Spatially covariant gravity with 2 degrees of freedom: Perturbative analysis

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(Received 20 May 2021; accepted 6 October 2021; published 2 November 2021)

We revisit the problem of building the Lagrangian of a large class of metric theories that respect spatial covariance, which propagate at most 2 degrees of freedom and in particular no scalar mode. The Lagrangians are polynomials built of the spatially covariant geometric quantities. By expanding the Lagrangian around a cosmological background and focusing on the scalar modes only, we find the conditions for the coefficients of the monomials in order to eliminate the scalar mode at the linear order in perturbations. We find the conditions up to $d = 4$, with d the total number of derivatives in the monomials, and determine the explicit Lagrangians for the cases of $d = 2$, $d = 3$ as well as the combination of $d = 2$ and $d = 3$. We also expand the Lagrangian of $d = 2$ to the cubic order in perturbations, and find additional conditions for the coefficients such that the scalar mode is eliminated up to the cubic order. This perturbative analysis can be performed order by order, and one expects to determine the final Lagrangian at some finite order such that the scalar mode is fully eliminated. Our analysis provides an alternative and complimentary approach to building spatially covariant gravity with only tensorial degrees of freedom. The resulting theories can be used as alternatives to the general relativity to describe the tensorial gravitational waves in a cosmological setting.

DOI: 10.1103/PhysRevD.104.104007

I. INTRODUCTION

Recently, there has been a revival of interest in questioning the uniqueness of general relativity (GR) as the theory of 2 tensorial degrees of freedom (TTDOFs). The Lovelock theorem [1] is an answer to this question, which claims that GR is the unique four-dimensional theory for the spacetime metric with the second order equations of motion, which obeys the general covariance and locality. As a result, GR is a unique theory for TTDOFs if all the assumptions of Lovelock are preserved.

From the field theoretical point of view, the idea of embedding the gravitational degrees of freedom in a field theory of metric variables was also explored. By coupling a massless spin-2 field to the energy-momentum tensors of matter field(s) as well as of its own, it was arguably believed that the Einstein-Hilbert action is the unique theory one will arrive at. This approach can be traced back to the Fierz-Pauli theory [2] and was further widely developed in [3–17] (see [18] for a review).[1]

In this work, we shall examine the conditions of propagating only TTDOFs in a large class of metric theories respecting spatial covariance, which we dub the spatially covariant gravity (SCG). The SCG can be traced back to the ghost condensation [26] and was developed in the effective field theory of inflation [27,28] as well as in the Hořava gravity [29,30]. It was further generalized in [31] in which a large class of SCG theories was proposed and was extended in [32] by including the dynamical lapse function and in [33] with an auxiliary scalar field.

Theories different from GR while propagating only TTDOFs first arose in the so-called cuscuton theory [34,35] and in a subclass of Hořava gravity [36,37]. In [38] a class of SCG theories with only TTDOFs was proposed as the minimal modification of the GR (MMG). The cuscuton and MMG theories have been further extended [39–45] and their applications on cosmology and black holes have been widely studied [46–54]. A class of four-dimensional Einstein-Gauss-Bonnet gravity was also proposed recently as an arguable TTDOF theory [55].

We shall employ the framework of SCG due to the following reasons.

(a) The Lagrangians of SCG theories are automatically written in the spacetime-split form, which is convenient for analysing the time evolution and the degrees of freedom using either the equations of motion or Hamiltonian constraint analysis.
(b) The SCG can be viewed as the gauge-fixed version of the scalar-tensor theory with a single timelike scalar field. By choosing the time coordinate as the scalar field $t = t(\phi)$, which is dubbed the unitary gauge in the literature, the generally covariant single field scalar-tensor theory can be naturally recast in the form of a

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[1]Recently the "bootstrap" approach also provided new understanding of the uniqueness of GR [19–24] (see [25] for a review).

SCG theory. Therefore the SCG can be used as a "generator" of the scalar-tensor theory, especially when the higher derivatives are present. The generally covariant scalar-tensor monomials and SCG monomials have been classified and their correspondence has been investigated in [56–58].[2] This may help one to build well-behaved higher derivative scalar-tensor theories after the (re)construction of the theory of Horndeski [60] in its modern form [61–63] and the degenerate higher order derivative scalar-tensor theory [64–67] (see [68,69] and references therein for reviews).

(c) Thanks to the spacetime-splitting nature of the SCG, the construction of SCG with only a single scalar degree of freedom becomes virtually trivial. Indeed, as shown in [70–72], the SCG without the dynamical lapse function automatically propagates at most at 3 degrees of freedom. When the lapse function becomes dynamical, more conditions must be imposed [32].

Counting the number of degrees of freedom can be well performed through Dirac's Hamiltonian constraint analysis (see [73] for a comprehensive review). In the framework of SCG, there are in principle two approaches to finding the conditions of propagating at most at 2 degrees of freedom.

(a) The traditional and conservative approach is to start from the Lagrangian and perform the Legendre transformation to derive the Hamiltonian, then to find the conditions for the Lagrangian by performing the Hamiltonian constraint analysis. In [74], starting from a general local Lagrangian of SCG, conditions of propagating at most at 2 degrees of freedom have been derived. The analysis was also generalized in [75] with a dynamical lapse function.

(b) The other approach is to start with the Hamiltonian directly, and to determine the conditions for the Hamiltonian instead of the Lagrangian. Indeed, simplified structure and condition(s) are found at the level of Hamiltonian in [42] in a class of SCG theories. This approach can be made even more "trivial" by imposing additional constraint(s) in the phase space through auxiliary variable(s) [76].

Both approaches have their merits and shortcomings. For the "Lagrangian" approach, it is more convenient to deal with the *local* Lagrangian, while the conditions are functional differential equations for the Lagrangian and mathematically complicated to solve. For the "Hamiltonian" approach, one is able to determine the Hamiltonian in a simple manner, while the corresponding Lagrangian is involved and typically *nonlocal* due to the presence of extra auxiliary variables.

In view of the above considerations, in this work we employ an alternative approach to constructing the SCG with only TTDOFs. We shall deal with the Lagrangian directly and determine the conditions at the level of equations of motion. In fact, constraint analysis as well as counting the number of degrees of freedom can be equivalently performed at the level of the Lagrangian and the equations of motion [77,78].

The idea is based on the fact that if the Lagrangian propagates at most at 2 DOFs—and in particular, no scalar mode—at the nonperturbative level, the scalar mode must not show up at any finite order in the perturbative expansion around a spatially homogeneous and isotropic background. In particular, the conditions can be determined order by order in a perturbative analysis, which may be relatively easier to manage. This is also the approach in [40] to building the so-called "extended cuscuton" theory. The same idea was also employed in [79] to find conditions for the SCG Lagrangians quadratic in the extrinsic curvature and in the velocity of the lapse function to propagate at most at 3 degrees of freedom.

This work is organized as follows. In Sec. II we briefly review the spatially covariant gravity and the general conditions to have at most 2 degrees of freedom. In Sec. III we describe our perturbative approach and derive the degeneracy condition in order to eliminate the scalar mode at linear order in perturbations. In Sec. IV we use the degeneracy condition to find the conditions for the Lagrangians up to $d = 4$ and give the explicit Lagrangians for $d = 2$, $d = 3$ as well as the combination of $d = 2, 3$. In Sec. V we use the Lagrangian of $d = 2$ as an illustrative example to show how to eliminate the scalar mode at the next order in perturbations. We summarize our results in Sec. VI.

II. SPATIALLY COVARIANT GRAVITY WITH 2 DEGREES OF FREEDOM

In this section, we make a brief review of the framework of SCG theory, and in particular the classification of the SCG polynomials. We also briefly summarize the conditions of having only 2 degrees of freedom, which are derived in [74].

The action of the spatially covariant gravity theories takes the general form

$$S = \int \mathrm{d}t\mathrm{d}^3xN\sqrt{h}\mathcal{L}(t, N, h_{ij}, K_{ij}, R_{ij}, \nabla_i, \varepsilon_{ijk}), \quad (1)$$

where N is the lapse function, h_{ij} is the three-dimensional spatial metric, K_{ij} is the extrinsic curvature defined by

$$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - \pounds_{\vec{N}} h_{ij}), \quad (2)$$

with $\pounds_{\vec{N}}$ the Lie derivative with respect to the shift vector N^i, R_{ij} is the three-dimensional spatial Ricci tensor, and ∇_i is the covariant derivative compatible with the spatial metric h_{ij}. The spatial Levi-Civita tensor $\varepsilon_{ijk} = \sqrt{h}\epsilon_{ijk}$

[2] The correspondence is subtle when the unitary gauge is not accessible, see [59] for a discussion.

with $\epsilon_{123} = 1$ is allowed, thus the parity-violating terms can be constructed by those building blocks with ε_{ijk}.[3] Note in principle the lapse function N may also acquire a kinetic term through $\frac{1}{N}(\partial_t N - N^i \nabla_i N)$, which has been considered in [32]. The shift vector N_i by itself is not a genuine geometric quantity of the spacetime foliation structure, which merely encodes the gauge freedom of choosing the spatial coordinates.

The SCG Lagrangians, although by themselves respecting only the spatial diffeomorphism, can be viewed as the "gauge-fixed" version of Lagrangians respecting the full general covariance of the spacetime. Such correspondence can be made easily by the Stueckelberg mapping [57]. The basic building blocks in the corresponding generally covariant theory are the timelike vector that is proportional to the gradient of the scalar field, and the induced metric on the hypersurface of the constant scalar field. This is very similar to the construction of nonrelativistic models by localizing the Galileon symmetry and the explicit realization of the Newton-Cartan space [see (e.g.) [88–90]].

In this work, instead of analyzing a general Lagrangian as in Eq. (1), we concentrate on polynomial-type Lagrangians, which are linear combinations of the SCG monomials. The irreducible SCG monomials are exhausted and classified up to $d = 4$ in [57] with d the total number of derivatives in the monomials. Here we briefly describe the construction with improved notation following [58]. We assign each SCG model a set of integers $(c_0; d_2, d_3)$ according to their corresponding monomials in the scalar-tensor theories after the Stueckelberg mapping. Precisely, c_0 is the number of spacetime Riemann curvature tensor, d_2, d_3 are numbers of the second and third order generally covariant derivatives of the scalar field ϕ, respectively. In fact we have the simple correspondences

$$K_{ij} \sim a_i \sim (0; 1, 0), \tag{3}$$

$$R_{ij} \sim (1; 0, 0), \tag{4}$$

$$\nabla_k K_{ij} \sim \nabla_i a_j \sim (0; 0, 1), \tag{5}$$

and thus d can be expressed by

$$d = \sum_{n=0} [(n+2)c_n + (n+1)d_{n+2}]. \tag{6}$$

We thus classify the various SCG monomials with d and then with the categories labeled by $(c_0; d_2, d_3)$.

Up to $d = 4$, the Lagrangian built of the irreducible SCG monomials is

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)}, \tag{7}$$

where the parity-preserving terms are

$$\mathcal{L}^{(0)} = c_1^{(0;0,0)}, \tag{8}$$

$$\mathcal{L}^{(1)} = c_1^{(0;1,0)} K, \tag{9}$$

$$\mathcal{L}^{(2)} = c_1^{(0;2,0)} K_{ij}K^{ij} + c_2^{(0;2,0)} a_i a^i + c_3^{(0;2,0)} K^2 + c_1^{(1;0,0)} R, \tag{10}$$

$$\begin{aligned}\mathcal{L}^{(3)} = {} & c_1^{(0;3,0)} K_{ij}K^{jk}K_k^i + c_2^{(0;3,0)} K_{ij}a^i a^j + c_3^{(0;3,0)} K_{ij}K^{ij}K + c_4^{(0;3,0)} K a_i a^i + c_5^{(0;3,0)} K^3 \\ & + c_1^{(0;1,1)} K_{ij}\nabla^i a^j + c_2^{(0;1,1)} K \nabla_i a^i + c_1^{(1;1,0)} R^{ij}K_{ij} + c_2^{(1;1,0)} RK,\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{L}^{(4)} = {} & c_1^{(0;4,0)} K_{ik}K_j^k a^i a^j + c_2^{(0;4,0)} K_{ij}K^{jk}K_k^i K + c_3^{(0;4,0)} K_{ij}a^i a^j K + c_4^{(0;4,0)} (K_{ij}K^{ij})^2 + c_5^{(0;4,0)} K_{ij}K^{ij} a_k a^k \\ & + c_6^{(0;4,0)} (a_i a^i)^2 + c_7^{(0;4,0)} K_{ij}K^{ij}K^2 + c_8^{(0;4,0)} a_i a^i K^2 + c_9^{(0;4,0)} K^4 + c_1^{(0;2,1)} K_i^k K_{jk} \nabla^i a^j + c_2^{(0;2,1)} K_j^i a^j \nabla_k K_i^k \\ & + c_3^{(0;2,1)} K_j^i a^j \nabla_i K + c_4^{(0;2,1)} K_{ij}\nabla^i a^j K + c_5^{(0;2,1)} K_{ij}K^{ij}\nabla_k a^k + c_6^{(0;2,1)} a_i a^i \nabla_j a^j + c_7^{(0;2,1)} K^2 \nabla_i a^i \\ & + c_1^{(0;0,2)} \nabla_k K_{ij} \nabla^k K^{ij} + c_2^{(0;0,2)} \nabla_i K^{ij} \nabla_k K_j^k + c_3^{(0;0,2)} \nabla_i K^{ij} \nabla_j K + c_4^{(0;0,2)} \nabla_i K \nabla^i K + c_5^{(0;0,2)} \nabla_i a_j \nabla^i a^j \\ & + c_6^{(0;0,2)} (\nabla_i a^i)^2 + c_1^{(1;2,0)} R_{ij}K_k^i K^{jk} + c_2^{(1;2,0)} R_{ij}a^i a^j + c_3^{(1;2,0)} R_{ij}K^{ij}K + c_4^{(1;2,0)} RK_{ij}K^{ij} + c_5^{(1;2,0)} R a_i a^i \\ & + c_6^{(1;2,0)} RK^2 + c_1^{(2;0,0)} R_{ij}R^{ij} + c_2^{(2;0,0)} R^2 + c_1^{(1;0,1)} R \nabla_i a^i,\end{aligned} \tag{12}$$

and the parity-violating terms are

[3]The spatially covariant parity violating terms and their cosmological implications have been widely investigated, see (e.g.) [80–87].

$$\tilde{\mathcal{L}}^{(3)} = \tilde{c}_1^{(0;1,1)} \varepsilon_{ijk} K^i_l \nabla^j K^{kl}, \tag{13}$$

$$\begin{aligned}\tilde{\mathcal{L}}^{(4)} = {}& \tilde{c}_1^{(0;2,1)} \varepsilon_{ijk} K^{im} K^{jn} \nabla_m K^k_n + \tilde{c}_2^{(0;2,1)} \varepsilon_{ijk} K^{mn} K^i_m \nabla^j K^k_n \\ &+ \tilde{c}_3^{(0;2,1)} \varepsilon_{ijk} K^i_l a^j \nabla^k a^l + \tilde{c}_4^{(0;2,1)} \varepsilon_{ijk} K^i_l \nabla^j K^{kl} K \\ &+ \tilde{c}_1^{(1;2,0)} \varepsilon_{ijk} R^i_l K^{jl} a^k + \tilde{c}_1^{(1;0,1)} \varepsilon_{ijk} R^i_l \nabla^j K^{kl}. \end{aligned} \tag{14}$$

In the above, $a_i \equiv \nabla_i \ln N$ is the acceleration. The coefficients $c_m^{(c_0;d_2,d_3)}$ and $\tilde{c}_m^{(c_0;d_2,d_3)}$ are generally functions of t and N without spatial derivatives. Note the spatial derivatives of Ricci tensor, like $\nabla_i R_{jk}$, are not included in our model as they are higher order in d.

Without any specific fine-tuning of the coefficients, in [31] it has been shown through a Hamiltonian analysis that the action (1) propagates 3 DOFs in general, of which one is a scalar mode and two are tensor modes.[4] Therefore, in order to eliminate one of the 3 DOFs, in particular, the scalar mode, additional conditions must be imposed.

For the action in Eq. (1), the general conditions of propagating at most TTDOFs have been derived in [74], which can be written as

$$\mathcal{S}(\vec{x},\vec{y}) \approx 0, \qquad \mathcal{J}(\vec{x},\vec{y}) \approx 0, \tag{15}$$

where

$$\mathcal{S}(\vec{x},\vec{y}) := \frac{\delta^2 S}{\delta N(\vec{x}) \delta N(\vec{y})} - \int \mathrm{d}^3x' \int \mathrm{d}^3y' N(\vec{x}') \frac{\delta}{\delta N(\vec{x})} \left(\frac{1}{N(\vec{x}')} \frac{\delta S}{\delta K_{i'j'}(\vec{x}')} \right) \tag{16}$$

$$\times \mathcal{G}_{i'j',k'l'}(\vec{x}',\vec{y}') N(\vec{y}') \frac{\delta}{\delta N(\vec{y})} \left(\frac{1}{N(\vec{y}')} \frac{\delta S}{\delta K_{i'j'}(\vec{y}')} \right), \tag{17}$$

with $\mathcal{G}_{ij,kl}(\vec{x},\vec{y})$ the inverse of the Hessian with respect to K_{ij} satisfying

$$\int \mathrm{d}^3z \mathcal{G}_{ij,mn}(\vec{x},\vec{z}) \frac{\delta^2 S}{\delta K_{mn}(\vec{z}) \delta K_{kl}(\vec{y})} \equiv \frac{1}{2} (\delta^k_i \delta^l_j + \delta^l_i \delta^k_j) \delta^3(\vec{x}-\vec{y}), \tag{18}$$

and

$$\begin{aligned}\mathcal{J}(\vec{x},\vec{y}) := {}& \int \mathrm{d}^3x' \int \mathrm{d}^3y' \int \mathrm{d}^3x'' \int \mathrm{d}^3y'' \frac{\delta C(\vec{x})}{\delta K_{ij}(\vec{x}')} \mathcal{G}_{i'j',k'l'}(\vec{x}',\vec{x}'') N(\vec{x}'') \frac{\delta^2 S}{\delta h_{i'j'}(\vec{x}'') \delta K_{k'l'}(\vec{y}'')} \mathcal{G}_{k'l',kl}(\vec{y}'',\vec{y}') \frac{\delta C(\vec{y})}{\delta K_{ij}(\vec{y}')} \\ &- \int \mathrm{d}^3x' \int \mathrm{d}^3y' \frac{\delta C'(\vec{x})}{\delta K_{ij}(\vec{x}')} \mathcal{G}_{ij,kl}(\vec{x}',\vec{y}') N(\vec{y}') \frac{\delta C(\vec{y})}{\delta h_{kl}(\vec{y}')} - (\vec{x} \leftrightarrow \vec{y}), \end{aligned} \tag{19}$$

with

$$C(\vec{x}) := -\frac{\delta S}{\delta N(\vec{x})} + \frac{1}{N(\vec{x})} \frac{\delta S}{\delta K_{ij}(\vec{x})} K_{ij}(\vec{x}). \tag{20}$$

These two TTDOF conditions, which are dubbed the degenerate condition and the consistency condition, are the necessary and sufficient conditions for the action (1) to propagate at most at two DOFs. When the lapse function becomes dynamical, the generalized conditions have also been derived in [75].

Although these TTDOF conditions are general and conceptually simple, they are mathematically involved to be solved to yield concrete Lagrangians. This is one of the motivations of this work to look for an alternative and more practical approach.

III. THE PERTURBATIVE APPROACH AND DEGENERACY CONDITION

The unwanted degree of freedom, if not contained in the theory, will never show up at any order in perturbations around some background. Thus one may tune the coefficients in the Lagrangian such that the unwanted scalar mode is eliminated order by order in perturbations. Since there is a finite number of conditions in the nonperturbative sense, one will stop at some finite order and get the final Lagrangian in which the scalar mode is fully eliminated. The perturbative approach can be a possible candidate method to bypass the mathematical difficulties in dealing with the nonperturbative conditions gotten in a Hamiltonian analysis. For a class of SCG theories with the dynamical lapse function, this perturbative analysis has been used to

[4]In [31] only the parity-preserving Lagrangian is considered, while from the analysis it is clear that the presence of ε_{ijk} would not change the constraint structure and thus the number of DOFs.

reduce the number of DOFs from 4 to 3 [79]. It was interesting that even at the cubic order in perturbations around a cosmological background, one could reproduce the fully nonperturbative conditions to eliminate the unwanted mode.

We consider perturbations around a Friedmann-Robertson-Walker background. For our purpose, we focus on the scalar perturbations only. After fixing the gauge freedom of the spatial diffeomorphism, the usual Arnowitt-Deser-Misner variables correspond to

$$N = \bar{N} e^{A}, \tag{21}$$

$$N_i = \bar{N} a \partial_i B, \tag{22}$$

$$h_{ij} = a^2 e^{2\zeta} \delta_{ij}, \tag{23}$$

where $a = a(t)$ is the scale factor and $\bar{N} = \bar{N}(t)$ is the background value of the lapse function.

Contrary to what one usually does in generally covariant theories, here we do not set $\bar{N} = 1$, since there is no time-reparametrization symmetry in our theory in general. In particular, we assume that the Lagrangian depends on the lapse function N explicitly while not on the time. As a result, generally the lapse function N has a nonunity background value $\bar{N}(t)$. On the other hand, setting $\bar{N} = 1$ implicitly redefines the time parameter t, which reintroduces the time dependence of the Lagrangian.

The quadratic action for the scalar perturbations takes the form (we follow the notation in [79])

$$S_2[\zeta, A, B] = \int \mathrm{d}t \mathrm{d}^3x \bar{N} a^3 \mathcal{L}_2. \tag{24}$$

The quadratic Lagrangian can be split into two parts

$$\mathcal{L}_2 = \mathcal{L}_2^{(\mathrm{I})} + \mathcal{L}_2^{(\mathrm{II})}, \tag{25}$$

in which $\mathcal{L}_2^{(\mathrm{I})}$ stands for terms relevant to counting the number of DOFs,

$$\mathcal{L}_2^{(\mathrm{I})} = \dot{\zeta}\hat{\mathcal{C}}_{\dot{\zeta}\dot{\zeta}}\dot{\zeta} + \dot{\zeta}\hat{\mathcal{C}}_{\dot{\zeta}A}A + \dot{\zeta}\hat{\mathcal{C}}_{\dot{\zeta}B}B + A\hat{\mathcal{C}}_{AA}A + A\hat{\mathcal{C}}_{AB}B + B\hat{\mathcal{C}}_{BB}B, \tag{26}$$

and $\mathcal{L}_2^{(\mathrm{II})}$ stands for terms irrelevant to counting the number of DOFs,

$$\mathcal{L}_2^{(\mathrm{II})} = \zeta\hat{\mathcal{C}}_{\zeta\zeta}\zeta + \zeta\hat{\mathcal{C}}_{\zeta A}A + \zeta\hat{\mathcal{C}}_{\zeta B}B. \tag{27}$$

In the above $\hat{\mathcal{C}}_{\dot{\zeta}\dot{\zeta}}$, $\hat{\mathcal{C}}_{\dot{\zeta}A}$, etc., are time-dependent operators which may contain spatial derivatives. Following [91], throughout this paper we shall use the shorthand

$$\dot{X} \equiv \frac{1}{\bar{N}}\frac{\partial X}{\partial t}, \qquad f' \equiv \bar{N}\frac{\partial f}{\partial N}\bigg|_{N=\bar{N}}, \qquad f'' \equiv \bar{N}^2\frac{\partial^2 f}{\partial N^2}\bigg|_{N=\bar{N}}. \tag{28}$$

At this point, note the quadratic Lagrangian for the scalar modes contains no parity-violating term. In other words, the parity-violating terms in Eqs. (13) and (14) do not contribute to the quadratic order Lagrangian for the scalar modes and have nothing to do with eliminating the scalar modes at least at the linear order in perturbations. Mathematically, this is simply because it is not possible to build a term quadratic in the scalar modes with ε_{ijk}. If we go to higher order, the parity-violating terms do contribute to the scalar modes.

It is clear that in the quadratic action (24), A and B act as the auxiliary variables (i.e., without the time derivatives). We may solve A and B formally from their equations of motion

$$2\frac{1}{\bar{N}}\partial_t(a^3\hat{\mathcal{C}}_{\dot{\zeta}\dot{\zeta}}\dot{\zeta}) + \frac{1}{\bar{N}}\partial_t(a^3\hat{\mathcal{C}}_{\dot{\zeta}A}A) + \frac{1}{\bar{N}}\partial_t(a^3\hat{\mathcal{C}}_{\dot{\zeta}B}B) - 2a^3\hat{\mathcal{C}}_{\zeta\zeta}\zeta - a^3\hat{\mathcal{C}}_{\zeta A}A - a^3\hat{\mathcal{C}}_{\zeta B}B = 0, \tag{29}$$

$$2\hat{\mathcal{C}}_{AA}A + \hat{\mathcal{C}}_{AB}B + \hat{\mathcal{C}}_{\dot{\zeta}A}\dot{\zeta} + \hat{\mathcal{C}}_{\zeta A}\zeta = 0, \tag{30}$$

$$\hat{\mathcal{C}}_{AB}A + 2\hat{\mathcal{C}}_{BB}B + \hat{\mathcal{C}}_{\dot{\zeta}B}\dot{\zeta} + \hat{\mathcal{C}}_{\zeta B}\zeta = 0. \tag{31}$$

The solutions for A and B can be formally written as

$$A = \frac{(\frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{\dot{\zeta}B} - \hat{\mathcal{C}}_{BB}\hat{\mathcal{C}}_{\dot{\zeta}A})\dot{\zeta} + (\frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{\zeta B} - \hat{\mathcal{C}}_{BB}\hat{\mathcal{C}}_{\zeta A})\zeta}{2\hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{BB} - \frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{AB}} \tag{32}$$

and

$$B = \frac{(\frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{\dot{\zeta}A} - \hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{\dot{\zeta}B})\dot{\zeta} + (\frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{\zeta A} - \hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{\zeta B})\zeta}{2\hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{BB} - \frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{AB}}, \tag{33}$$

with $2\hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{BB} - \frac{1}{2}\hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{AB} \neq 0$. Since $\hat{\mathcal{C}}_{\dot{\zeta}\dot{\zeta}}$, $\hat{\mathcal{C}}_{\dot{\zeta}A}$, etc., may contain spatial derivatives, the above solutions may be better understood in the Fourier space. Plugging the above solutions into Eq. (29) yields the equation of motion for the single variable ζ. If in the equation of motion ζ acquires a second derivative term $\ddot{\zeta}$, ζ is dynamical. Therefore in order to have no scalar mode propagating at the linear order, we have to "kill" the coefficient of $\ddot{\zeta}$ in its equation of motion. After some manipulations, one find that this implies

$$\Delta \coloneqq 2\hat{\mathcal{C}}_{\dot{\zeta}\dot{\zeta}}\left(2\hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{BB} - \frac{1}{2}\hat{\mathcal{C}}_{AB}^2\right) + \hat{\mathcal{C}}_{AB}\hat{\mathcal{C}}_{\dot{\zeta}A}\hat{\mathcal{C}}_{\dot{\zeta}B} - \hat{\mathcal{C}}_{BB}\hat{\mathcal{C}}_{\dot{\zeta}A}^2 - \hat{\mathcal{C}}_{AA}\hat{\mathcal{C}}_{\dot{\zeta}B}^2 = 0. \tag{34}$$

We may refer to Eq. (34) as the degeneracy condition. The main task in this work is thus to use Eq. (34) as our starting point to find the conditions for the various coefficients $c_m^{(c_0;d_2,d_3)}$ and $\tilde{c}_m^{(c_0;d_2,d_3)}$ such that no scalar mode is propagating at the linear order in perturbations.

In the above, we derive the degeneracy condition (24) by solving the auxiliary variables and looking at the coefficient of the kinetic term in the effective Lagrangian for the single variable ζ, which is the standard operation in calculating the cosmological perturbations. We emphasize that it is not trivial to count the number of dynamical degree of freedom even for the quadratic Lagrangian in Eq. (24). In the Appendix, we make a thorough analysis of a point particle model.

IV. ELIMINATE THE SCALAR MODE AT THE LINEAR ORDER

In this section, we shall find the conditions for the coefficients in the Lagrangian such that the degeneracy condition (24) is satisfied, and thus no scalar mode propagates at the linear order in perturbations.

A. $d = 2$

We consider the model constructed by all the terms of $d = 2$:

$$S = \int \mathrm{d}t\mathrm{d}^3 x N\sqrt{h}(\mathcal{L}^{(2)} - \Lambda), \tag{35}$$

where $\mathcal{L}^{(2)}$ is given in Eq. (10). We have introduced a positive cosmological constant $\Lambda > 0$ in order to have an expanding background solution. Equivalently, the above Lagrangian can be regarded as the linear combination of $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(2)}$ with $c_1^{(0;0,0)} = \Lambda$.

Expanding the action to the first order in perturbations yields

$$S_1 = \int \mathrm{d}t\mathrm{d}^3 x \mathcal{L}_1, \tag{36}$$

with

$$\begin{aligned}\mathcal{L}_1 \simeq{}& \bar{N}a^3[-3H^2(b_2 - b_2') - \Lambda]A \\ &+ 3\bar{N}a^3[-(3H^2 + 2\dot{H})b_2 - \Lambda - 2H\dot{b}_2]\zeta,\end{aligned} \tag{37}$$

where we define

$$b_2 \equiv c_1^{(0,2,0)} + 3c_3^{(0,2,0)}, \tag{38}$$

for short. The background equations of motion are determined by requiring $S_1 = 0$, which are

$$-3H^2(b_2 - b_2') - \Lambda = 0, \tag{39}$$

$$-(3H^2 + 2\dot{H})b_2 - \Lambda - 2H\dot{b}_2 = 0, \tag{40}$$

for A and ζ, respectively. At this point, keep in mind that $\dot{f}$ and f' are defined as in Eq. (28). The Hubble parameter is defined to be $H \coloneqq \frac{\dot{a}}{a} \equiv \frac{1}{\bar{N}a}\frac{\partial a}{\partial t}$. From Eq. (39) it is transparent that we get a generally expanding background only with a nonvanishing cosmological constant Λ.

Expanding the action to the second order in perturbations yields

$$S_2 = \int \mathrm{d}t\mathrm{d}^3 x\bar{N}a^3\mathcal{L}_2. \tag{41}$$

According to Eq. (24), the relevant coefficients are

$$\begin{aligned}\hat{\mathcal{C}}^{(2)}_{\dot{\zeta}\dot{\zeta}} &= 3b_2, \\ \hat{\mathcal{C}}^{(2)}_{\dot{\zeta}A} &= 6H(b_2' - b_2),\end{aligned} \tag{42}$$

$$\hat{\mathcal{C}}^{(2)}_{\dot{\zeta}B} = -2b_2\frac{\partial^2}{a}, \tag{43}$$

$$\hat{\mathcal{C}}^{(2)}_{AA} = \frac{3}{2}H^2(b_2'' - 2b_2' + 2b_2) - \tilde{b}_2\frac{\partial^2}{a^2}, \tag{44}$$

$$\hat{\mathcal{C}}^{(2)}_{AB} = 2H(-b_2' + b_2)\frac{\partial^2}{a}, \tag{45}$$

$$\hat{\mathcal{C}}^{(2)}_{BB} = \frac{1}{3}(2w_2 + b_2)\frac{\partial^4}{a^2}, \tag{46}$$

where we denote

$$\tilde{b}_2 \coloneqq c_2^{(0,2,0)} \tag{47}$$

and

$$w_2 \coloneqq c_1^{(0,2,0)}, \tag{48}$$

for short. For later convenience, we also have

$$\hat{\mathcal{C}}^{(2)}_{\zeta A} = -4(h_2' + h_2)\frac{\partial^2}{a^2}, \tag{49}$$

with

$$h_2 \coloneqq c_1^{(1,0,0)}, \tag{50}$$

for short, and

$$\hat{\mathcal{C}}^{(2)}_{\zeta B} = 0. \tag{51}$$

In the above we have made use of the background equations of motion (39) and (40) to eliminate Λ and to simplify the expressions of the coefficients. From Eqs. (32) and (33), the solutions for A and B are

$$A = \frac{4[3a^2H(b_2 - b_2')w_2\dot{\zeta} + (h_2 + h_2')(b_2 + 2w_2)\partial^2\zeta]}{3a^2H^2[-2(b_2')^2 + b_2(2b_2' + b_2'') + 2(2b_2 - 2b_2' + b_2'')w_2] - 2\tilde{b}_2(b_2 + 2w_2)\partial^2} \tag{52}$$

and

$$B = \frac{3a\{4H(b_2 - b_2')(h_2 + h_2')\partial^2\zeta - 3a^2H^2(-2(b_2')^2 + b_2(2b_2' + b_2''))\dot{\zeta} + 2b_2\tilde{b}_2\partial^2\dot{\zeta}\}}{\{-3a^2H^2[-2(b_2')^2 + b_2(2b_2' + b_2'') + 2(2b_2 - 2b_2' + b_2'')w_2] + 2\tilde{b}_2(b_2 + 2w_2)\partial^2\}\partial^2}. \tag{53}$$

By plugging the above results into Eq. (34), we get the degeneracy condition

$$\Delta^{(2)} = 12H^2w_2[-2(b_2')^2 + b_2(2b_2' + b_2'')]\frac{\partial^4}{a^2} - 8b_2\tilde{b}_2w_2\frac{\partial^6}{a^4}. \tag{54}$$

At this point, note we need to require that $w_2 \neq 0$, otherwise there will be no kinetic term for the gravitational waves [87]. In order to have $\Delta^{(2)} = 0$ so that there is no scalar mode propagating, one special solution is

$$b_2 \equiv c_1^{(0,2,0)} + 3c_3^{(0,2,0)} = 0. \tag{55}$$

However, this choice is conflict with the background equation of motion (39). Therefore we must have $b_2 \neq 0$, which is also the case of GR. Then first we have to require that

$$\tilde{b}_2 \equiv c_2^{(0,2,0)} = 0, \tag{56}$$

so that the $\sim\partial^6$ term in Eq. (54) vanishes, since we have assumed $w_2 \neq 0$. This indicates that the acceleration a_i should not appear explicitly. We also need to require

$$-2(b_2')^2 + b_2(2b_2' + b_2'') = 0. \tag{57}$$

Note Eq. (57) must hold for any value of $\bar{N}$. Therefore Eq. (57) is regarded as a homogeneous differential equation for b_2 as a function of N, in which b_2' and b_2'' are defined as in Eq. (28). For later convenience, the solutions for A and B (52) in Eq. (53) under the conditions (56) and (57) get simplified to

$$A = \frac{b_2}{H(b_2 - b_2')}\dot{\zeta} + \frac{b_2(h_2 + h_2')(b_2 + 2w_2)}{3a^2H^2(b_2 - b_2')^2w_2}\partial^2\zeta \tag{58}$$

and

$$B = -\frac{b_2(h_2 + h_2')}{aH(b_2 - b_2')w_2}\zeta, \tag{59}$$

which involves no $\dot{\zeta}$. In Eqs. (58) and (59) we have made use of Eq. (57) to replace b_2'' in terms of b_2 and b_2'.

The general solution for b_2 to Eq. (57) is

$$b_2 = \frac{C_1N}{1 + C_2N}, \tag{60}$$

where C_1, C_2 are two constants. This solution is also consistent with the analysis in [74] [see Eqs. (110) and (111) therein]. Obviously, $b_2 = \text{const}$ is a trivial solution, which corresponds to the limit $C_1, C_2 \to \infty$ by keeping $\frac{C_1}{C_2}$ finite. To conclude, the Lagrangian

$$\mathcal{L}^{(2)} = w_2\hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_1N}{1 + C_2N}K^2 + h_2R, \tag{61}$$

with w_2 and h_2 being general functions of N, contains no dynamical scalar degrees of freedom at linear order in a cosmological background. Here $\hat{K}_{ij}$ is the traceless part of K_{ij} defined by

$$\hat{K}_{ij} \coloneqq K_{ij} - \frac{1}{3}Kh_{ij}. \tag{62}$$

With the form of the Lagrangian (61), GR is a special case with the choice

$$c_1^{(0;2,0)} = c_1^{(1;0,0)} = 1, \tag{63}$$

and thus corresponds to

$$|C_1|, |C_2| \to \infty, \qquad \text{keeping} \qquad \frac{C_1}{C_2} = -2. \tag{64}$$

B. $d = 3$

Next we consider

$$S = \int \mathrm{d}t\mathrm{d}^3xN\sqrt{h}(\mathcal{L}^{(3)} - \Lambda), \tag{65}$$

where $\mathcal{L}^{(3)}$ is given in Eq. (11), and again we include a positive cosmological constant Λ in order to have a nonvanishing H. Expanding the action to the first order in perturbations yields

$$S_1 = \int \mathrm{d}t\mathrm{d}^3x\mathcal{L}_1, \tag{66}$$

with

$$\mathcal{L}_1 = \bar{N}a^3[3H^3(b_3' - 2b_3) - \Lambda]A + 3\bar{N}a^3[-6H(H^2+\dot{H})b_3 - \Lambda - 3H^2\dot{b}_3]\zeta, \quad (67)$$

where we define

$$b_3 \equiv c_1^{(0,3,0)} + 3c_3^{(0,3,0)} + 9c_5^{(0,3,0)}, \quad (68)$$

for short. Thus the background equations of motion are

$$-3H^3(2b_3 - b_3') - \Lambda = 0, \quad (69)$$

$$-6H(H^2+\dot{H})b_3 - \Lambda - 3H^2\dot{b}_3 = 0, \quad (70)$$

for A and ζ, respectively. Note we must have $b_3 \neq 0$ in order to make Eq. (69) consistent with a nonvanishing Λ.

Expanding the action to the second order in perturbations yields

$$S_2 = \int \mathrm{d}t\mathrm{d}^3x\bar{N}a^3\mathcal{L}_2. \quad (71)$$

According to Eq. (24), the relevant coefficients are

$$\hat{\mathcal{C}}^{(3)}_{\dot{\zeta}\dot{\zeta}} = 9Hb_3, \quad (72)$$

$$\hat{\mathcal{C}}^{(3)}_{\dot{\zeta}A} = -9H^2(-b_3' + 2b_3) + f_3\frac{\partial^2}{a^2}, \quad (73)$$

$$\hat{\mathcal{C}}^{(3)}_{\dot{\zeta}B} = -6Hb_3\frac{\partial^2}{a}, \quad (74)$$

$$\hat{\mathcal{C}}^{(3)}_{AA} = \frac{3}{2}H^3(b_3'' - 4b_3' + 6b_3) + H(f_3' - \tilde{b}_3)\frac{\partial^2}{a^2}, \quad (75)$$

$$\hat{\mathcal{C}}^{(3)}_{AB} = 3H^2(-b_3' + 2b_3)\frac{\partial^2}{a} - \tilde{f}_3\frac{\partial^4}{a^3}, \quad (76)$$

$$\hat{\mathcal{C}}^{(3)}_{BB} = H(2w_3 + b_3)\frac{\partial^4}{a^2}, \quad (77)$$

where we denote

$$f_3 \coloneqq c_1^{(0,1,1)} + 3c_2^{(0,1,1)}, \quad (78)$$

$$\tilde{b}_3 \coloneqq c_2^{(0,3,0)} + 3c_4^{(0,3,0)}, \quad (79)$$

$$\tilde{f}_3 \coloneqq c_1^{(0,1,1)} + c_2^{(0,1,1)}, \quad (80)$$

$$w_3 \coloneqq c_1^{(0,3,0)} + c_3^{(0,3,0)}, \quad (81)$$

as shorthand. We have made use of the background equations of motion (69) and (70) to simply the coefficients.

After some manipulations, the degeneracy condition (34) now becomes

$$\Delta^{(3)} = 54H^5w_3[-3(b_3')^2 + 2b_3(2b_3' + b_3'')]\frac{\partial^4}{a^2} + 36H^3w_3[-b_3'f_3 + 2b_3(f_3 + f_3' - \tilde{b}_3)]\frac{\partial^6}{a^4} - H[b_3(f_3 - 3\tilde{f}_3)^2 + 2(f_3)^2w_3]\frac{\partial^8}{a^6}. \quad (82)$$

Similar to the case of $d = 2$, we require $w_3 \neq 0$, otherwise there will be no gravitational waves [87]. Thus the degeneracy condition $\Delta^{(3)} = 0$ yields a set of three equations,

$$-3(b_3')^2 + 2b_3(2b_3' + b_3'') = 0, \quad (83)$$

$$-b_3'f_3 + 2b_3(f_3 + f_3' - \tilde{b}_3) = 0, \quad (84)$$

$$b_3(f_3 - 3\tilde{f}_3)^2 + 2(f_3)^2w_3 = 0. \quad (85)$$

Recall that there are nine free coefficients in the original Lagrangian $\mathcal{L}^{(3)}$ (11), which are subject to the above three equations in order to eliminate the scalar degree of freedom at the linear order. We can solve b_3 from Eq. (83) to be

$$b_3 = \frac{D_1N^2}{(1 + D_2N)^2}, \quad (86)$$

with D_1, D_2 being two constants. At this point, note that in order to make the background equation of motion (69) consistent, which now reads

$$6H^3\frac{D_1D_2N^3}{(1 + D_2N)^3} + \Lambda = 0, \quad (87)$$

we have to require that (since $N > 0$)

$$\frac{D_1D_2}{(1 + D_2N)^3} < 0. \quad (88)$$

By using the solution (86), we then solve $\tilde{b}_3$ from Eq. (84) to be

$$\tilde{b}_3 = \frac{D_2N}{1 + D_2N}f_3 + f_3'. \quad (89)$$

As for Eq. (85), according to whether f_3 is vanishing or not, we discuss two cases.

1. Case 1

If $f_3 = 0$, since we assume $b_3 \neq 0$, from Eq. (85) we must also have $\tilde{f}_3 = 0$, which implies that

$$c_1^{(0;1,1)} = 0, \qquad c_2^{(0;1,1)} = 0, \quad (90)$$

and w_3 can be chosen freely (but nonvanishing) in general. In this case, there is no ∇a term in the Lagrangian. As a result, $\tilde{b}_3 = 0$ and thus we may solve

$$c_4^{(0;3,0)} = -\frac{1}{3} c_2^{(0;3,0)}. \tag{91}$$

In this case, the Lagrangian is given by

$$\mathcal{L}^{(3),\mathrm{I}} = c_1^{(0;3,0)} \hat{K}_{ij} \hat{K}^{jk} \hat{K}^i_k + w_3 \hat{K}_{ij} \hat{K}^{ij} K + \frac{1}{9} \frac{D_1 N^2}{(1+D_2 N)^2} K^3 + c_2^{(0;3,0)} \hat{K}_{ij} a^i a^j + c_1^{(1;1,0)} R^{ij} K_{ij} + c_2^{(1;1,0)} R K, \tag{92}$$

where the coefficients $c_1^{(0;3,0)}$, w_3, etc., are generally functions of N.

2. Case 2

If $f_3 \neq 0$, we can solve w_3 or more conveniently $c_3^{(0;3,0)}$ from Eq. (85) to be

$$c_3^{(0;3,0)} = -c_1^{(0;3,0)} - 2 \frac{D_1 N^2}{(1+D_2 N)^2} \left(\frac{c_1^{(0;1,1)}}{f_3} \right)^2. \tag{93}$$

As a result, by making use of Eqs. (68) and (86), we may solve

$$c_5^{(0;3,0)} = \frac{2}{9} c_1^{(0;3,0)} + \frac{2}{3} \frac{D_1 N^2}{(1+D_2 N)^2} \left(\frac{c_1^{(0;1,1)}}{f_3} \right)^2 + \frac{1}{9} \frac{D_1 N^2}{(1+D_2 N)^2}. \tag{94}$$

In this case the Lagrangian reduces to

$$\begin{aligned} \mathcal{L}^{(3),\mathrm{II}} = {} & c_1^{(0;3,0)} \hat{K}_{ij} \hat{K}^{jk} \hat{K}^i_k + \frac{1}{9} \frac{D_1 N^2}{(1+D_2 N)^2} K^3 \\ & - 2 \frac{D_1 N^2}{(1+D_2 N)^2} \left(\frac{c_1^{(0;1,1)}}{f_3} \right)^2 \hat{K}_{ij} \hat{K}^{ij} K \\ & + c_2^{(0;3,0)} \hat{K}_{ij} a^i a^j + \frac{1}{3} \left(\frac{D_2 N}{1+D_2 N} f_3 + f_3' \right) K a_i a^i \\ & + c_1^{(0;1,1)} \hat{K}_{ij} \nabla^i a^j + \frac{1}{3} f_3 K \nabla_i a^i \\ & + c_1^{(1;1,0)} R^{ij} K_{ij} + c_2^{(1;1,0)} R K, \end{aligned} \tag{95}$$

which contains the spatial derivative terms of the acceleration ∇a.

We thus conclude that Eqs. (92) and (95) are two viable Lagrangians that do not propagate any scalar modes at the linear order in a cosmological background.

C. $d = 2$ with $d = 3$

In the above we have determined the viable Lagrangians when only $d = 2$ or $d = 3$ terms are present. It should not be surprising that although the scalar mode is eliminated for $d = 2$ and $d = 3$ individually, the scalar mode will reappear if we naively combine them. This also happens in the investigation of degenerate higher order scalar-tensor theories [66,68,92]. Fortunately in our case, viable Lagrangians with the combination of $d = 2$ and $d = 3$ terms do exist, after imposing additional conditions on the coefficients.

We consider the combined Lagrangian

$$S = \int \mathrm{d}t \mathrm{d}^3 x N \sqrt{h} (\mathcal{L}^{(2)} + \mathcal{L}^{(3)} - \Lambda), \tag{96}$$

in which $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)}$ are given in Eqs. (10) and (11), respectively.

The analysis is completely parallel to the above. Expanding the action to the first order in perturbations yields

$$S_1 = \int \mathrm{d}t \mathrm{d}^3 x \mathcal{L}_1, \tag{97}$$

where it follows from Eqs. (37) and (67) that

$$\begin{aligned} \mathcal{L}_1 = {} & \bar{N} a^3 [-3H^2 (b_2 - b_2') + 3H^3 (b_3' - 2b_3) - \Lambda] A \\ & + 3\bar{N} a^3 [-(3H^2 + 2\dot{H}) b_2 - 2H\dot{b}_2 \\ & - 6H(H^2 + \dot{H}) b_3 - \Lambda - 3H^2 \dot{b}_3] \zeta. \end{aligned} \tag{98}$$

The coefficients in the quadratic Lagrangian for the scalar modes read

$$\begin{aligned} \hat{\mathcal{C}}^{(2)+(3)}_{\dot{\zeta}\dot{\zeta}} &= \hat{\mathcal{C}}^{(2)}_{\dot{\zeta}\dot{\zeta}} + \hat{\mathcal{C}}^{(3)}_{\dot{\zeta}\dot{\zeta}}, \\ \hat{\mathcal{C}}^{(2)+(3)}_{\dot{\zeta}A} &= \hat{\mathcal{C}}^{(2)}_{\dot{\zeta}A} + \hat{\mathcal{C}}^{(3)}_{\dot{\zeta}A}, \end{aligned} \tag{99}$$

$$\hat{\mathcal{C}}^{(2)+(3)}_{\dot{\zeta}B} = \hat{\mathcal{C}}^{(2)}_{\dot{\zeta}B} + \hat{\mathcal{C}}^{(3)}_{\dot{\zeta}B}, \tag{100}$$

$$\hat{\mathcal{C}}^{(2)+(3)}_{AA} = \hat{\mathcal{C}}^{(2)}_{AA} + \hat{\mathcal{C}}^{(3)}_{AA}, \tag{101}$$

$$\hat{\mathcal{C}}^{(2)+(3)}_{AB} = \hat{\mathcal{C}}^{(2)}_{AB} + \hat{\mathcal{C}}^{(3)}_{AB}, \tag{102}$$

$$\hat{\mathcal{C}}^{(2)+(3)}_{BB} = \hat{\mathcal{C}}^{(2)}_{BB} + \hat{\mathcal{C}}^{(3)}_{BB}. \tag{103}$$

The degeneracy condition thus becomes

$$\begin{aligned}\Delta^{(2)+(3)} = & \{12H^2[-2(b_2')^2 + b_2(2b_2' + b_2'')]w_2 + 54H^5[-3(b_3')^2 + 2b_3(2b_3' + b_3'')]w_3 \\ & + H^3[12(3b_3(b_2'' + 2b_2') + b_2(2b_3' + b_3'') - 6b_2'b_3')w_2 + 36(-2(b_2')^2 + b_2(2b_2' + b_2''))w_3] \\ & + H^4[18(-3(b_3')^2 + 2b_3(2b_3' + b_3''))w_2 + 36(3b_3(b_2'' + 2b_2') + b_2(2b_3' + b_3'') - 6b_2'b_3')w_3]\}\frac{\partial^4}{a^2} \\ & + \{-8b_2\tilde{b}_2w_2 + 36H^3[-b_3'f_3 + 2b_3(f_3 + f_3' - \tilde{b}_3)]w_3 \\ & + H[8(-b_2'f_3 - 3b_3\tilde{b}_2 + b_2(f_3 + f_3' - \tilde{b}_3))w_2 - 24b_2\tilde{b}_2w_3] \\ & + H^2[12(-b_3'f_3 + 2b_3(f_3 + f_3' - \tilde{b}_3))w_2 + 24(-b_2'f_3 - 3b_3\tilde{b}_2 + b_2(f_3 + f_3' - \tilde{b}_3))w_3]\}\frac{\partial^6}{a^4} \\ & + \left\{-\frac{1}{3}(b_2(f_3 - 3\tilde{f}_3)^2 + 2(f_3)^2w_2) - H[b_3(f_3 - 3\tilde{f}_3)^2 + 2(f_3)^2w_3]\right\}\frac{\partial^8}{a^6}. \end{aligned} \tag{104}$$

Note we have used the background equations of motion to simply the expressions for the coefficients. We require that the degeneracy condition should be satisfied for any power of ∂ and H, thus we get the set of constraints

$$-2(b_2')^2 + b_2(2b_2' + b_2'') = 0, \tag{105}$$

$$-3(b_3')^2 + 2b_3(2b_3' + b_3'') = 0, \tag{106}$$

$$3b_3(b_2'' + 2b_2') + b_2(2b_3' + b_3'') - 6b_2'b_3' = 0, \tag{107}$$

$$\tilde{b}_2 = 0, \tag{108}$$

$$-b_3'f_3 + 2b_3(f_3 + f_3' - \tilde{b}_3) = 0, \tag{109}$$

$$-b_2'f_3 - 3b_3\tilde{b}_2 + b_2(f_3 + f_3' - \tilde{b}_3) = 0, \tag{110}$$

$$b_2(f_3 - 3\tilde{f}_3)^2 + 2(f_3)^2w_2 = 0, \tag{111}$$

$$b_3(f_3 - 3\tilde{f}_3)^2 + 2(f_3)^2w_3 = 0. \tag{112}$$

Compared with the constraints in $d = 2$ and $d = 3$, respectively, we find that additional constraints should be imposed in the combined case to ensure the degeneracy.

First of all, Eqs. (105) and (106) are exactly Eqs. (57) and (83) in the cases of $d = 2$ and $d = 3$ individually. The general solutions for b_2 and b_3 are given in Eqs. (60) and (86), respectively, from which we may solve

$$c_3^{(0,2,0)} = -\frac{1}{3}c_1^{(0,2,0)} + \frac{1}{3}\frac{C_1N}{1 + C_2N} \tag{113}$$

and

$$c_5^{(0;3,0)} = -\frac{1}{9}c_1^{(0;3,0)} - \frac{1}{3}c_3^{(0;3,0)} + \frac{1}{9}\frac{D_1N^2}{(1 + D_2N)^2}. \tag{114}$$

If Eqs. (105) and (106) are both satisfied, Eq. (107) reduces to

$$\frac{3}{2b_2b_3}(2b_3b_2' - b_2b_3')^2 = 0, \tag{115}$$

which yields a constraint between b_2 and b_3. By plugging in the solutions (60) and (86), we may solve

$$C_2 = D_2, \tag{116}$$

which implies that b_3 is determined by b_2 through

$$b_3 = \frac{D_1}{C_1^2}b_2^2. \tag{117}$$

As a result, Eq. (114) reduces to

$$c_5^{(0;3,0)} = -\frac{1}{9}c_1^{(0;3,0)} - \frac{1}{3}c_3^{(0;3,0)} + \frac{1}{9}\frac{D_1N^2}{(1 + C_2N)^2}. \tag{118}$$

Equation (108) implies

$$c_2^{(0,2,0)} = 0. \tag{119}$$

With Eq. (117), the left-hand side of Eq. (109) becomes

$$\text{lhs} = 2\frac{D_1}{C_1^2}b_2[-b_2'f_3 + b_2(f_3 + f_3' - \tilde{b}_3)], \tag{120}$$

while due to Eq. (108), the left-hand side of Eq. (110) becomes

$$\text{lhs} \equiv -b_2'f_3 + b_2(f_3 + f_3' - \tilde{b}_3), \tag{121}$$

which is proportional to Eq. (109). Generally, we look for solutions with $b_2 \neq 0$ (since GR belongs to the case), thus Eqs. (109) and (110) are satisfied only if

$$-b_2' f_3 + b_2(f_3 + f_3' - \tilde{b}_3) = 0. \tag{122}$$

We thus solve $\tilde{b}_3$ as

$$\tilde{b}_3 = \frac{C_2 N}{1 + C_2 N} f_3 + f_3',$$

which also implies

$$c_4^{(0;3,0)} = -\frac{1}{3} c_2^{(0;3,0)} + \frac{1}{3}\left(\frac{C_2 N}{1 + C_2 N} f_3 + f_3'\right). \tag{123}$$

In order to make Eqs. (111) and (112) have solutions, f_3 and $f_3 - 3\tilde{f}_3$ must be either nonvanishing or vanishing simultaneously. Therefore we have the following two cases.

1. Case 1

If $f_3 = 0$ and $f_3 - 3\tilde{f}_3 = 0$ (and thus $\tilde{f}_3 = 0$), Eqs. (111) and (112) are both automatically satisfied. There is no restriction on b_2, b_3, w_2, w_3. In this case, since $f_3 = \tilde{f}_3 = 0$ we have

$$c_1^{(0;1,1)} = 0, \qquad c_2^{(0;1,1)} = 0. \tag{124}$$

As a result, Eq. (123) reduces to

$$c_4^{(0;3,0)} = -\frac{1}{3} c_2^{(0;3,0)}. \tag{125}$$

In this case, the Lagrangian is given by

$$\begin{aligned}\mathcal{L}^{(2)+(3),\mathrm{I}} = {} & c_1^{(0;2,0)} \hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_1 N}{1 + C_2 N} K^2 + c_1^{(1;0,0)} R \\ & + c_1^{(0;3,0)} \hat{K}_{ij}\hat{K}^{jk}\hat{K}^i_k + w_3 \hat{K}_{ij}\hat{K}^{ij} K \\ & + \frac{1}{9}\frac{D_1 N^2}{(1 + C_2 N)^2} K^3 + c_2^{(0;3,0)} \hat{K}_{ij} a^i a^j \\ & + c_1^{(1;1,0)} R^{ij} K_{ij} + c_2^{(1;1,0)} R K,\end{aligned} \tag{126}$$

which contains no spatial derivative of the acceleration ∇a.

2. Case 2

If $f_3 \neq 0$ and $f_3 - 3\tilde{f}_3 \neq 0$, from Eq. (111) we solve

$$w_2 = -\frac{1}{2} b_2 \left(1 - 3\frac{\tilde{f}_3}{f_3}\right)^2, \tag{127}$$

i.e.,

$$c_1^{(0,2,0)} = -2\frac{C_1 N}{1 + C_2 N}\left(\frac{c_1^{(0;1,1)}}{f_3}\right)^2, \tag{128}$$

and from Eq. (112) we solve

$$w_3 = -\frac{1}{2} b_3 \left(1 - 3\frac{\tilde{f}_3}{f_3}\right)^2, \tag{129}$$

i.e.,

$$c_3^{(0;3,0)} = -c_1^{(0;3,0)} - 2\frac{D_1 N^2}{(1 + C_2 N)^2}\left(\frac{c_1^{(0;1,1)}}{f_3}\right)^2. \tag{130}$$

As a result, Eq. (118) reduces to

$$c_5^{(0;3,0)} = \frac{2}{9} c_1^{(0;3,0)} + \frac{1}{9}\frac{D_1 N^2}{(1 + C_2 N)^2}\left[1 + 6\left(\frac{c_1^{(0;1,1)}}{f_3}\right)^2\right]. \tag{131}$$

In this case, the Lagrangian is given by

$$\begin{aligned}\mathcal{L}^{(2)+(3),\mathrm{II}} = {} & -2\frac{C_1 N}{1 + C_2 N}\left(\frac{c_1^{(0;1,1)}}{f_3}\right)^2 \hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_1 N}{1 + C_2 N} K^2 + c_1^{(1;0,0)} R \\ & + c_1^{(0;3,0)} \hat{K}_{ij}\hat{K}^{jk}\hat{K}^i_k - 2\frac{D_1 N^2}{(1 + C_2 N)^2}\left(\frac{c_1^{(0;1,1)}}{f_3}\right)^2 \hat{K}_{ij}\hat{K}^{ij} K + \frac{1}{9}\frac{D_1 N^2}{(1 + C_2 N)^2} K^3 \\ & + c_2^{(0;3,0)} \hat{K}_{ij} a^i a^j + \frac{1}{3}\left(\frac{C_2 N}{1 + C_2 N} f_3 + f_3'\right) K a_i a^i + c_1^{(0;1,1)} \hat{K}_{ij} \nabla^i a^j + \frac{1}{3} f_3 K \nabla_i a^i \\ & + c_1^{(1;1,0)} R^{ij} K_{ij} + c_2^{(1;1,0)} R K,\end{aligned} \tag{132}$$

which contains spatial derivatives of the acceleration ∇a.

We conclude that Eqs. (126) and (132) are two viable Lagrangians in which $d=2$ and $d=3$ terms are both present, and propagate no scalar mode at the linear order in perturbations around a cosmological background.

D. $d=4$

Now we consider the most involved case of $d=4$. The action is

$$S=\int \mathrm{d}t\mathrm{d}^3xN\sqrt{h}(\mathcal{L}^{(4)}-\Lambda), \tag{133}$$

with $\mathcal{L}^{(4)}$ given in Eq. (12). Expanding the action to the first order in perturbations yields

$$S_1=\int \mathrm{d}t\mathrm{d}^3x\mathcal{L}_1, \tag{134}$$

with

$$\begin{aligned}\mathcal{L}_1=&\bar{N}a^3[-9H^4(-b_4'+3b_4)-\Lambda]A\\&+3\bar{N}a^3[-9H^2(3H^2+4\dot{H})b_4-\Lambda-12H^3\dot{b}_4]\zeta,\end{aligned} \tag{135}$$

where we denote

$$b_4\coloneqq c_2^{(0,4,0)}+c_4^{(0,4,0)}+3c_7^{(0,4,0)}+9c_9^{(0,4,0)}, \tag{136}$$

for short. The background equations of motion are

$$-9H^4(-b_4'+3b_4)-\Lambda=0, \tag{137}$$

$$-9H^2(3H^2+4\dot{H})b_4-\Lambda-12H^3\dot{b}_4=0, \tag{138}$$

for A and ζ, respectively.

The relevant coefficients in the quadratic Lagrangian for the scalar modes are

$$\hat{\mathcal{C}}^{(4)}_{\zeta\zeta}=54H^2b_4-d_4\frac{\partial^2}{a^2}, \tag{139}$$

$$\hat{\mathcal{C}}^{(4)}_{\zeta A}=-36H^3(-b_4'+3b_4)+H(2d_4+2f_4-\tilde{f}_4)\frac{\partial^2}{a^2}, \tag{140}$$

$$\hat{\mathcal{C}}^{(4)}_{\zeta B}=-36H^2b_4\frac{\partial^2}{a}+2\tilde{d}_4\frac{\partial^4}{a^3}, \tag{141}$$

$$\begin{aligned}\hat{\mathcal{C}}^{(4)}_{AA}=&\frac{9}{2}H^4(b_4''-6b_4'+12b_4)\\&+H^2(f_4'-d_4-\tilde{b}_4-f_4+\tilde{f}_4)\frac{\partial^2}{a^2}+\bar{d}_4\frac{\partial^4}{a^4},\end{aligned} \tag{142}$$

$$\hat{\mathcal{C}}^{(4)}_{AB}=12H^3(-b_4'+3b_4)\frac{\partial^2}{a}+H(-2\tilde{d}_4-\hat{f}_4)\frac{\partial^4}{a^3}, \tag{143}$$

$$\hat{\mathcal{C}}^{(4)}_{BB}=2H^2(w_4+3b_4)\frac{\partial^4}{a^2}-\hat{d}_4\frac{\partial^6}{a^4}, \tag{144}$$

where we denote

$$d_4\coloneqq 3c_1^{(0,0,2)}+c_2^{(0,0,2)}+3c_3^{(0,0,2)}+9c_4^{(0,0,2)}, \tag{145}$$

$$f_4\coloneqq c_1^{(0,2,1)}+3c_4^{(0,2,1)}+3c_5^{(0,2,1)}+9c_7^{(0,2,1)}, \tag{146}$$

$$\tilde{f}_4\coloneqq c_2^{(0,2,1)}+3c_3^{(0,2,1)}, \tag{147}$$

$$\tilde{d}_4\coloneqq c_1^{(0,0,2)}+c_2^{(0,0,2)}+2c_3^{(0,0,2)}+3c_4^{(0,0,2)}. \tag{148}$$

$$\tilde{b}_4\coloneqq c_1^{(0,4,0)}+3c_3^{(0,4,0)}+3c_5^{(0,4,0)}+9c_8^{(0,4,0)}, \tag{149}$$

$$\bar{d}_4\coloneqq c_5^{(0,0,2)}+c_6^{(0,0,2)}, \tag{150}$$

$$\begin{aligned}\hat{f}_4\coloneqq &2c_1^{(0,2,1)}-c_2^{(0,2,1)}-c_3^{(0,2,1)}+4c_4^{(0,2,1)}\\&+2c_5^{(0,2,1)}+6c_7^{(0,2,1)},\end{aligned} \tag{151}$$

$$w_4\coloneqq 3c_2^{(0,4,0)}+2c_4^{(0,4,0)}+3c_7^{(0,4,0)}, \tag{152}$$

$$\hat{d}_4\coloneqq c_1^{(0,0,2)}+c_2^{(0,0,2)}+c_3^{(0,0,2)}+c_4^{(0,0,2)}, \tag{153}$$

as shorthand.

After some manipulation, the degeneracy condition (34) is found to be

$$\begin{aligned}\Delta^{(4)}=&H^8\Delta_4^{(4)}\frac{\partial^4}{a^2}+H^6\Delta_6^{(4)}\frac{\partial^6}{a^4}+H^4\Delta_8^{(4)}\frac{\partial^8}{a^6}\\&+H^2\Delta_{10}^{(4)}\frac{\partial^{10}}{a^8}+\Delta_{12}^{(4)}\frac{\partial^{12}}{a^{10}},\end{aligned} \tag{154}$$

where

$$\Delta_4^{(4)}=648[-4(b_4')^2+3b_4(2b_4'+b_4'')]w_4, \tag{155}$$

$$\Delta_6^{(4)}=36[-(2b_4'+b_4'')d_4+12b_4(f_4+f_4'-\tilde{b}_4)+4b_4'(-2f_4+\tilde{f}_4)]w_4+36[4(b_4')^2-3b_4(2b_4'+b_4'')](d_4+9\hat{d}_4-6\tilde{d}_4), \tag{156}$$

$$\begin{aligned}\Delta_8^{(4)}=&-18(2b_4'+b_4'')(\tilde{d}_4)^2+24d_4[-b_4(f_4+f_4'-\tilde{b}_4)+b_4'\hat{f}_4]-6b_4(-2f_4+3\hat{f}_4+\tilde{f}_4)^2\\&+24\tilde{d}_4[6b_4(f_4+f_4'-\tilde{b}_4)+b_4'(-2f_4-3\hat{f}_4+\tilde{f}_4)]+18\hat{d}_4[(2b_4'+b_4'')d_4-12b_4(f_4+f_4'-\tilde{b}_4)-4b_4'(-2f_4+\tilde{f}_4)]\\&+2[-4d_4(f_4+f_4'-\tilde{b}_4)+216b_4\bar{d}_4-(-2f_4+\tilde{f}_4)^2]w_4,\end{aligned} \tag{157}$$

$$\Delta_{10}^{(4)} = -4(f_4 + f_4' - \tilde{b}_4)(\tilde{d}_4)^2 + d_4[-24b_4\bar{d}_4 + (\hat{f}_4)^2] + 2\tilde{d}_4[72b_4\bar{d}_4 + \hat{f}_4(-2f_4 + \tilde{f}_4)] + \hat{d}_4[4d_4(f_4 + f_4' - \tilde{b}_4) - 216b_4\bar{d}_4 + (-2f_4 + \tilde{f}_4)^2] - 8d_4\bar{d}_4w_4, \tag{158}$$

and

$$\Delta_{12}^{(4)} = 4\bar{d}_4[d_4\hat{d}_4 - (\tilde{d}_4)^2]. \tag{159}$$

It is interesting that $\tilde{b}_4$ always arises in terms of $(f_4 + f_4' - \tilde{b}_4)$, and $\tilde{f}_4$ always arises in terms of $(-2f_4 + \tilde{f}_4)$.

Following the analysis in the above, we may solve the coefficients such that $\Delta^{(4)} = 0$. For example, from Eqs. (155) and (156) we have two constraints

$$-4(b_4')^2 + 3b_4(2b_4' + b_4'') = 0 \tag{160}$$

and

$$-(2b_4' + b_4'')d_4 + 12b_4(f_4 + f_4' - \tilde{b}_4) + 4b_4'(-2f_4 + \tilde{f}_4) = 0, \tag{161}$$

which when combined yield

$$-(b_4')^2 d_4 + 9(b_4)^2(f_4 + f_4' - \tilde{b}_4) + 3b_4b_4'(-2f_4 + \tilde{f}_4) = 0. \tag{162}$$

The full treatment of the case of $d = 4$, however, is involved and out of the scope of the present work.

V. ELIMINATE THE SCALAR MODE AT THE SECOND ORDER: $d = 2$

In the previous section, we eliminated the scalar mode at linear order in perturbations by making use of the degeneracy condition (34). Clearly the conditions for the coefficients derived in the above are merely necessary conditions, which means that the scalar mode will reappear if we go to higher orders. Thus one needs to find the conditions for the coefficients such that the scalar mode is eliminated order by order. In this section we take the case of $d = 2$ as an illustrative example.

Expanding the action (35) to the cubic order in perturbations yields

$$\begin{aligned}\mathcal{L}_3 = {}& 9\bar{N}a^3b_2\zeta\dot{\zeta}^2 - \bar{N}a^2Hb_2\zeta^2\partial^2B - 2\bar{N}ah_2\zeta^2\partial^2\zeta + a\bar{N}\tilde{b}_2\zeta\partial_iA\partial^iA - 2a^2H\bar{N}b_2\zeta\partial_i\zeta\partial^iB \\ & - 2a\bar{N}h_2\zeta\partial_i\zeta\partial^i\zeta + \frac{1}{3}a\bar{N}(-b_2 + w_2)\zeta(\partial^2B)^2 + \frac{2}{3}a\bar{N}(b_2 + 2w_2)\partial_i\zeta\partial^iB\partial^2B \\ & - 2a\bar{N}w_2\partial_i\partial_jB\partial^iB\partial^j\zeta - 2a\bar{N}w_2\partial^iB\partial_j\partial_iB\partial^j\zeta - a\bar{N}w_2\zeta\partial_j\partial_iB\partial^j\partial^iB \\ & + 27\bar{N}a^3Hb_2\zeta^2\dot{\zeta} - 2a^2b_2\zeta\partial^2B\partial_t\zeta - 2\bar{N}a^2b_2\partial_i\zeta\partial^iB\dot{\zeta} + 2a^2H\bar{N}(b_2 - b_2')A\zeta\partial^2B \\ & + 2a^2H\bar{N}(b_2 - b_2')A\partial_i\zeta\partial^iB - 18a^3HA\zeta\partial_t\zeta(b_2 - b_2') + 2\bar{N}a^2(b_2 - b_2')A\partial^2B\dot{\zeta} \\ & - \frac{27}{2}\bar{N}a^3H^2(-2b_2 + b_2')\zeta^3 + 3\bar{N}a^3(-b_2 + b_2')A\dot{\zeta}^2 + a\bar{N}(\tilde{b}_2 + \tilde{b}_2')A\partial_iA\partial^iA \\ & - 4a\bar{N}(h_2 + h_2')A\zeta\partial^2\zeta - 2a\bar{N}(h_2 + h_2')A\partial_i\zeta\partial^i\zeta + \frac{1}{3}a\bar{N}(-b_2 + w_2 + b_2' - w_2')A(\partial^2B)^2 \\ & + a\bar{N}(-w_2 + w_2')A\partial_j\partial_iB\partial^j\partial^iB + \frac{9}{2}\bar{N}a^3H^2(2b_2 - 2b_2' + b_2'')A^2\zeta \\ & - \bar{N}a^2H(b_2 - b_2' + b_2'')A^2\partial^2B + 3\bar{N}a^3H(b_2 - b_2' + b_2'')A^2\dot{\zeta} \\ & - 2\bar{N}a(h_2 + 3h_2' + h_2'')A^2\partial^2\zeta + \frac{1}{2}\bar{N}a^3H^2b_2^{(3)}A^3,\end{aligned} \tag{163}$$

where we have used the background equations of motion to simplify the coefficients. No integration by parts has been performed at this point.

By making use of the degeneracy conditions (56) and (57), and plugging the solutions for A and B into Eqs. (58) and (59), after some manipulations, we get the induced cubic action $S_3[\zeta]$ for the single variable ζ. We tend not to present the full and explicit expression of $S_3[\zeta]$ due to its length. We pay special attention to the terms which are relevant to eliminating the scalar mode (i.e., ζ), and have the following observations.

(a) First, we found that there is no $\dot{\zeta}^3$, i.e., there are no terms with three time derivatives.

(b) Second, there is one "dangerous" term with two time derivatives:

$$-2\bar{N}a\frac{b_2^2(2h_2'+h_2'')}{H^2(b_2-b_2')^2}\dot{\zeta}^2\partial^2\zeta, \tag{164}$$

and thus we need to require (since $b_2 \neq 0$)

$$2h_2'+h_2''=0, \tag{165}$$

with $b_2 - b_2' \neq 0$. The general solution for h_2 is

$$h_2(N) = C_3 + \frac{C_4}{N}, \tag{166}$$

where C_3, C_4 are constants.

(c) Third, there is one dangerous term with one time derivative:

$$\frac{2b_2(h_2+h_2')^2[-2b_2w_2^2+2w_2^2b_2'+b_2^2(-w_2+w_2')]}{3aH^3w_2^2(b_2-b_2')^3} \times \bar{N}\dot{\zeta}(\partial^2\zeta)^2, \tag{167}$$

which implies that w_2 must be related to b_2 by

$$\frac{1}{w_2^2}(w_2-w_2') = -2\frac{1}{b_2^2}(b_2-b_2'). \tag{168}$$

With the solution for b_2 in Eq. (60), the general solution for w_2 is

$$w_2 = \frac{C_5C_1N}{1-2C_5C_2N}, \tag{169}$$

where C_5 is another constant.

By combining Eqs. (56), (60), (166) and (169), the Lagrangian (61) is further reduced to

$$\mathcal{L}^{(2)} = \frac{C_5C_1N}{1-2C_5C_2N}\hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_1N}{1+C_2N}K^2 + \left(C_3+\frac{C_4}{N}\right)R. \tag{170}$$

We conclude that the Lagrangian (170) propagates no scalar degrees of freedom up to the second order in perturbations on a cosmological background.

VI. CONCLUSION

In this work, we revisited the problem of propagating at most 2 tensorial degrees of freedom in a large class of spatially covariant gravity theories, of which the Lagrangians are polynomials built of spatial geometric quantities. Although the general conditions have been derived in [74,75], these conditions are mathematically involved to be solved to yield concrete Lagrangians.

We thus take an alternative and complementary approach in this work based on a perturbative analysis. The idea is simple: if the Lagrangian has no scalar degrees of freedom in a fully nonlinear sense, the scalar mode must not show up at any finite order if we perturbatively expand the Lagrangian around a cosmological background. This perturbative analysis allows us to determine the coefficients in the Lagrangian order by order. Since at the fully nonlinear level, there is a finite number of conditions imposed on the functional form of the Lagrangian, this perturbative analysis must stop at some *finite* order. In other words, there must be a finite order up to which we kill the scalar mode, and then the scalar mode is eliminated at fully nonlinear order. In fact, as shown in [79] in a specific example, it is sufficient to tune the coefficients up to the cubic order in the Lagrangian such that the unwanted scalar mode is fully removed.

In this work, we mainly focused on the linear cosmological perturbations. In Sec. III we showed that in order to eliminate the unwanted scalar mode at the linear order, the degeneracy condition (34) must be imposed. This is also supported by a more rigorous Lagrangian constraint analysis in the Appendix. We then used Eq. (34) as the starting point to determine the coefficients of the Lagrangians for $d = 2$, 3, 4 in Sec. IV, where d is the total number of derivatives in a SCG monomial. In particular, we determined the concrete form of the Lagrangians for $d = 2$ in Eq. (61) and for $d = 3$ in Eq. (92) in the absence of ∇a terms and in Eq. (95) in the presence of ∇a terms, respectively. We thus concluded that Eqs. (61), (92) and (95) propagate no scalar modes at the linear order in perturbations around the cosmological background. The scalar mode will rearise in the naive combination of the scalar-mode-free Lagrangians for $d = 2$ and $d = 3$. Therefore one needs more restrictions on the coefficients in order to eliminate the scalar mode if $d = 2$ and $d = 3$ Lagrangians are present simultaneously. The final results are given in Eq. (126) in the absence of ∇a terms and in Eq. (132) in the presence of ∇a terms, respectively.

It is not surprising that although the scalar mode has been eliminated at linear order, it may reappear at nonlinear orders. In Sec. V we expanded the Lagrangian up to the cubic order for $d = 2$ and found the conditions for the coefficients to eliminate the scalar mode up to the cubic order. The result is given in Eq. (170). In principle this procedure can be performed order by order, and one expects to determine the Lagrangian at some finite order, such that the scalar mode has been fully eliminated.

ACKNOWLEDGMENTS

This work was partly supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11975020.

APPENDIX: CLASSICAL MECHANICS WITH ONE DYNAMICAL AND TWO AUXILIARY VARIABLES

In this Appendix, we make a thorough analysis of a classical mechanics system with one dynamical and two auxiliary variables. We shall classify various cases according to the number of DOFs as well as the nature of constraints and gauge identities.

The most general quadratic Lagrangian for three variables $\{q^1, q^2, q^3\}$, of which one is dynamical and two are auxiliary variables, takes the form

$$\begin{aligned} L = &\frac{1}{2}g_{11}(\dot{q}^1)^2 + f_{12}\dot{q}^1q^2 + f_{13}\dot{q}^1q^3 \\ &+ \frac{1}{2}w_{22}(q^2)^2 + \frac{1}{2}w_{33}(q^3)^2 + w_{23}q^2q^3 \\ &+ \frac{1}{2}w_{11}(q^1)^2 + w_{12}q^1q^2 + w_{13}q^1q^3. \end{aligned} \tag{A1}$$

The coefficients g_{11}, f_{12}, etc., are assumed to be constants for simplicity. We assume $g_{11} \neq 0$ so that q^1 acquires an apparent kinetic term, while q^2 and q^3 do not have explicit time derivatives and act as the auxiliary variables. Our task is to search for cases in which there are no dynamics in the Lagrangian (A1).

Varying the Lagrangian yields

$$\delta L \simeq -\mathcal{E}_i^{(0)}\delta q^i, \tag{A2}$$

where the equations of motion take the form

$$\mathcal{E}_i^{(0)} \coloneqq W_{ij}^{(0)}\ddot{q}^j + V_j^{(0)} \approx 0, \tag{A3}$$

with

$$W_{ij}^{(0)} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A4}$$

and

$$V_j^{(0)} = \begin{pmatrix} f_{12}\dot{q}^2 + f_{13}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - w_{13}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - w_{22}q^2 - w_{23}q^3 \\ -f_{13}\dot{q}^1 - w_{13}q^1 - w_{23}q^2 - w_{33}q^3 \end{pmatrix}. \tag{A5}$$

In this Appendix, for clarity we use "≈" to denote on-shell equalities, i.e., those hold only when the equations of motion are satisfied. Here and in what follows the superscript "(n)" stands for "level n," whose meaning will be clear soon.

1. Level 0

Since $\mathrm{rank}(W_{ij}^{(0)}) = 1$, there are two null eigenvectors for $W_{ij}^{(0)}$:

$$u_{1,i}^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_{2,i}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{A6}$$

which when contracted with $\mathcal{E}_i^{(0)}$ yield

$$u_1^{(0)i}\mathcal{E}_i^{(0)} = -f_{12}\dot{q}^1 - w_{12}q^1 - w_{22}q^2 - w_{23}q^3 \equiv \mathcal{E}_2^{(0)}, \tag{A7}$$

$$u_2^{(0)i}\mathcal{E}_i^{(0)} = -f_{13}\dot{q}^1 - w_{13}q^1 - w_{23}q^2 - w_{33}q^3 \equiv \mathcal{E}_3^{(0)}. \tag{A8}$$

According to the algorithm of detecting constraints in the Lagrangian formalism, at each level, we have to examine whether the contractions lead to constraints or identities. We have three cases according to how many constraints/identities we get.

a. Case 1: Two identities

If the two contractions are vanishing identically, we get two gauge identities $G_1^{(0)} \coloneqq \mathcal{E}_2^{(0)} \equiv 0$ and $G_2^{(0)} \coloneqq \mathcal{E}_3^{(0)} \equiv 0$ at "level 0." In this Appendix "≡" stands for off-shell identities, which always hold no matter whether the equations of motion are satisfied or not. This requires $f_{12} = f_{13} = w_{12} = w_{13} = w_{22} = w_{23} = w_{33} = 0$. This case, however, is trivial since terms involving q^2, q^3 in the Lagrangian (A1) completely drop out and the Lagrangian reduces to that of a single variable q^1. Then the algorithm ends. We include this case merely for completeness.

b. Case 2: One constraint and one identity

Without loss of generality, we assume at least one of $\{f_{12}, w_{12}, w_{22}, w_{23}\}$ is not vanishing, and denote the constraint at level 0 as

$$\phi^{(0)} \coloneqq u_1^{(0)i}\mathcal{E}_i^{(0)} \approx 0. \tag{A9}$$

Then that $u_2^{(0)i}\mathcal{E}_i^{(0)}$ leads to an identity implies that

$$u_2^{(0)i}\mathcal{E}_i^{(0)} = \lambda u_1^{(0)i}\mathcal{E}_i^{(0)}, \tag{A10}$$

with some constant λ, i.e.,

$$f_{13} = \lambda f_{12}, \tag{A11}$$

$$w_{13} = \lambda w_{12}, \tag{A12}$$

$$w_{23} = \lambda w_{22}, \tag{A13}$$

$$w_{33} = \lambda w_{23} \equiv \lambda^2 w_{22}. \tag{A14}$$

We then get one gauge identity at level 0:

$$G^{(0)} \coloneqq \lambda \mathcal{E}_2^{(0)} - \mathcal{E}_3^{(0)} \equiv 0. \tag{A15}$$

Note that $\lambda = 0$ is trivial, since in this case the q^3 sector completely drops out, and the original Lagrangian reduces to that of two variables q^1 and q^2.

c. Case 3: Two constraints

This is of course the general case. As long as at least one of Eqs. (A11)–(A14) is not satisfied, we get two constraints at level 0:

$$\phi_1^{(0)} \coloneqq \mathcal{E}_2^{(0)} \approx 0, \tag{A16}$$

$$\phi_2^{(0)} \coloneqq \mathcal{E}_3^{(0)} \approx 0. \tag{A17}$$

2. Case 2: Level 1

Using Eqs. (A11)–(A14) to replace $\{f_{13}, w_{13}, w_{23}, w_{33}\}$ in terms of $\{f_{12}, w_{12}, w_{22}, w_{23}\}$, the constraint $\phi^{(0)}$ in Eq. (A9) becomes

$$\phi^{(0)} \to -f_{12}\dot{q}^1 - w_{12}q^1 - w_{22}q^2 - \lambda w_{22}q^3. \tag{A18}$$

According the standard algorithm, we build the enlarged "vector" of equations of motion as

$$\mathcal{E}_{i_1}^{(1)} \coloneqq \begin{pmatrix} \mathcal{E}_i^{(0)} \\ \dot{\phi}^{(0)} \end{pmatrix} = W_{i_1 j}^{(1)} \ddot{q}^j + V_{i_1}^{(1)}, \tag{A19}$$

with

$$W_{i_1 j}^{(1)} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \end{pmatrix} \tag{A20}$$

and

$$V_{i_1}^{(1)} = \begin{pmatrix} f_{12}\dot{q}^2 + \lambda f_{12}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - \lambda w_{12}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - w_{22}q^2 - \lambda w_{22}q^3 \\ -\lambda f_{12}\dot{q}^1 - \lambda w_{12}q^1 - \lambda w_{22}q^2 - \lambda^2 w_{22}q^3 \\ -w_{12}\dot{q}^1 - w_{22}\dot{q}^2 - \lambda w_{22}\dot{q}^3 \end{pmatrix}. \tag{A21}$$

There are two trivial null eigenvectors for $W_{i_1 j}^{(1)}$, which are merely $u_{1,i}^{(0)}$, $u_{2,i}^{(0)}$ in Eq. (A6) augmented by zeros. On the other hand, there is a nontrivial null eigenvector

$$u_{i_1}^{(1)} = \begin{pmatrix} f_{12} \\ 0 \\ 0 \\ g_{11} \end{pmatrix}, \tag{A22}$$

which is valid whether $f_{12} = 0$ or not. Contracting $u_{i_1}^{(1)}$ with $\mathcal{E}_{i_1}^{(1)}$ yields

$$\begin{aligned} u^{(1)i_1}\mathcal{E}_{i_1}^{(1)} &= -g_{11}w_{12}\dot{q}^1 + (f_{12}^2 - g_{11}w_{22})\dot{q}^2 \\ &\quad + \lambda(f_{12}^2 - g_{11}w_{22})\dot{q}^3 \\ &\quad - f_{12}w_{11}q^1 - f_{12}w_{12}q^2 - \lambda f_{12}w_{12}q^3. \end{aligned} \tag{A23}$$

We need to check whether $u^{(1)i_1}\mathcal{E}_{i_1}^{(1)}$ leads to an identity or a new constraint. There are two subcases.

a. Case 2.1: One identity

Up to "level 1," we have only one constraint $\phi^{(0)}$ given in Eq. (A18). If $u^{(1)i_1}\mathcal{E}_{i_1}^{(1)}$ is not an independent constraint, it implies that

$$u^{(1)i_1}\mathcal{E}_{i_1}^{(1)} \propto \phi_1^{(0)}, \tag{A24}$$

which puts restrictions on the coefficients. After some manipulation, the necessary and sufficient condition for Eq. (A24) can be written as

$$w_{12} = \sqrt{w_{11}}\frac{f_{12}}{\sqrt{g_{11}}}, \qquad w_{22} = \frac{f_{12}^2}{g_{11}}, \tag{A25}$$

with $f_{12} \neq 0$. Note we must have $f_{12} \neq 0$ since if $f_{12} = 0$, Eq. (A24) implies $w_{12} = w_{22} = w_{23} = 0$, which conflicts with the assumption that at least one of $\{f_{12}, w_{12}, w_{22}, w_{23}\}$ is not vanishing in order to have the constraint $\phi^{(0)}$. Then we get one gauge identity at level 1:

$$\begin{aligned} G^{(1)} &\coloneqq u^{(1)i_1}\mathcal{E}_{i_1}^{(1)} - g_{11}w_{11}\phi_1^{(0)} \\ &= f_{12}\mathcal{E}_1^{(0)} + g_{11}\frac{\mathrm{d}\mathcal{E}_2^{(0)}}{\mathrm{d}t} - g_{11}w_{11}\mathcal{E}_2^{(0)} \equiv 0. \end{aligned} \tag{A26}$$

Then the algorithm ends.

In this case, we have one constraint $\phi_1^{(0)}$ in Eq. (A9), two gauge identities $G^{(0)}$ and $G^{(1)}$ in Eqs. (A15) and (A26), respectively. It is easy to show that in this case there is no dynamical degree of freedom.

b. Case 2.2: One constraint

As long as Eq. (A25) (at least one of the two equalities) is not satisfied, $u^{(1)i_1}\mathcal{E}_{i_1}^{(1)}$ leads to a new independent constraint:

$$\phi^{(1)} := u^{(1)i_1}\mathcal{E}^{(1)}_{i_1} \approx 0. \tag{A27}$$

Then we go to the next level.

3. Case 2.2: Level 2

By appending $\dot{\phi}^{(1)}$ [with $\phi^{(1)}$ in Eq. (A27)] to $\mathcal{E}^{(1)}_{i_1}$, we build the enlarged vector of equations of motion:

$$\mathcal{E}^{(2)}_{i_2} = W^{(2)}_{i_2 j}\ddot{q}^j + V^{(2)}_{i_2}, \tag{A28}$$

with

$$W^{(2)}_{i_2 j} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -g_{11}w_{12} & f_{12}^2 - g_{11}w_{22} & \lambda(f_{12}^2 - g_{11}w_{22}) \end{pmatrix} \tag{A29}$$

and

$$V^{(2)}_{i_2} = \begin{pmatrix} f_{12}\dot{q}^2 + \lambda f_{12}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - \lambda w_{12}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - w_{22}q^2 - \lambda w_{22}q^3 \\ -\lambda f_{12}\dot{q}^1 - \lambda w_{12}q^1 - \lambda w_{22}q^2 - \lambda^2 w_{22}q^3 \\ -w_{12}\dot{q}^1 - w_{22}\dot{q}^2 - \lambda w_{22}\dot{q}^3 \\ -f_{12}w_{11}\dot{q}^1 - f_{12}w_{12}\dot{q}^2 - \lambda f_{12}w_{12}\dot{q}^3 \end{pmatrix}. \tag{A30}$$

Since we have assumed $\lambda \neq 0$, we have two subcases.

a. Case 2.2.1: $w_{22} \neq \frac{f_{12}^2}{g_{11}}$

In this case, $W^{(2)}_{i_2 j}$ do not possess any further nontrivial left null eigenvectors. The algorithm therefore ends.

In this case, we have two constraints $\phi^{(0)}$, $\phi^{(1)}$ and one gauge identity $G^{(0)}$. One can show that there is 1 dynamical degree of freedom.

b. Case 2.2.2: $w_{22} = \frac{f_{12}^2}{g_{11}}$ while $w_{12} \neq \sqrt{w_{11}}\frac{f_{12}}{\sqrt{g_{11}}}$

Since $w_{22} = \frac{f_{12}^2}{g_{11}}$, we have to require that the other equality in Eq. (A25) is not satisfied, i.e.,

$$w_{12} \neq \sqrt{w_{11}}\frac{f_{12}}{\sqrt{g_{11}}}. \tag{A31}$$

In this case, $W^{(2)}_{i_2 j}$ possesses a new nontrivial null eigenvector

$$u^{(2)}_{i_2} = \begin{pmatrix} w_{12} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A32}$$

Contracting $u^{(2)}_{i_2}$ with $\mathcal{E}^{(2)}_{i_2}$ yields

$$u^{(2)i_2}\mathcal{E}^{(2)}_{i_2} = -f_{12}w_{11}\dot{q}^1 - w_{12}w_{11}q^1 - w_{12}^2 q^2 - \lambda w_{12}^2 q^3. \tag{A33}$$

Recall that we have two constraints $\phi^{(0)}_1$ in Eq. (A18) and $\phi^{(1)}$ in Eq. (A27), which in our case reduce to

$$\phi^{(0)} \to -f_{12}\dot{q}^1 - w_{12}q^1 - \frac{f_{12}^2}{g_{11}}q^2 - \lambda\frac{f_{12}^2}{g_{11}}q^3, \tag{A34}$$

$$\phi^{(1)} \to -g_{11}w_{12}\dot{q}^1 - f_{12}w_{11}q^1 - f_{12}w_{12}q^2 - \lambda f_{12}w_{12}q^3. \tag{A35}$$

One can show that $u^{(2)i_2}\mathcal{E}^{(2)}_{i_2}$ is linearly independent of $\phi^{(0)}$ and $\phi^{(1)}$ as long as Eq. (A31) is satisfied. Therefore $u^{(2)i_2}\mathcal{E}^{(2)}_{i_2}$ leads to a new constraint,

$$\phi^{(2)} := u^{(2)i_2}\mathcal{E}^{(2)}_{i_2} \approx 0. \tag{A36}$$

4. Case 2.2.2: Level 3

Appending $\dot{\phi}^{(2)}$ [with $\phi^{(2)}$ given in Eq. (A36)] to $\mathcal{E}^{(2)}_{i_2}$ yields

$$\mathcal{E}^{(3)}_{i_3} = W^{(3)}_{i_3 j}\ddot{q}^j + V^{(3)}_{i_3}, \tag{A37}$$

with

$$W^{(3)}_{i_3 j} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -g_{11}w_{12} & 0 & 0 \\ -f_{12}w_{11} & 0 & 0 \end{pmatrix} \tag{A38}$$

and

$$V^{(3)}_{i_3} = \begin{pmatrix} f_{12}\dot{q}^2 + \lambda f_{12}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - \lambda w_{12}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - \frac{f_{12}^2}{g_{11}}q^2 - \lambda\frac{f_{12}^2}{g_{11}}q^3 \\ -\lambda f_{12}\dot{q}^1 - \lambda w_{12}q^1 - \lambda\frac{f_{12}^2}{g_{11}}q^2 - \lambda^2\frac{f_{12}^2}{g_{11}}q^3 \\ -w_{12}\dot{q}^1 - \frac{f_{12}^2}{g_{11}}\dot{q}^2 - \lambda\frac{f_{12}^2}{g_{11}}\dot{q}^3 \\ -f_{12}w_{11}\dot{q}^1 - f_{12}w_{12}\dot{q}^2 - \lambda f_{12}w_{12}\dot{q}^3 \\ -w_{11}w_{12}\dot{q}^1 - w_{12}^2\dot{q}^2 - \lambda w_{12}^2\dot{q}^3 \end{pmatrix}. \tag{A39}$$

$W^{(3)}_{i_3 j}$ possesses a nontrivial null eigenvector

$$u^{(3)}_{i_3} = \begin{pmatrix} f_{12}w_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ g_{11} \end{pmatrix}, \tag{A40}$$

which when contracted with $\mathcal{E}^{(3)}_{i_3}$ yields

$$u^{(3)i_3}\mathcal{E}^{(3)}_{i_3} = -g_{11}w_{11}w_{12}\dot{q}^1 + (w_{11}f_{12}^2 - g_{11}w_{12}^2)\dot{q}^2 + \lambda(w_{11}f_{12}^2 - g_{11}w_{12}^2)\dot{q}^3 - f_{12}w_{11}^2q^1 - f_{12}w_{11}w_{12}q^2 - \lambda f_{12}w_{11}w_{12}q^3. \tag{A41}$$

Comparing this with the previous constraints $\phi^{(0)}, \phi^{(1)}, \phi^{(2)}$ in Eqs. (A9), (A27) and (A36), clearly $u^{(3)i_3}\mathcal{E}^{(3)}_{i_3}$ leads to a new constraint

$$\phi^{(3)} \coloneqq u^{(3)i_3}\mathcal{E}^{(3)}_{i_3} \approx 0, \tag{A42}$$

since $w_{11}f_{12}^2 - g_{11}w_{12}^2 \neq 0$.

5. Case 2.2.2: Level 4

Appending $\dot{\phi}^{(3)}$ [with $\phi^{(3)}$ given in Eq. (A42)] to $\mathcal{E}^{(3)}_{i_3}$ yields

$$\mathcal{E}^{(4)}_{i_4} = W^{(4)}_{i_4 j}\ddot{q}^j + V^{(4)}_{i_4}, \tag{A43}$$

with

$$W^{(4)}_{i_4 j} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -g_{11}w_{12} & 0 & 0 \\ -f_{12}w_{11} & 0 & 0 \\ -g_{11}w_{12}w_{11} & w_{11}f_{12}^2 - g_{11}w_{12}^2 & \lambda(w_{11}f_{12}^2 - g_{11}w_{12}^2) \end{pmatrix} \tag{A44}$$

and

$$V^{(4)}_{i_4} = \begin{pmatrix} f_{12}\dot{q}^2 + \lambda f_{12}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - \lambda w_{12}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - \frac{f_{12}^2}{g_{11}}q^2 - \lambda\frac{f_{12}^2}{g_{11}}q^3 \\ -\lambda f_{12}\dot{q}^1 - \lambda w_{12}q^1 - \lambda\frac{f_{12}^2}{g_{11}}q^2 - \lambda^2\frac{f_{12}^2}{g_{11}}q^3 \\ -w_{12}\dot{q}^1 - \frac{f_{12}^2}{g_{11}}\dot{q}^2 - \lambda\frac{f_{12}^2}{g_{11}}\dot{q}^3 \\ -f_{12}w_{11}\dot{q}^1 - f_{12}w_{12}\dot{q}^2 - \lambda f_{12}w_{12}\dot{q}^3 \\ -w_{11}w_{12}\dot{q}^1 - w_{12}^2\dot{q}^2 - \lambda w_{12}^2\dot{q}^3 \\ -f_{12}w_{11}^2\dot{q}^1 - f_{12}w_{11}w_{12}\dot{q}^2 - \lambda f_{12}w_{11}w_{12}\dot{q}^3 \end{pmatrix}. \tag{A45}$$

Clearly, since $w_{11}f_{12}^2 - g_{11}w_{12}^2 \neq 0$, $W^{(4)}_{i_4 j}$ does not possess a nontrivial null eigenvector. The algorithm ends.

To summarize, in this case, we have four constraints, $\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \phi^{(3)}$, given in Eqs. (A34), (A35), (A36) and (A42), and one gauge identity $G^{(0)}$ given in Eq. (A15). As a result, there are no dynamical degree of freedom.

6. Case 3: Level 1

From Eqs. (A16) and (A17), the two independent constraints at level 0 are

$$\dot{\phi}^{(0)}_1 = -f_{12}\ddot{q}^1 - w_{12}\dot{q}^1 - w_{22}\dot{q}^2 - w_{23}\dot{q}^3, \tag{A46}$$

$$\dot{\phi}^{(0)}_2 = -f_{13}\ddot{q}^1 - w_{13}\dot{q}^1 - w_{23}\dot{q}^2 - w_{33}\dot{q}^3. \tag{A47}$$

In "case 3," the enlarged vector of equations of motion is

$$\mathcal{E}^{(1)}_{i_1} = \begin{pmatrix} \mathcal{E}^{(0)}_i \\ \dot{\phi}^{(0)}_1 \\ \dot{\phi}^{(0)}_2 \end{pmatrix} = W^{(1)}_{i_1 j}\ddot{q}^j + V^{(1)}_{i_1}, \tag{A48}$$

with

$$W^{(1)}_{i_1 j} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -f_{13} & 0 & 0 \end{pmatrix} \tag{A49}$$

and

$$V^{(1)}_{i_1} = \begin{pmatrix} f_{12}\dot{q}^2 + f_{13}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - w_{13}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - w_{22}q^2 - w_{23}q^3 \\ -f_{13}\dot{q}^1 - w_{13}q^1 - w_{23}q^2 - w_{33}q^3 \\ -w_{12}\dot{q}^1 - w_{22}\dot{q}^2 - w_{23}\dot{q}^3 \\ -w_{13}\dot{q}^1 - w_{23}\dot{q}^2 - w_{33}\dot{q}^3 \end{pmatrix}. \tag{A50}$$

In this case $W^{(1)}_{i_1 j}$ possesses two nontrivial null eigenvectors:

$$u^{(1)}_{1,i_1} = \begin{pmatrix} f_{12} \\ 0 \\ 0 \\ g_{11} \\ 0 \end{pmatrix}, \qquad u^{(1)}_{2,i_1} = \begin{pmatrix} f_{13} \\ 0 \\ 0 \\ 0 \\ g_{11} \end{pmatrix}. \tag{A51}$$

Contracting $u^{(1)}_{1,i_1}$ and $u^{(1)}_{2,i_1}$ with $\mathcal{E}^{(1)}_{i_1}$ yields

$$\begin{aligned} u_1^{(1)i_1}\mathcal{E}^{(1)}_{i_1} &= -g_{11}w_{12}\dot{q}^1 + (f_{12}^2 - g_{11}w_{22})\dot{q}^2 \\ &\quad + (f_{12}f_{13} - g_{11}w_{23})\dot{q}^3 - f_{12}w_{11}q^1 \\ &\quad - f_{12}w_{12}q^2 - f_{12}w_{13}q^3 \end{aligned} \tag{A52}$$

and

$$\begin{aligned} u_2^{(1)i_1}\mathcal{E}^{(1)}_{i_1} &= -g_{11}w_{13}\dot{q}^1 + (f_{13}f_{12} - g_{11}w_{23})\dot{q}^2 \\ &\quad + (f_{13}^2 - g_{11}w_{33})\dot{q}^3 - f_{13}w_{11}q^1 \\ &\quad - f_{13}w_{12}q^2 - f_{13}w_{13}q^3. \end{aligned} \tag{A53}$$

We need to check whether $u_1^{(1)i_1}\mathcal{E}^{(1)}_{i_1}$ and $u_2^{(1)i_1}\mathcal{E}^{(1)}_{i_1}$ lead to new constraints or identities. To this end, together with $\phi_1^{(0)}$ and $\phi_2^{(0)}$ in Eqs. (A16) and (A17), we write

$$\begin{pmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ u_1^{(1)i_1}\mathcal{E}^{(1)}_{i_1} \\ u_2^{(1)i_1}\mathcal{E}^{(1)}_{i_1} \end{pmatrix} = \boldsymbol{M}^{(1)} \begin{pmatrix} \dot{q}^1 \\ \dot{q}^2 \\ \dot{q}^3 \\ q^1 \\ q^2 \\ q^3 \end{pmatrix}, \tag{A54}$$

with the 4×6 matrix

$$\boldsymbol{M}^{(1)} \coloneqq \begin{pmatrix} -f_{12} & 0 & 0 & -w_{12} & -w_{22} & -w_{23} \\ -f_{13} & 0 & 0 & -w_{13} & -w_{23} & -w_{33} \\ -g_{11}w_{12} & f_{12}^2 - g_{11}w_{22} & f_{12}f_{13} - g_{11}w_{23} & -f_{12}w_{11} & -f_{12}w_{12} & -f_{12}w_{13} \\ -g_{11}w_{13} & f_{12}f_{13} - g_{11}w_{23} & f_{13}^2 - g_{11}w_{33} & -f_{13}w_{11} & -f_{13}w_{12} & -f_{13}w_{13} \end{pmatrix}. \tag{A55}$$

Then the question is equivalent to checking the rank of $\boldsymbol{M}^{(1)}$. For later convenience, we define the submatrix

$$\boldsymbol{\Delta} \coloneqq \begin{pmatrix} f_{12}^2 - g_{11}w_{22} & f_{12}f_{13} - g_{11}w_{23} \\ f_{12}f_{13} - g_{11}w_{23} & f_{13}^2 - g_{11}w_{33} \end{pmatrix}, \tag{A56}$$

for which the determinant is

$$\begin{aligned} \det\boldsymbol{\Delta} &= g_{11}[-f_{12}^2 w_{33} + 2f_{12}f_{13}w_{23} - f_{13}^2 w_{22} \\ &\quad + g_{11}(w_{22}w_{33} - w_{23}^2)]. \end{aligned} \tag{A57}$$

a. Two identities (impossible)

First we shall show that it is impossible to have two identities. In fact, in order to have two new identities, we have to require that the entries of the submatrix $\boldsymbol{\Delta}$ in Eq. (A56) are vanishing identically. This can also be understood in that since there are neither $\dot{q}^2$ nor $\dot{q}^3$ terms in $\phi_1^{(0)}$, $\phi_2^{(2)}$, as long as at least one of the coefficients of $\dot{q}^2$ or $\dot{q}^3$ in $u_1^{(1)i_1}\mathcal{E}^{(1)}_{i_1}$ and $u_2^{(1)i_1}\mathcal{E}^{(1)}_{i_1}$ is not vanishing, we get a new constraint. To conclude, the necessary conditions to have two new identities are

$$w_{22} = \frac{f_{12}^2}{g_{11}}, \qquad w_{23} = \frac{f_{12}f_{13}}{g_{11}}, \qquad w_{33} = \frac{f_{13}^2}{g_{11}}. \tag{A58}$$

On the other hand, comparing this with Eqs. (A11)–(A14), we have to require

$$f_{12}w_{13} - f_{13}w_{12} \neq 0 \tag{A59}$$

in order not to go back to "case 2."

With these considerations, $u_1^{(1)i_1}\mathcal{E}^{(1)}_{i_1}$ and $u_2^{(1)i_1}\mathcal{E}^{(1)}_{i_1}$ reduce to

$$\begin{aligned} u_1^{(1)i_1}\mathcal{E}^{(1)}_{i_1} &\to -g_{11}w_{12}\dot{q}^1 - f_{12}w_{11}q^1 \\ &\quad - f_{12}w_{12}q^2 - f_{12}w_{13}q^3, \end{aligned} \tag{A60}$$

$$u_2^{(1)i_1}\mathcal{E}_{i_1}^{(1)} \to -g_{11}w_{13}\dot{q}^1 - f_{13}w_{11}q^1 - f_{13}w_{12}q^2 - f_{13}w_{13}q^3. \tag{A61}$$

In order to have two identities among the four equalities $\{\phi_1^{(0)}, \phi_2^{(0)}, u_1^{(1)i_1}\mathcal{E}_{i_1}^{(1)}, u_2^{(1)i_1}\mathcal{E}_{i_1}^{(1)}\} \approx 0$ we have to make sure that the rank of the 4×4 matrix

$$\begin{pmatrix} -f_{12} & -w_{12} & -\frac{f_{12}^2}{g_{11}} & -\frac{f_{12}f_{13}}{g_{11}} \\ -f_{13} & -w_{13} & -\frac{f_{12}f_{13}}{g_{11}} & -\frac{f_{13}^2}{g_{11}} \\ -g_{11}w_{12} & -f_{12}w_{11} & -f_{12}w_{12} & -f_{12}w_{13} \\ -g_{11}w_{13} & -f_{13}w_{11} & -f_{13}w_{12} & -f_{13}w_{13} \end{pmatrix} \tag{A62}$$

is 2. However, the determinant of the above 4×4 matrix is

$$-(f_{13}w_{12} - f_{12}w_{13})^3 \neq 0, \tag{A63}$$

since we must have Eq. (A59). Therefore it is impossible to have two independent constraints $\{\phi_1^{(0)}, \phi_2^{(0)}\}$ at level 0, and in the meanwhile to have two new gauge identities $\{G_1^{(1)}, G_2^{(1)}\}$ at level 1.

b. Case 3.1: One constraint and one identity

The necessary condition is that the submatrix $\mathbf{\Delta}$ defined in Eq. (A56) is degenerate but not identically vanishing, i.e.,

$$\mathrm{rank}\mathbf{\Delta} = 1. \tag{A64}$$

This is because
(a) if the rank $\mathbf{\Delta} = 2$, there will be two new constraints, and
(b) if the rank $\mathbf{\Delta} = 0$ (then $\mathbf{\Delta} \equiv 0$), according to the analysis in Sec. 6 a, either there are still two new constraints when Eq. (A59) is satisfied or we go back to "case 2.1."

Without loss of generality, the submatrix $\mathbf{\Delta}$ can be written in the form

$$\mathbf{\Delta} = \omega\begin{pmatrix} 1 & \eta \\ \eta & \eta^2 \end{pmatrix}, \tag{A65}$$

with ω, η being constants. Note that we require $\omega \neq 0$ in order to have rank $\mathbf{\Delta} = 1$. With Eq. (A65), $\boldsymbol{M}^{(1)}$ reduces to

$$\boldsymbol{M}^{(1)} \to \begin{pmatrix} -f_{12} & 0 & 0 & -w_{12} & \frac{\omega - f_{12}^2}{g_{11}} & \frac{\eta\omega - f_{12}f_{13}}{g_{11}} \\ -f_{13} & 0 & 0 & -w_{13} & \frac{\eta\omega - f_{12}f_{13}}{g_{11}} & \frac{\eta^2\omega - f_{13}^2}{g_{11}} \\ -g_{11}w_{12} & \omega & \omega\eta & -f_{12}w_{11} & -f_{12}w_{12} & -f_{12}w_{13} \\ -g_{11}w_{13} & \omega\eta & \omega\eta^2 & -f_{13}w_{11} & -f_{13}w_{12} & -f_{13}w_{13} \end{pmatrix}, \tag{A66}$$

and our question thus reduces to checking if it is possible to have rank$\boldsymbol{M}^{(1)} = 3$.

One can show that the necessary condition to have rank$\boldsymbol{M}^{(1)} = 3$ is to require

$$\mathcal{D}^{(1)} \coloneqq -\omega(f_{13}w_{12} - f_{12}w_{13})^2 + \omega^2\left[(\eta w_{12} - w_{13})^2 - \frac{w_{11}}{g_{11}}(\eta f_{12} - f_{13})^2\right] = 0. \tag{A67}$$

If

$$f_{13}w_{12} - f_{12}w_{13} = 0, \tag{A68}$$

then we need to require

$$f_{13} - \eta f_{12} \neq 0, \tag{A69}$$

and Eq. (A67) yields

$$w_{12} = \frac{f_{12}\sqrt{w_{11}}}{\sqrt{g_{11}}}. \tag{A70}$$

On the other hand, if

$$f_{13}w_{12} - f_{12}w_{13} \neq 0, \tag{A71}$$

Eq. (A67) also implies one constraint among the coefficients. In both cases we have rank$\boldsymbol{M}^{(1)} = 3$.

To conclude, it is possible to have rank$\boldsymbol{M}^{(1)} = 3$ so that we get one constraint and one gauge identity on level 1. Without loss of generality, we may choose the new constraint to be

$$\phi^{(1)} \coloneqq u_1^{(1)i_1}\mathcal{E}_{i_1}^{(1)} \approx 0. \tag{A72}$$

The gauge identity must be the form

$$G^{(1)} \coloneqq a\phi_1^{(0)} + b\phi_2^{(1)} - \eta\phi_1^{(1)} + u_2^{(1)i_1}\mathcal{E}_{i_1}^{(1)}, \tag{A73}$$

where the constants a, b are not vanishing simultaneously. a and b are determined by the concrete form of the null eigenvector of $\boldsymbol{M}^{(1)}$, which we do not show here explicitly.

c. Case 3.2: Two new constraints

Generally, one of the following applies:
(1) $\det\boldsymbol{\Delta}\neq 0$,
(2) $\operatorname{rank}\boldsymbol{\Delta}=1$, $\mathcal{D}^{(1)}\neq 0$ [with $\mathcal{D}^{(1)}$ defined in Eq. (A67)], or
(3) $\boldsymbol{\Delta}=0$ and $f_{13}w_{12}-f_{12}w_{13}\neq 0$,
and we have two constraints at level 1:

$$\phi_1^{(1)} \coloneqq u_1^{(1)i_1}\mathcal{E}_{i_1}^{(1)}\approx 0 \tag{A74}$$

and

$$\phi_2^{(1)} \coloneqq u_2^{(1)i_1}\mathcal{E}_{i_1}^{(1)}\approx 0. \tag{A75}$$

7. Case 3.1: Level 2

By appending $\dot{\phi}^{(1)}$ [with $\phi^{(1)}$ given in Eq. (A72)] to $\mathcal{E}_{i_1}^{(1)}$ we have

$$\mathcal{E}_{i_2}^{(2)} = W_{i_2 j}^{(2)}\ddot{q}^j + V_{i_2}^{(2)}, \tag{A76}$$

with

$$W_{i_2 j}^{(2)} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -f_{13} & 0 & 0 \\ -g_{11}w_{12} & f_{12}^2-g_{11}w_{22} & f_{12}f_{13}-g_{11}w_{23} \end{pmatrix}. \tag{A77}$$

Since

$$f_{12}^2-g_{11}w_{22}\neq 0,$$

$W_{i_2 j}^{(2)}$ has no nontrivial null eigenvector. The algorithm ends.

In this case, we have three constraints $\phi_1^{(0)}$, $\phi_2^{(0)}$, and $\phi^{(1)}$ given in Eqs. (A16), (A17) and (A72) and one gauge identity $G^{(1)}$ given in Eq. (A73). Therefore there are no dynamical degree of freedom.

8. Case 3.2: Level 2

From Eqs. (A74) and (A75), by appending $\dot{\phi}^{(1)}$ and $\dot{\phi}^{(2)}$ to $\mathcal{E}_{i_1}^{(1)}$ we get

$$\mathcal{E}_{i_2}^{(2)} = W_{i_2 j}^{(2)}\ddot{q}^j + V_{i_2}^{(2)}, \tag{A78}$$

with

$$W_{i_2 j}^{(2)} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -f_{13} & 0 & 0 \\ -g_{11}w_{12} & f_{12}^2-g_{11}w_{22} & f_{12}f_{13}-g_{11}w_{23} \\ -g_{11}w_{13} & f_{13}f_{12}-g_{11}w_{23} & f_{13}^2-g_{11}w_{33} \end{pmatrix} \tag{A79}$$

and

$$V_{i_2}^{(2)} = \begin{pmatrix} f_{12}\dot{q}^2+f_{13}\dot{q}^3-w_{11}q^1-w_{12}q^2-w_{13}q^3 \\ -f_{12}\dot{q}^1-w_{12}q^1-w_{22}q^2-w_{23}q^3 \\ -f_{13}\dot{q}^1-w_{13}q^1-w_{23}q^2-w_{33}q^3 \\ -w_{12}\dot{q}^1-w_{22}\dot{q}^2-w_{23}\dot{q}^3 \\ -w_{13}\dot{q}^1-w_{23}\dot{q}^2-w_{33}\dot{q}^3 \\ -f_{12}w_{11}\dot{q}^1-f_{12}w_{12}\dot{q}^2-f_{12}w_{13}\dot{q}^3 \\ -f_{13}w_{11}\dot{q}^1-f_{13}w_{12}\dot{q}^2-f_{13}w_{13}\dot{q}^3 \end{pmatrix}. \tag{A80}$$

We have to examine whether $W_{i_2 j}^{(2)}$ possesses new nontrivial null eigenvectors. According to the rank of the matrix $\boldsymbol{\Delta}$ in Eq. (A56), we have to discuss three subcases.

a. Case 3.2.1: $\det\boldsymbol{\Delta}\neq 0$

In this case, clearly there are no nontrivial eigenvectors of $W_{i_2 j}^{(2)}$. The algorithm ends.

b. Case 3.2.2: $\operatorname{rank}\boldsymbol{\Delta}=1$ and $\mathcal{D}^{(1)}\neq 0$

In this case $\det\boldsymbol{\Delta}=0$ but $\boldsymbol{\Delta}\neq 0$. Similar to the discussion in Sec. 6 b, we make use of the form (A65) and keep in mind that $\omega\neq 0$. Then $W_{i_2 j}^{(2)}$ reduces to

$$W_{i_2 j}^{(2)} \to \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -f_{13} & 0 & 0 \\ -g_{11}w_{12} & \omega & \omega\eta \\ -g_{11}w_{13} & \omega\eta & \omega\eta^2 \end{pmatrix}. \tag{A81}$$

$W_{i_2 j}^{(2)}$ possesses a single nontrivial null eigenvector, which can be chosen to be

$$u_{i_2}^{(2)} \coloneqq \begin{pmatrix} -\eta w_{12} + w_{13} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\eta \\ 1 \end{pmatrix}. \tag{A82}$$

Contracting $u_{i_2}^{(2)}$ with $\mathcal{E}_{i_2}^{(2)}$ yields

$$\begin{aligned} u^{(2)i_2}\mathcal{E}_{i_2}^{(2)} = {} & (\eta f_{12}w_{11} - f_{13}w_{11})\dot{q}^1 + (f_{12}w_{13} - f_{13}w_{12})\dot{q}^2 \\ & + \eta(f_{12}w_{13} - f_{13}w_{12})\dot{q}^3 \\ & + w_{11}(\eta w_{12} - w_{13})q^1 + w_{12}(\eta w_{12} - w_{13})q^2 \\ & + w_{13}(\eta w_{12} - w_{13})q^3. \end{aligned} \tag{A83}$$

Comparing this with $\phi_1^{(0)}$, $\phi_2^{(0)}$ in Eqs. (A16) and (A17) and $\phi_1^{(1)}$, $\phi_2^{(1)}$ in Eqs. (A74) and (A75), after some manipulations, one finds that in this case $u^{(2)i_2}\mathcal{E}_{i_2}^{(2)}$ always leads to a new constraint at "level 2":

$$\phi^{(2)} \coloneqq u^{(2)i_2}\mathcal{E}_{i_2}^{(2)} \approx 0. \tag{A84}$$

c. Case 3.2.3: $\boldsymbol{\Delta} = 0$

In this case clearly there are two new nontrivial null eigenvectors for $W_{i_2 j}^{(2)}$:

$$u_{1,i_2}^{(2)} = \begin{pmatrix} w_{12} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_{2,i_2}^{(2)} = \begin{pmatrix} w_{13} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A85}$$

Contracting $u_{1,i_2}^{(2)}$ and $u_{2,i_2}^{(2)}$ with $\mathcal{E}_{i_2}^{(2)}$ yields

$$\begin{aligned} u_1^{(2)i_2}\mathcal{E}_{i_2}^{(2)} = {} & -f_{12}w_{11}\dot{q}^1 + (w_{12}f_{13} - f_{12}w_{13})\dot{q}^3 \\ & - w_{12}w_{11}q^1 - w_{12}^2 q^2 - w_{12}w_{13}q^3 \end{aligned} \tag{A86}$$

and

$$\begin{aligned} u_2^{(2)i_2}\mathcal{E}_{i_2}^{(2)} = {} & -f_{13}w_{11}\dot{q}^1 + (w_{13}f_{12} - f_{13}w_{12})\dot{q}^2 \\ & - w_{13}w_{11}q^1 - w_{13}w_{12}q^2 - w_{13}^2 q^3. \end{aligned} \tag{A87}$$

Since we have already assumed $\boldsymbol{\Delta} = 0$, Eq. (A59) must be satisfied, i.e., $w_{12}f_{13} - f_{12}w_{13} \neq 0$.

Comparing this with the constraints $\phi_1^{(0)}$, $\phi_2^{(0)}$ in Eqs. (A16) and (A17), $\phi_1^{(1)}$, $\phi_2^{(1)}$ in Eqs. (A74) and (A75), since the determinant of the 6×6 matrix [after making use of Eq. (A58)]

$$\begin{pmatrix} -f_{12} & 0 & 0 & -w_{12} & -\frac{f_{12}^2}{g_{11}} & -\frac{f_{12}f_{13}}{g_{11}} \\ -f_{13} & 0 & 0 & -w_{13} & -\frac{f_{12}f_{13}}{g_{11}} & -\frac{f_{13}^2}{g_{11}} \\ -g_{11}w_{12} & 0 & 0 & -f_{12}w_{11} & -f_{12}w_{12} & -f_{12}w_{13} \\ -g_{11}w_{13} & 0 & 0 & -f_{13}w_{11} & -f_{13}w_{12} & -f_{13}w_{13} \\ -f_{12}w_{11} & 0 & w_{12}f_{13} - f_{12}w_{13} & -w_{12}w_{11} & -w_{12}^2 & -w_{12}w_{13} \\ -f_{13}w_{11} & w_{13}f_{12} - f_{13}w_{12} & 0 & -w_{13}w_{11} & -w_{13}w_{12} & -w_{13}^2 \end{pmatrix} \tag{A88}$$

is

$$-(w_{12}f_{13} - f_{12}w_{13})^5 \neq 0, \tag{A89}$$

$u_1^{(2)i_2}\mathcal{E}_{i_2}^{(2)}$ and $u_2^{(2)i_2}\mathcal{E}_{i_2}^{(2)}$ lead to two new constraints at level 2:

$$\phi_1^{(2)} \coloneqq u_1^{(2)i_2}\mathcal{E}_{i_2}^{(2)} \approx 0, \tag{A90}$$

$$\phi_2^{(2)} \coloneqq u_2^{(2)i_2}\mathcal{E}_{i_2}^{(2)} \approx 0. \tag{A91}$$

9. Case 3.2.2: Level 3

From Eq. (A84), we have

$$\mathcal{E}^{(3)}_{i_3} = W^{(3)}_{i_3 j}\ddot{q}^j + V^{(3)}_{i_3}, \tag{A92}$$

with

$$W^{(3)}_{i_3 j} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -f_{13} & 0 & 0 \\ -g_{11}w_{12} & \omega & \omega\eta \\ -g_{11}w_{13} & \omega\eta & \omega\eta^2 \\ \eta f_{12}w_{11} - f_{13}w_{11} & f_{12}w_{13} - f_{13}w_{12} & \eta(f_{12}w_{13} - f_{13}w_{12}) \end{pmatrix} \tag{A93}$$

and

$$V^{(3)}_{i_3 j} = \begin{pmatrix} f_{12}\dot{q}^2 + f_{13}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - w_{13}q^3 \\ \frac{q^2(\omega - f_{12}^2)}{g_{11}} + \frac{q^3(\eta\omega - f_{12}f_{13})}{g_{11}} - f_{12}\dot{q}^1 - w_{12}q^1 \\ \frac{q^2(\eta\omega - f_{12}f_{13})}{g_{11}} + \frac{q^3(\eta^2\omega - f_{13}^2)}{g_{11}} - f_{13}\dot{q}^1 - w_{13}q^1 \\ \frac{\dot{q}^2(\omega - f_{12}^2)}{g_{11}} + \frac{\dot{q}^3(\eta\omega - f_{12}f_{13})}{g_{11}} - w_{12}\dot{q}^1 \\ \frac{\dot{q}^2(\eta\omega - f_{12}f_{13})}{g_{11}} + \frac{\dot{q}^3(\eta^2\omega - f_{13}^2)}{g_{11}} - w_{13}\dot{q}^1 \\ -f_{12}w_{11}\dot{q}^1 - f_{12}w_{12}\dot{q}^2 - f_{12}w_{13}\dot{q}^3 \\ -f_{13}w_{11}\dot{q}^1 - f_{13}w_{12}\dot{q}^2 - f_{13}w_{13}\dot{q}^3 \\ w_{11}\dot{q}^1(\eta w_{12} - w_{13}) + w_{12}\dot{q}^2(\eta w_{12} - w_{13}) - w_{13}\dot{q}^3(w_{13} - \eta w_{12}) \end{pmatrix}. \tag{A94}$$

There is one nontrivial null eigenvector for $W^{(3)}_{i_3 j}$, which can be chosen to be

$$u^{(3)}_{i_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{g_{11}w_{12}}{f_{12}}(f_{12}w_{13} - f_{13}w_{12}) + w_{11}\omega(\eta - \frac{f_{13}}{f_{12}}) \\ 0 \\ f_{13}w_{12} - f_{12}w_{13} \\ 0 \\ \omega \end{pmatrix} \tag{A95}$$

for $f_{12} \neq 0$, and to be

$$u_{i_3}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ g_{11}\frac{w_{12}}{f_{13}}(f_{12}w_{13} - f_{13}w_{12}) + \omega\frac{w_{11}}{f_{13}}(\eta f_{12} - f_{13}) \\ f_{13}w_{12} - f_{12}w_{13} \\ 0 \\ \omega \end{pmatrix} \tag{A96}$$

for $f_{13} \neq 0$, and to be

$$u_{i_3}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{A97}$$

for $f_{12} = f_{13} = 0$. In all cases, the contraction of $u_{i_3}^{(3)}$ with $\mathcal{E}_{i_3}^{(3)}$ can be shown to be a new constraint at "level 3,"

$$\phi^{(3)} \coloneqq u^{(3)i_3}\mathcal{E}_{i_3}^{(3)} \approx 0, \tag{A98}$$

since the 6×6 matrix $\boldsymbol{M}^{(3)}$ defined by

$$\begin{pmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \\ \phi^{(2)} \\ u^{(3)i_3}\mathcal{E}_{i_3}^{(3)} \end{pmatrix} = \boldsymbol{M}^{(3)} \begin{pmatrix} \dot{q}^1 \\ \dot{q}^2 \\ \dot{q}^3 \\ q^1 \\ q^2 \\ q^3 \end{pmatrix} \tag{A99}$$

always possesses a nonvanishing determinant.

10. Case 3.2.2: Level 4

There is no nontrivial null eigenvector of $W_{i_4 j}^{(4)}$. The algorithm ends.

To summarize, we have six constraints, $\phi_1^{(0)}$, $\phi_2^{(0)}$, $\phi_1^{(1)}$, $\phi_2^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, and thus there is no dynamical degree of freedom.

11. Case 3.2.3: Level 3

From Eqs. (A90) and (A91), by appending $\dot{\phi}_1^{(2)}$ and $\dot{\phi}_2^{(2)}$ to $\mathcal{E}_{i_2}^{(2)}$, we get

$$\mathcal{E}_{i_3}^{(3)} = W_{i_3 j}^{(3)}\ddot{q}^j + V_{i_3}^{(3)}, \tag{A100}$$

with [we have used Eq. (A58) to replace w_{22}, w_{23} and w_{33}]

$$W_{i_3 j}^{(3)} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_{12} & 0 & 0 \\ -f_{13} & 0 & 0 \\ -g_{11}w_{12} & 0 & 0 \\ -g_{11}w_{13} & 0 & 0 \\ -f_{12}w_{11} & 0 & w_{12}f_{13} - f_{12}w_{13} \\ -f_{13}w_{11} & w_{13}f_{12} - f_{13}w_{12} & 0 \end{pmatrix} \tag{A101}$$

and

$$V_{i_2}^{(2)} = \begin{pmatrix} f_{12}\dot{q}^2 + f_{13}\dot{q}^3 - w_{11}q^1 - w_{12}q^2 - w_{13}q^3 \\ -f_{12}\dot{q}^1 - w_{12}q^1 - \frac{f_{12}^2}{g_{11}}q^2 - \frac{f_{12}f_{13}}{g_{11}}q^3 \\ -f_{13}\dot{q}^1 - w_{13}q^1 - \frac{f_{12}f_{13}}{g_{11}}q^2 - \frac{f_{13}^2}{g_{11}}q^3 \\ -w_{12}\dot{q}^1 - \frac{f_{12}^2}{g_{11}}\dot{q}^2 - \frac{f_{12}f_{13}}{g_{11}}\dot{q}^3 \\ -w_{13}\dot{q}^1 - \frac{f_{12}f_{13}}{g_{11}}\dot{q}^2 - \frac{f_{13}^2}{g_{11}}\dot{q}^3 \\ -f_{12}w_{11}\dot{q}^1 - f_{12}w_{12}\dot{q}^2 - f_{12}w_{13}\dot{q}^3 \\ -f_{13}w_{11}\dot{q}^1 - f_{13}w_{12}\dot{q}^2 - f_{13}w_{13}\dot{q}^3 \\ -w_{12}w_{11}\dot{q}^1 - w_{12}^2\dot{q}^2 - w_{12}w_{13}\dot{q}^3 \\ -w_{13}w_{11}\dot{q}^1 - w_{13}w_{12}\dot{q}^2 - w_{13}^2\dot{q}^3 \end{pmatrix}. \tag{A102}$$

Since $w_{12}f_{13} - f_{12}w_{13} \neq 0$, there is no nontrivial eigenvector for $W_{i_3 j}^{(3)}$. The algorithm ends.

To summarize, in this case, we have six constraints: $\phi_1^{(0)}$, $\phi_2^{(0)}$ in Eqs. (A16) and (A17), $\phi_1^{(1)}$, $\phi_2^{(1)}$ in Eqs. (A74) and (A75), and $\phi_1^{(2)}$, $\phi_2^{(2)}$ in Eqs. (A90) and (A91). Therefore there is no dynamical degree of freedom.

12. Summary

The classification of Lagrangians with one dynamical and two auxiliary variables is summarized in Table I. According to the types of constraints/identities as well as to that at which level these constraints/identities arise, there are in total eight cases (in some sense eight types of

TABLE I. Classification of the quadratic Lagrangians with one dynamical and two auxiliary variables. Here $\cdots$ indicates that there are neither constraints nor gauge identities available at the corresponding level.

	Level 0	Level 1	Level 2	Level 3	$\#_{\rm DOF}$
Case 1	$G_1^{(0)}$, $G_2^{(0)}$	$\cdots$	$\cdots$	$\cdots$	1
Case 2.1	$\phi^{(0)}$, $G^{(0)}$	$G^{(1)}$	$\cdots$	$\cdots$	0
Case 2.2.1		$\phi^{(1)}$	$\cdots$	$\cdots$	1
Case 2.2.2			$\phi^{(2)}$	$\phi^{(3)}$	0
Case 3.1	$\phi_1^{(0)}$, $\phi_2^{(0)}$	$\phi^{(1)}$, $G^{(1)}$	$\cdots$	$\cdots$	0
Case 3.2.1			$\cdots$	$\cdots$	1
Case 3.2.2		$\phi_1^{(1)}$, $\phi_2^{(1)}$	$\phi^{(2)}$	$\phi^{(3)}$	0
Case 3.2.3			$\phi_1^{(2)}$, $\phi_2^{(2)}$	$\cdots$	0

theories). Counting the number of degrees of freedom in the Lagrangian approach is discussed in [93–96], which is given by the simple formula [97]

$$\#_{\rm DOF} = N - \frac{1}{2}(l + g + e), \tag{A103}$$

in which N, l and g are the total numbers of the variables, the Lagrangian constraints and the gauge identities, respectively. e is the total number of the gauge parameters plus its successive derivatives. With this classification, it is transparent that the linear scalar perturbations in GR belong to "case 3.1." While what we explored in this work corresponds to "case 3.2.2" and "case 3.2.3" (together with case 3.1).

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