{(2)} = -P_{X_1} - X_1 + X_5 + i\pi, \tag{B12}$$

$$\mathcal{X}_3 = \chi_{15}^{(3)} = C_1 - C_5 + P_{X_5} - X_1 + X_5 - Y_1 + Y_5 - Z_1 + Z_5 + i\pi, \tag{B13}$$

$$\mathcal{X}_4 = \chi_{15}^{(4)} = -C_1 - P_{X_5} + P_{Y_5} + P_{Z_1} + X_1 - X_5 + Y_1 + Z_1 + 2i\pi, \tag{B14}$$

$$\mathcal{X}_5 = \chi_{14}^{(5)} = -C_4 - P_{X_1} + P_{X_4} + P_{Y_1} - X_1 + X_4 + Y_4 + Z_4 + 2i\pi. \tag{B15}$$

$$\mathcal{T}_{12} = \frac{1}{2}(X_2 - X_3 - X_4 + Y_1 + Y_2 - Y_3 - Y_4 + Z_2), \tag{B16}$$

$$\mathcal{T}_{13} = \frac{1}{2}(-X_2 + X_3 - Y_2 + Y_3 - Y_5 + Z_1 - Z_2 - Z_5), \tag{B17}$$

$$\mathcal{T}_{14} = \frac{1}{2}(-Y_2 - Z_2 - Z_3 + Z_5), \tag{B18}$$

$$\mathcal{T}_{15} = \frac{1}{2}(-X_2 - Y_2 - Z_2 - Z_4), \tag{B19}$$

$$\mathcal{T}_{23} = \frac{1}{2}(-X_4 + Y_1 + Y_4 - Y_5 + Z_1 - Z_5), \tag{B20}$$

$$\mathcal{T}_{24} = \frac{1}{2}(X_2 + X_3 - X_4 + Y_1 + Y_3 - Y_4 + Z_3 + Z_5), \tag{B21}$$

$$\mathcal{T}_{25} = \frac{1}{2}(-X_3 - X_4 + Y_1 + Y_3 - Y_4 - Z_4), \tag{B22}$$

$$\mathcal{T}_{34} = \frac{1}{2}(-X_2 + X_3 + Y_3 - Y_5 + Z_1 + Z_3), \tag{B23}$$

$$\mathcal{T}_{35} = \frac{1}{2}(X_3 + Y_3 - Y_5 + Z_1 - Z_4 - Z_5), \tag{B24}$$

$$\mathcal{T}_{45} = \frac{1}{2}(X_2 + Z_3 + Z_4 + Z_5). \tag{B25}$$

$$\mathcal{Y}_1 = \chi_{23}^{(1)} = Z_2 + Z_3, \tag{B26}$$

$$\mathcal{Y}_2 = \chi_{14}^{(2)} = X_1 + X_4, \tag{B27}$$

$$\mathcal{Y}_3 = \chi_{45}^{(3)} - 2\pi i = -X_5 + Y_4 - Y_5 - Z_5, \tag{B28}$$

$$\mathcal{Y}_4 = -\chi_{35}^{(4)} + 2\pi i = X_3 + Y_3 - Y_5 + Z_3, \tag{B29}$$

$$\mathcal{Y}_5 = \chi_{34}^{(5)} - 2\pi i = -X_4 + Y_3 - Y_4 - Z_4. \tag{B30}$$

We impose $C_a = 2\pi i$ on all $2L_{ab}$ and $\mathcal{X}_a$. We check that (45) implies

$$\{\mathcal{Q}_I, \mathcal{P}_J\}_\Omega = \delta_{IJ}, \qquad \{\mathcal{Q}_I, \mathcal{Q}_J\}_\Omega = \{\mathcal{P}_I, \mathcal{P}_J\}_\Omega = 0. \qquad I, J = (\ell_{ab}, \mathcal{S}_a). \tag{B31}$$

APPENDIX C: SYMPLECTIC TRANSFORMATION

The linear symplectic transformation from $\vec{\Phi} \equiv (X_a, Y_a, Z_a)_{a=1}^5$ and $\vec{\Pi} \equiv (P_{X_a}, P_{Y_a}, P_{Z_a})_{a=1}^5$ to $\vec{\mathcal{Q}}$, $\vec{\mathcal{P}}$ is given by

$$\begin{pmatrix} \vec{\mathcal{Q}} \\ \vec{\mathcal{P}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -(\mathbf{B}^T)^{-1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\Phi} \\ \vec{\Pi} \end{pmatrix} + i\pi \begin{pmatrix} \vec{t} \\ \vec{0} \end{pmatrix}. \tag{C1}$$

Explicitly, $\mathbf{A}$, $\mathbf{B}$, $\vec{t}$ are given below:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{C2}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C3}$$

$$\vec{t} = (-3, -3, -2, -4, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0)^T. \tag{C4}$$

APPENDIX D: PROOF OF LEMMA III.1

Lemma D.1. $H_{i=1,...,4} \in \mathrm{SU}(2)$ satisfy $H_4H_3H_2H_1 = 1$, which exist if and only if $j'_{i=1,...,4}$ satisfy the triangle inequality, i.e., there exists J such that

$$|j'_1 - j'_2| \leq J \leq \min(j'_1 + j'_2, k - j'_1 - j'_2), \quad \text{(D1)}$$

$$|j'_3 - j'_4| \leq J \leq \min(j'_3 + j'_4, k - j'_3 - j'_4). \quad \text{(D2)}$$

Proof: We denote $\frac{4\pi}{k} j'_i = r_i \in [0, 2\pi)$. Here, $H_i = \cos(r_i/2) + i\vec{n}'_i \cdot \vec{\sigma} \sin(r_i/2)$, where $\vec{n}'$ is a unit vector in $\mathbb{R}^3$. Note that $\vec{n}' = -\vec{n}$ when there is a minus sign in (100) and $\vec{n}' = \vec{n}$ when there is a plus sign. We denote $H_2H_1 = \cos(R/2) + i\vec{N} \cdot \vec{\sigma} \sin(R/2)$ with $R = \frac{4\pi}{k} J \in [0, 2\pi)$; then $H_4H_3 = \cos(R/2) - i\vec{N} \cdot \vec{\sigma} \sin(R/2)$. Taking the trace gives

$$\cos\left(\frac{R}{2}\right) = \cos\left(\frac{r_1}{2}\right)\cos\left(\frac{r_2}{2}\right) - \vec{n}'_1 \cdot \vec{n}'_2 \sin\left(\frac{r_1}{2}\right)\sin\left(\frac{r_2}{2}\right), \quad \text{(D3)}$$

$$\cos\left(\frac{R}{2}\right) = \cos\left(\frac{r_3}{2}\right)\cos\left(\frac{r_4}{2}\right) - \vec{n}'_3 \cdot \vec{n}'_4 \sin\left(\frac{r_3}{2}\right)\sin\left(\frac{r_4}{2}\right). \quad \text{(D4)}$$

Since $\sin(\frac{r_i}{2}) \geq 0$, unit vectors $\vec{n}'_{i=1,...,4}$ exist if and only if

$$\cos\left(\frac{r_1 + r_2}{2}\right) \leq \cos\left(\frac{R}{2}\right) \leq \cos\left(\frac{r_1 - r_2}{2}\right), \quad \text{(D5)}$$

$$\cos\left(\frac{r_3 + r_4}{2}\right) \leq \cos\left(\frac{R}{2}\right) \leq \cos\left(\frac{r_3 - r_4}{2}\right), \quad \text{(D6)}$$

which is equivalent to

$$|r_1 - r_2| \leq R \leq \min(r_1 + r_2, 4\pi - r_1 - r_2), \quad \text{(D7)}$$

$$|r_3 - r_4| \leq R \leq \min(r_3 + r_4, 4\pi - r_3 - r_4). \quad \text{(D8)}$$

Conversely, Eqs. (D5) and (D6) or Eqs. (D7) and (D8) imply the existence of two spherical triangles in S^3 sharing a common edge. The spherical triangles form a 4-gon whose edges are geodesics in S^3 with length $r_i/2$ ($i = 1, …, 4$). The diagonal of the 4-gon is a geodesic whose length is $R/2$. The 4-gon in S^3 implies the existence of $H_{i=1,...,4} \in \mathrm{SU}(2)$, which satisfy $H_4H_3H_2H_1 = 1$ by the argument in Sec. III B. ∎

APPENDIX E: DETERMINING ξ_i's FROM θ AND ϕ

It is useful to consider $\cos(\theta_{24}) = -\frac{1}{2}[\mathrm{Tr}(H_1H_4^{-1}) - \mathrm{Tr}(H_1)\mathrm{Tr}(H_4)] = \frac{1}{2}\mathrm{Tr}(H_4H_1) = \frac{1}{2}\mathrm{Tr}(H_2H_3)$. The following relation holds between ϕ and θ_{24} [37]:

$$2\cos(\theta_{24}) = \frac{(e^{i\phi} + e^{-i\phi})\sqrt{c_{12}(A)c_{34}(A)} - 2(m_2m_3 + m_1m_4) + A(m_1m_3 + m_2m_4)}{A^2 - 4}, \quad \text{(E1)}$$

where $m_i = \mathrm{Tr}(H_i)$ and

$$A = e^{i\theta} + e^{-i\theta}, \qquad c_{ij}(A) = A^2 + m_i^2 + m_j^2 - Am_im_j - 4. \quad \text{(E2)}$$

For SU(2) flat connections satisfying $H_4H_3H_2H_1 = 1$, we make a partial gauge fixing that $H_4 = \mathrm{diag}(e^{ia_4}, e^{-ia_4})$, $a_4 \in [0, \pi)$.[13] Thus, as a unit vector in Euclidean $\mathbb{R}^4$, $v_j = (v_j^0, v_j^1, v_j^2, v_j^3)$,

$$v_4 = (\cos(a_4), 0, 0, -\sin(a_4)) \quad \text{(E3)}$$

representing H_4^{-1}. For the triangle (v_1, v_3, v_4), we use $v_1 = (1, 0, 0, 0)$, $\langle v_1, v_3\rangle = \cos(\theta_{13})$ ($\theta_{13} = \theta$), $\langle v_3, v_4\rangle = \cos(a_3)$, and $\langle v_4, v_4\rangle = 1$ to determine v_3,

$$\begin{aligned} v_3 &= (\cos(\theta_{13}), 0, v_3^2, v_3^3), \\ v_3^2 &= \sqrt{-(\csc^2(a_4)(\cos^2(a_3) + \cos^2(\theta_{13}))) + 2\cos(a_3)\cot(a_4)\csc(a_4)\cos(\theta_{13}) + 1}, \\ v_3^3 &= \csc(a_4)(\cos(a_4)\cos(\theta_{13}) - \cos(a_3)), \end{aligned} \quad \text{(E4)}$$

where we use the remaining rotational symmetry (of the 1-2 plane) to fix $v_3^1 = 0$ and $v_3^2 > 0$. Then we use $\langle v_1, v_2\rangle = \cos(a_1)$, $\langle v_2, v_3\rangle = \cos(a_2)$, $\langle v_2, v_4\rangle = \cos(\theta_{24})$, and $\langle v_2, v_2\rangle = 1$ to determine v_2,

[13] We use the conjugation $\varepsilon \mathrm{diag}(\lambda, \lambda^{-1})\varepsilon^{-1} = \mathrm{diag}(\lambda^{-1}, \lambda)$, where $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ and $\det(\varepsilon) = 1$, to fix $a_4 \in [0, \pi)$ in $\lambda = e^{ia_4}$.

$$
\begin{aligned}
v_2 &= (\cos(a_1), v_2^1, v_2^2, \csc(a_4)(\cos(a_1)\cos(a_4) - \cos(\theta_{24}))), \\
v_2^1 &= \pm(2\cos(a_2)\csc(a_4)(\cot(a_4)(\cos(a_1)\cos(a_3) + \cos(\theta_{13})\cos(\theta_{24})) - \csc(a_4)(\cos(a_1)\cos(\theta_{13}) \\
&\quad + \cos(a_3)\cos(\theta_{24}))) + \csc(a_4)(-2\cos(a_1)\cot(a_4)\cos(\theta_{24}) + \csc(a_4)(\cos^2(\theta_{13}) + \sin^2(\theta_{13})\cos^2(\theta_{24})) \\
&\quad -2\cos(a_3)\cos(\theta_{13})(\cot(a_4) - \cos(a_1)\csc(a_4)\cos(\theta_{24})) + \cos^2(a_3)\csc(a_4) + \sin^2(a_3)\cos^2(a_1)\csc(a_4)) \\
&\quad + \cos^2(a_2) - 1)^{\frac{1}{2}}(\csc^2(a_4)(\cos^2(a_3) + \cos^2(\theta_{13})) - 2\cos(a_3)\cot(a_4)\csc(a_4)\cos(\theta_{13}) - 1)^{-\frac{1}{2}}, \\
v_2^2 &= 2(\cos(a_1)(\cos(\theta_{13}) - \cos(a_3)\cos(a_4)) + \cos(\theta_{24})(\cos(a_3) - \cos(a_4)\cos(\theta_{13})) + \sin^2(a_4)(-\cos(a_2))) \\
&\quad \times \sqrt{-(\csc^2(a_4)(\cos^2(a_3) + \cos^2(\theta_{13}))) + 2\cos(a_3)\cot(a_4)\csc(a_4)\cos(\theta_{13}) + 1} \\
&\quad \times (-4\cos(a_3)\cos(a_4)\cos(\theta_{13}) + \cos(2a_3) + \cos(2a_4) + \cos(2\theta_{13}) + 1)^{-1},
\end{aligned} \tag{E5}
$$

where $\pm$ of v_2^1 corresponds to the parity symmetry with respect to the plane of F_{134} (spanned by the x^0, x^2, x^3 directions in $\mathbb{R}^4$) where v_1, v_3, v_4 leave. Choosing $+$ or $-$ of v_2^1 is equivalent to fixing the orientation of $n_{123} \wedge n_{134}$ since $v_2^1 \to -v_2^1$ transforms as

$$
n_{123} \wedge n_{134} \to -n_{123} \wedge n_{134}, \quad \text{where } n_{ijk}^d \|n_{ijk}\| = \epsilon_{abcd} v_i^a v_j^b v_k^c. \tag{E6}
$$

Now all $\{H_i\}_{i=1}^4$ are fixed by

$$
H_1 = v_2, \qquad H_4 = v_4^{-1}, \qquad H_3 = v_4 v_3^{-1}, \qquad H_2 = v_3 v_2^{-1}, \tag{E7}
$$

$$
\text{where } v_j = v_j^0 I + i\sum_{a=1}^{3} v_j^a \boldsymbol{\sigma}_a. \tag{E8}
$$

Every H_i is uniquely determined by $(a_i, \theta_{13}, \theta_{24})$, where θ_{24} relates to ϕ by (E1); then ξ_i is determined up to a scaling as the eigenvector of H_i for the eigenvalue whose square is $e^{2L_{ab}}$.

APPENDIX F: CRITICAL EQUATIONS

Derivatives of $S_{\vec{\mathcal{P}}}$ are given by

$$
-\frac{2\pi(1+b^2)}{ik}\partial_{\vec{\mathcal{P}}} S_0 = \mathbf{AB}^T \cdot \vec{\mathcal{P}} + \vec{\mathcal{Q}} + \frac{k}{(1+b^2)}\mathbf{AB}^T \cdot (\vec{\mathcal{P}} - b^2\vec{\tilde{\mathcal{P}}}), \tag{F1}
$$

$$
-\frac{2\pi(1+b^{-2})}{ik}\partial_{\vec{\tilde{\mathcal{P}}}} S_0 = \mathbf{AB}^T \cdot \vec{\tilde{\mathcal{P}}} + \vec{\tilde{\mathcal{Q}}} - \frac{k}{(1+b^2)}\mathbf{AB}^T \cdot (\vec{\mathcal{P}} - b^2\vec{\tilde{\mathcal{P}}}), \tag{F2}
$$

$$
-\frac{2\pi(1+b^2)}{ik}\partial_{\vec{\mathcal{P}}} S_1 = -\mathbf{B} \cdot (P_{X_{a=1,\ldots,5}}, P_{Y_{a=1,\ldots,5}}, P_{Z_{a=1,\ldots,5}})^T, \tag{F3}
$$

$$
\text{e.g.} \qquad P_{Z_a} \equiv \log(1 - e^{-Z_a}) - \log(1 - e^{X_a+Y_a+Z_a}), \tag{F4}
$$

$$
-\frac{2\pi(1+b^{-2})}{ik}\partial_{\vec{\tilde{\mathcal{P}}}} \tilde{S}_1 = -\mathbf{B} \cdot (\tilde{P}_{X_{a=1,\ldots,5}}, \tilde{P}_{Y_{a=1,\ldots,5}}, \tilde{P}_{Z_{a=1,\ldots,5}})^T, \tag{F5}
$$

$$
\text{e.g.} \qquad \tilde{P}_{X_a} = \log(1 - e^{-\tilde{Z}_a}) - \log(1 - e^{\tilde{X}_a+\tilde{Y}_a+\tilde{Z}_a}), \tag{F6}
$$

$$
\vec{\mathcal{P}} = -(\mathbf{B}^T)^{-1} \cdot (X_{a=1,\ldots,5}, Y_{a=1,\ldots,5}, Z_{a=1,\ldots,5})^T, \tag{F7}
$$

$$
\vec{\tilde{P}} = -(\mathbf{B}^T)^{-1} \cdot (\tilde{X}_{a=1,\ldots,5}, \tilde{Y}_{a=1,\ldots,5}, \tilde{Z}_{a=1,\ldots,5})^T, \tag{F8}
$$

where the branches of the logarithms are the same as the canonical lift in (13).

We define

$$X''_a \coloneqq \log(1-e^{-X_a}), \qquad Y''_a \coloneqq \log(1-e^{-Y_a}),$$
$$Z''_a \coloneqq \log(1-e^{-Z_a}), \qquad W''_a \coloneqq \log(1-e^{-W_a}), \tag{F9}$$

$$\tilde{X}''_a \coloneqq \log(1-e^{-\tilde{X}_a}), \qquad \tilde{Y}''_a \coloneqq \log(1-e^{-\tilde{Y}_a}),$$
$$\tilde{Z}''_a \coloneqq \log(1-e^{-\tilde{Z}_a}), \qquad \tilde{W}''_a \coloneqq \log(1-e^{-\tilde{W}_a}), \tag{F10}$$

such that e.g., $z = e^Z$ and $z'' = e^{Z''}$ reproduce $z^{-1} + z'' - 1 = 0$, i.e., the Lagrangian submanifold $\mathcal{L}_\Delta \subset \mathcal{P}_{\partial\Delta}$ of framed flat $\mathrm{PSL}(2,\mathbb{C})$ connections on the ideal tetrahedron Δ. Here, $W_a, \tilde{W}_a$ are given by (133). The above logarithms are defined with the same canonical lifts as in (13). We define P_{X_a}, P_{Y_a}, P_{Z_a} and $\tilde{P}_{X_a}$, $\tilde{P}_{Y_a}$, $\tilde{P}_{Z_a}$ ($a = 1, \ldots, 5$) in the same way as (44). Note that X_a, Y_a, Z_a, P_{X_a}, P_{Y_a}, P_{Z_a} with Eqs. (F9), (F10), and (133) parametrize the moduli space of framed flat $\mathrm{PSL}(2,\mathbb{C})$ connections on the ideal octahedron $\mathrm{oct}(a)$ made by gluing four ideal tetrahedra.

The critical equations $\partial_{X_I} S_{\vec{p}} = \partial_{\tilde{X}_I} S_{\vec{p}} = 0$ can be written in terms of $\vec{\Phi} \equiv (X_a, Y_a, Z_a)_{a=1}^5$ and $\vec{\Pi} \equiv (P_{X_a}, P_{Y_a}, P_{Z_a})_{a=1}^5$:

$$\vec{\mathcal{Q}}' = \mathbf{A} \cdot \vec{\Phi} + \mathbf{B} \cdot \vec{\Pi} + 2\pi i(\vec{n} + \vec{p}), \tag{F11}$$

$$\vec{\tilde{\mathcal{Q}}}' = \mathbf{A} \cdot \vec{\tilde{\Phi}} + \mathbf{B} \cdot \vec{\tilde{\Pi}} - 2\pi i(\vec{n} + \vec{p}), \tag{F12}$$

where $\vec{p} \in \mathbb{Z}^{15}$. Up to $2\pi i(\vec{n} + \vec{p})$, the critical equations (136) and (137) reproduce the $\mathcal{Q}$ part of (53), whereas here $\vec{\Phi}$ and $\vec{\Pi}$ are related by (F9), (F10), and (44). Note that the $\mathcal{P}$ part of (53) has been reproduced by the relation between $(X_a, Y_a, Z_a)_{a=1}^5$ and $\vec{\mathcal{P}}$ [see above (133)].

For the vertex amplitude A_v, the critical equations $\partial_{\mathcal{Q}} \mathcal{I}_{\vec{p},\vec{s}} = \partial_{\tilde{\mathcal{Q}}} \mathcal{I}_{\vec{p},\vec{s}} = 0$ give

$$\frac{2\pi(1+b^2)}{k} \partial_{\mathcal{Q}_a} \mathcal{I}_{\vec{p},\vec{s}} = -i\mathcal{P}_a + \sqrt{2} b z_a - \frac{b^2(\mathcal{Q}'_a + \tilde{\mathcal{Q}}'_a)}{1+b^2} + \frac{\mathcal{Q}'_a - b^2 \tilde{\mathcal{Q}}'_a}{1+b^2} + y_a + i x_a - 2\pi s_a = 0 \tag{F13}$$

$$= -i\mathcal{P}_a + \sqrt{2} b z_a - \frac{2\pi b}{k} \mu_a - \frac{2\pi i}{k} m_a + y_a + i x_a - 2\pi s_a = 0, \tag{F14}$$

$$\frac{2\pi(1+b^{-2})}{k} \partial_{\tilde{\mathcal{Q}}_a} \mathcal{I}_{\vec{p},\vec{s}} = -i\tilde{\mathcal{P}}_a + \sqrt{2} b^{-1} z_a - \frac{(\mathcal{Q}'_a + \tilde{\mathcal{Q}}'_a)}{1+b^2} - \frac{\mathcal{Q}'_a - b^2 \tilde{\mathcal{Q}}'_a}{1+b^2} - y_a - i x_a + 2\pi s_a = 0 \tag{F15}$$

$$= -i\tilde{\mathcal{P}}_a + \sqrt{2} b^{-1} z_a - \frac{2\pi b^{-1}}{k} \mu_a + \frac{2\pi i}{k} m_a - y_a - i x_a + 2\pi s_a = 0, \tag{F16}$$

where μ_a and m_a relate to $\mathcal{Q}'_a$ and $\tilde{\mathcal{Q}}'_a$ by (69). The above equations are solved as

$$\frac{2\pi}{k} \mu_a = \sqrt{2}\mathrm{Re}(z_a), \qquad \frac{2\pi}{k} \nu_a = \sqrt{2}\mathrm{Im}(z_a), \qquad \frac{2\pi}{k} m_a = x_a, \qquad \frac{2\pi}{k} n_a = y_a - 2\pi s_a, \tag{F17}$$

where ν_a and n_a relate to X_a and $\tilde{X}_a$ by (69). Although μ_a, ν_a have nonzero imaginary parts, $\alpha_a = \mathrm{Im}(\mu_a), \beta_a = \mathrm{Im}(\nu_a)$ are fixed and do not scale as $k \to \infty$ [whereas $\mathrm{Re}(\mu_a), \mathrm{Re}(\nu_a)$ are not fixed and need to be determined by the critical equations]; thus, we can view μ_a, ν_a as real in (F17) as far as the semiclassical limit is concerned. The domain of n_a has been restricted to the single period $n_a \in [-\delta, k - \delta]$ by (126) ($\delta > 0$ is arbitrarily small), so the last equation implies

$$s_a = 0 \tag{F18}$$

when $y_a \in [0, 2\pi)$ and y_a is not infinitesimally close to 0 or 2π.

[1] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2014), https://doi.org/10.1017/CBO9781107706910.

[2] A. Perez, The spin foam approach to quantum gravity, Living Rev. Relativity **16**, 3 (2013).

[3] M. Smerlak, Divergences in spinfoam quantum gravity, Ph.D. thesis, University of Mediterranee, Aix-Marseille II, 2012.

[4] A. Riello, Self-energy of the Lorentzian Engle-Pereira-Rovelli-Livine and Freidel-Krasnov model of quantum gravity, Phys. Rev. D **88**, 024011 (2013).
[5] P. Donà, Infrared divergences in the EPRL-FK spin foam model, Classical Quant. Grav. **35**, 175019 (2018).
[6] V. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, Topology **31**, 865 (1992).
[7] S. Mizoguchi and T. Tada, Three-Dimensional Gravity from the Turaev-Viro Invariant, Phys. Rev. Lett. **68**, 1795 (1992).
[8] K. Noui and P. Roche, Cosmological deformation of Lorentzian spin foam models, Classical Quant. Grav. **20**, 3175 (2003).
[9] M. Han, 4-dimensional spin-foam model with quantum Lorentz group, J. Math. Phys. (N.Y.) **52**, 072501 (2011).
[10] W. J. Fairbairn and C. Meusburger, Quantum deformation of two four-dimensional spin foam models, J. Math. Phys. (N.Y.) **53**, 022501 (2012).
[11] M. Dupuis and F. Girelli, Observables in loop quantum gravity with a cosmological constant, Phys. Rev. D **90**, 104037 (2014).
[12] J. Lewandowski and A. Okolow, Quantum group connections, J. Math. Phys. (N.Y.) **50**, 123522 (2009).
[13] H. M. Haggard, M. Han, W. Kaminski, and A. Riello, SL(2,C) Chern-Simons theory, a non-planar graph operator, and 4d loop quantum gravity with a cosmological constant: Semiclassical geometry, Nucl. Phys. **B900**, 1 (2015).
[14] J. Engle, E. Livine, R. Pereira, and C. Rovelli, LQG vertex with finite Immirzi parameter, Nucl. Phys. **B799**, 136 (2008).
[15] T. Dimofte, Quantum Riemann surfaces in Chern-Simons theory, Adv. Theor. Math. Phys. **17**, 479 (2013).
[16] T. Dimofte, Complex Chern-Simons theory at level k via the 3d-3d correspondence, Commun. Math. Phys. **339**, 619 (2015).
[17] J. Ellegaard Andersen and R. Kashaev, Complex quantum Chern-Simons, arXiv:1409.1208.
[18] M. Culler and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. Math. **117**, 109 (1983).
[19] M. Han, 4d quantum geometry from 3d supersymmetric gauge theory and holomorphic block, J. High Energy Phys. 01 (2016) 065.
[20] Y. Ding, M. Han, and C. Rovelli, Generalized spinfoams, Phys. Rev. D **83**, 124020 (2011).
[21] H. M. Haggard, M. Han, and A. Riello, Encoding curved tetrahedra in face holonomies: Phase space of shapes from group-valued moment maps, Ann. Inst. Henri Poincaré **17**, 2001 (2016).
[22] J. W. Barrett, R. Dowdall, W. J. Fairbairn, F. Hellmann, and R. Pereira, Lorentzian spin foam amplitudes: Graphical calculus and asymptotics, Classical Quant. Grav. **27**, 165009 (2010).
[23] H. M. Haggard, M. Han, W. Kaminski, and A. Riello, Four-dimensional quantum gravity with a cosmological constant from three-dimensional holomorphic blocks, Phys. Lett. B **752**, 258 (2016).
[24] H. M. Haggard, M. Han, W. Kaminski, and A. Riello, SL(2, C) Chern-Simons theory, flat connections, and four-dimensional quantum geometry, Adv. Theor. Math. Phys. **23**, 1067 (2019).
[25] H.-J. Chung, T. Dimofte, S. Gukov, and P. Sulkowski, 3d-3d correspondence revisited, J. High Energy Phys. 04 (2016) 140.
[26] T. Dimofte, D. Gaiotto, and S. Gukov, Gauge theories labelled by three-manifolds, Commun. Math. Phys. **325**, 367 (2014).
[27] T. Dimofte, D. Gaiotto, and R. van der Veen, RG domain walls and hybrid triangulations, Adv. Theor. Math. Phys. **19**, 137 (2015).
[28] D. Gaiotto, G. W. Moore, and A. Neitzke, Wall-crossing, Hitchin systems, and the WKB approximation, arXiv:0907.3987.
[29] V. V. Fock and A. B. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, arXiv:math/0311149.
[30] Y. Imamura, H. Matsuno, and D. Yokoyama, Factorization of the $S^3/\mathbb{Z}_n$ partition function, Phys. Rev. D **89**, 085003 (2014).
[31] L. Faddeev, Discrete Heisenberg-Weyl group and modular group, Lett. Math. Phys. **34**, 249 (1995).
[32] R. M. Kashaev, The hyperbolic volume of knots from quantum dilogarithm, arXiv:q-alg/9601025.
[33] J. Gazeau, *Coherent States in Quantum Physics* (Wiley, New York, 2009).
[34] H. Minkowski, *Ausgewählte Arbeiten zur Zahlentheorie und zur Geometrie*, Vol. 12 of Teubner-Archiv zur Mathematik (Springer Vienna, Vienna, 1989).
[35] L. Freidel and K. Krasnov, A new spin foam model for 4d gravity, Classical Quant. Grav. **25**, 125018 (2008).
[36] M. Han and Z. Huang, Loop-quantum-gravity simplicity constraint as surface defect in complex Chern-Simons theory, Phys. Rev. D **95**, 104031 (2017).
[37] N. Nekrasov, A. Rosly, and S. Shatashvili, Darboux coordinates, Yang-Yang functional, and gauge theory, Nucl. Phys. B, Proc. Suppl. **216**, 69 (2011).
[38] M. Kapovich and J. J. Millson, The symplectic geometry of polygons in Euclidean space, J. Diff. Geom. **44**, 479 (1996).
[39] M. Kapovich, J. J. Millson, and T. Treloar, The symplectic geometry of polygons in hyperbolic 3-space, Asian J. Math. **4**, 123 (2000).
[40] E. R. Livine and S. Speziale, A new spinfoam vertex for quantum gravity, Phys. Rev. D **76**, 084028 (2007).
[41] E. Bianchi, P. Dona, and S. Speziale, Polyhedra in loop quantum gravity, Phys. Rev. D **83**, 044035 (2011).
[42] F. Conrady and L. Freidel, Quantum geometry from phase space reduction, J. Math. Phys. (N.Y.) **50**, 123510 (2009).
[43] K. Gawedzki, Conformal field theory: A case study, arXiv:hep-th/9904145.
[44] J. Engle, K. Noui, A. Perez, and D. Pranzetti, The SU(2) black hole entropy revisited, J. High Energy Phys. 05 (2011) 016.
[45] Y. Ding and M. Han, On the asymptotics of quantum group spinfoam model, arXiv:1103.1597.
[46] M. Han and Z. Huang, SU(2) flat connection on a Riemann surface and 3D twisted geometry with a cosmological constant, Phys. Rev. D **95**, 044018 (2017).
[47] W. Kaminski, M. Kisielowski, and J. Lewandowski, The EPRL intertwiners and corrected partition function, Classical Quant. Grav. **27**, 165020 (2010); Erratum, Classical Quant. Grav. **29**, 049501 (2012).

[48] E. Bianchi, D. Regoli, and C. Rovelli, Face amplitude of spinfoam quantum gravity, Classical Quant. Grav. **27**, 185009 (2010).
[49] E. Witten, Quantization of Chern-Simons gauge theory with complex gauge group, Commun. Math. Phys. **137**, 29 (1991).
[50] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. **121**, 351 (1989).
[51] L. D. Faddeev and R. M. Kashaev, Quantum dilogarithm, Mod. Phys. Lett. A **09**, 427 (1994).
[52] E. Suárez-Peiró, A Schläfli differential formula for simplices in semi-Riemannian hyperquadrics, Gauss-Bonnet formulas for simplices in the de Sitter sphere and the dual volume of a hyperbolic simplex, Pac. J. Math. **194**, 229 (2000).
[53] A. Y. Alekseev, H. Grosse, and V. Schomerus, Combinatorial quantization of the Hamiltonian Chern-Simons theory, Commun. Math. Phys. **172**, 317 (1995).
[54] A. Y. Alekseev, H. Grosse, and V. Schomerus, Combinatorial quantization of the Hamiltonian Chern-Simons theory II, Commun. Math. Phys. **174**, 561 (1996).
[55] T. Thiemann, Gauge field theory coherent states (GCS): 1. General properties, Classical Quant. Grav. **18**, 2025 (2001).
[56] B. Bahr, G. Rabuffo, and S. Steinhaus, Renormalization of symmetry restricted spin foam models with curvature in the asymptotic regime, Phys. Rev. D **98**, 106026 (2018).