Fokker-Planck equation for black holes in thermal potential

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We construct a kind of thermal potential and then put the black hole thermodynamic system in it. In this regard, some thermodynamic properties of the black hole are related to the geometric characteristics of the thermal potential. Driven by the intrinsic thermodynamic fluctuations, the behavior of the black hole in the thermal potential is stochastic. With the help of solving the Fokker-Planck equation analytically, we obtain the discrete energy spectrum of Schwarzschild and Banados-Teitelboim-Zanelli (BTZ) black holes in the thermal potential. For Schwarzschild black holes, the energy spectrum is proportional to the temperature of the ensemble, which is an external parameter, and the ground state is nonzero. For BTZ black hole, the energy spectrum only depends on the anti–de Sitter radius, which is the intrinsic parameter. Moreover, the ground state of the BTZ black hole in thermal potential is zero. This also reflects the difference between three-dimensional and four-dimensional gravity.

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I. INTRODUCTION

As an important bridge between general relativity and quantum mechanics, the black hole has attracted a lot of attention in its related physical properties. The intriguing discovery of the semiclassical description on black hole temperature and entropy [1,2] revealed the thermal nature of black holes and established a profound relationship between gravity and thermodynamics. Thus, a black hole is mapped to a thermodynamic system. Thermodynamics or statistical physics may provide a potentially complementary description of black hole physics; it can even provide some original insights into the quantum nature of gravity. Subsequently, some abundant properties of black holes have been investigated [3–5].

Based on the remarkable observation that the horizon area of nonextremal black holes behaves as a classical adiabatic invariant, Bekenstein conjectured that the horizon area of a nonextremal quantum black hole should have a discrete eigenvalue spectrum [6]. After that, within the discussion of the quasinormal modes and Bohr's correspondence principle, some works [7,8] have shown that the mass and horizon area of black holes have a discrete spectrum. Afterwards, many studies suggest that black holes exhibit some kind of quantum behavior and make the relationship between the black hole and the quantization of gravity closer [9–13].

Nowadays, with the development of research in this respect, a black hole—a unique thermodynamic system—presents incredible and peculiar thermodynamic properties with some quantum characteristics, such as Hawking-Page phase transition [14] corresponding to confinement-deconfinement transition in gauge theory [15] with the AdS/CFT correspondence, large and small black hole phase transition similar to gas-liquid phase transition [16–21], information loss of black hole [22], chaotic effect [23–29], etc. Some phenomenological schemes have been proposed to analyze the thermodynamic properties of black holes. Black hole molecular hypothesis [30] based on thermodynamics geometry [31] is used to analyze some possible microbehaviors of black holes [32–37]. With the idea of stochastic process [38], the dynamic process of black hole phase transition have been discussed [39–42], where the on-shell Gibbs free energy is generalized to the off-shell Gibbs free energy by replacing the Hawking temperature with the ensemble temperature. In addition, in our previous work [43], we introduced the general Landau potential to analyze the process of black hole phase transition dynamically.

In this paper, our main motivations are as follows:

(a) Based on the spirit of free-energy landscape [39], we want to find a more intuitive way to analyze some dynamic behaviors of black holes. Therefore, we construct a kind of thermal potential by using the temperature of the ensemble. We put the black hole system in such a unique thermal potential that some geometric characteristics of the thermal potential can

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directly reflect the thermodynamic properties of the black hole. Driven by thermal fluctuations, the black hole system moves in such a thermal potential. Moreover, we find that the thermal potential constructed in the current way is equivalent (in form) to the off-shell free energy in the free-energy landscape, but both of them have some certain different physical connotations.

(b) We are also interested in the dynamical scenario of the black hole in thermal potential, but we are more concerned about whether we can get an analytic result. Fortunately, in the process of the stochastic effect of thermal fluctuation, we obtain the analytic dynamic description of Schwarzschild and Banados-Teitelboim-Zanelli (BTZ) black holes by solving the Fokker-Planck equation analytically.

(c) Our results also suggest an essential difference between three-dimensional gravity and four-dimensional gravity. The four-dimensional Schwarzschild black hole is in the inverted harmonic oscillator potential, so its thermal stability is self-evident. The eigenvalue spectrum of Schwarzschild black hole in the inverted harmonic oscillator potential is proportional to the temperature of the ensemble. At the same time, the system has a nonzero ground state. Once the ensemble temperature is exactly in accordance with the expression of the Schwarzschild black hole temperature (i.e., the thermal potential reaches the maximum) the eigenvalue is proportional to the reciprocal of the mass of Schwarzschild black hole while for the three-dimensional BTZ black hole, it is in harmonic oscillator potential and its energy spectrum only depends on the anti–de Sitter (AdS) radius. Furthermore, the ground state of the BTZ black hole in harmonic oscillator potential is zero.

This paper is organized as follows. In Sec. II we construct the thermal potential for black holes by introducing the canonical ensemble. In Sec. III we give a brief introduction of the Fokker-Planck equation and its transformation to the eigenvalue problem. Taking two specific black holes (Schwarzschild and BTZ) as examples, we solve the Fokker-Planck equations of these two black holes analytically, and give the corresponding energy spectrum and related discussions. Finally, Sec. IV is devoted to summary and some future prospects. Throughout this paper, we adopt the units $\hbar = c = k_B = G = 1$.

II. THERMAL POTENTIAL

Consider a canonical ensemble at temperature T composed of a large number of states in which one, or a group of them, can represent a real black hole. The real black hole state (on shell) is the solution of the Einstein field equation while others (off shell) are not. When the ensemble temperature T is equal to the Hawking temperature T_h, the ensemble is made up of real black hole states, which is in equilibrium. For a specific black hole thermodynamic system, we can construct the thermal potential

$$U = \int (T_h - T)\mathrm{d}S, \tag{1}$$

where the thermodynamic entropy S of the black hole is seen as a variable. For black holes, we know $T_h = t(S, Y)$, where the function $t(S, Y)$ is the relation satisfied by thermodynamic entropy S and other parameters Y of the black hole, like the AdS radius l, charge Q, angular momentum J, etc. The ensemble temperature T now here is treated as an independent constant, which can take any positive value in any way. Note that $T = T_h$ mentioned in this paper is just one of the ways to get the value of the ensemble temperature T.

The integrand in the above definition (1) of thermal potential can be understood as the deviation of all possible states in the canonical ensemble from the real black hole state (or the equilibrium state). In other words, in the equilibrium state, the thermal potential will show extreme behavior, i.e.,

$$\frac{\mathrm{d}U}{\mathrm{d}S} = 0 \Rightarrow T = T_h. \tag{2}$$

Physically, through the construction of such a thermal potential (1), we put a black hole thermodynamic system in the potential field U. Due to the thermodynamic fluctuations, the stochastic behavior of black holes in such a potential field can reflect some thermodynamic characteristics of black holes.

Moreover, another advantage of the thermal potential (1) is that the concavity and convexity at the extreme point are related to the stability of the thermodynamic system,

$$\delta\left(\frac{\mathrm{d}U}{\mathrm{d}S}\bigg|_{T=T_h}\right) = \frac{\partial t(S, Y)}{\partial S}\bigg|_{Y} \delta S. \tag{3}$$

When $\partial t(S, Y)/\partial S > 0$, the thermodynamic system is in a stable state, while $\partial t(S, Y)/\partial S < 0$ corresponds to an unstable state.

For a simple thermodynamic system, according to the first law of thermodynamics $dE = T_h dS - PdV$, where E is the internal energy, P is the pressure, and V is the thermodynamic volume of the system, we have

$$U = \int (T_h - T)\mathrm{d}S = E + PV - TS. \tag{4}$$

Formally, we can see that the thermal potential constructed in this paper is equivalent to the off-shell free energy in the free energy landscape [39].

III. FOKKER-PLANCK EQUATION

Due to thermal fluctuations, the black hole moves stochastically in the thermal potential, which leads to different phase transition characteristics. In fact, this is a kind of stochastic process. The probability distribution $W(x,t)$ of these black hole states (including on-shell states and off-shell states) evolving in time under the thermal fluctuation should be described by the probabilistic Fokker-Planck equation (or in mathematical literature, it is also called a forward Kolmogorov equation). The Fokker-Planck equation provides a powerful tool with which the effects of fluctuations close to transition points can be adequately treated. It is not restricted to systems near thermal equilibrium, and can be applied to systems far from thermal equilibrium (like the laser) as well.

The one-variable Fokker-Planck equation with time-independent drift coefficient $D^{(1)}(x)$ and constant diffusion coefficient D is [38,44]

$$\begin{aligned}\frac{\partial W(x,t)}{\partial t} &= \left[\frac{\partial}{\partial x} f'(x) + D\frac{\partial^2}{\partial x^2}\right] W(x,t) \\ &= L_{\mathrm{FP}} W(x,t) = -\frac{\partial}{\partial x} S(x,t), \end{aligned} \tag{5}$$

where $S(x,t)$ is the probability current, the potential is given by $f(x) = -\int^x D^{(1)}(y)dy$ and $f'(x) \coloneqq df(x)/dx$. A separation ansatz for probability density $W(x,t) = \varphi(x)e^{-\varepsilon t}$ leads to the eigenvalue equation for the Fokker-Planck equation with appropriate boundary conditions

$$L_{\mathrm{FP}}\varphi(x) = -\varepsilon\varphi(x). \tag{6}$$

For convenience, we introduce $\Phi(x) = f(x)/D$ resulting that the Fokker-Planck operator L_{FP} which can be written as

$$L_{\mathrm{FP}} = \frac{\partial}{\partial x} D e^{-\Phi(x)} \frac{\partial}{\partial x} e^{\Phi(x)}. \tag{7}$$

Easily, we can obtain a Hermitian operator $L \coloneqq -e^{\Phi(x)/2} L_{\mathrm{FP}} e^{-\Phi(x)/2}$ and the eigenvalue equation (6) becomes [44]

$$L\psi(x) = \varepsilon\psi(x), \tag{8}$$

where $\psi(x) = e^{\Phi(x)/2}\varphi(x)$ and the Hermitian operator L has the same form as the single-particle Hamilton operator in quantum mechanics,

$$L = -D\frac{\partial^2}{\partial x^2} + V_s(x), \quad V_s(x) = \frac{1}{4D}[f'(x)]^2 - \frac{1}{2}f''(x). \tag{9}$$

Now we talk about the boundary conditions for the above eigenvalue problem of the Fokker-Planck equation. We note that we study in this paper the motion behavior of all possible states in the canonical ensemble in the thermal potential and the thermal potential is constructed from the black hole background. Once the thermal potential is determined, this becomes an usual quantum mechanical problem. Hence the boundary conditions below are natural. For all possible states in the canonical ensemble, the real black hole states are only some special cases of them.

(a) Reflecting boundary condition (RBC): in the region $x > x_{\max}$ or $x < x_{\min}$, the potential $\Phi(x)$ tends to an infinitely high positive value, which requires $S = 0$.
(b) Absorbing boundary condition (ABC): in the region $x > x_{\max}$ or $x < x_{\min}$, the potential $\Phi(x)$ tends to an infinitely large negative value, which requires $e^{\Phi}W = 0$.
(c) Natural boundary condition (NBC): for $x_{\max} \to +\infty$ and $x_{\min} \to -\infty$, we have $S = 0$ or $e^{\Phi}W = 0$.

In the next discussion, we will see the stochastic behaviors of two different black holes (the four-dimensional Schwarzschild black hole and the three-dimensional BTZ black hole) in different thermal potentials. By solving the Fokker-Planck equation, we will see the application of the above boundary conditions.

A. Application 1: Schwarzschild black hole

For a four-dimensional Schwarzschild black hole, its metric reads [3]

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2), \tag{10}$$

where M is the Arnowitt-Deser-Misner mass of the black hole. Correspondingly, the Hawking temperature and Bekenstein-Hawking entropy of Schwarzschild black hole take the forms (in equilibrium) [3] in terms of the radius of the event horizon r_h

$$T_h = \frac{1}{4\pi r_h}, \qquad S = \pi r_h^2. \tag{11}$$

In the light of Eq. (1), we can obtain the thermal potential of the Schwarzschild black hole easily

$$U = \frac{1}{2} r_h - \pi T r_h^2. \tag{12}$$

It is the inverted harmonic oscillator potential or parabolic potential barrier [45], which plays indispensable roles in physics particularly in unstable systems. If the ensemble temperature T is exactly in accordance with the temperature expression (11) of a Schwarzschild black hole, i.e., $T = T_h$, we can clearly see that the thermal potential reaches the maximum.

For a Schwarzschild black hole, it is now in the potential field (12). Next, in order to analyze some dynamical behaviors of the black hole, we set the potential $f(x) = x/2 - \pi T x^2$, and thus the effective potential is

$$V_s(x) = \frac{\pi^2 T^2}{D} z^2 + \pi T, \qquad z = x - \frac{1}{4\pi T}. \tag{13}$$

The Fokker-Planck equation (8) becomes the following simple form with the help of auxiliary variable $\xi = \sqrt{\pi T/D} z$

$$\frac{\partial^2}{\partial \xi^2} \psi(x) + \left(\frac{\varepsilon}{\pi T} - 1 - \xi^2 \right) \psi(x) = 0. \tag{14}$$

Hence we can get the corresponding eigenvalues and eigenfunctions

$$\varepsilon_n = 2\pi T(n+1), \quad n = 0, 1, 2, \cdots \tag{15}$$

$$\psi_n(x) = \left(\frac{T}{D} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}, \tag{16}$$

where $H_n(\xi)$ are the Hermite polynomials. Now, we discuss the above results.

(a) We find that, due to thermal fluctuations, the Schwarzschild black hole in the potential field has discrete eigenvalues, which are proportional to the ensemble temperature under the ABC and NBC at $x \to x_{\max}, x_{\min} \to \pm\infty$. The higher the ensemble temperature is, the greater its eigenvalue is.

(b) The system has nonzero ground state, which is characteristic of the inverted harmonic oscillator potential. The ground state of a Schwarzschild black hole in thermal potential is

$$\varepsilon_0 = 2\pi T. \tag{17}$$

When the ensemble temperature is higher, the corresponding ground state is larger.

(c) The difference between the two energy levels of the system is also proportional to the ensemble temperature,

$$\varepsilon_{n+1} - \varepsilon_n = 2\pi T. \tag{18}$$

That is to say, the difference between two adjacent energy levels is always the same as the ground state energy of the system.

(d) If the inverted harmonic oscillator described by (12) is in one-dimensional quantum mechanics, the Lyapunov exponent in both classical and quantum mechanics is $\lambda = \sqrt{2\pi T}$ [23,46,47]. Therefore, the eigenvalue of a Schwarzschild black hole in the thermal potential is proportional to the square of the Lyapunov exponent of the inverse harmonic oscillator [48],

$$\varepsilon_n \propto \lambda^2, \tag{19}$$

which is a very meaningful result and helpful for us to understand some chaotic effects of the black hole system.

(e) We have known that for the inverted harmonic oscillator potential, no stationary point exists for the Fokker-Planck equation. However, when we consider ABC and NBC at $x \to x_{\max}, x_{\min} \to \pm\infty$, eigenfunctions do exist (the probability current S for these eigenfunctions is finite), and they can be used to calculate the transition probability. Immediately, we obtain the transition probability into eigenfunctions $(t \geq t')$

$$\begin{aligned} P(z, t|z', t') = {} & \sqrt{\frac{T}{D[1 - e^{-4\pi T(t-t')}]}} \\ & \times \exp\left(-\frac{\pi T[z - e^{-2\pi T(t-t')} z']^2}{D[1 - e^{-4\pi T(t-t')}]} \right) \\ & \times e^{-2\pi T(t-t')}. \end{aligned} \tag{20}$$

(f) If the ensemble temperature is exactly in accordance with the temperature expression of a Schwarzschild black hole, that is $T = T_h = 1/(8\pi M)$, we can find that the eigenvalue of a Schwarzschild black hole in the thermal potential is proportional to the inverse of the mass

$$\varepsilon_n = \frac{n+1}{4M}, \quad n = 0, 1, 2, \ldots, \tag{21}$$

indicating that the larger the mass of a black hole is, the smaller its eigenvalues is in the thermal potential.

B. Application 2: BTZ black hole

Next, we take a look at a three-dimensional black hole. As a simple example, we consider a neutral and nonrotating BTZ black hole. Its metric is [49,50]

$$\mathrm{d}s^2 = -\left(-2m + \frac{r^2}{l^2} \right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{-2m + r^2/l^2} + r^2 \mathrm{d}\varphi^2, \tag{22}$$

where m is related to the black hole mass, l is the AdS radius which is connected with the negative cosmological constant Λ via $\Lambda = -1/l^2$. Naturally, some basic thermodynamic properties of the BTZ black hole in terms of the event horizon radius r_h are

$$T_h = \frac{r_h}{2\pi l^2}, \qquad S = \frac{1}{2}\pi r_h. \tag{23}$$

With the help of Eq. (1), we can obtain the thermal potential of the BTZ black hole

$$U = \frac{r_h^2}{8l^2} - \frac{\pi T}{2} r_h. \tag{24}$$

Obviously, the BTZ black hole is in the harmonic oscillator potential [51] and it is always thermodynamically stable. Formally, we set $f(x) = x^2/(8l^2) - \pi T x/2$, and then the effective potential is

$$V_s(x) = \frac{z^2}{64Dl^4} - \frac{1}{8l^2}, \qquad z = x - 2\pi T l^2. \tag{25}$$

By substituting the auxiliary variable $\xi = \sqrt{1/(8Dl^2)}z$, the thermodynamic evolution equation (8) of the BTZ black hole can be written in the following simple form

$$\frac{\partial^2}{\partial \xi^2}\psi(x) + (8l^2\varepsilon + 1 - \xi^2)\psi(x) = 0. \tag{26}$$

Hence, we can get the corresponding eigenvalues and eigenfunctions about the thermodynamic evolution of a BTZ black hole in the thermal potential

$$\varepsilon_n = \frac{n}{4l^2}, \quad n = 0, 1, 2, \cdots \tag{27}$$

$$\psi_n(x) = \left(\frac{1}{8\pi D l^2}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}. \tag{28}$$

(a) When we get the above eigenvalues, we use RBC and NBC at $x \to x_{\max}, x_{\min} \to \pm\infty$. Because the BTZ black hole is in the harmonic oscillator potential, we get the stationary solution.
(b) The energy spectrum of a BTZ black hole only depends on the parameter of the black hole itself: the AdS radius l. This is different from the case of a Schwarzschild black hole, where the energy spectrum is related to the temperature of the ensemble. Furthermore, the energy spectrum is also discrete.
(c) Although the BTZ black hole is in the harmonic oscillator potential, its behavior is different from that of a quantum harmonic oscillator. The most prominent difference is that the ground state is zero, i.e., $\varepsilon_0 = 0$.
(d) The difference between two adjacent energy levels is always $1/(4l^2)$.

IV. SUMMARY

We consider a canonical ensemble composed of a large number of states with the same structure under the same macrocondition, i.e., at the same temperature T. One, or a group of states, can represent the real black hole systems and its temperature can be labeled as the Hawking temperature T_h. The real black hole states (on-shell states) are solutions of the Einstein field equation, while others (off-shell states) are not. When the ensemble temperature T is equal to the Hawking temperature T_h, the ensemble is made up of real black hole states, which is in equilibrium. When the ensemble temperature T is not equal to the Hawking temperature T_h, we can say that all possible states in the canonical ensemble deviate from the real black hole state (or the equilibrium state). Note that the ensemble temperature T can take any positive value in any way and $T = T_h$ is just one of the ways to get the value of the ensemble temperature T.

Usually for thermodynamics in free energy, the degree of the thermal motion is measured by the product of temperature and the entropy. Therefore, we simply introduce a thermal potential (1) to roughly reflect the degree of deviation. In the equilibrium state, the thermal potential shows extreme behavior. Meanwhile the concavity and convexity of potential can be related to the stability of the thermodynamic system. In this way, we can observe the thermal-motion behavior of states of the canonical ensemble in such a thermal potential due to the thermal fluctuation.

With the help of solving the Fokker-Planck equation analytically in such a thermal potential, we investigate the motion behavior of all possible states in the canonical ensemble in the thermal potential. This becomes the barrier penetration problem (for the thermal potential of a Schwarzschild black hole) in quantum mechanics or the motion of states in the potential well (for thermal potential of a BTZ black hole). Therefore, it is natural to calculate the eigenvalues and transition probability of states in the potential barrier or potential well.

For the Schwarzschild black hole, it is in inverted harmonic oscillator potential. We consider the absorbing and natural boundary conditions to obtain eigenvalues. The calculation shows that the motion behavior of the states in the canonical ensemble that we consider in the Schwarzschild thermal potential [Eq. (12)] is discrete, and its eigenvalue depends on the temperature T of the canonical ensemble. The higher the ensemble temperature is, the larger the eigenvalue is. At the same time, the system has a nonzero ground state. When the ensemble temperature T is exactly in accordance with the temperature T_h of Schwarzschild black hole, i.e., states in the canonical ensemble are at the highest point of the barrier [or the thermal potential Eq. (12) reaches the maximum], the ensemble is made up of real black hole states and the eigenvalue of the Schwarzschild black hole state in such thermal potential is proportional to the inverse of the mass, i.e., Eq. (21). That is to say, the larger the black hole mass is, the smaller the eigenvalue is and the smaller the energy value of the ground state is.

TABLE I. The expressions of thermal potential of several simple black holes.

	Thermal potential $f(x) =$
Schwarzschild-AdS black hole	$\frac{1}{2}x + \frac{4\pi P}{3}x^3 - \pi T x^2$
Reissner-Nordström black hole	$\frac{1}{2}x + \frac{Q^2}{2x} - \pi T x^2$
Charged AdS black hole	$\frac{1}{2}x + \frac{4\pi P}{3}x^3 + \frac{Q^2}{2x} - \pi T x^2$
Charged BTZ black hole	$\frac{1}{8l^2}x^2 - \frac{\pi T}{2}x - \frac{Q^2}{16}\ln x$
Rotating BTZ black hole	$\frac{1}{8l^2}x^2 - \frac{\pi T}{2}x + \frac{J^2}{32x^2}$

For the BTZ black hole, it is in harmonic oscillator potential. We consider the reflecting and natural boundary conditions to obtain eigenvalues. Compared with the Schwarzschild black hole and quantum harmonic oscillator, a BTZ black hole presents two major differences. One is that the energy spectrum only depends on the intrinsic parameter—the AdS radius—of the black hole; the other is that the ground state is zero. This further reflects the unique nature of three-dimensional gravity.

Naturally, the current method can be extended to other black hole models, such as the thermal potential of several simple black holes listed in Table I. Although the expression of thermal potential of other models is somewhat complex, we can use the perturbation method to calculate the eigenvalues and eigenfunctions of the system, and then obtain some quantum properties of thermodynamic system due to thermal fluctuation. In addition, we believe that the current way can also be extended to various statistical physical models to obtain many properties of the system.

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$$U = -\pi T \tilde{r}_h^{\,2} + \frac{1}{16\pi T}, \qquad \tilde{r}_h = r_h - \frac{1}{4\pi T}.$$

Hence we can called it as inverted harmonic oscillator (IHO) potential once the constant term $\frac{1}{16\pi T}$ in the above formula is absorbed within U.

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$$U = \frac{\tilde{r}_h^{\,2}}{8l^2} - \frac{\pi^2 T^2}{2l^2}, \qquad \tilde{r}_h = r_h - 2\pi T l^2.$$

Hence we can called it as harmonic oscillator potential once the constant term $-\frac{\pi^2 T^2}{2l^2}$ in the above formula is absorbed within U.