Quantum Hŏrava-Lifshitz cosmology in the de Broglie–Bohm interpretation

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Classical and quantum Friedmann-Lemaître-Robertson-Walker universes filled with noninteracting radiation and dust fluids are considered in the framework of Hořava-Lifshitz gravity theory. The Hořava-Lifshitz theory is set in its projectable version and without the detailed balance condition. Canonical quantization is performed in the Wheeler-DeWitt approach of quantum cosmology for a minisuperspace model in light of the de Broglie–Bohm interpretation of quantum mechanics. The main results are analytical solutions for nonsingular quantum bounce and cyclic universes for open and closed spatial sections in terms of the parameters of Hořava-Lifshitz theory.

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I. INTRODUCTION

The standard model of particle physics successfully describes electroweak and strong interactions, which operate at the quantum level. General Relativity (GR), which is a classical theory of space-time and matter, is the most successful theory of gravitation. A first principles description of nature seems to be quantum mechanical, making it a natural way to assume that gravity must be described by a quantum theory as well.

Applying the canonical quantization scheme to GR, we achieve the so-called canonical quantum gravity theory [1], which is a constrained system governed by the Wheeler-DeWitt equation [2]. The solution of the Wheeler-DeWitt equation yields the quantum state of the 3-geometry and matter fields by a wave functional (on superspace), which describes the entire Universe. However, this formulation faces some drawbacks. First, we expect the wave function to constrain the dynamics of the Universe, but the absence of a momentum which is canonically conjugate to the time variable implies that the wave function is static. This is called the problem of time [3]. Second, the wave function describes no particular metric, but all spacelike metrics (superspace). Both issues compromise the definition and identification of space-time singularities. Finally, measurement demands a collapse of the wave function by an external observer, which is a feature of the Copenhagen interpretation of quantum theory. This is called the measurement problem.

In order to circumvent these issues, we may consider the de Broglie–Bohm (dBB) interpretation of quantum theory [4,5] instead of the orthodox one. The time degree of freedom can now be defined from matter degrees of freedom if it is described by a hydrodynamical perfect fluid. This can be done using the Schutz formalism [6], where the time variable is associated with the fluid's potentials. We can also define a metric which evolves in time according to guidance equations. The measurement problem is either eliminated, because in the ontological interpretation the evolution of the Universe is deterministic and does not demand a collapse of the wave function, or the action of an external observer. Therefore, the dBB interpretation of canonical quantum gravity [7–11] is a suitable way to establish a quantum cosmology theory. In this context, one can define contraction/expansion as well as singularities in space-time.

In order to solve the Wheeler-DeWitt equation in a simple form, we restrict the superspace to the minisuperspace [12–14], where the degrees of freedom are reduced to a finite number while still preserving the main qualitative aspects of the full picture. This is a reasonable framework for developing quantum cosmology.

Quantum cosmology models in minisuperspace in the dBB interpretation are vast in the literature [7,15–28]. In particular, bouncing models are a very relevant feature of quantum cosmology, where quantum effects are responsible for avoiding the big bang (and eventually big crunch) singularity. In this context, the authors of Refs. [19,28–30] presented bouncing models for perfect fluids using the Schutz formalism, whereas in Ref. [31] a scalar field is considered as the matter content. In the case of perfect fluids, a very interesting result is given in Ref. [30], which is the quantum version of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe filled with radiation and dust, where radiation dominates around the bounce and dust dominates in the contraction and expansion phases far from the bounce. Cyclic universe solutions are also present for a closed spatial section, where multiple bounces are present. These results will be very important in this work.

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It is important to mention that a cosmological bounce can solve the standard cosmological puzzles [32], like the horizon and flatness problems, and are also robust for generating primordial cosmological perturbations with an almost scale invariant spectrum [28,32]. However, although an inflationary phase can be avoided for these models, in principle, it can still happen. Finally, dark matter has been also considered in this context through the dynamics of a scalar field [33].

Recently, a new theory of gravity was proposed by Hořava [34,35] which was based on the introduction of an anisotropic scaling between space and time. The theory was inspired by Lifshitz's work on critical exponents in phase transitions in the context of condensed matter physics, then referred to as Hořava-Lifshitz (HL) theory. The anisotropy is introduced in order to generalize GR in the high energy ultraviolet (UV) scale while recovering GR in the low energy infrared (IR) scale. This generalization is due to Lorentz symmetry breaking in the UV scale through a Lifshitz-like process, while this symmetry is preserved in the IR scale. Because of the anisotropy in space and time, HL theory is usually represented in the Arnowitt-Deser-Misner (ADM) formalism [36], which splits the 4-metric $g_{\mu\nu}$ into a 3-metric h_{ij}, a shift 3-vector N_i, and a lapse function N, where the last item parametrizes time.

There are two important issues to consider in HL theory. First, the variables in the ADM formalism may depend on both space and time. However, Hořava pointed out [34] that the lapse function N should depend only on time, which is reasonable in the cosmological setting. This is called the *projectable condition*. On the other hand, some authors have considered the general case in which N depends on both space and time, i.e., a nonprojectable condition. Both projectable and nonprojectable theories suffer from problems such as ghost scalar modes and instabilities [37–39]. However, a consistent projectable HL gravity theory with a local $U(1)$ symmetry can be shown to eliminate the ghost scalar modes [40,41] so that instability and strong coupling does not happen in the gravitational sector. On the other hand, nonprojectable HL gravity theories can also be consistent whether the local $U(1)$ symmetry is present [42–44] or not [45], where the latter theory is the so-called healthy extension. A review of the aforementioned issues is given in Ref. [37]. From another point of view, in the cosmological setting, it seems risky to impose the condition that the lapse function depend only on time. However, in this case the Hamiltonian constraint is no longer local but instead integrated over all spatial volume. This result corresponds to Friedmann equations with an additional cold dark matter–like component [46]. However, in homogeneous models these spatial integrals are simply the spatial volume then the additional matter content vanishes [47,48]. These considerations can be extended to the quantum realm in minisuperspace cosmology models [49,50], which is described by the projectable version of HL theory. It is then reasonable to consider the projectable HL in this context. It is important to mention that the aforementioned theory with local $U(1)$ symmetry can also be applied to the cosmological setting [51].

Second, Hořava also considered a simplification in order to reduce the number of terms of his theory which is called the *detailed balance condition*. However, although the detailed balance is a simplifying assumption, it is not really necessary [47,48]. Therefore, in this paper I consider the projectable version of HL theory without the detailed balance condition.

Nonsingular classical HL cosmological models have been considered in the literature for the matter bounce scenario [52], where the nonsingular behavior is due to the presence of spatial curvature. However, nonsingular classical cosmologies can also be achieved by other mechanisms, such as from matter contents [53] and other modified gravity theories [54]. On the other hand, in addition to the aforementioned quantum nonsingular cosmologies in minisuperspace in the dBB interpretation, the literature is vast for nonsingular quantum cosmological models [55] and also heuristically motivated ones [56]. In particular, there is also a great literature for quantum nonsingular cosmologies in the framework of Loop Quantum Cosmology [57–61].

Hořava-Lifshitz quantum cosmology in minisuperspace has already been considered in the literature [47,49,50,62–65]. In particular, exact solutions were obtained in Refs. [62,63] for perfect fluids for some values of the equation of state parameter and for some of the parameters of HL theory. In this work, I present exact solutions for nonsingular universes in the cases of closed and open FLRW quantum cosmologies in HL theory, where the matter content is composed of noninteracting radiation and dust fluids.

This paper is organized as follows. In Sec. II, I introduce the FLRW cosmology in HL gravity in the ADM formalism and the Schutz formalism for the matter content. The gravitational and matter Hamiltonians are presented and the full Hamiltonian for a universe filled with noninteracting radiation and dust fluids is obtained. In Sec. III, classical FLRW cosmology in HL theory is considered and analytical solutions for the scale factor are presented for open and closed universes. Some particular cases are also shown. These solutions contain all parameters of HL theory except the cosmological constantlike term, which is neglected in this work. In Sec. IV, I obtain nonsingular solutions for quantum FLRW cosmology in HL theory, which are the results of this paper. From the canonical quantization of the Hamiltonian, a Wheeler-DeWitt equation is obtained and analytically solved. From the solutions for the wave function, using the dBB interpretation of quantum mechanics, I obtain analytical solutions for the scale factor for noninteracting radiation and dust fluids for open and closed universes. Single (contracting/expanding) bounce universe solutions were obtained as well as (multiple bounce) cyclic ones. The analytical quantum potential was also derived for these solutions, which is responsible for the occurrence of nonsingular behavior in the scale factor evolution. In the Appendix, the motivation for the initial condition used in Sec. IV is discussed. Finally, the concluding remarks and future perspectives are presented in Sec. V. Throughout this work, I am using the natural units system, in which $c = \hbar = 1$.

II. FLRW COSMOLOGY IN HOŘAVA-LIFSHITZ GRAVITY

The FLRW metric for a homogeneous and isotropic space-time is written as

$$ds^2 = -N(t)dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \qquad (2.1)$$

where $N(t)$ is the lapse function, $a(t)$ is the scale factor, $d\Omega^2$ is the line element of a 2-sphere with unitary radius, and k is the curvature constant of spatial sections, which is $k = 1$, 0, and -1 for closed, flat and open universes, respectively.[1] The 4-metric is defined as $g_{\mu\nu}$, where greek indices run from 0 to 3, and the spatial 3-metric is defined as h_{ij}, where the latin indices run from 1 to 3.

In the following subsections, I introduce the Hamiltonian densities for the gravitational and matter sections in HL theory.

A. Gravitational Hamiltonian

In HL theory, an anisotropic scaling between space and time is introduced as

$$t \to b^z t, \qquad \vec{x} \to b\vec{x}, \qquad (2.2)$$

where b is a scale parameter and z is a dynamical critical exponent. While the UV sector requires $z = 3$, which breaks Lorentz invariance, in the IR sector it is recovered for $z = 1$ (see Refs. [34,35] for details).

The gravitational action of HL theory is composed of kinetic and potential parts. The former generalizes the Einstein-Hilbert action and is given in terms of the extrinsic curvature (and derivatives) and a free parameter λ, which reduces to the GR kinetic term in the limit $\lambda \to 1$. On the other hand, the potential part depends only on the 3-metric (and spatial derivatives). The detailed balance condition refers to the choice of potential part, which simplifies the theory by reducing the number of terms and facilitating its renormalization. However, Sotiriou *et al.* [47,48] showed that one should avoid this condition.

Following the lines of Refs. [62–64], in this paper I consider the projectable HL gravity without the detailed balance condition, for $z = 3$ in $(3+1)$ dimensions [49], whose action reads

$$\begin{aligned} S_{\mathrm{HL}} = \frac{M_{\mathrm{Pl}}^2}{2}\int_{\mathcal{M}} d^3x dt N\sqrt{h}[&K_{ij}K^{ij} - \lambda K^2 + \\ &- g_0 M_{\mathrm{Pl}}^2 - g_1 R - M_{\mathrm{Pl}}^{-2}(g_2 R^2 + g_3 R_{ij}R^{ij}) + \\ &- M_{\mathrm{Pl}}^{-4}(g_4 R^3 + g_5 R R^i_j R^j_i + g_6 R^i_j R^j_k R^k_i + \\ &+ g_7 R\nabla^2 R + g_8 \nabla_i R_{jk}\nabla^i R^{jk})] \\ &+ M_{\mathrm{Pl}}^2 \int_{\partial\mathcal{M}} d^3x\sqrt{h}K, \end{aligned} \qquad (2.3)$$

where $M_{\mathrm{Pl}} = 1/\sqrt{8\pi G}$ is the Planck mass, h_{ij} is defined on the boundary $\partial\mathcal{M}$ of the 4-dimensional manifold $\mathcal{M}$, h is the determinant of h_{ij}, K_{ij} is the extrinsic curvature tensor, K is the trace of K_{ij}, R_{ij} is the Ricci tensor, R is the Ricci scalar, and λ and g_i $(i = 0, \ldots, 8)$ are HL parameters involved in corrections to GR.

In the case of FLRW space-time, the metric and the action are given by Eqs. (2.1) and (2.3), respectively. From the results of Ref. [62], the Lagrangian density in HL theory reads

$$\mathcal{L}_{\mathrm{HL}} = N\left(-\frac{a\dot{a}^2}{N^2} + g_c k a - g_\Lambda a^3 - \frac{g_r k^2}{a} - \frac{g_s k}{a^3}\right), \qquad (2.4)$$

where the HL parameters are defined as [68]

$$\begin{aligned} g_c &= \frac{2}{3\lambda - 1}, & g_\Lambda &= \frac{2\Lambda}{3(3\lambda-1)}, \\ g_r &= \frac{4(3g_2 + g_3)}{(3\lambda-1)M_{\mathrm{Pl}}^2}, & g_s &= \frac{8(9g_4 + 3g_5 + g_6)}{(3\lambda-1)M_{\mathrm{Pl}}^4}, \end{aligned} \qquad (2.5)$$

and the parameter g_c is positive definite. The subscripts refer to the fluidlike behavior of each term: $(g_c, g_\Lambda, g_r, g_s) =$ (curvature, cosmological constant, radiation, stiff matter)–like terms.

We note that the scale factor is the only variable in Eq. (2.4). In order to obtain the Hamiltonian formulation, we must compute P_a, the momentum canonically conjugated to a, which reads

$$P_a = \frac{\partial\mathcal{L}_{\mathrm{HL}}}{\partial\dot{a}} = -\frac{2a\dot{a}}{N}. \qquad (2.6)$$

Using the definition of P_a, the Hamiltonian density reads

$$H_{\mathrm{HL}} = N\left(-\frac{P_a^2}{4a} - g_c k a + g_\Lambda a^3 + \frac{g_r k^2}{a} + \frac{g_s k}{a^3}\right). \qquad (2.7)$$

[1] I will not discuss the possibility of the Universe being open or closed. However, Refs. [66,67] provide more details about the possibility of a closed Universe.

The lapse function N appears in $H_{\rm HL}$ as a Lagrangian multiplier (there is no momentum canonically conjugated to N). Rewriting $H_{\rm HL}$ as $H_{\rm HL} = N\mathcal{H}_{\rm HL}$, when we vary $H_{\rm HL}$ with respect to N, we obtain [11] $\mathcal{H}_{\rm HL} \approx 0$ ($\approx$ means weakly zero). This is called the *super-Hamiltonian constraint*.

B. Matter Hamiltonian

A perfect fluid can be described by a Hamiltonian using the Schutz formalism [6]. The action for a perfect fluid in this formalism reads

$$S_{\rm m} = \int d^4x\sqrt{-g}(16\pi p), \tag{2.8}$$

where g is the determinant of $g_{\mu\nu}$ and p is the pressure of the fluid. The equation of state for a perfect fluid is written as

$$p = \omega\rho, \tag{2.9}$$

where ρ is the energy density and ω is the equation of state parameter, which is subject to $-1 \leq \omega \leq 1$.

The Schutz formalism consists of writing the fluid 4-velocity U_μ in terms of six velocity potentials, where in FRLW cosmology these potentials are reduced to three. Substituting the FLRW metric (2.1) into the $S_{\rm m}$ action (2.8) and identifying the canonical variables and performing some canonical transformations [69], we obtain a simple Hamiltonian for a single perfect fluid which reads

$$H_{\rm m} = N\frac{P_T}{a^{3\omega}}, \tag{2.10}$$

where P_T is the momentum canonically conjugated to the fluid degree of freedom T, which can be interpreted as a time variable. The Hamiltonian $H_{\rm m}$ can also be written as $H_{\rm m} = N\mathcal{H}_{\rm m}$, where $\mathcal{H}_{\rm m}$ is a Lagrangian multiplier similar to $\mathcal{H}_{\rm HL}$ of the gravitational part.

C. Full Hamiltonian

From the gravitational and matter Hamiltonians, Eqs. (2.7) and (2.10), respectively, the minisuperspace Hamiltonian for a single perfect fluid in HL theory reads

$$H = N\left(-\frac{P_a^2}{4a} - g_c ka + g_\Lambda a^3 + \frac{g_r k^2}{a} + \frac{g_s k}{a^3} + \frac{P_T}{a^{3\omega}}\right), \tag{2.11}$$

from which we can also define the super-Hamiltonian constraint $\mathcal{H} = \mathcal{H}_{\rm HL} + \mathcal{H}_m$, which satisfies

$$\mathcal{H} \approx 0. \tag{2.12}$$

In the following sections, I will show that this constraint leads to the Friedmann equation in the classical level, whereas at the quantum level it gives the Wheeler-DeWitt equation. In the latter, the super-Hamiltonian constraint is essential in the procedure of canonical quantization.

In this work, I will consider a HL quantum FLRW universe filled with noninteracting radiation ($\omega = 1/3$) and dust ($\omega = 0$) fluids. However, Eq. (2.11) is valid only for a single fluid. I will set $\omega = 1/3$ in order to describe a radiation fluid and, following Ref. [30], I will also include dust as a second decoupled fluid by introducing another term of the type of Eq. (2.10) for $\omega = 0$. From these considerations, the Hamiltonian for radiation and dust fluids in HL theory reads

$$\begin{aligned} H_{\rm rd} = N\mathcal{H}_{\rm rd} = N\bigg(&-\frac{P_a^2}{4a} - g_c ka + g_\Lambda a^3 \\ &+ \frac{g_r k^2}{a} + \frac{g_s k}{a^3} + \frac{P_T}{a} + P_\varphi\bigg), \end{aligned} \tag{2.13}$$

where P_φ is a constant stemming from Eq. (2.10) for $\omega = 0$ and $\mathcal{H}_{\rm rd}$ is the super-Hamiltonian related to $H_{\rm rd}$.

Additionally, for the potential part of HL theory, I will consider from now on the particular case of no cosmological constantlike term ($g_\Lambda = 0$).

III. CLASSICAL DYNAMICS HŎRAVA-LIFSHITZ COSMOLOGY

In this section, I consider analytical solutions of the HL classical cosmology in minisuperspace for noninteracting radiation and dust fluids. The equations of motion for each system variable and its canonically conjugate momentum are then obtained from the evaluation of the Poisson brackets of each of them with the Hamiltonian.

The HL super-Hamiltonian constraint containing radiation and dust fluids, Eq. (2.13), together with the equation for P_a, Eq. (2.6), result in the classical Friedmann equation,

$$\left(\frac{\dot a}{a}\right)^2 = N^2\left(\frac{g_s k}{a^6} + \frac{g_r k^2}{a^4} - \frac{g_c k}{a^2} + \frac{P_T}{a^4} + \frac{P_\varphi}{a^3}\right). \tag{3.1}$$

From the Hamiltonian, Eq. (2.13), the equations of motion for T and φ read

$$\dot T = \{T, H_{\rm rd}\} = \frac{N}{a} \to adT = Ndt,$$
$$\dot\varphi = \{\varphi, H_{\rm rd}\} = N \to d\varphi = Ndt.$$

One needs to choose a gauge, which corresponds to choosing the lapse function N in order to define the time variable. In principle, both T and φ can be the time variable. If we choose $N = 1$, φ is the cosmic time, whereas for $N = a$ we obtain the result that T is the conformal time. Analytical solutions of the Friedmann equation, Eq. (3.1), can be obtained only for the latter, so I will consider the gauge $N = a$ throughout this paper, which means that

$T = \eta$, where η is the conformal time. Therefore, Eq. (3.1) now reads

$$\left(\frac{a'}{a^2}\right)^2 = \frac{g_s k}{a^6} + \frac{g_r k^2}{a^4} - \frac{g_c k}{a^2} + \frac{P_\eta}{a^4} + \frac{P_\varphi}{a^3}, \tag{3.2}$$

where the prime denotes the derivative with respect to conformal time.

The analytical solutions for Eq. (3.2) with the initial condition $a(0) = 0$ read

$$a(\eta) = \begin{cases} \sqrt{\sqrt{\frac{g_s}{g_c}}\sin(2\sqrt{g_c}\eta) + \frac{(P_\eta + g_r)}{g_c}\sin^2(\sqrt{g_c}\eta)} + \frac{P_\varphi}{2g_c}[1 - \cos(\sqrt{g_c}\eta)], & k = 1, \\ \sqrt{\sqrt{\frac{|g_s|}{g_c}}\sinh(2\sqrt{g_c}\eta) + \frac{(P_\eta + g_r)}{g_c}\sinh^2(\sqrt{g_c}\eta)} + \frac{P_\varphi}{2g_c}[\cosh(\sqrt{g_c}\eta) - 1], & k = -1, \end{cases} \tag{3.3}$$

where g_c and the product $g_s k$ are both positive definite. The latter is positive when $g_s > 0$ ($g_s < 0$) for $k = 1$ ($k = -1$). From now on, the notation $g_s k = |g_s|$ will represent the case $g_s < 0, k = -1$ or both the $g_s > 0, k = 1$ and $g_s < 0, k = -1$ cases when the results for both $k = 1$ and $k = -1$ can be written in a single expression. From Eq. (3.2), the stiff matterlike term involving g_s initially dominates near the singularity at $\eta = 0$, followed by radiation dominance, which consists of a radiation fluid plus a "HL radiation" term involving g_r. Far from the singularity, radiation domination is followed by dust domination and ends up with domination of the curvature term involving g_c. If the cosmological constant term were not neglected, as in the Hamiltonian given by Eq. (2.13), far from the singularity it would dominate after the curvature term.

Now I consider some particular cases. In the limit where the curvaturelike term is negligible ($g_c \to 0$), Eqs. (3.3) read

$$a(\eta) = \sqrt{2\sqrt{|g_s|}\eta + (P_\eta + g_r)\eta^2} + \frac{P_\varphi \eta^2}{4}, \tag{3.4}$$

which is valid for both $k = 1$ and $k = -1$. Also, when we additionally consider that the stiff matterlike time is negligible ($g_s \to 0$), the latter result reduces to

$$a(\eta) = \sqrt{P_\eta + g_r k^2}\eta + \frac{P_\varphi}{4}\eta^2. \tag{3.5}$$

The remaining term from HL theory is the radiationlike constant, g_r, which adds up to usual radiation. When $k = 0$ or in the limit $g_r \to 0$, HL reduces to GR and one obtains a flat universe filled with radiation and dust fluids, where radiation dominates near the singularity and dust dominates far from it.

On the other hand, when stiff matter like time is negligible ($g_s \to 0$) in Eqs. (3.3), one obtains

$$a(\eta) = \begin{cases} \sqrt{\frac{(P_\eta + g_r)}{g_c}}\sin(\sqrt{g_c}\eta) + \frac{P_\varphi}{2g_c}[1 - \cos(\sqrt{g_c}\eta)], & k = 1, \\ \sqrt{\frac{(P_\eta + g_r)}{g_c}}\sinh(\sqrt{g_c}\eta) + \frac{P_\varphi}{2g_c}[\cosh(\sqrt{g_c}\eta) - 1], & k = -1. \end{cases} \tag{3.6}$$

In the following, I will consider the HL quantum solutions.

IV. QUANTUM DYNAMICS HOŘAVA-LIFSHITZ COSMOLOGY

In this section, I consider analytical solutions of the HL quantum cosmology in minisuperspace for noninteracting radiation and dust fluids. The cyclic and bounce universe solutions presented in this section are obtained for the first time for HL theory for nonzero g_c, g_r, and g_s and for radiation and dust fluids.

The HL super-Hamiltonian constraint containing radiation and dust fluids, from Eq. (2.13), reads

$$\mathcal{H}_{\rm rd} = -\frac{P_a^2}{4a} - g_c k a + \frac{g_r k^2}{a} + \frac{g_s k}{a^3} + \frac{P_\eta}{a} + P_\varphi \approx 0, \tag{4.1}$$

where I have considered the lapse function $N = a$ and $g_\Lambda = 0$ as in Sec. III. Using the Dirac formalism for constrained systems, the super-Hamiltonian constraint is promoted to an operator, which annihilates the quantum wave function of the Universe, Ψ, in the form

$$\hat{\mathcal{H}}_{\rm rd}\Psi = 0. \tag{4.2}$$

This is the so-called Wheeler-DeWitt equation. From Eqs. (4.1) and (4.2), one obtains

$$i\partial_\eta \Psi = \left(\frac{1}{4}\partial_a^2 - g_c k a^2 + g_r k^2 + \frac{g_s k}{a^2} - i a \partial_\varphi\right)\Psi, \quad (4.3)$$

where $\hat{P}_a = -i\partial_a$, $\hat{P}_\eta = -i\partial_\eta$, and $\hat{P}_\varphi = -i\partial_\varphi$ (see Refs. [30,69]) and $\Psi = \Psi(a,\varphi,\eta)$. However, two important considerations must be made. First, there is an operator-ordering ambiguity in the Wheeler-DeWitt equation, which is related to the choice of measure in the path integral in the canonical quantization procedure when one replaces the momentum P_a with its corresponding operator [70]. In order to account for this ambiguity, one must rewrite the kinetic term properly. Second, in order to get rid of the derivative ∂_φ, following Ref. [30] I will consider that the wave function Ψ is an eigenstate of the dust matter operator, such that $\hat{P}_\varphi|\Psi\rangle = P_\varphi|\Psi\rangle$, which implies that dust matter is conserved.[2] Therefore, the wave function Ψ in an eigenstate of $\hat{P}_\varphi$ with eigenvalue P_φ and the wave function $\Psi = \Psi(a,\varphi,\eta)$ can be written as

$$\Psi(a,\varphi,\eta) = \Psi(a,\eta)e^{iP_\varphi \varphi}. \quad (4.4)$$

From these considerations, Eq. (4.3) now reads

$$i\partial_\eta \Psi(a,\eta) = \left(-\frac{1}{4}\partial_a^2 + \frac{\alpha}{4a}\partial_a + g_c k a^2 - g_r k^2 - \frac{g_s k}{a^2} - P_\varphi a\right)\Psi(a,\eta), \quad (4.5)$$

where the parameter α represents the ambiguity in the ordering of a and P_a in the kinetic term of Eq. (4.1) and the transformation $\eta \to -\eta$ is also considered. The appropriate choice of α will be useful in the following calculations, although the results must not depend on it. On the other hand, the sign change in the time variable was done in order to write the Wheeler-DeWitt equation as a Schrödinger-type equation (except for the term involving α at this point).

A. De Broglie–Bohm interpretation

The wave function Ψ can be written in the polar form as $\Psi = Re^{iS}$, such that the imaginary and real parts give evolution equations for R and S, respectively, which are real functions. In order to introduce the dBB interpretation, I write the Lagrangian for Eq. (4.5), which reads

$$\mathcal{L}_{\rm rd} = a^{-\alpha}\left[i\Psi^*\partial_\eta\Psi - \frac{1}{4}\partial_a\Psi^*\partial_a\Psi - V\Psi^*\Psi\right], \quad (4.6)$$

where

$$V = g_c k a^2 - \frac{g_s k}{a^2} - g_r k^2 - P_\varphi a \quad (4.7)$$

is an external classical potential. From Noether's theorem [71], one knows that the invariance of Ψ under internal symmetry ($\Psi \to e^{i\theta}\Psi$) results in a conserved charge ρ and a conserved current J which are related by the following continuity equation:

$$\partial_\eta(a^{-\alpha}R^2) + \partial_a\left(a^{-\alpha}R^2\frac{\partial_a S}{2}\right) = 0, \quad (4.8)$$

where $\rho = a^{-\alpha}R^2$ and $J = a^{-\alpha}R^2(\partial_a S)/2$. This equation is the exact imaginary part of Eq. (4.5) mentioned before and is interpreted as the equation of conservation of probability.

In the dBB interpretation, particles have a deterministic trajectory, which is given by $a(\eta)$ in the present case. Therefore, an equation of motion must be postulated. This equation can be built out of $J = \rho v$, where $v = a'$ is the velocity of the particle. Therefore,

$$a' = \frac{\partial_a S}{2}, \quad (4.9)$$

which is known as the *guidance equation*.[3] From the knowledge of S and an initial condition for a, one can integrate it to obtain $a(\eta)$. Also, one can notice the trajectory of the particle is independent of the choice of α.

On the other hand, the real part of Eq. (4.5) in terms of R and S reads

$$\partial_\eta S + \frac{(\partial_a S)^2}{4} + V + Q = 0. \quad (4.10)$$

This equation is a Hamilton-Jacobi-type equation, where the last term,

$$Q = -\frac{1}{4}\frac{\partial_a^2(a^{-\alpha}R)}{(a^{-\alpha}R)}, \quad (4.11)$$

is the so-called quantum potential. Therefore, from Eq. (4.10), one concludes that deterministic trajectories are subject to classical and quantum potentials.

Therefore, in order to obtain a solution for $a(\eta)$ from Eq. (4.9), one needs to solve both Eq. (4.8) and Eq. (4.10), which are coupled equations for R and S.

B. Analytical results

The main goal of this paper is to present analytical solutions for Eq. (4.5). There is a solution for a similar problem in the literature, which can be adapted to the

[2]As a first attempt, I consider this simple case, where the evolution is nonunitary. However, one can manage to obtain a unitary solution in which dust matter creation is possible [30].

[3]For arbitrary lapse function N and equation of state parameter ω, and undoing the transformation $\eta \to -\eta$, $a' = -\frac{N}{2a}(\partial_a S)$.

present case. However, some changes of variables must be done in order to use this solution. These details will be presented in the following. From the analytical solutions of the Wheeler-DeWitt equation, analytical solutions can be obtained for the scale factor, thereby solving Eq. (4.9).

The aforementioned solution is given in Ref. [30]. This solution was given for the particular case of Eq. (4.5) when $g_c = 1$, $g_r = 0$, and $g_s = 0$. Additionally, Falciano *et al.* set $\alpha = 0$. The Wheeler-DeWitt equation for this case [Eq. (25) from Ref. [30]] reads

$$i\partial_\eta \psi(a,\eta) = \left(-\frac{1}{2m}\partial_a^2 + \frac{m\omega_0^2}{2}a^2 - P_\varphi a\right)\psi(a,\eta). \tag{4.12}$$

This is a one-dimensional Schrödinger equation for a particle with mass m and a potential which contains a harmonic oscillator term with a frequency ω_0 and a constant force P_φ. In order to use this solution for the present case, one needs to reduce Eq. (4.5) to Eq. (4.12) while performing some changes of variables. One also needs to set $\alpha \neq 0$, which is in fact the key point here and, as shown in Sec. IV A, it does not affect the result for the scale factor.

I consider the following change of variables for the wave function of Eq. (4.5):

$$\Psi(a,\eta) = e^{ig_r k^2 \eta} a^{\alpha/2} \mu(a,\eta), \tag{4.13}$$

where $\alpha = -1 \pm \sqrt{1 + 16 g_s k}$. Both values of α are suitable to absorb the term g_s/a^2. I choose

$$\alpha = -1 + \sqrt{1 + 16 g_s k}, \tag{4.14}$$

where one obtains $\alpha \to 0$ when $g_s \to 0$. The term $-g_r k^2$ is also absorbed into the complex exponential. From these changes of variables, Eq. (4.5) is now an equation for $\mu(a,\eta)$ which reads

$$i\partial_\eta \mu(a,\eta) = \left(-\frac{1}{4}\partial_a^2 + g_c k a^2 - P_\varphi a\right)\mu(a,\eta). \tag{4.15}$$

This equation is exactly Eq. (4.12) for the particular case when

$$m = 2, \tag{4.16}$$

$$\omega_0 = \sqrt{g_c k}. \tag{4.17}$$

For these values of ω_0 and m, the analytical result for the wave function given by Eq. (41) of Ref. [30] reads

$$\begin{aligned}
\mu(a,\eta) &= \left(\frac{8\bar{\sigma}}{\pi}\right)^{1/4} \sqrt{\frac{1}{\cos(\sqrt{g_c k}\eta)\left[1 + i\bar{\sigma}\frac{\tan(\sqrt{g_c k}\eta)}{\sqrt{g_c k}}\right]}} \\
&\times \exp\left\{\frac{i\sqrt{g_c k}}{\tan(\sqrt{g_c k}\eta)} \left[a^2 - \frac{\left(a - \frac{P_\varphi}{2}\frac{[1-\cos(\sqrt{g_c k}\eta)]}{g_c k}\right)^2}{\cos(\sqrt{g_c k}\eta)^2\left[1 + i\bar{\sigma}\frac{\tan(\sqrt{g_c k}\eta)}{\sqrt{g_c k}}\right]}\right.\right. \\
&+ a P_\varphi \frac{1 - \cos(\sqrt{g_c k}\eta)}{g_c k \cos(\sqrt{g_c k}\eta)} \\
&\left.\left. + \frac{P_\varphi^2}{2 g_c k}\left(\eta \frac{\tan(\sqrt{g_c k}\eta)}{\sqrt{g_c k}} - \frac{1 - \cos(\sqrt{g_c k}\eta)}{g_c k \cos(\sqrt{g_c k}\eta)}\right)\right]\right\}.
\end{aligned} \tag{4.18}$$

The solution for $\mu(a,\eta)$ was obtained using the propagator of the forced quantum harmonic oscillator using the following initial state:

$$\mu_0(a,0) = \left(\frac{8\bar{\sigma}}{\pi}\right)^{1/4} e^{-\bar{\sigma}a^2}, \tag{4.19}$$

where $\bar{\sigma} = \sqrt{\sigma^2 - g_s k} + i\sqrt{g_s k}$. Notice that both the solution and its initial condition contain the parameter g_s in the constant $\bar{\sigma}$, although there is no term involving g_s in Eq. (4.15). In fact, only the parameters σ and P_φ, which are related to radiation and dust fluids, respectively, were expected. Additionally, the dynamics for this matter content is radiation domination near $\eta = 0$ and dust domination far from it. Therefore, an initial condition at $\eta = 0$ must involve only the parameter σ. However, the full problem considered in this section before the changes of variables involves a g_s term, which, from the classical dynamics of Sec. III, behaves as a stiff matterlike fluid and dominates over radiation near $\eta = 0$. One should then expect that an initial condition for this problem uniquely involves g_s. Nonetheless, owing to the fact that the g_s term does not represent a fluid, radiation fluid degrees of freedom must also be present in the initial condition, i.e., this term must be a function of both g_s and σ. In other words, although changes of variables are performed and g_s is no longer explicit in Eq. (4.15), the g_s information is encoded in $\bar{\sigma}$ due to its relevance near $\eta = 0$. In the Appendix, I derive an ansatz solution for the case of a radiation dominated quantum FLRW universe in the framework of HL theory where only the g_s parameter is nonzero. The obtained ansatz justifies this choice of initial condition. Analogously, the g_r radiation fluidlike term is encoded in the radiation fluid degree of freedom, σ.

From this analytical result, the expression for $\Psi(a,\varphi,\eta)$ from Eqs. (4.4), (4.13), and (4.14) reads

$$\Psi(a,\varphi,\eta) = e^{ig_r k^2\eta} e^{iP_\varphi\varphi} a^{(-1+\sqrt{1+16g_s k})/2}\mu(a,\eta), \tag{4.20}$$

where $\mu(a,\eta)$ is given by the solution of Eq. (4.18).

From Eq. (4.20), one can obtain $S(a,\eta)$ and substitute it into Eq. (4.9). Solving this equation for the initial condition $a(0) = a_B$, one obtains

$$a(\eta) = \begin{cases} a_B\sqrt{\cos^2(\sqrt{g_c}\eta) + \sqrt{\frac{g_s}{g_c}}\sin(2\sqrt{g_c}|\eta|) + \frac{\sigma^2}{g_c}\sin^2(\sqrt{g_c}\eta) + \frac{P_\varphi}{2g_c}[1-\cos(\sqrt{g_c}\eta)]}, & k=1, \\ a_B\sqrt{\cosh^2(\sqrt{g_c}\eta) + \sqrt{\frac{|g_s|}{g_c}}\sinh(2\sqrt{g_c}|\eta|) + \frac{\sigma^2}{g_c}\sinh^2(\sqrt{g_c}\eta) + \frac{P_\varphi}{2g_c}[\cosh(\sqrt{g_c}\eta)-1]}, & k=-1. \end{cases} \tag{4.21}$$

These are quantum bounce solutions in which, in contrast to the classical solutions, the squared trigonometric and hyperbolic cosine novel terms play the role of avoiding singularities. These results enable the study of the background quantum cosmology for this model from an analytical perspective.

The positive spatial section solution represents a cyclic universe (no big bang or big crunch singularities), whereas the negative spatial section solution has a unique bounce which connects a contraction and an expansion phase. However, one must consider some limits for the constants g_c and g_s in order to appreciate the overlap of these effects. Some plots will be presented later in this section in order to illustrate the behavior of these solutions for different values of its constants.

Comparing Eqs. (4.21) to the classical solutions, Eqs. (3.3), one obtains that in the classical limit $\sigma^2 = P_\eta + g_r k^2$, which means that σ is a degree of freedom related to a radiation fluid. Therefore, g_r contributes only to an effective radiation fluid density and does not appear explicitly in $a(\eta)$. Also, the classical behavior is obtained when

$$\left.\begin{array}{l} |\tan(\sqrt{g_c}\eta)| \\ |\tanh(\sqrt{g_c}\eta)| \end{array}\right\} \gg \frac{\sqrt{g_c}(\sqrt{|g_s|+\sigma^2} - \sqrt{|g_s|})}{\sigma^2} \tag{4.22}$$

for $k=1$ and $k=-1$, respectively. In the limit where $g_c \to 1$ and $g_s \to 0$, these results reduce to $1/\sigma$ for both values of k, where σ contains the $g_r k^2$ term. Also, both results reduce to $|\eta| \gg 1/\sigma$ when one additionally considers the limit $k \to 0$ (see Ref. [30]).

One must draw attention to the module $|\eta|$ in the terms involving g_s in Eqs. (4.21). This is because the wave function given in Eq. (4.18) was obtained by the propagation of the initial condition only for negative values of η. Therefore, one needs to also solve for positive values of η [setting $\eta \to -\eta$ in both Eq. (4.9) and Eq. (4.18)] and incorporate it into the solutions. Owing to the presence of odd trigonometric and hyperbolic sines, the module ensures that the solution is symmetric, which can be confirmed from the classical solutions of Sec. III. This observation was not necessary in the calculations of Ref. [30], because only even functions appear in the results for $a(\eta)$.

As in Sec. III, I will consider some particular cases of Eqs. (4.21). In the limit where $g_c \to 0$, for both values of k it reduces to

$$a(\eta) = a_B\sqrt{1 + 2\sqrt{|g_s|}|\eta| + \sigma^2\eta^2 + \frac{P_\varphi\eta^2}{4}}. \tag{4.23}$$

Also considering in the latter equation the limit where $g_s \to 0$, it reduces to

$$a(\eta) = a_B\sqrt{1 + \sigma^2\eta^2 + \frac{P_\varphi\eta^2}{4}}, \tag{4.24}$$

where now $\bar{\sigma} = \sigma$. This result resembles the case of a FLRW flat universe in GR with radiation and dust [30], except for the HL radiation term correction for $k \neq 0$. When $k = 0$, HL theory reduces to GR and the results are exactly the same.

On the other hand, when $g_s \to 0$, Eqs. (4.21) reduce to

$$a(\eta) = \begin{cases} a_B\sqrt{\cos^2(\sqrt{g_c}\eta) + \frac{\sigma^2}{g_c}\sin^2(\sqrt{g_c}\eta) + \frac{P_\varphi}{2g_c}[1-\cos(\sqrt{g_c}\eta)]}, & k=1, \\ a_B\sqrt{\cosh^2(\sqrt{g_c}\eta) + \frac{\sigma^2}{g_c}\sinh^2(\sqrt{g_c}\eta) + \frac{P_\varphi}{2g_c}[\cosh(\sqrt{g_c}\eta)-1]}, & k=-1. \end{cases} \tag{4.25}$$

These results are qualitatively similar to those of Ref. [30] when $g_c = 1$, which are given by

$$a(\eta) = \begin{cases} a_B\sqrt{\cos^2(\eta) + \sigma^2\sin^2(\eta) + \frac{P_\varphi}{2}[1-\cos(\eta)]}, & k=1, \\ a_B\sqrt{\cosh^2(\eta) + \sigma^2\sinh^2(\eta) + \frac{P_\varphi}{2}[\cosh(\eta)-1]}, & k=-1. \end{cases} \tag{4.26}$$

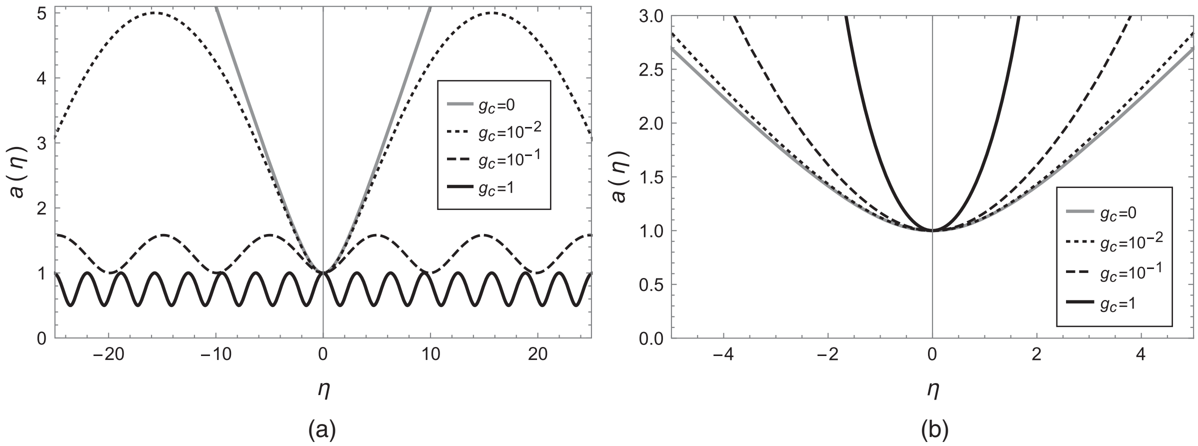

FIG. 1. Analytical scale factor in quantum HL theory as a function of the conformal time in the limit $g_s \to 0$. Plots are given for $a_B = 1$, $P_\varphi = 0$, $\sigma = 0.5$, and certain representative values of g_c. In (a) and (b), the spatial sections $k = 1$ and $k = -1$, respectively, are considered.

In order to illustrate the results, I present some plots of the scale factor for representative values, which capture the effect of each constant in its evolution. I will consider $P_\varphi = 0$ for all plots and focus on the effects of g_c and g_s, which are introduced by HL theory. Additionally, in all plots there is a *gray curve* which corresponds to the solution with respect to GR, where the HL parameters are null, that is given by Eq. (4.24).

In Fig. 1, the scale factor is considered in the limit where $g_s \to 0$, which is given by Eqs. (4.25). In Fig. 1(a), one notices that for $k = 1$ nonzero values of g_c produce oscillating universes which are cyclic and which have multiple bounces. Starting from $g_c = 0$, for increasing values of g_c the oscillation frequency grows and the oscillation amplitude becomes smaller. In Fig. 1(b), for $k = -1$, increasing values of g_c make the bounce more curved.

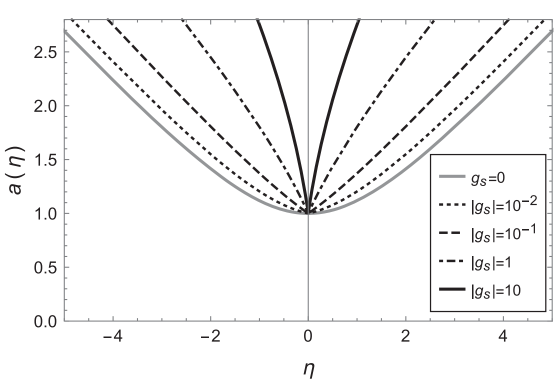

FIG. 2. Analytical scale factor in quantum HL theory as a function of the conformal time in the limit $g_c \to 0$. The plots are given for $a_B = 1$, $P_\varphi = 0$, $\sigma = 0.5$, and certain representative values of g_s. The plots are valid for both spatial sections, $k = \pm 1$, where $g_s > 0\,(g_s < 0)$ for $k = 1\,(k = -1)$.

In Fig. 2, the scale factor is considered in the limit where $g_c \to 0$, which is given by Eq. (4.23). In this case, for both values of the spatial section, the bouncing universes become more curved for increasing values of g_s. This is qualitatively similar to the case $g_s \to 0$ for $k = -1$ of Fig. 1(b), where it is the increase of g_c that progressively curves the universes. However, their difference is that in the present case $a(\eta)/a_B \approx 1 + \sqrt{g_s}|\eta| + \mathcal{O}(\eta^2)$ near the bounce, whereas $a(\eta)/a_B \approx 1 + (g_c + \sigma^2)\eta^2/2 + \mathcal{O}(\eta^3)$ in Fig. 1(b). Therefore, the bounce solution is more abrupt in the present case.

In Fig. 3, the scale factor is considered for nonzero values of both g_c and g_s, which are given by Eqs. (4.21). The effects are overlapping, which is the more general case. In Fig. 3(a), where $k = 1$, one notices that the oscillatory behavior due to nonzero values of g_c is affected by nonzero values of g_s in two aspects: (i) in the oscillatory regime, the frequency and the oscillation amplitude increase (far from the bounce), and (ii) the bounce around $\eta = 0$ is linear, whereas the other bounces are qualitatively similar and deeper. In Fig. 3(b), where $k = -1$, the effects of both increasing values of g_c and g_s, as I show in Fig. 1(b) and Fig. 2, make the bounce more curved. Therefore, the effects add up. However, near the bounce the scale factor reads $a(\eta)/a_B \approx 1 + \sqrt{g_s}|\eta| + (g_c - |g_s| + \sigma^2)\eta^2/2 + \mathcal{O}(\eta^3)$, which means that g_s dominates near the bounce, whereas as we move away from the bounce both effects become relevant and g_c eventually dominates over g_s.

From the solution for the wave function, Eq. (4.20), one can also obtain the quantum potential, Eq. (4.11), which is responsible for the quantum effects. The full analytical expression is lengthy, but one can obtain a reasonable approximation while noticing from Eqs. (4.21) that the dust fluid contribution dominates only far from the bounce. The quantum effects are present in the entire evolution for $k = 1$ when $g_c \neq 0$, but the dust fluid contribution does not

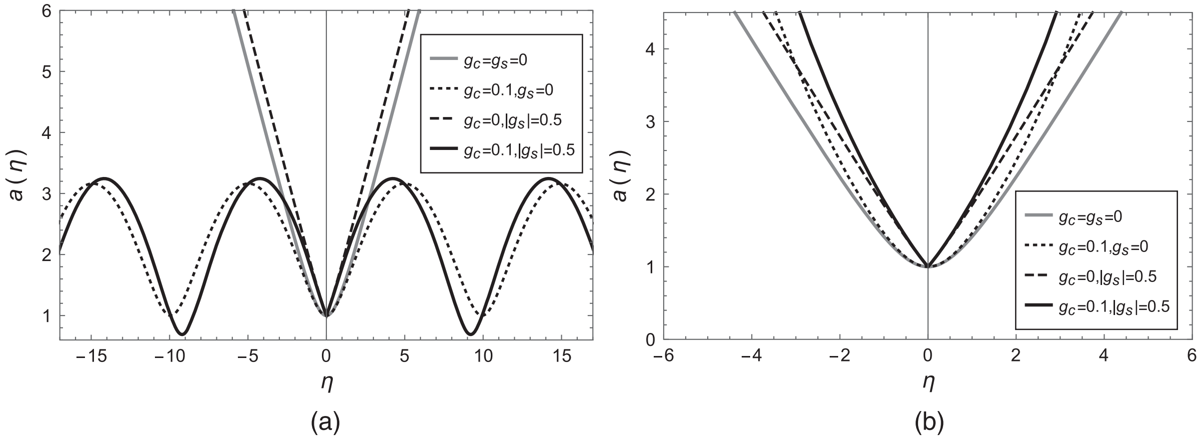

FIG. 3. Analytical scale factor in quantum HL theory as a function of the conformal time for representative values of g_c and g_s. The plots are given for $a_B = 1$, $P_\varphi = 0$, and $\sigma = 0.5$. In (a) and (b), the spatial sections $k = 1$ and $k = -1$, respectively, are considered.

significantly affect the qualitative quantum potential behavior. Therefore, it is a good approximation to consider Eqs. (4.21) when P_φ is negligible. From these considerations, the quantum potential, Eq. (4.11), reads

$$Q(\eta) = \frac{a_B^2[\sqrt{\sigma^2 - |g_s|} - 4a_B^2(\sigma^2 - |g_s|)]}{a(\eta)^2}, \tag{4.27}$$

where $a(\eta)$ is given by Eqs. (4.21) for negligible P_φ. This result is a generalization of Eq. (22) of Ref. [63]. One explicitly notices that when $a_B = 0$, i.e., the universe is singular, the quantum potential vanishes.

In Fig. 4, one can notice that the behavior of $Q(\eta)$ for both spatial sections is in agreement with the panels of Fig. 3. For $k = 1$, the cyclic universe solutions with multiple bounces show that the quantum potential is also oscillatory and non-negligible in the entire time evolution. On the other hand, for $k = -1$ there is a unique bounce at $\eta = 0$ that is dominated by the g_s effect, which becomes more curved as we increase the values of g_c and g_s.

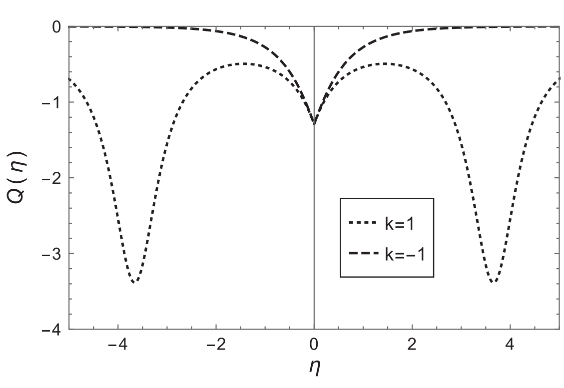

FIG. 4. Quantum potential in HL theory as a function of the conformal time for both $k = 1$ and $k = -1$. Plots are given for the representative values $a_B = 1$, $P_\varphi = 0$, and $\sigma = g_c = |g_s| = 0.5$.

V. CONCLUSIONS

In this work, I presented a quantum minisuperspace model of FLRW cosmology in the framework of the HL theory of gravity. For this purpose, I considered the HL theory in its projectable version and without the detailed balance condition. The Universe is filled with noninteracting radiation and dust fluids, which were introduced using the Schutz approach. Canonical quantization was then performed and a Wheeler-DeWitt equation, Eq. (4.5), was obtained. Performing a separation of variables and a suitable choice of parameters, an analytical solution of Eq. (4.5) for the wave function was first obtained in this context for nonzero values of g_c, g_r, and g_s present in the HL Hamiltonian, Eq. (2.13). I set $g_\Lambda = 0$, which will be considered in a future work.

From the solution of the wave function, I considered the dBB interpretation of quantum mechanics in order to derive analytical solutions for the scale factor, which are given by Eqs. (4.21). These are quantum bounce solutions in the cases of closed and open FLRW quantum cosmologies in HL theory. The closed universe solutions are cyclic and have multiple bounces which avoid big bang and big crunch singularities. On the other hand, the open universe solutions are contracting/expanding unique bounce solutions, which get more curved as one increases the values of the g_c and g_s HL parameters.

I also obtained the quantum potential, Eq. (4.27), whose qualitative behavior confirms its role in the quantum evolution of each solution, i.e., that it is responsible for the avoidance of the singularities. Classical results were also presented in Sec. III, which were important for identifying $\bar{\sigma}$ and constructing the quantum results with a well-defined classical limit. In the Appendix, an ansatz for the wave function was derived in the case where only the radiation fluid degree of freedom and the g_s parameter of HL gravity are nonzero, which confirms the value of $\bar{\sigma}$.

As future perspectives, these results will be important for studying gravitational particle production and baryogenesis in this context. On the other hand, it would also be interesting to consider a stiff matter fluid ($\omega = 1$) in this context as well as when $g_\Lambda \neq 0$, as previously mentioned.

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APPENDIX: ANSATZ SOLUTION FOR A QUANTUM FLRW UNIVERSE FILLED WITH RADIATION AND NONZERO g_s HL PARAMETER

In this section, I consider the quantum cosmology for HL theory in the particular case where $g_c = g_r = 0$ and $g_s \neq 0$ (except for g_Λ, which is neglected in this paper) filled with a radiation fluid. This case corresponds to a particular case of Eq. (4.5) which reads

$$i\partial_\eta \Psi(a,\eta) = \left(-\frac{1}{4}\partial_a^2 + \frac{\alpha}{4a}\partial_a - \frac{g_s k}{a^2}\right)\Psi(a,\eta). \tag{A1}$$

A solution for this equation can be obtained from the following ansatz:

$$\Psi_{\mathrm{ansatz}}(a,\eta) = a^{\alpha/2} f(\eta) e^{-g(\eta)a^2}, \tag{A2}$$

where $f(\eta)$ and $g(\eta)$ are arbitrary functions of η. From Sec. IV, the quantum analytical solution for this case, where the choice $\alpha = -1 + \sqrt{1 + 16 g_s k}$ was considered, is given by

$$a(\eta) = a_B \sqrt{1 + 2\sqrt{g_s k}|\eta| + \sigma^2 \eta^2}. \tag{A3}$$

The expressions for $f(\eta)$ and $g(\eta)$ of the wave-function ansatz which reproduce this analytical solution read

$$f(\eta) = \frac{1}{\sqrt{1 + i\bar{\sigma}\eta}}, \qquad g(\eta) = \frac{\bar{\sigma}}{1 + i\bar{\sigma}\eta}, \tag{A4}$$

where $\bar{\sigma} = \sqrt{\sigma^2 - g_s k} + i\sqrt{g_s k}$. In terms of these functions, the ansatz results as follows:

$$\Psi_{\mathrm{ansatz}}(a,\eta) \propto a^{(-1+\sqrt{1+16 g_s k})/2} \frac{e^{-\frac{\bar{\sigma} a^2}{1 + i\bar{\sigma}\eta}}}{\sqrt{1 + i\bar{\sigma}\eta}}. \tag{A5}$$

From the ansatz, the initial condition at $\eta = 0$ reads

$$\Psi_{\mathrm{ansatz}}(a,0) \propto a^{(-1+\sqrt{1+16 g_s k})/2} e^{-\bar{\sigma} a^2}. \tag{A6}$$

Comparing this result to Eqs. (4.13) and (4.18), this is the exact initial dependence on the scale factor a. One can also notice the presence of the parameter $\bar{\sigma}$ in the exponential. Therefore, Eq. (4.19) is the appropriate initial condition, as we notice that the term $a^{(-1+\sqrt{1+16 g_s k})/2}$ was factorized by a previous change of variables.

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