Higher-order tail contributions to the energy and angular momentum fluxes in a two-body scattering process

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The need for more and more accurate gravitational-wave templates requires taking into account all possible contributions to the emission of gravitational radiation from a binary system. Therefore, working within a multipolar-post-Minkowskian framework to describe the gravitational-wave field in terms of the source multipole moments, the dominant instantaneous effects should be supplemented by hereditary contributions arising from nonlinear interactions between the multipoles. The latter effects include tails and memories and are described in terms of integrals depending on the past history of the source. We compute higher-order tail (i.e., tail-of-tail, tail-squared, and memory) contributions to both energy and angular momentum fluxes and their averaged values along hyperboliclike orbits at the leading post-Newtonian approximation, using harmonic coordinates and working in the Fourier domain. Because of the increasing level of accuracy recently achieved in the determination of the scattering angle in a two-body system by several complementary approaches, the knowledge of these terms will provide useful information to compare results from different formalisms.

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I. INTRODUCTION

Tail effects in a two-body interaction are generated in the wave zone far from the system, where the latter can be described as a single object endowed with multipoles. The gravitational-wave generation formalism developed by Blanchet and Damour [1–5] combines a multipolar-post-Minkowskian (MPM) expansion in the far zone with a post-Newtonian (PN) expansion in the near zone to relate the gravitational radiation emitted by the binary system to the PN expansion in the near zone, where the two constituents of the binary can be resolved as individual sources. A matching procedure in the overlapping region where both expansions are valid allows for expressing the radiative moments as nonlinear functionals of two infinite sets of time-varying source multipole moments. The latter moments mix with each other as the waves propagate, so that the relation between radiative and source moments includes many nonlinear interactions, which are called hereditary effects [6–11], depending on the full past history of the source.

Starting from the 4PN level of accuracy, the Hamiltonian governing the conservative two-body dynamics acquires a nonlocal part summarizing several such hereditary effects, including tails, tails of tails, tail squared, memory, etc. The dominant tails are due to the quadratic nonlinear interaction between higher-order multipole moments and the mass monopole, namely, the total Arnowitt-Deser-Misner (ADM) mass. The nonlinear memory effect also arises at the quadratic level due to the interaction between two quadrupole moments. The tail-of-tail and tail-squared contributions are cubic nonlinear effects caused by the interaction between the tail itself and the ADM mass and the self-interaction of the tail itself, respectively [11].

Tail-transported nonlocal dynamical correlations lead to a nonlocal action, so that the instantaneous interaction terms in the Hamiltonian are complemented by a (time-symmetric) nonlocal-in-time interaction [12–14]. The nonlocal Hamiltonian has been recently determined up to 6PN by using a time-split version of the gravitational-wave energy flux, including both first-order ($4+5+6$PN) and second-order (5.5PN) tail effects [15–18]. The formalism developed there has allowed the computation of both local and nonlocal parts of the conservative scattering angle up to the seventh order in G by using a combined PM-PN expansion [19]. However, radiation-reaction effects as well as tail effects also enter the problem starting at $O(G^3)$ and $O(G^4)$, respectively. This fact has both conservative and dissipative aspects, as discussed in Ref. [20]. Having already taken into account the time-symmetric aspect of tail interaction in the nonlocal contribution to the conservative dynamics, the only tail-related effect to be added to radiation-reaction is the time-antisymmetric one, as explicitly shown in Ref. [12] at 4PN level.

In order to evaluate the radiation-reaction contributions to the scattering angle, one needs the radiative losses of energy, angular momentum, and linear momentum. While there exists a rich literature for the gravitational-wave tails in the case of coalescing black holes [21–26], the companion situation of black holes undergoing a scattering process is less studied. We have already computed in Ref. [20] the (integrated) leading-order tail contributions to the loss of energy, angular momentum, and linear momentum along hyperboliclike orbits, limiting to the leading PN term for each of them. We will refer to these terms as "past tails," to be distinguished from the time-symmetric tails entering the nonlocal part of the Hamiltonian. In the present paper, we evaluate the orbital average of higher-order energy and angular momentum past tails (tail of tail and tail squared) as well as the corresponding time-symmetric tails.

The paper is organized as follows. In Sec. II, we recall the main definitions of the various past tail integrals computed in this work. These integrals are more conveniently computed in the frequency domain. Section III provides all necessary information to get the final expressions for the hereditary contributions to the orbital-averaged energy and angular momentum fluxes, using a Fourier decomposition of the multipole moments. The explicit evaluation for hyperboliclike motion is done in Sec. IV, where the results are expressed as an expansion in the large angular momentum parameter. Time-symmetric tails are instead computed in Sec. V, using a slight extension of the general formulas introduced in Sec. III. Finally, in Sec. VI, we summarize our results and discuss their relevance for recent developments in the relativistic two-body scattering problem.

We will denote the masses of the two bodies as m_1 and m_2 with $m_2 > m_1$ and define the symmetric mass ratio $\nu = \mu/M$ as the ratio of the reduced mass $\mu \equiv m_1 m_2/(m_1+m_2)$ to the total mass $M = m_1 + m_2$, as standard. We will use the following dimensionless energy and angular momentum parameters:

$$\bar{E} \equiv \frac{E_{\rm tot} - Mc^2}{\mu c^2} \tag{1.1}$$

and

$$j \equiv \frac{cJ}{G m_1 m_2} = \frac{cJ}{GM\mu}, \tag{1.2}$$

where $E_{\rm tot}$ and J are the total center-of-mass energy and angular momentum of the binary system, respectively.

II. ENERGY AND ANGULAR MOMENTUM TAIL INTEGRALS

The total contribution to the energy and angular momentum fluxes

$$\mathcal{F} \equiv \left(\frac{dE}{dt}\right)^{\rm GW}, \qquad \mathcal{G}_i \equiv \left(\frac{dJ_i}{dt}\right)^{\rm GW} \tag{2.1}$$

can be split as the sum of instantaneous and hereditary terms. The latter can be further decomposed as tail, tail-of-tail, tail-squared, and higher nonlinear interaction terms. We will consider below quadratic and cubic-in-G interactions only at their leading PN level of accuracy.

The hereditary part of the gravitational-wave energy flux reads

$$\mathcal{F}_{\rm hered}(t) = \mathcal{F}_{\rm tail}(t) + \mathcal{F}_{\rm tail(tail)}(t) + \mathcal{F}_{({\rm tail})^2}(t), \tag{2.2}$$

where the quadratic and cubic tails (according to the terminology introduced in Ref. [11]) are given by

$$\mathcal{F}_{\rm tail}(t) = \frac{G^2\mathcal{M}}{c^8}\left\{\frac{4}{5}I^{(3)}_{ij}(t)\int_{-\infty}^t d\tau I^{(5)}_{ij}(\tau)\left[\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{11}{12}\right]+O\left(\frac{1}{c^2}\right)\right\} \tag{2.3}$$

and

$$\begin{aligned}\mathcal{F}_{\rm tail(tail)}(t) &= \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}I^{(3)}_{ij}(t)\int_{-\infty}^t d\tau I^{(6)}_{ij}(\tau)\left[\ln^2\left(\frac{t-\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{124627}{44100}\right],\\ \mathcal{F}_{({\rm tail})^2}(t) &= \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\left(\int_{-\infty}^t d\tau I^{(5)}_{ij}(\tau)\left[\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{11}{12}\right]\right)^2,\end{aligned} \tag{2.4}$$

respectively. Here $\mathcal{M}$ denotes the total ADM mass of the system (which can be set equal to M at the leading order in the PN expansion) and $\tau_0 = c r_0$, with r_0 a constant length scale entering the relation between the retarded time in radiative coordinates and the corresponding retarded time in harmonic coordinates. The quadratic term (2.3) is the dominant tail at order 1.5PN, while the two cubic-order tails (2.4) are both at 3PN order.

Similarly, the hereditary part of the angular momentum flux is decomposed as

$$\mathcal{G}_i^{\rm hered}(t) = \mathcal{G}_i^{\rm tail}(t) + \mathcal{G}_i^{\rm tail(tail)}(t) + \mathcal{G}_i^{({\rm tail})^2}(t) + \mathcal{G}_i^{\rm memory}(t), \tag{2.5}$$

where

$$\mathcal{G}_i^{\rm tail}(t) = \frac{G^2\mathcal{M}}{c^8}\epsilon_{iab}\left\{\frac{4}{5}I_{aj}^{(2)}(t)\int_{-\infty}^{t}d\tau\left[\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{11}{12}\right]I_{bj}^{(5)}(\tau)\right.$$
$$\left.+\frac{4}{5}I_{bj}^{(3)}(t)\int_{-\infty}^{t}d\tau\left[\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{11}{12}\right]I_{aj}^{(4)}(\tau)+O\left(\frac{1}{c^2}\right)\right\}, \tag{2.6}$$

starting at 1.5PN order, and

$$\mathcal{G}_i^{\rm tail(tail)}(t) = \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\epsilon_{iab}\left\{I_{aj}^{(2)}(t)\int_{-\infty}^{t}d\tau\left[\ln^2\left(\frac{t-\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{124627}{44100}\right]I_{bj}^{(6)}(\tau)\right.$$
$$\left.+I_{bj}^{(3)}(t)\int_{-\infty}^{t}d\tau\left[\ln^2\left(\frac{t-\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{124627}{44100}\right]I_{aj}^{(5)}(\tau)\right\},$$
$$\mathcal{G}_i^{({\rm tail})^2}(t) = \frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\epsilon_{iab}\left(\int_{-\infty}^{t}d\tau\left[\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{11}{12}\right]I_{aj}^{(4)}(\tau)\right)\left(\int_{-\infty}^{t}d\tau\left[\ln\left(\frac{t-\tau}{2\tau_0}\right)+\frac{11}{12}\right]I_{bj}^{(5)}(\tau)\right), \tag{2.7}$$

which are both 3PN order.

At the quadratic-in-G order, one also have the following nonlinear memory integral:

$$\mathcal{G}_i^{\rm memory}(t) = \frac{4}{35}\frac{G^2}{c^{10}}\epsilon_{iab}I_{aj}^{(3)}(t)\int_{-\infty}^{t}d\tau I_{cb}^{(3)}(\tau)I_{jc}^{(3)}(\tau), \tag{2.8}$$

which is 2.5PN order. Notice that there is no memory contribution in the case of the energy flux, because the memory integral is time differentiated and, therefore, becomes instantaneous, as discussed in Ref. [21], and then taken into account in the instantaneous part.

It is convenient to introduce the following notation:

$$\mathcal{T}_{\ln^m}[X_L^{(n)};C_{X_L}](t) = \int_{-\infty}^{t}d\tau X_L^{(n)}(\tau)\ln^m\left(\frac{t-\tau}{C_{X_L}}\right), \tag{2.9}$$

where $X_L^{(n)}$ denotes a generic multipolar moment with L (either electric-type or magnetic-type) tensorial indices and differentiated n times with respect to time and C_{X_L} is a constant which depends on the multipolar moment considered. The integral $\mathcal{T}_{\ln^m}(X_L^{(n)})$, thus, represents the m-type past tail associated with the history of $X_L^{(n)}$, from past infinity to the present time. For the purpose of the present work, we need only $m = 1, 2$.

The energy and angular momentum past tails [Eqs. (2.3)–(2.4) and (2.6)–(2.7), respectively] can then be written as

$$\mathcal{F}_{\rm tail}(t) = \frac{4}{5}\frac{G^2\mathcal{M}}{c^8}I_{ij}^{(3)}(t)\mathcal{T}_{\ln}[I_{ij}^{(5)};C_{I_2}](t),$$
$$\mathcal{F}_{\rm tail(tail)}(t) = \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}I_{ij}^{(3)}(t)\left[\mathcal{T}_{\ln^2}[I_{ij}^{(6)};C_{I_2}](t)-\frac{107}{105}\mathcal{T}_{\ln}[I_{ij}^{(6)};\tilde{C}_{I_2}](t)\right],$$
$$\mathcal{F}_{({\rm tail})^2}(t) = \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}(\mathcal{T}_{\ln}[I_{ij}^{(5)};C_{I_2}](t))^2 \tag{2.10}$$

and

$$\mathcal{G}_i^{\rm tail}(t) = \frac{4}{5}\frac{G^2\mathcal{M}}{c^8}\epsilon_{iab}[I_{aj}^{(2)}(t)\mathcal{T}_{\ln}[I_{bj}^{(5)};C_{I_2}](t)+I_{bj}^{(3)}(t)\mathcal{T}_{\ln}[I_{aj}^{(4)};C_{I_2}](t)],$$
$$\mathcal{G}_i^{\rm tail(tail)}(t) = \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\epsilon_{iab}\left\{I_{aj}^{(2)}(t)\left[\mathcal{T}_{\ln^2}[I_{bj}^{(6)};C_{I_2}](t)-\frac{107}{105}\mathcal{T}_{\ln}[I_{bj}^{(6)};\tilde{C}_{I_2}](t)\right]\right.$$
$$\left.+I_{bj}^{(3)}(t)\left[\mathcal{T}_{\ln^2}[I_{aj}^{(5)};C_{I_2}](t)-\frac{107}{105}\mathcal{T}_{\ln}[I_{aj}^{(5)};\tilde{C}_{I_2}](t)\right]\right\},$$
$$\mathcal{G}_i^{({\rm tail})^2}(t) = \frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\epsilon_{iab}\mathcal{T}_{\ln}[I_{aj}^{(4)};C_{I_2}](t)\mathcal{T}_{\ln}[I_{bj}^{(5)};C_{I_2}](t), \tag{2.11}$$

respectively, with

$$C_{I_2} = 2\tau_0 e^{-\frac{11}{12}}, \qquad \tilde{C}_{I_2} = C_{I_2}e^{\frac{515063}{179760}} \approx 17.55C_{I_2}. \tag{2.12}$$

In order to compute the above tail integrals at their leading PN approximation, one needs only the quadrupole moment and its time derivatives evaluated at the Newtonian level.

We will evaluate below the leading-order contribution to the orbital average of the tail integrals

$$(\Delta E)_{\rm X}=\int_{-\infty}^{\infty}dt\mathcal{F}_{\rm X}(t),$$
$$(\Delta J_i)_{\rm X}=\int_{-\infty}^{\infty}dt\mathcal{G}_i^{\rm X}(t), \tag{2.13}$$

with $X=[\text{tail},\text{tail(tail)},(\text{tail})^2,\text{memory}]$, along hyperboliclike orbits.

III. COMPUTING THE TAIL INTEGRALS IN THE FOURIER DOMAIN

Each of the integrals above is conveniently computed in the Fourier domain. Inserting in Eq. (2.9) the Fourier expansion of $X_L(\tau)$, i.e.,

$$X_L(\tau)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\hat{X}_L(\omega), \tag{3.1}$$

and changing the integration variable as $t-\tau=\xi$ yields

$$\begin{aligned}\mathcal{T}_{\ln^m}[X_L^{(n)};C_{X_L}](t)&=\int_0^{\infty}d\xi X_L^{(n)}(t-\xi)\ln^m\left(\frac{\xi}{C_{X_L}}\right)\\&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}(-i\omega)^n\hat{X}_L(\omega)A_m(\omega,C_{X_L}),\end{aligned} \tag{3.2}$$

with

$$A_m(\omega,C_{X_L})=\int_0^{\infty}d\xi e^{i\omega\xi}\ln^m\left(\frac{\xi}{C_{X_L}}\right). \tag{3.3}$$

For $m=1$, 2 we have the relations [see Eqs. (4.7) and (4.13) in Ref. [21]]

$$\begin{aligned}A_1(\omega,C_{X_L})&=-\frac{\pi}{2|\omega|}-\frac{i}{|\omega|}\text{sgn}(\omega)\ln(C_{X_L}|\omega|e^{\gamma}),\\A_2(\omega,C_{X_L})&=\frac{\pi}{|\omega|}\ln(C_{X_L}|\omega|e^{\gamma})\\&\quad+\frac{i}{|\omega|}\text{sgn}(\omega)\left[\ln^2(C_{X_L}|\omega|e^{\gamma})-\frac{\pi^2}{12}\right],\end{aligned} \tag{3.4}$$

with the properties

$$\begin{aligned}A_1(\omega,C_{X_L})+A_1(-\omega,C_{X_L})&=-\frac{\pi}{|\omega|},\\A_1(\omega,C_{X_L})-A_1(-\omega,C_{X_L})&=-2\frac{i}{\omega}\ln(C_{X_L}|\omega|e^{\gamma}),\\A_1(-\omega,C_{X_L})A_1(\omega,C_{X_L})&=\frac{1}{\omega^2}\left(\frac{\pi^2}{4}+\ln^2(C_{X_L}|\omega|e^{\gamma})\right),\end{aligned} \tag{3.5}$$

and

$$A_2(\omega,C_{X_L})=-i\omega\left(A_1(\omega,C_{X_L})^2-\frac{\pi^2}{6\omega^2}\right). \tag{3.6}$$

Taking the orbital averages (2.13) leads to integrals of the type

$$\begin{aligned}F_m[Y_M^{(p)},X_L^{(n)};C_{X_L}]&=\int_{-\infty}^{\infty}dtY_M^{(p)}(t)\mathcal{T}_{\ln^m}[X_L^{(n)};C_{X_L}](t)\\&=\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega' t}(-i\omega')^p\hat{Y}_M(\omega')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}(-i\omega)^n\hat{X}_L(\omega)A_m(\omega,C_{X_L})\\&=(-1)^n\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(i\omega)^{n+p}\hat{Y}_M(-\omega)\hat{X}_L(\omega)A_m(\omega,C_{X_L})\\&=\int_0^{\infty}\frac{d\omega}{2\pi}(i\omega)^{n+p}[(-1)^n\hat{Y}_M(-\omega)\hat{X}_L(\omega)A_m(\omega,C_{X_L})+(-1)^p\hat{Y}_M(\omega)\hat{X}_L(-\omega)A_m(-\omega,C_{X_L})],\end{aligned} \tag{3.7}$$

which in the special case $Y=X$ and $L=M$ becomes

$$F_m[X_L^{(p)},X_L^{(n)};C_{X_L}]=\int_0^{\infty}\frac{d\omega}{2\pi}(i\omega)^{n+p}\hat{X}_L(-\omega)\hat{X}_L(\omega)[(-1)^nA_m(\omega,C_{X_L})+(-1)^pA_m(-\omega,C_{X_L})]. \tag{3.8}$$

Using this result, the energy and angular momentum past tails (2.10) and (2.11) become

$$\begin{aligned}(\Delta E)_{\rm tail}&=\frac{2}{5}\frac{G^2\mathcal{M}}{c^8}\int_0^{\infty}d\omega\omega^7\kappa(\omega),\\(\Delta E)_{\rm tail(tail)}&=-\frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_0^{\infty}\frac{d\omega}{2\pi}\omega^8\kappa(\omega)\left[\ln^2(C_{I_2}\omega e^{\gamma})-\frac{\pi^2}{12}+\frac{107}{105}\ln(C_{I_2}\omega e^{\gamma})+\frac{515063}{176400}\right],\\(\Delta E)_{({\rm tail})^2}&=\frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_0^{\infty}\frac{d\omega}{2\pi}\omega^8\kappa(\omega)\left[\ln^2(C_{I_2}\omega e^{\gamma})+\frac{\pi^2}{4}\right]\end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
(\Delta J_i)_{\rm tail} &= \frac{2}{5}\frac{G\mathcal{M}^2}{c^8}\int_0^\infty d\omega\omega^6\kappa_i(\omega),\\
(\Delta J_i)_{\rm tail(tail)} &= -\frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_0^\infty \frac{d\omega}{2\pi}\omega^7\kappa_i(\omega)\left[\ln^2(C_{I_2}\omega e^\gamma)-\frac{\pi^2}{12}+\frac{107}{105}\ln(C_{I_2}\omega e^\gamma)+\frac{515063}{176400}\right],\\
(\Delta J_i)_{({\rm tail})^2} &= \frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_0^\infty \frac{d\omega}{2\pi}\omega^7\kappa_i(\omega)\left[\ln^2(\omega C_{I_2}e^\gamma)+\frac{\pi^2}{4}\right],
\end{aligned}
\tag{3.10}
$$

respectively, where we have introduced the notation

$$
\begin{aligned}
\kappa_{ab}(\omega) &= \hat{I}_{aj}(\omega)\hat{I}_{bj}(-\omega)=\kappa_{ba}(-\omega),\\
\kappa(\omega) &= {\rm Tr}[\kappa_{ab}(\omega)],\\
\kappa_i(\omega) &= 2i\epsilon_{iab}\kappa_{ab}(\omega).
\end{aligned}
\tag{3.11}
$$

It is also useful to introduce the magnitude $\mathcal{N}(\omega)$ and the direction n_i of the vector $\kappa_i(\omega)$, so that

$$
\kappa_i(\omega)\equiv\mathcal{N}(\omega)n_i. \tag{3.12}
$$

Notice that (i) in both cases the contributions from logarithms squared cancel out once the tail-of-tail and tail-square terms are summed up; for example,

$$
\begin{aligned}
(\Delta E)_{\rm tail(tail)+(tail)^2} &= +\frac{8}{5}\left(-\frac{107}{105}\right)\frac{G^3\mathcal{M}^2}{c^{11}}\\
&\quad\times\int_0^\infty\frac{d\omega}{2\pi}\omega^8\kappa(\omega)\ln\left(\frac{\omega}{\rm scale}\right),
\end{aligned}
\tag{3.13}
$$

where

$$
\ln\left(\frac{\rm scale}{C_{I_2}e^\gamma}\right)=\frac{105}{107}\left(\frac{\pi^2}{3}-\frac{515065}{176400}\right); \tag{3.14}
$$

(ii) the dimensions of κ (or κ_i) are obtained recalling the dimensions of the quadrupolar moment in the Fourier space are not the same as in the ordinary space, namely,

$$
I_{ab}(t)\sim\frac{1}{T}\hat{I}_{ab}(\omega), \tag{3.15}
$$

with an obvious use of notation. Therefore,

$$
\kappa(\omega)\sim\hat{I}^2\sim(TML^2)^2, \tag{3.16}
$$

which implies, for example,

$$
\begin{aligned}
(\Delta E)_{\rm tail} &\sim \frac{G^2M}{c^8}\frac{\kappa(\omega)}{T^8}\\
&\sim\frac{G^2M}{c^8}\frac{T^2M^2L^4}{T^8}\\
&\sim Mc^2.
\end{aligned}
\tag{3.17}
$$

A direct comparison between energy and angular momentum past tails shows that the following simple relation holds between the corresponding densities:

$$
(\Delta E)_{\rm X}=\int_0^\infty d\omega\frac{dE^{\rm X}}{d\omega},\qquad (\Delta J)_{\rm X}=\int_0^\infty d\omega\frac{dJ^{\rm X}}{d\omega}, \tag{3.18}
$$

such that

$$
\omega\kappa(\omega)\frac{dJ_i^{\rm X}}{d\omega}-\kappa_i(\omega)\frac{dE^{\rm X}}{d\omega}=0, \tag{3.19}
$$

for all different tail terms, $X={\rm tail},{\rm tail(tail)},{\rm and}\,({\rm tail})^2$. For example,

$$
\frac{dE^{\rm tail}}{d\omega}=\frac{2}{5}\frac{G^2\mathcal{M}}{c^8}\omega^7\kappa(\omega), \tag{3.20}
$$

etc. More precisely,

$$
\frac{dJ_i^{\rm X}}{d\omega}=\mathcal{P}(\omega)\frac{dE^{\rm X}}{d\omega}n_i, \tag{3.21}
$$

with

$$
\mathcal{P}(\omega)\equiv\frac{\mathcal{N}(\omega)}{\omega\kappa(\omega)}, \tag{3.22}
$$

determining both the direction and magnitude of the angular momentum flow (in terms of energy flow) in the Fourier space. The loss of angular momentum rate per unit frequency, thus, dominates with respect to the energy one in the range of frequencies wherein $\mathcal{P}(\omega)>1$, and vice versa for $\mathcal{P}(\omega)<1$.

Equation (3.21) connecting the loss of energy and angular momentum rates per unit frequency closely resembles the proportionality relation between the gravitational-wave energy and angular momentum fluxes for circular orbits, satisfying the first law of binary black hole dynamics in the adiabatic approximation [25].

IV. EXPLICIT RESULTS FOR HYPERBOLICLIKE ORBITS

Let us evaluate the leading-order contribution to the orbital average of the tail integrals (2.13) in the case of hyperboliclike motion. We need only the Newtonian description of the dynamics of a binary system. The corresponding Keplerian parametrization of the hyperbolic motion in harmonic coordinates in terms of dimensionless variables (and $c=1$)—i.e., $r = r^{\rm phys}/(GM)$, $t = t^{\rm phys}/(GM)$—is [27]

$$
\begin{aligned}
r &= \bar a_r(e_r\cosh v - 1),\\
\bar n t &= e_r \sinh v - v,\\
\phi &= 2\arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\frac{v}{2}\right].
\end{aligned} \tag{4.1}
$$

We will assume the motion to be confined in the $x-y$ plane, so that (r,ϕ) are polar coordinates on that plane. The expressions of the orbital parameters $\bar n$, $\bar a_r$, and e_r as functions of the specific binding energy $\bar E$ [Eq. (1.1)] and of the dimensionless angular momentum j [Eq. (1.2)] of the system are given by

$$
\bar n = (2\bar E)^{3/2}, \qquad \bar a_r = \frac{1}{2\bar E}, \qquad e_r = \sqrt{1+2\bar E j^2}, \tag{4.2}
$$

with $\bar E > 0$ also expressed in terms of the relative momentum for infinite separation p_∞ as $2\bar E \equiv p_\infty^2$. The parametric equations (4.1) are obtained through analytic continuation of the corresponding elliptic motion ($\bar E < 0$) by replacing $v \to iv$. Therefore, $\bar n$ and $\bar a_r$ are the hyperbolic counterparts of the inverse radial period and the semimajor axis, respectively, whereas e_r still has the meaning of an eccentricity parameter. Notice that this property is lost from 2PN on [28].

The first step consists in Fourier transforming the quadrupole moment, i.e.,

$$
\hat I_{ab}(\omega) = \int \frac{dt}{dv} e^{i\omega t(v)} I_{ab}(t)|_{t=t(v)} dv. \tag{4.3}
$$

This is done by using the integral representation of the Hankel functions of the first kind of order $p \equiv \frac{q}{e_r}$ and argument $q \equiv iu$, with $u \equiv \omega e_r \bar a_r^{3/2}$,

$$
H_p^{(1)}(q) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{q\,\sinh\,v - pv} dv. \tag{4.4}
$$

As the argument $q = iu$ of the Hankel function is purely imaginary, the Hankel function becomes converted into a modified Bessel function of the first kind (Bessel K function), according to the relation

$$
H_p^{(1)}(iu) = \frac{2}{\pi} e^{-i\frac{\pi}{2}(p+1)} K_p(u). \tag{4.5}
$$

The typical term is of the kind $e^{q\,\sinh\,v-(p+k)v}$, the Fourier transform of which is

$$
e^{q\,\sinh\,v-(p+k)v} \to 2e^{-i\frac{\pi}{2}(p+k)} K_{p+k}(u), \tag{4.6}
$$

involving Bessel functions having the same argument u but various orders differing by integers. However, standard identities valid for Bessel functions allow one to reduce the orders to either p or $p+1$.

All energy and angular momentum tail integrals (3.9) and (3.10) are defined in terms of the trace $\kappa(\omega)$ of the tensor $\kappa_{ab}(\omega)$ [Eq. (3.11)] and the magnitude $\mathcal{N}(\omega)$ of its associated vector $\kappa_i(\omega)$ (proportional to its dual and orthogonal to the orbital plane, i.e., with $n_i = \delta_{iz}$) [Eq. (3.12)], which are given by

$$
\begin{aligned}
\kappa(u) &= 32\frac{\nu^2\bar a_r^7}{p^4u^4} e^{-i\pi p}\Big\{u^2(p^2+u^2+1)(p^2+u^2)K_{p+1}^2(u)\\
&\quad - u[(2p-3)u^2+2p(p-1)^2](p^2+u^2)K_p(u)K_{p+1}(u)\\
&\quad + \left[u^6 + \left(4p^2-3p+\frac{1}{3}\right)u^4 + (5p^2-7p+2)p^2u^2 + 2p^4(p-1)^2\right]K_p^2(u)\Big\},\\
\mathcal{N}(u) &= 128\frac{\nu^2\bar a_r^7}{p^4u^4} e^{-i\pi p}\sqrt{p^2+u^2}[uK_{p+1}(u) + (p^2+u^2-p)K_p(u)]\\
&\quad \times\left\{(p^2+u^2)uK_{p+1}(u) - \left[\left(p-\frac{1}{2}\right)u^2 + p^2(p-1)\right]K_p(u)\right\},
\end{aligned} \tag{4.7}
$$

as functions of the frequency-related variable u introduced above. For convenience, with an abuse of notation, we denoted here as $\kappa(u)$ the dimensionless (rescaled) quantity

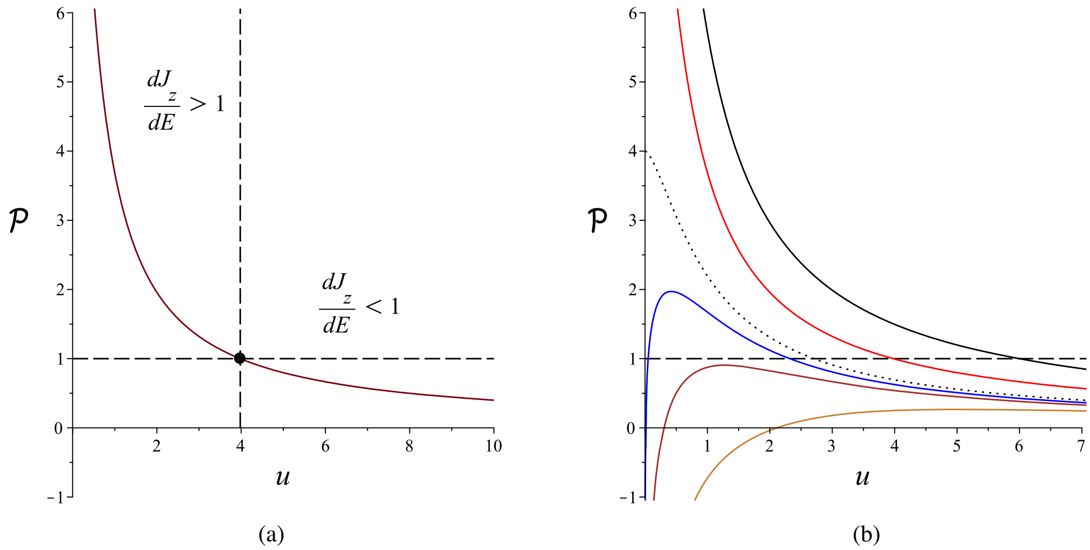

FIG. 1. Behavior of the proportionality factor $\mathcal{P}$ [see Eq. (3.22)] as a function of the frequency-related dimensionless variable u, showing the regions of energy versus angular momentum dominance. In (a) the orbital parameters have been set as $\bar{a}_r = 1$ and $e_r = 2$, implying that $\mathcal{P}(u) = 1$ at $u = u_*(e_r, \bar{a}_r) \approx 3.979$. This is the typical behavior for $e_r > e_r^{\rm sep} = \sqrt{2}$, for which $\mathcal{P}$ positively diverges as $u \to 0$ and monotonically decreases for increasing frequencies, crossing the horizontal line $\mathcal{P}(u) = 1$ at some value of $u = u_*(e_r, \bar{a}_r)$. The intersection point moves to the right for increasing values of the eccentricity [see also (b)]. The curves in (b) correspond instead to different values of e_r (and the same value of $\bar{a}_r = 1$), moving from bottom to top for increasing eccentricity $e_r = [1.1, 1.2, 1.3, \sqrt{2}, 2, 3]$. The dotted curve is the separatrix between the two different behaviors (see the text).

$$\kappa(u) \to \frac{\kappa^{\rm phys}(u)}{[(\frac{GM}{c^2})^3 \frac{M}{c}]^2}, \tag{4.8}$$

and similarly for $\mathcal{N}(u)$.

Figure 1 shows the frequency regions of energy versus angular momentum dominance for selected values of the eccentricity parameter and fixed semimajor axis. At high frequencies ($u \to \infty$), the proportionality factor $\mathcal{P}$ [defined in Eq. (3.22)] goes to zero for every value of the eccentricity. For instance, at the leading order in the large-eccentricity expansion limit, we find

$$[\mathcal{P}(u)]^{\rm LO}_{u\to\infty} \sim \frac{2}{u} \bar{a}_r^{3/2} e_r, \tag{4.9}$$

where the asymptotic relation $\bar{a}_r^{3/2} e_r \sim j/p_\infty^2$ also holds. In the limit of low frequencies ($u \to 0$), instead its behavior strongly depends on the chosen value of e_r. In fact, for $u \to 0$, one has

$$\mathcal{P}(u)_{u\to 0} \sim 2\bar{a}_r^{3/2}\left[-\frac{e_r^2-2}{\sqrt{e_r^2-1}} \ln\left(\frac{ue^\gamma}{2}\right) + 2\sqrt{e_r^2-1}\right], \tag{4.10}$$

implying that there exists a critical value of the eccentricity $e_r = e_r^{\rm sep} = \sqrt{2}$ such that in this limit $\mathcal{P}(u)$ gets the finite value $4\bar{a}_r^{3/2}$, whereas it logarithmically diverges assuming either positive or negative values depending on whether e_r is greater or smaller than $e_r^{\rm sep}$.

The tail integrals cannot be performed in closed analytical form due to the dependence of the order of the Bessel K functions on the integration variable. However, the order p tends to zero when $e_r \to \infty$, allowing for the explicit computation in a large-eccentricity expansion [18,19,29]. In fact, Taylor expanding the Bessel functions around $p = 0$ leads to integrals involving Bessel functions $K_\nu(u)$ and their derivatives $\frac{\partial^n K_\nu(u)}{\partial \nu^n}$ with respect to the order ν evaluated at $\nu = (0, 1)$ only. Therefore, one is left with integrals of the type

$$\int_0^\infty du\, f(u) \ln^m(u), \tag{4.11}$$

with $m = 0, 1, 2$, which are conveniently computed by using the Mellin transform [19]. The latter is defined as

$$g(s) = \int_0^\infty du\, u^{s-1} f(u), \tag{4.12}$$

so that

$$g(1) = \int_0^\infty du\, f(u),$$
$$\left.\frac{dg(s)}{ds}\right|_{s=1} = \int_0^\infty du\, f(u) \ln(u),$$
$$\left.\frac{d^2g(s)}{ds^2}\right|_{s=1} = \int_0^\infty du\, f(u) \ln^2(u). \tag{4.13}$$

We list below the results of the computation, by using the equivalent large-j expansion limit in place of the large-eccentricity limit:

$$
\begin{aligned}
(\Delta E)_{\rm tail} &= Mc^2\nu^2\left[\frac{3136}{45}\frac{p_\infty^6}{j^4}+\frac{297\pi^2}{20}\frac{p_\infty^5\pi}{j^5}+\left(\frac{9344}{45}+\frac{88576}{675}\pi^2\right)\frac{p_\infty^4}{j^6}+\left(-\frac{2755}{64}\pi^4+\frac{1579}{3}\pi^2\right)\frac{p_\infty^3\pi}{j^7}+O\left(\frac{1}{j^8}\right)\right],\\
(\Delta E)_{\rm tail(tail)} &= Mc^2\nu^2\left[\left(-\frac{297}{10}\mathcal{L}^2-\frac{1709227}{6125}-\frac{130071}{700}\mathcal{L}-\frac{297}{40}\pi^2\right)\frac{p_\infty^8\pi}{j^5}\right.\\
&\quad+\left(-\frac{177152}{225}\mathcal{L}^2-\frac{841216}{945}\mathcal{L}-\frac{2060423552}{826875}-\frac{1417216}{225}\mathcal{L}\ln(2)+\frac{44288}{675}\pi^2-\frac{2834432}{225}\ln(2)^2\right.\\
&\quad\left.-\frac{3364864}{945}\ln(2)\right)\frac{p_\infty^7}{j^6}\\
&\quad+\left(-\frac{405611}{32}\zeta(3)-\frac{57855}{16}\zeta(3)\mathcal{L}-\frac{3158}{3}\mathcal{L}^2-\frac{403863077}{105840}-\frac{7898921}{2520}\mathcal{L}-\frac{1579}{6}\pi^2+\frac{8265}{128}\pi^4\right)\frac{p_\infty^6\pi}{j^7}\\
&\quad\left.+O\left(\frac{1}{j^8}\right)\right],\\
(\Delta E)_{({\rm tail})^2} &= Mc^2\nu^2\left[\left(-\frac{297}{5}\gamma^2+420\gamma+1260\ln(2)-\frac{71707}{160}-\frac{1782}{5}\gamma\ln(2)+\frac{297}{10}\mathcal{L}^2+\frac{3111}{20}\mathcal{L}-\frac{99}{40}\pi^2\right.\right.\\
&\quad\left.-\frac{2673}{5}\ln(2)^2\right)\frac{p_\infty^8\pi}{j^5}\\
&\quad+\left(\frac{708608}{225}\gamma\ln(2)+\frac{177152}{225}\mathcal{L}^2+\frac{98816}{1125}\mathcal{L}-\frac{27545792}{16875}+\frac{1417216}{225}\mathcal{L}\ln(2)+\frac{44288}{225}\pi^2\right.\\
&\quad\left.-\frac{354304}{225}\gamma^2+\frac{2480128}{225}\ln(2)^2+\frac{10336256}{3375}\gamma-\frac{9150464}{3375}\ln(2)\right)\frac{p_\infty^7}{j^6}\\
&\quad+\left(-\frac{558789}{32}\zeta(3)-\frac{6316}{3}\gamma^2+\frac{1447691}{180}\gamma+\frac{1447691}{60}\ln(2)-\frac{2354665}{432}+\frac{173565}{8}\ln(2)\zeta(3)+\frac{57855}{8}\gamma\zeta(3)\right.\\
&\quad\left.-12632\gamma\ln(2)+\frac{57855}{16}\zeta(3)\mathcal{L}+\frac{3158}{3}\mathcal{L}^2+\frac{82471}{40}\mathcal{L}-\frac{1579}{18}\pi^2+\frac{8265}{128}\pi^4-18948\ln(2)^2\right)\frac{p_\infty^6\pi}{j^7}\\
&\quad\left.+O\left(\frac{1}{j^8}\right)\right],
\end{aligned}
\tag{4.14}
$$

for the energy, and

$$
\begin{aligned}
(\Delta J_z)_{\rm tail} &= \frac{GM^2}{c}\nu^2\left[\frac{448}{5}\frac{p_\infty^4}{j^3}+\frac{69}{5}\pi^2\frac{p_\infty^3\pi}{j^4}+\left(\frac{4352}{45}\pi^2+\frac{128}{15}\right)\frac{p_\infty^2}{j^5}+\left(-\frac{423}{16}\pi^4+303\pi^2\right)\frac{p_\infty\pi}{j^6}+O\left(\frac{1}{j^7}\right)\right],\\
(\Delta J_z)_{\rm tail(tail)} &= \frac{GM^2}{c}\nu^2\left[\left(-\frac{69}{10}\pi^2-\frac{3997468}{18375}-\frac{27637}{175}\mathcal{L}-\frac{138}{5}\mathcal{L}^2\right)\frac{p_\infty^6\pi}{j^4}\right.\\
&\quad+\left(-\frac{96745024}{55125}-\frac{8704}{15}\mathcal{L}^2-\frac{662528}{1575}\mathcal{L}+\frac{2176\pi^2}{45}-\frac{139264}{15}\ln(2)^2-\frac{2650112}{1575}\ln(2)-\frac{69632}{15}\mathcal{L}\ln(2)\right)\frac{p_\infty^5}{j^5}\\
&\quad\left.+\left(-\frac{303}{2}\pi^2-\frac{5175383}{2940}+\frac{1269}{32}\pi^4-\frac{104051}{70}\mathcal{L}-606\mathcal{L}^2-\frac{8883}{4}\zeta(3)\mathcal{L}-\frac{296757}{40}\zeta(3)\right)\frac{p_\infty^4\pi}{j^6}+O\left(\frac{1}{j^7}\right)\right],\\
(\Delta J_z)_{({\rm tail})^2} &= \frac{GM^2}{c}\nu^2\left[\left(\frac{161}{10}\pi^2+\frac{2833}{40}+\frac{138}{5}\mathcal{L}^2+\frac{649}{5}\mathcal{L}\right)\frac{p_\infty^6\pi}{j^4}\right.\\
&\quad+\left(\frac{8704}{15}\mathcal{L}^2-\frac{512}{3}\mathcal{L}+\frac{19936}{135}+\frac{69632}{15}\mathcal{L}\ln(2)+\frac{2176}{15}\pi^2+\frac{139264}{15}\ln(2)^2-\frac{2048}{3}\ln(2)\right)\frac{p_\infty^5}{j^5}\\
&\quad\left.+\left(\frac{707}{2}\pi^2-\frac{18073}{40}-\frac{1269}{32}\pi^4+\frac{8883}{4}\zeta(3)\mathcal{L}+606\mathcal{L}^2+\frac{8689}{10}\mathcal{L}+\frac{31437}{5}\zeta(3)\right)\frac{p_\infty^4\pi}{j^6}+O\left(\frac{1}{j^7}\right)\right],
\end{aligned}
\tag{4.15}
$$

for the angular momentum, with

$$\mathcal{L} = \ln\left(\frac{r_0 p_\infty^2}{4j}\right). \tag{4.16}$$

The angular momentum memory integral (2.8) requires a separate treatment. Let us denote

$$F_{bj}(t) = \int_{-\infty}^{t} d\tau I_{cb}^{(3)}(\tau) I_{jc}^{(3)}(\tau) = \int_0^\infty dt' I_{cb}^{(3)}(t-t') I_{jc}^{(3)}(t-t'), \tag{4.17}$$

so that

$$\mathcal{G}_i^{\rm memory}(t) = \frac{4}{35}\frac{G^2}{c^{10}} \epsilon_{iab} I_{aj}^{(3)}(t) F_{bj}(t). \tag{4.18}$$

The integral (4.17) does not depend on time. In fact, inserting the Fourier transform of the quadrupole moment yields

$$\begin{aligned} F_{bj}(t) &= \int_0^\infty dt' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} (-i\omega)^3 (-i\omega')^3 \\ &\quad \times e^{-i\omega(t-t')} e^{-i\omega'(t-t')} \hat{I}_{cb}(\omega) \hat{I}_{jc}(\omega') \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} (-i\omega)^3 (-i\omega')^3 \\ &\quad \times e^{-i(\omega+\omega')t} \hat{I}_{cb}(\omega) \hat{I}_{jc}(\omega') \pi\delta(\omega+\omega') \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^6 \hat{I}_{cb}(\omega) \hat{I}_{jc}(-\omega). \end{aligned} \tag{4.19}$$

Recalling the definition of the tensor $\kappa_{ab}(\omega)$ [Eq. (3.11)] and restricting the range of frequencies between $[0, \infty)$ then gives

$$F_{bj} = \int_0^\infty \frac{d\omega}{2\pi} \omega^6 \kappa_{(bj)}(\omega), \tag{4.20}$$

with $\kappa_{(bj)}(\omega) = \frac{1}{2}(\kappa_{bj}(\omega) + \kappa_{jb}(\omega))$. Finally, the orbital average (2.13) reads

$$(\Delta J_i)_{\rm memory} = \frac{4}{35}\frac{G^2}{c^{10}} \epsilon_{iab} H_{aj} F_{bj}, \tag{4.21}$$

where

$$H_{aj} = \int_{-\infty}^{\infty} dt I_{aj}^{(3)}(t). \tag{4.22}$$

The latter integral turns out to be

$$\begin{aligned} H_{aj} &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (-i\omega)^3 e^{-i\omega t} \hat{I}_{aj}(\omega) \\ &= \int_{-\infty}^{\infty} d\omega (-i\omega)^3 \delta(\omega) \hat{I}_{aj}(\omega) \\ &= -\frac{4\nu\sqrt{e_r^2-1}}{\bar{a}_r e_r^2} (\delta_{ax}\delta_{jy} + \delta_{ay}\delta_{jx}), \end{aligned} \tag{4.23}$$

so that the averaged memory integral (4.21) becomes

$$(\Delta J_i)_{\rm memory} = -\frac{4\nu\sqrt{e_r^2-1}}{\bar{a}_r e_r^2} (\epsilon_{ixy} F_{yy} + \epsilon_{iyx} F_{xx}), \tag{4.24}$$

with only nonvanishing component

$$(\Delta J_z)_{\rm memory} = -\frac{4}{35}\frac{G^2}{c^{10}} \frac{4\nu\sqrt{e_r^2-1}}{\bar{a}_r e_r^2} (F_{yy} - F_{xx}), \tag{4.25}$$

the large-j expansion of which reads

$$(\Delta J_z)_{\rm memory} = -\frac{GM^2}{c}\nu^3 \left[\frac{16}{105}\frac{p_\infty^5 \pi}{j^4} + \frac{128}{63}\frac{p_\infty^4}{j^5} + \frac{8}{7}\frac{p_\infty^3 \pi}{j^6} + O\left(\frac{1}{j^7}\right)\right]. \tag{4.26}$$

V. TIME-SYMMETRIC TAILS

The tails defined above should be more properly termed “past tails,” since they refer to the past interaction between the two bodies, in the sense that the integration variable $\xi = t - \tau$ in the typical tail integral (3.2) takes values in the interval $\tau \in [0, \infty)$, namely,

$$\mathcal{T}_{\ln^m}[X_L^{(n)}; C_{X_L}](t) = \int_0^\infty d\tau X_L^{(n)}(t-\tau) \ln^m\left(\frac{\tau}{C_{X_L}}\right), \tag{5.1}$$

implying contributions from $X_L^{(n)}(\xi)$ with ξ varying in the range $\xi \in (-\infty, t]$.

Previous works focusing on ellipticlike motion used the tails in this precise form. The meaning and importance of time-symmetric tails was proven at the 4PN level in Ref. [12], where tail effects on the dynamics were decomposed in a time-symmetric action contribution and a time-antisymmetric radiation-reaction force. Such a decomposition seems clearly extendable when considering effects which are linear in radiation reaction, as recently accomplished in Ref. [20]. It is well known that the radiation-reaction force starts at 2.5PN, so that quadratic effects in radiation reaction start affecting the dynamics of the system beyond the 4PN order. More precisely, one expects that second-order effects will enter the dynamics at order $\frac{G^4}{c^{10}}$, i.e., at the 4PM level and the 5PN level (see the discussion in Sec. X in Ref. [20]). No complete treatment of the energy flux (as well as angular and linear momentum fluxes) exists at such a level yet. It is reasonable to expect that the contribution of higher-order time-symmetric tails becomes relevant as soon as the PN accuracy increases, as they did at 4PN. Further investigation is necessary to systematically include in the dynamics time-symmetric tails which are nonlinear in radiation reaction, whatever approach one uses (e.g., the effective field theory approach [30–32]).

Let us replace $X_L^{(n)}(t-\tau)$ by the sum of its symmetric (sym) and antisymmetric (asym) parts,

$$\begin{aligned} X_L^{(n)}(t-\tau) &= \frac{1}{2}[X_L^{(n)}(t-\tau)+X_L^{(n)}(t+\tau)] \\ &\quad +\frac{1}{2}[X_L^{(n)}(t-\tau)-X_L^{(n)}(t+\tau)] \\ &\equiv X_{L,\mathrm{sym}}^{(n)}(t,\tau)+X_{L,\mathrm{asym}}^{(n)}(t,\tau). \end{aligned} \tag{5.2}$$

The time-symmetric part only is used as the proper tail contribution, since the time-antisymmetric one is already included in the nonlocal part of the Hamiltonian. The time-symmetric (ts) version of $\mathcal{T}_{\ln^m}[X_L^{(n)};C_{X_L}](t)$, thus, reads

$$\begin{aligned} &\mathcal{T}^{\mathrm{ts}}_{\ln^m}[X_L^{(n)};C_{X_L}](t) \\ &\quad = \int_0^\infty d\tau X^{(n)}_{\mathrm{L\,sym}}(t,\tau)\ln^m\left(\frac{\tau}{C_{X_L}}\right). \end{aligned} \tag{5.3}$$

Passing then to the Fourier domain, the above expression becomes

$$\begin{aligned} \mathcal{T}^{\mathrm{ts}}_{\ln^m}[X_L^{(n)};C_{X_L}](t) &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n\hat{X}_L(\omega) \\ &\quad \times e^{-i\omega t}[A_m(\omega,C_{X_L})+A_m(-\omega,C_{X_L})]. \end{aligned} \tag{5.4}$$

The final expressions for the (averaged) energy and angular momentum time-symmetric tails are given by Eq. (2.13) with the replacement $\mathcal{T}_{\ln^m}\to\mathcal{T}^{\mathrm{ts}}_{\ln^m}$ in the fluxes (2.10) and (2.11).

Consider now the time-symmetric version of the basic integral (3.7), i.e.,

$$\begin{aligned} F^{\mathrm{ts}}_m[Y_M^{(p)},X_L^{(n)};C_{X_L}] &= \int_{-\infty}^{\infty} dt Y_M^{(p)}(t)\mathcal{T}^{\mathrm{ts}}_{\ln^m}[X_L^{(n)};C_{X_L}](t) \\ &= \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega' t}(-i\omega')^p\hat{Y}_M(\omega') \\ &\quad \times\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n\hat{X}_L(\omega)e^{-i\omega t}[A_m(\omega,C_{X_L})+A_m(-\omega,C_{X_L})] \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n(i\omega)^p\hat{Y}_M(-\omega)\hat{X}_L(\omega)[A_m(\omega,C_{X_L})+A_m(-\omega,C_{X_L})], \end{aligned} \tag{5.5}$$

which for $m=1$ and $m=2$ becomes

$$\begin{aligned} F^{\mathrm{ts}}_1[Y_M^{(p)},X_L^{(n)};C_{X_L}] &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n(i\omega)^p\hat{Y}_M(-\omega)\hat{X}_L(\omega)[A_1(\omega,C_{X_L})+A_1(-\omega,C_{X_L})] \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n(i\omega)^p\hat{Y}_M(-\omega)\hat{X}_L(\omega)(-)\frac{\pi}{|\omega|} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} F^{\mathrm{ts}}_2[Y_M^{(p)},X_L^{(n)};C_{X_L}] &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n(i\omega)^p\hat{Y}_M(-\omega)\hat{X}_L(\omega)[A_2(\omega,C_{X_L})+A_2(-\omega,C_{X_L})] \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^n(i\omega)^p\hat{Y}_M(-\omega)\hat{X}_L(\omega)\frac{2\pi}{|\omega|}\ln(C_{X_L}|\omega|e^\gamma), \end{aligned} \tag{5.7}$$

respectively, having used Eq. (3.5). In the special case $Y=X$ and $L=M$, Eqs. (5.6) and (5.7) simplify as

$$F^{\mathrm{ts}}_1[X_L^{(p)},X_L^{(n)};C_{X_L}] = \frac{1}{2}(-1)^{n+1}i^{n+p}\int_{-\infty}^{\infty}\frac{d\omega}{2}\omega^{n+p}\hat{X}_L(-\omega)\hat{X}_L(\omega)\frac{1}{|\omega|}, \tag{5.8}$$

$$F^{\mathrm{ts}}_2[X_L^{(p)},X_L^{(n)};C_{X_L}] = \frac{1}{2}(-1)^n i^{n+p}\int_{-\infty}^{\infty}d\omega\omega^{n+p}\hat{X}_L(-\omega)\hat{X}_L(\omega)\frac{1}{|\omega|}\ln(C_{X_L}|\omega|e^\gamma), \tag{5.9}$$

respectively. Therefore, when $n+p$ is odd, both $F_1^{\mathrm{ts}}[X_L^{(p)}, X_L^{(n)}; C_{X_L}]$ and $F_2^{\mathrm{ts}}[X_L^{(p)}, X_L^{(n)}; C_{X_L}]$ vanish identically.

The time-symmetric energy tail turns out to be

$$\begin{aligned}(\Delta E)_{\mathrm{tail,ts}} &= \int_{-\infty}^{\infty} dt \mathcal{F}_{\mathrm{tail,ts}}(t) = \frac{4}{5}\frac{G^2\mathcal{M}}{c^8} F_1^{\mathrm{ts}}[I_{ij}^{(3)}, I_{ij}^{(5)}; C_{I_2}] \\ &= \frac{4}{5}\frac{G^2\mathcal{M}}{c^8}\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^5(i\omega)^3\hat{I}_{ij}(-\omega)\hat{I}_{ij}(\omega)(-)\frac{\pi}{|\omega|} \\ &= \frac{1}{5}\frac{G^2\mathcal{M}}{c^8}\int_{-\infty}^{\infty} d\omega\omega^8\hat{I}_{ij}(-\omega)\hat{I}_{ij}(\omega)\frac{1}{|\omega|} \\ &= \frac{2}{5}\frac{G^2\mathcal{M}}{c^8}\int_0^{\infty} d\omega\omega^7\hat{I}_{ij}(-\omega)\hat{I}_{ij}(\omega),\end{aligned} \tag{5.10}$$

coinciding with the analogous result for past tails. The time-symmetric part of the tail-of-tail integral is instead identically vanishing:

$$\begin{aligned}(\Delta E)_{\mathrm{tail(tail),ts}} &= \int_{-\infty}^{\infty} dt \mathcal{F}_{\mathrm{tail(tail),ts}}(t) = \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\left(F_2^{\mathrm{ts}}[I_{ij}^{(3)}, I_{ij}^{(6)}; C_{I_2}] - \frac{107}{105}F_1^{\mathrm{ts}}[I_{ij}^{(3)}, I_{ij}^{(6)}; \tilde{C}_{I_2}]\right) \\ &= 0,\end{aligned} \tag{5.11}$$

due to the general property shown above with $n+p=9$. Finally, the time-symmetric tail-squared integral reads

$$\begin{aligned}(\Delta E)_{(\mathrm{tail})^2,\mathrm{ts}} &= \int_{-\infty}^{\infty} dt \mathcal{F}_{(\mathrm{tail})^2,\mathrm{ts}}(t) \\ &= \frac{1}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^5\hat{I}_{ij}(\omega)e^{-i\omega t}(-)\frac{\pi}{|\omega|}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}(-i\omega')^5\hat{I}_{ij}(\omega')e^{-i\omega' t}(-)\frac{\pi}{|\omega'|} \\ &= \frac{1}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^5(i\omega)^5\hat{I}_{ij}(\omega)\hat{I}_{ij}(-\omega)\frac{\pi^2}{\omega^2} \\ &= \frac{1}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\pi\int_0^{\infty} d\omega\omega^8\hat{I}_{ij}(\omega)\hat{I}_{ij}(-\omega),\end{aligned} \tag{5.12}$$

differently from the analogous past tail case.

The time-symmetric angular momentum tails turn out to be

$$\begin{aligned}(\Delta J_i)_{\mathrm{tail,ts}} &= \int_{-\infty}^{\infty} dt \mathcal{G}_i^{\mathrm{tail,ts}} = \frac{4}{5}\frac{G^2\mathcal{M}}{c^8}\epsilon_{iab}[F_1^{\mathrm{ts}}[I_{aj}^{(2)}, I_{bj}^{(5)}; C_{I_2}] + F_1^{\mathrm{ts}}[I_{bj}^{(3)}, I_{aj}^{(4)}; C_{I_2}]] \\ &= \frac{2}{5}\frac{G^2\mathcal{M}}{c^8}\int_0^{\infty} d\omega\omega^6\kappa_i(\omega), \\ (\Delta J_i)_{\mathrm{tail(tail),ts}} &= \int_{-\infty}^{\infty} dt \mathcal{G}_i^{\mathrm{tail(tail),ts}} = \frac{4}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\epsilon_{iab}[F_2^{\mathrm{ts}}[I_{aj}^{(2)}, I_{bj}^{(6)}; C_{I_2}] + F_2^{\mathrm{ts}}[I_{aj}^{(3)}, I_{bj}^{(5)}; C_{I_2}] \\ &\quad - \frac{107}{105}(F_1^{\mathrm{ts}}[I_{aj}^{(2)}, I_{bj}^{(6)}; \tilde{C}_{I_2}] + F_1^{\mathrm{ts}}[I_{aj}^{(3)}, I_{bj}^{(5)}; \tilde{C}_{I_2}])] = 0, \\ (\Delta J_i)_{(\mathrm{tail})^2,\mathrm{ts}} &= \int_{-\infty}^{\infty} dt \mathcal{G}_i^{(\mathrm{tail})^2,\mathrm{ts}} = \frac{8}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\epsilon_{iab}\int_{-\infty}^{\infty} dt\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-i\omega)^4\hat{I}_{aj}(\omega)e^{-i\omega t}(-)\frac{\pi}{|\omega|}\right] \\ &\quad \times\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}(-i\omega')^5\hat{I}_{bj}(\omega')e^{-i\omega' t}(-)\frac{\pi}{|\omega'|}\right] \\ &= \frac{\pi}{10}\frac{G^3\mathcal{M}^2}{c^{11}}\int_{-\infty}^{\infty} d\omega\omega^7\kappa_i(\omega) \\ &= \frac{\pi}{5}\frac{G^3\mathcal{M}^2}{c^{11}}\int_0^{\infty} d\omega\omega^7\kappa_i(\omega),\end{aligned} \tag{5.13}$$

where we have used the property that $\kappa_i(\omega)$ is an odd function of the frequency [see Eq. (3.11)]:

$$\kappa_i(-\omega) = 2i\epsilon_{iab}\hat{I}_{aj}(-\omega)\hat{I}_{bj}(\omega) = -2i\epsilon_{iba}\hat{I}_{bj}(-\omega)\hat{I}_{aj}(\omega) = -\kappa_i(\omega). \quad (5.14)$$

In summary, direct comparison between the orbital averages of the energy and angular momentum tail integrals [Eqs. (3.9) and (3.10)] and their time-symmetric counterparts [Eqs. (5.10)–(5.12) and (5.13), respectively] shows that

$$(\Delta E)_{\rm tail,ts} = (\Delta E)_{\rm tail}, \qquad (\Delta E)_{\rm tail(tail),ts} = 0, \qquad (\Delta E)_{({\rm tail})^2,{\rm ts}} = (\Delta E)^{\rm nolog}_{({\rm tail})^2},$$
$$(\Delta J_i)_{\rm tail,ts} = (\Delta J_i)_{\rm tail}, \qquad (\Delta J_i)_{\rm tail(tail),ts} = 0, \qquad (\Delta J_i)_{({\rm tail})^2,{\rm ts}} = (\Delta J_i)^{\rm nolog}_{({\rm tail})^2}, \quad (5.15)$$

where "nolog" stands for the nonlogarithmic term of the corresponding (past tail) quantity. The property (3.21) and related discussion apply also to this case.

The explicit computation of the time-symmetric tail-squared integrals in the large-j expansion limit gives

$$(\Delta E)_{({\rm tail})^2,{\rm ts}} = Mc^2\nu^2\pi^2\left[\frac{297}{40}\frac{p_\infty^8\pi}{j^5} + \frac{44288}{225}\frac{p_\infty^7}{j^6} + \frac{1579}{6}\frac{p_\infty^6\pi}{j^7} + O\left(\frac{1}{j^8}\right)\right] \quad (5.16)$$

and

$$(\Delta J_z)_{({\rm tail})^2,{\rm ts}} = \frac{GM^2}{c}\nu^2\pi^2\left[\frac{69}{10}\frac{p_\infty^6\pi}{j^4} + \frac{2176}{15}\frac{p_\infty^5}{j^5} + \frac{303}{2}\frac{p_\infty^4\pi}{j^6} + O\left(\frac{1}{j^7}\right)\right]. \quad (5.17)$$

Noticeably, the coefficients of the previous expansions [Eqs. (5.16) and (5.17)] coincide with the corresponding coefficients of the $\mathcal{L}^2$ terms in Eqs. (4.14) and (4.15) divided by a factor of 4.

VI. CONCLUDING REMARKS

We have computed higher-order tail (i.e., tail-of-tail and tail-squared) contributions to both the energy and angular momentum losses averaged along hyperboliclike orbits at their leading PN approximation, using harmonic coordinates and working in the Fourier domain. These terms are conveniently denoted as past tails, since they are determined by the full past interaction among the bodies. We have also evaluated the time-symmetric counterpart of these tail integrals, which plays a key role in the construction of the nonlocal part of the conservative two-body dynamics starting from the 4PN level. All results have been expressed as an expansion in the large-eccentricity parameter and then converted in a large angular momentum expansion. It is interesting to note that we have obtained a nonvanishing value for the (averaged) nonlinear angular momentum memory integral, differently from the bound case [22].

We have also found the interesting result (valid at the same approximation level in which the tail integrals are computed) that there exists a direct proportionality between the loss of energy and angular momentum rates per unit frequency by a frequency-dependent factor which is the same for tails of any kind, generalizing similar links known for circular orbits only. The ranges of frequencies wherein such a factor is smaller (greater) than one correspond to those regions in the spectrum of energy (angular momentum) loss dominance for fixed values of the orbital parameters.

The inclusion of tail effects is necessary for the construction of the two-body Hamiltonian at high PN orders as well as for the evaluation of more and more accurate expressions for the radiative losses of energy, angular momentum, and linear momentum. The latter are essential for computing the radiation-reaction contribution to the scattering angle in the relativistic two-body problem [20]. We leave for a forthcoming study the computation of higher-order tails of the linear momentum flux, following the same lines outlined in the present work.

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