

Weakly non-Gaussian formula for the Minkowski functionals in general dimensions

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The Minkowski functionals are useful statistics to quantify the morphology of various random fields. They have been applied to numerous analyses of geometrical patterns, including various types of cosmic fields, morphological image processing, etc. In some cases, including cosmological applications, small deviations from the Gaussianity of the distribution are of fundamental importance. Analytic formulas for the expectation values of Minkowski functionals with small non-Gaussianity have been derived in limited cases to date. We generalize these previous works to derive an analytic expression for expectation values of Minkowski functionals up to second-order corrections of non-Gaussianity in a space of general dimensions. The derived formula has sufficient generality to be applied to any random fields with weak non-Gaussianity in a statistically homogeneous and isotropic space of any dimensions.

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I. INTRODUCTION

Statistical analyses of random fields are of importance in a broad range of research fields. For example, cosmic structures observed in the Universe, such as the temperature fluctuations of the cosmic microwave background, the density fields in the large-scale structure, weak lensing fields, and so forth, are considered as realizations of random fields. Only the statistical properties of the fields can be predicted using cosmological theories of the very early Universe, such as the theory of inflation [1–4]. Although numerous inflationary theories have been proposed to date [5], it is still not clear which theory is relevant to our Universe. Alternatives to the theory of inflation, such as an ekpyrotic scenario [6], have also been proposed. Different theories predict different statistical properties of the initial density field, and thus the observational constraints against properties of cosmic fields are crucial in discriminating between the theories of the very early Universe.

The maximal extraction of statistical information from the observed fields is one of the most important challenges in cosmology. The most fundamental statistic is the spatial correlation function, or its Fourier counterpart, the power spectrum, which characterizes the clustering strength of random fields as a function of scale [7]. The two-point

correlation function (or the power spectrum) completely characterizes the statistical information of Gaussian random fields. However, much of the information in generally non-Gaussian random fields cannot be captured solely using this statistic. Specifically, while simple models of single-field inflation with a minimal kinetic term and a smooth potential predict a negligible level of non-Gaussianities [8,9], various other inflationary models can predict various types of non-negligible non-Gaussianities [5,10]. Therefore, it is crucial to determine whether non-Gaussianities are contained in the initial density field or not and the type of non-Gaussianity if it exists. Even if the initial density field is purely Gaussian, gravitationally nonlinear evolution induces the non-Gaussianity in the cosmic fields, from which one can also extract information on the evolution of the Universe. For these reasons, non-Gaussianities, which cannot be probed using the correlation function or the power spectrum, play important roles in cosmology.

Straightforward statistics beyond the two-point correlation function are the higher-order correlation functions, such as the three-point correlation function, four-point correlation function, and so on [7]. The Fourier counterparts of higher-order correlation functions are the polyspectra, such as the bispectrum, trispectrum, and so on [11]. All the statistical information of random fields is contained in the hierarchy of these higher-order statistics. It is relatively straightforward to theoretically predict the polyspectra of a given model of non-Gaussian cosmic fields. Higher-order correlation functions

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have many arguments because they represent spatial correlations among many separations. Therefore, it is challenging to measure their accurate functional forms based on observational data.

Statistical tools for probing non-Gaussianities of random fields are not confined to higher-order correlation functions and the polyspectra. Among various statistical approaches, the characterization of the morphological structures of random fields is a unique way to probe non-Gaussianities. The Minkowski functionals [12,13] comprise a set of statistics that quantitatively characterize the stochastic geometry. According to Hadwiger's theorem [14,15], the $d + 1$ numbers of the Minkowski functionals in d spatial dimensions completely characterize the global morphological properties that satisfy motional invariance and additivity.

The Minkowski functionals were first introduced into cosmology by Mecke *et al.* [16] for the analysis of point sets, such as the positions of galaxies in the Universe. Later, Schmalzing and Buchert [17] consider the Minkowski functionals of excursion sets in smoothed cosmic fields. One of the Minkowski functionals is the Euler characteristic, or equivalently, the genus statistic of isocontours. Prior to the introduction of the Minkowski functionals in cosmology, the genus statistic [18] was applied to smoothed cosmic fields such as the distribution of galaxies [19–33], fluctuations of the cosmic microwave background [34–40], weak lensing fields [41,42], intergalactic medium [43–45], and the x-ray remnant of supernovae [46]. Applications of the Minkowski functionals to cosmology are currently quite popular, including the analyses of the large-scale structure [47–57], cosmic microwave background [58–70], weak lensing fields [71–83], reionization epochs [84–91], and so forth.

The Minkowski functionals are also applied to other fields of research, such as in morphological image analysis [92], to describe porous media and complex fluids [93,94], magnetic resonance imaging [95,96], the structure of human radial peripapillary capillaries [97], mammary gland tissue [98], spinodal decomposition [99], quantum motion in billiards [100], regional seismicity realizations [101], microemulsions [102], thin polymer films [103], the internal structure of bimetallic nanocomposites [104], the thermodynamics of two-phase systems [105], and many others.

A striking feature of the Minkowski functionals of smoothed fields is that the shapes of the functional dependencies of the isocontour threshold are universal for Gaussian random fields. Analytic expressions of the Minkowski functionals for Gaussian random fields were derived by Tomita [106]. Deviations from Tomita's formula imply the non-Gaussianity of the distribution. Thus, the Minkowski functionals are considered as probes for non-Gaussianities in cosmic fields. Analytic expressions of the Minkowski functionals for weakly non-Gaussian random fields in two- and three-dimensional space were derived by one of the authors

of this paper [107,108]. The expressions can be generalized to include the anisotropic effects of redshift-space distortions of the large-scale structure [109,110]. These formulas assume that the non-Gaussianity is sufficiently weak and the non-Gaussian corrections are given by linear terms of the skewness parameters. We refer to these lowest-order corrections due to the skewness parameters as first-order corrections of non-Gaussianity. The next-order corrections, which we call second-order corrections of non-Gaussianity, are given by quadratic terms of the skewness parameters and linear terms of the kurtosis parameters. Analytic expressions of the Minkowski functionals with second-order corrections of non-Gaussianity in a two-dimensional space were also derived by one of the authors of this paper [111]. Analytic expressions of the genus statistic with second-order corrections in two- and three-dimensional spaces were derived in Refs. [110,112,113]. Remarkably, formal expressions of the genus statistic in two- and three-dimensional spaces using the Gram-Charlier expansion to all orders are also known [112,113].

The purpose of this paper is to derive analytic expressions of the Minkowski functionals with first- and second-order corrections of non-Gaussianity in general dimensions for the first time. In a previous paper, Ref. [108], analytic expressions of the Minkowski functionals with first-order corrections of non-Gaussianity in general dimensions were conjectured based on results for two and three dimensions. We provide a proof of this conjecture in this paper. In addition, we derive the second-order corrections in general dimensions for the first time. The derived formula is given by Eq. (91) in the following, which is the main result of this paper. The known formulas in the literature indicated above are reproduced as special cases of the general formula.

In cosmology, the newly derived formula with second-order corrections in three dimensions should be useful for future applications. Mathematically, it is interesting that there exist analytic expressions in general dimensions. Moreover, the approach used for the derivation in this study is instructive. It is straightforward to derive formulas for third- and higher-order corrections of non-Gaussianity using the method presented in this work.

The authors are preparing an accompanying paper [114], where an equivalent formula to this paper is mathematically derived by an approach different from the one used in this paper. The parametrizations of the formula in these two approaches are also different. We have confirmed that the two expressions are actually equivalent, and thus the derived formula is cross-checked in two independent ways.

This paper is organized as follows. In Sec. II, a theory to derive non-Gaussian correction terms for a given statistic is reviewed. In Sec. III, the general properties and formulas for the Minkowski functionals are reviewed. After necessary preparations in the preceding sections, we describe the calculation of the non-Gaussian corrections of Minkowski functionals and present the main result of this paper in

Sec. IV. The main conclusions are summarized in Sec. V. Appendixes A and B are devoted to the proofs of important formulas that are used in this work.

II. A GENERAL THEORY OF NON-GAUSSIAN CORRECTIONS TO A MEAN VALUE

In this section, a theory to derive non-Gaussian correction terms for a given statistic is reviewed. The same method is already described in Ref. [108].

We consider a function $F(f_\mu)$ which depends on the field value $f(\mathbf{x})$ and its spatial derivatives up to the second order

$$(f_\mu) = (f, f_i, f_{ij}), \quad (1)$$

where $f_i \equiv \partial f / \partial x_i$, $f_{ij} \equiv \partial^2 f / \partial x_i \partial x_j$ ($i \leq j$), and x_i are the spatial coordinates. In d dimensions, the number of elements of the vector f_μ is $N = 1 + d + d(d+1)/2 = (d+1)(d+2)/2$. We assume that the field f has a zero mean, $\langle f \rangle = 0$, which implies $\langle f_\mu \rangle = 0$. The joint probability distribution function of the variables f_μ is denoted by $P(\mathbf{f})$, and the partition function is defined by

$$Z(\mathbf{J}) = \int_{-\infty}^{\infty} d^N f P(\mathbf{f}) e^{i\mathbf{J}\mathbf{f}}, \quad (2)$$

where $\mathbf{J} = (J^\mu)$ is an N -dimensional vector. According to the cumulant expansion theorem [115], we have

$$\ln Z(\mathbf{J}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{\mu_1=1}^N \cdots \sum_{\mu_n=1}^N \langle f_{\mu_1} \cdots f_{\mu_n} \rangle_c J^{\mu_1} \cdots J^{\mu_n}, \quad (3)$$

where $\langle \cdots \rangle_c$ denotes the cumulant. Denoting the covariance of \mathbf{f} as an $N \times N$ matrix \mathbf{M} with components given as $M_{\mu\nu} \equiv \langle f_\mu f_\nu \rangle_c$, we have

$$Z(\mathbf{J}) = \exp \left(-\frac{1}{2} \mathbf{J}^T \mathbf{M} \mathbf{J} + \sum_{n=3}^{\infty} \frac{i^n}{n!} \sum_{\mu_1, \dots, \mu_n} \langle f_{\mu_1} \cdots f_{\mu_n} \rangle_c J^{\mu_1} \cdots J^{\mu_n} \right). \quad (4)$$

Applying the inverse transform of Eq. (2), and substituting into the preceding equation, we obtain

$$P(\mathbf{f}) = \int_{-\infty}^{\infty} \frac{d^N J}{(2\pi)^N} Z(\mathbf{J}) e^{-i\mathbf{J}\mathbf{f}} = \exp \left[\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \langle f_{\mu_1} \cdots f_{\mu_n} \rangle_c \frac{\partial^n}{\partial f_{\mu_1} \cdots \partial f_{\mu_n}} \right] P_G(\mathbf{f}), \quad (5)$$

where

$$P_G(\mathbf{f}) = \int_{-\infty}^{\infty} \frac{d^N J}{(2\pi)^N} \exp \left(-i\mathbf{J} \cdot \mathbf{f} - \frac{1}{2} \mathbf{J}^T \mathbf{M} \mathbf{J} \right) = \frac{1}{(2\pi)^{N/2} \sqrt{\det \mathbf{M}}} \exp \left(-\frac{1}{2} \mathbf{f}^T \mathbf{M}^{-1} \mathbf{f} \right) \quad (6)$$

is the multivariate Gaussian distribution function of \mathbf{f} with the covariance matrix \mathbf{M} , and the repeated indices of μ_1, \dots, μ_n are assumed to be summed without summation symbols. Using the expression of Eq. (5), the mean value $\langle F \rangle$ of an arbitrary function $F(\mathbf{f})$ is formally given by

$$\langle F \rangle = \int_{-\infty}^{\infty} d^N f F(\mathbf{f}) P(\mathbf{f}) = \left\langle \exp \left(\sum_{n=3}^{\infty} \frac{1}{n!} \langle f_{\mu_1} \cdots f_{\mu_n} \rangle_c \frac{\partial^n}{\partial f_{\mu_1} \cdots \partial f_{\mu_n}} \right) F(\mathbf{f}) \right\rangle_G, \quad (7)$$

where

$$\langle \cdots \rangle_G = \int_{-\infty}^{\infty} d^N f \cdots P_G(\mathbf{f}) \quad (8)$$

denotes averaging over the Gaussian distribution function. Assuming the higher-order cumulants are small, Eq. (7) is formally expanded as

$$\begin{aligned} \langle F \rangle &= \langle F \rangle_G + \frac{1}{6} \langle f_{\mu_1} f_{\mu_2} f_{\mu_3} \rangle_c \left\langle \frac{\partial^3 F}{\partial f_{\mu_1} \partial f_{\mu_2} \partial f_{\mu_3}} \right\rangle_G \\ &+ \frac{1}{24} \langle f_{\mu_1} f_{\mu_2} f_{\mu_3} f_{\mu_4} \rangle_c \left\langle \frac{\partial^4 F}{\partial f_{\mu_1} \partial f_{\mu_2} \partial f_{\mu_3} \partial f_{\mu_4}} \right\rangle_G \\ &+ \frac{1}{72} \langle f_{\mu_1} f_{\mu_2} f_{\mu_3} \rangle_c \langle f_{\mu_4} f_{\mu_5} f_{\mu_6} \rangle_c \\ &\times \left\langle \frac{\partial^6 F}{\partial f_{\mu_1} \partial f_{\mu_2} \partial f_{\mu_3} \partial f_{\mu_4} \partial f_{\mu_5} \partial f_{\mu_6}} \right\rangle_G + \cdots \end{aligned} \quad (9)$$

It is useful to define dimensionless fields

$$\alpha(\mathbf{x}) = \frac{f(\mathbf{x})}{\sigma_0}, \quad \eta_i(\mathbf{x}) = \frac{f_i(\mathbf{x})}{\sigma_1}, \quad \zeta_{ij}(\mathbf{x}) = \frac{f_{ij}(\mathbf{x})}{\sigma_2}, \quad (10)$$

where σ_j are spectral parameters that are defined as

$$\sigma_j^2 \equiv \langle f(-\Delta)^j f \rangle, \quad (11)$$

and $\Delta = \partial^2 / \partial x_i \partial x_i$ is the Laplacian operator. In particular, $\sigma_0^2 \equiv \langle f^2 \rangle$ is the variance of the field. We denote the set of dimensionless variables as

$$(X_\mu) = (\alpha, \eta_i, \zeta_{ij}). \quad (12)$$

In most of the physical applications, it is often the case that the m -point cumulant of X_μ has the order of σ_0^{m-2} ,

$$\langle X_{\mu_1} \cdots X_{\mu_m} \rangle_c \sim \mathcal{O}(\sigma_0^{m-2}). \quad (13)$$

This ordering is called hierarchical ordering. We assume this type of ordering throughout this paper. In this case, the normalized cumulants

$$C_{\mu_1 \cdots \mu_m}^{(m)} \equiv \frac{\langle X_{\mu_1} \cdots X_{\mu_m} \rangle_c}{\sigma_0^{m-2}} \quad (14)$$

have order 1 in terms of σ_0 . Changing the variables from f_μ to X_μ in Eq. (9), we have a series expansion

$$\begin{aligned} \langle F \rangle &= \langle F \rangle_G + \frac{1}{6} C_{\mu_1 \mu_2 \mu_3}^{(3)} \left\langle \frac{\partial^3 F}{\partial X_{\mu_1} \partial X_{\mu_2} \partial X_{\mu_3}} \right\rangle_G \sigma_0 + \left[\frac{1}{24} C_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4)} \left\langle \frac{\partial^4 F}{\partial X_{\mu_1} \partial X_{\mu_2} \partial X_{\mu_3} \partial X_{\mu_4}} \right\rangle_G \right. \\ &\quad \left. + \frac{1}{72} C_{\mu_1 \mu_2 \mu_3}^{(3)} C_{\mu_4 \mu_5 \mu_6}^{(3)} \left\langle \frac{\partial^6 F}{\partial X_{\mu_1} \partial X_{\mu_2} \partial X_{\mu_3} \partial X_{\mu_4} \partial X_{\mu_5} \partial X_{\mu_6}} \right\rangle_G \right] \sigma_0^2 + \mathcal{O}(\sigma_0^3). \end{aligned} \quad (15)$$

III. MINKOWSKI FUNCTIONALS

In this section, we define Minkowski functionals and briefly review their properties and relations to the Euler characteristic. The Minkowski functionals comprise $d + 1$ numbers that characterize the morphological properties of random fields in a domain \mathcal{D} of d -dimensional space. For an excursion set \mathcal{F}_ν , a set of all points \mathbf{x} with $\alpha(\mathbf{x}) \geq \nu$, we denote the Minkowski functionals per unit volume as $V_k^{(d)}(\nu)$, where $k = 0, 1, \dots, d$, and their mean values as $\bar{V}_k^{(d)}(\nu) = \langle V_k^{(d)}(\nu) \rangle$.

For $k = 0$, the Minkowski functional $V_0^{(d)}$ corresponds to the volume fraction of the excursion set,

$$V_0^{(d)}(\nu) = \frac{1}{|\mathcal{D}|} \int_{\mathcal{F}_\nu} d^d x = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^d x \Theta[\alpha(\mathbf{x}) - \nu], \quad (16)$$

where $|\mathcal{D}|$ is the entire volume of the domain \mathcal{D} . The other Minkowski functionals with $k = 1, \dots, d$ correspond to surface integrals of the boundary $\partial\mathcal{F}_\nu$ of the excursion set

$$V_k^{(d)}(\nu) = \frac{1}{|\mathcal{D}|} \int_{\partial\mathcal{F}_\nu} d^{d-1} x v_k^{(d)}(\nu, \mathbf{x}), \quad (17)$$

where $v_k^{(d)}(\nu, \mathbf{x})$ are the local Minkowski functionals defined by

$$v_k^{(d)}(\nu, \mathbf{x}) = \frac{1}{\omega_k d} K_{k-1}^{(d)}(\nu, \mathbf{x}), \quad (18)$$

and

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2 + 1)} \quad (19)$$

is the volume of the unit ball in k dimensions. On the boundary hypersurface, $\mathbf{x} \in \partial\mathcal{F}_\nu$, $K_m^{(d)}(\nu, \mathbf{x})$ is the invariant obtained from the inverse radii of curvature R_1, R_2, \dots, R_{d-1}

of the hypersurface orientated toward the lower density regions [116]. That is,

$$K_m^{(d)}(\nu, \mathbf{x}) = \frac{1}{d-1} \sum_s \frac{1}{C_m R_{s(1)} R_{s(2)} \cdots R_{s(m)}}, \quad (20)$$

where \sum_s denotes the symmetric summation over $d-1$ $C_m = (d-1)!/n!(d-m-1)!$ combinations of m different components of $(R_1, R_2, \dots, R_{d-1})$. For example, in two-dimensional space with $d = 2$,

$$K_0^{(2)} = 1, \quad K_1^{(2)} = \frac{1}{R_1}, \quad (21)$$

and, in three-dimensional space with $d = 3$,

$$K_0^{(3)} = 1, \quad K_1^{(3)} = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right), \quad K_2^{(3)} = \frac{1}{R_1 R_2}. \quad (22)$$

Substituting Eqs. (20) and (21) into Eqs. (18) and (17), we obtain the formulas for the Minkowski functionals presented in Refs. [117,118], respectively.

The quantity $K_{d-1}^{(d)} = R_1^{-1} R_2^{-1} \cdots R_{d-1}^{-1}$ corresponds to the Gauss total curvature. Due to the Gauss-Bonnet theorem [119], the density of the Euler characteristic of \mathcal{F}_ν is given by

$$\chi^{(d)}(\mathcal{F}_\nu) = \frac{1}{\omega_d d} \frac{1}{|\mathcal{D}|} \int_{\partial\mathcal{F}_\nu} d^{d-1} x K_{d-1}^{(d)}(\nu, \mathbf{x}). \quad (23)$$

Therefore, the Minkowski functional with $k = d$ corresponds to the d -dimensional Euler characteristic, $V_d^{(d)} = \chi^{(d)}$. For analytic evaluations of the Minkowski functionals of an excursion set in random fields, Crofton's formula [117,120,121] in integral geometry serves as a powerful tool. This formula states that

$$V_k^{(d)}(\nu) = \frac{\omega_d}{\omega_{d-k} \omega_k} \int_{\mathcal{E}_k^{(d)}} d\mu_k(E) \chi^{(k)}(\mathcal{F}_\nu \cap E), \quad (24)$$

where E is an arbitrary k -dimensional hypersurface and $\chi^{(k)}$ is the density of the Euler characteristic of the intersection $\mathcal{F}_\nu \cap E$ in k dimensions. This quantity is integrated over the space $\mathcal{E}_k^{(d)}$ of all conceivable hypersurfaces, and the integration measure $d\mu_k(E)$ is normalized to give $\int_{\mathcal{E}_k^{(d)}} d\mu_k(E) = 1$. Using Crofton's formula of Eq. (24), and assuming statistical isotropy and homogeneity, the expectation values of the Minkowski functionals are given as

$$\bar{V}_k^{(d)}(\nu) = \frac{\omega_d}{\omega_{d-k}\omega_k} \langle \chi^{(k)}(\mathcal{F}_\nu \cap E) \rangle = \frac{\omega_d}{\omega_{d-k}\omega_k} \bar{V}_k^{(k)}(\nu), \quad (25)$$

where $V_k^{(k)}(\nu) = \chi^{(k)}(\mathcal{F}_\nu \cap E)$ is the k th-order Minkowski functional of the intersection $\mathcal{F}_\nu \cap E$. The expectation value of $V_k^{(k)}$ does not depend on the choice of the hypersurface E due to the statistical isotropy and homogeneity. Using this relation, the expectation values of the Minkowski functionals for each order can be estimated by only evaluating expectation values of the Euler characteristic in the spaces of lower dimensions.

It is convenient to use the Morse theorem [119,121,122] to evaluate the expectation value of the Euler characteristic. The Euler characteristic $\chi(\mathcal{F}_\nu)$ is given by an alternating sum of the number of critical points,

$$\chi(\mathcal{F}_\nu) = \sum_{m=0}^d (-1)^{d-m} C_m(\mathcal{F}_\nu), \quad (26)$$

where C_m is the number of critical points that satisfy $f_i = \partial f / \partial x_i = 0$ of index m , and the index m is the number of negative eigenvalues of the matrix $f_{ij} = \partial^2 f / \partial x_i \partial x_j$ at each critical point.

To count the number of critical points, we first consider the d -dimensional delta function $\delta^d(\mathbf{x} - \mathbf{x}_c)$ around the critical point \mathbf{x}_c . Applying the Taylor expansion, we have

$$f(\mathbf{x}) \simeq f(\mathbf{x}_c) + \frac{1}{2} f_{ij}(\mathbf{x}_c) (x_i - x_{ci})(x_j - x_{cj}), \quad (27)$$

up to the second order. The first-order term does not appear because a critical point \mathbf{x}_c satisfies $f_i(\mathbf{x}_c) = 0$. Taking spatial derivatives of this equation, we have

$$\eta_i(\mathbf{x}) \simeq \frac{\sigma_2}{\sigma_1} \zeta_{ij}(\mathbf{x}_c) (x_j - x_{cj}). \quad (28)$$

Therefore, the delta function of the critical point is given by

$$\delta^d(\mathbf{x} - \mathbf{x}_c) = \left(\frac{\sigma_2}{\sigma_1} \right)^d \delta^d(\boldsymbol{\eta}) |\det \zeta|, \quad (29)$$

near the critical point \mathbf{x}_c . When the right-hand side (rhs) is expanded to include the entire space, the left-hand side (lhs)

should be replaced by a summation of delta functions for all the critical points. From the definition of index m , we have $|\det \zeta| = (-1)^m \det \zeta$. Thus, we have

$$\sum_{\text{critical points: } i} (-1)^{d-m_i} \delta^d(\mathbf{x} - \mathbf{x}_c^{(i)}) = (-1)^d \left(\frac{\sigma_2}{\sigma_1} \right)^d \delta^d(\boldsymbol{\eta}) \det \zeta, \quad (30)$$

where m_i is the index of the i th critical point $\mathbf{x}_c^{(i)}$. Due to Eq. (26), the expectation value of the preceding equation with constraint $\alpha(\mathbf{x}_c) \geq \nu$ corresponds to the density of the Euler characteristic of the body \mathcal{F}_ν ,

$$n_\chi^{(d)}(\nu) = (-1)^d \left(\frac{\sigma_2}{\sigma_1} \right)^d \langle \Theta(\alpha - \nu) \delta^d(\boldsymbol{\eta}) \det \zeta \rangle, \quad (31)$$

where $\Theta(x)$ is the Heaviside step function.

IV. NON-GAUSSIAN CORRECTIONS TO THE MINKOWSKI FUNCTIONALS

A. Gaussian averages for derivatives of the Euler characteristic

The density of the Euler characteristic $n_\chi^{(d)}(\nu)$ of \mathcal{F}_ν in the d -dimensional space is given by Eq. (31). Instead of the Euler characteristic, we take the function F in Sec. II as a differential Euler characteristic density $-(d/d\nu)n_\chi(\nu)$, i.e.,

$$F \equiv (-1)^d \left(\frac{\sigma_2}{\sigma_1} \right)^d \delta(\alpha - \nu) \delta^d(\boldsymbol{\eta}) \det \zeta, \quad (32)$$

so that the integral of the expectation value $\langle F \rangle$ by ν should give the Euler characteristic density $n_\chi(\nu)$,

$$n_\chi(\nu) = \int_\nu^\infty d\nu \langle F \rangle. \quad (33)$$

In order to evaluate Eq. (15), we only need to calculate a Gaussian average of the function F and its derivatives with respect to X_μ . The Gaussian statistics are characterized only by the covariance matrix $M_{\mu\nu} = \langle X_\mu X_\nu \rangle$. Assuming statistical isotropy and rotational invariance of the field variables, they are given by [123]

$$\begin{aligned} \langle \alpha^2 \rangle &= 1, & \langle \alpha \eta_i \rangle &= 0, & \langle \alpha \zeta_{ij} \rangle &= -\frac{\gamma}{d} \delta_{ij}, & \langle \eta_i \eta_j \rangle &= \frac{1}{d} \delta_{ij}, \\ \langle \eta_i \zeta_{jk} \rangle &= 0, & \langle \zeta_{ij} \zeta_{kl} \rangle &= \frac{1}{d(d+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (34)$$

where

$$\gamma \equiv \frac{\sigma_1^2}{\sigma_0 \sigma_2}. \quad (35)$$

It is convenient to define a new variable

$$Z_{ij} \equiv \frac{d}{\gamma} \zeta_{ij} + \delta_{ij} \alpha, \quad (36)$$

instead of ζ_{ij} . The covariances of a new set of variables (α, η_i, Z_{ij}) are given by

$$\begin{aligned} \langle \alpha^2 \rangle &= 1, & \langle \alpha \eta_i \rangle &= 0, & \langle \alpha Z_{ij} \rangle &= 0, \\ \langle \eta_i \eta_j \rangle &= -\frac{1}{d} \delta_{ij}, & \langle \eta_i Z_{jk} \rangle &= 0, \\ \langle Z_{ij} Z_{kl} \rangle &= -\delta_{ij} \delta_{kl} + \frac{d}{(d+2)\gamma^2} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (37)$$

so that the variables α , η_i , and Z_{ij} are independent of each other for Gaussian statistics. In Eq. (15), we need to evaluate the Gaussian average of a type

$$\begin{aligned} & \left\langle \frac{\partial^{m_0+2m_1+m_2} F}{\partial \alpha^{m_0} \partial \eta_{i_1} \cdots \partial \eta_{i_{2m_1}} \partial \zeta_{j_1 k_1} \cdots \partial \zeta_{j_{m_2} k_{m_2}}} \right\rangle_G \\ &= \left(\frac{\sigma_1}{d\sigma_0} \right)^d \left\langle \frac{\partial^{2m_1} \delta^d(\boldsymbol{\eta})}{\partial \eta_{i_1} \cdots \partial \eta_{i_{2m_1}}} \right\rangle_G \left(\frac{\gamma}{d} \right)^{-m_2} \\ & \times \left(-\frac{d}{d\nu} \right)^{m_0} \left[\langle \delta(\alpha - \nu) \rangle_G \left\langle \frac{\partial^{m_2} \det(\nu I - Z)}{\partial Z_{j_1 k_1} \cdots \partial Z_{j_{m_2} k_{m_2}}} \right\rangle_G \right], \end{aligned} \quad (38)$$

where I is a $d \times d$ unit matrix and Z is a $d \times d$ symmetric matrix with $Z_{ji} = Z_{ij}$. Since the Gaussian distribution functions of the variables α and η_i are given by

$$P_G^{(0)}(\alpha) = \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}}, \quad P_G^{(1)}(\boldsymbol{\eta}) = \left(\frac{d}{2\pi} \right)^{d/2} e^{-d\boldsymbol{\eta}^2/2}, \quad (39)$$

we have

$$\langle \delta(\alpha - \nu) \rangle_G = \frac{e^{-\nu^2/2}}{\sqrt{2\pi}}, \quad (40)$$

$$\left\langle \frac{\partial^{2m_1} \delta^d(\boldsymbol{\eta})}{\partial \eta_{i_1} \cdots \partial \eta_{i_{2m_1}}} \right\rangle_G = \frac{d^{d/2+m_1}}{(2\pi)^{d/2}} H_{i_1 \cdots i_{2m_1}}(\mathbf{0}), \quad (41)$$

where

$$H_{i_1 \cdots i_m}(\mathbf{x}) = e^{|\mathbf{x}|^2/2} \frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}} e^{-|\mathbf{x}|^2/2} \quad (42)$$

is the multivariate Hermite polynomial. In particular,

$$H_{i_1 \cdots i_{2m}}(\mathbf{0}) = H_{2m}(0) \delta_{(i_1 i_2} \cdots \delta_{i_{2m-1} i_{2m})}, \quad (43)$$

where $H_{2m}(0) = (-1)^m (2m-1)!!$ is the zero-point value of the (probabilists') Hermite polynomial, $H_m(\nu) = e^{\nu^2/2} (-d/d\nu)^m e^{-\nu^2/2}$, and the round brackets in the indices of the Kronecker delta represent symmetrization of the indices inside the brackets.

B. A useful formula of Gaussian averages for derivatives of the determinant

Now, we consider the last factor of Eq. (38),

$$\left\langle \frac{\partial^{m_2} \det(\nu I - Z)}{\partial Z_{j_1 k_1} \cdots \partial Z_{j_{m_2} k_{m_2}}} \right\rangle_G, \quad (44)$$

For $m_2 = 0$, there is a simple identity

$$\langle \det(\nu I - Z) \rangle_G = H_d(\nu). \quad (45)$$

The proof of this equation is given in Appendix A.

For $m_2 \geq 1$, the partial derivatives $\partial/\partial Z_{ij}$ are performed under the condition that the variables with $i \leq j$ are the set of independent variables in Eq. (38). It is convenient to introduce a redundant set of independent variables

$$Y_{ij} \equiv \begin{cases} Z_{ij}, & (i \leq j), \\ Z_{ji}, & (i > j), \end{cases} \quad (46)$$

which is a symmetric tensor, $Y_{ij} = Y_{ji}$. Considering the variables Y_{ij} as independent variables, the partial derivatives with respect to Z_{ij} are given by the partial derivatives with respect to Y_{ij} as

$$\frac{\partial}{\partial Z_{ij}} = \begin{cases} \frac{\partial}{\partial Y_{ii}}, & (i = j), \\ \frac{\partial}{\partial Y_{ij}} + \frac{\partial}{\partial Y_{ji}}, & (i < j). \end{cases} \quad (47)$$

In Eq. (15), a type of differential operator,

$$\sum_{i \leq j} C_{ij} \frac{\partial}{\partial Z_{ij}} = \sum_{i,j} C_{ij} \frac{\partial}{\partial Y_{ij}} = C_{ij} D_Z^{ij}, \quad (48)$$

appears, where C_{ij} is an arbitrary symmetric tensor with $C_{ij} = C_{ji}$, and

$$D_Z^{ij} \equiv \frac{1}{2} \left(\frac{\partial}{\partial Y_{ij}} + \frac{\partial}{\partial Y_{ji}} \right) = \begin{cases} \frac{\partial}{\partial Z_{ii}}, & (i = j), \\ \frac{1}{2} \frac{\partial}{\partial Z_{ij}}, & (i < j), \\ \frac{1}{2} \frac{\partial}{\partial Z_{ji}}, & (i > j) \end{cases} \quad (49)$$

is a symmetric differential operator, and the summation over repeated indices is assumed in the last expression of Eq. (48).

Thus, when Eq. (38) is summed with the weight of the cumulants, we generally have

$$\begin{aligned} & \sum_{i_1, \dots, i_{2m_1}} \sum_{j_1 \leq k_1} \dots \sum_{j_{m_2} \leq k_{m_2}} \langle \alpha^{m_0} \eta_{i_1} \dots \eta_{i_{2m_1}} \zeta_{j_1 k_1} \dots \zeta_{j_{m_2} k_{m_2}} \rangle_c \\ & \times \left\langle \frac{\partial^{m_0+2m_1+m_2} F}{\partial \alpha^{m_0} \partial \eta_{i_1} \dots \partial \eta_{i_{2m_1}} \partial \zeta_{j_1 k_1} \dots \partial \zeta_{j_{m_2} k_{m_2}}} \right\rangle_G \\ & = \frac{1}{(2\pi)^{(d+1)/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^d d^{m_1} H_{m_1}(0) \left(\frac{\gamma}{d} \right)^{-m_2} \\ & \times \langle \alpha^{m_0} |\boldsymbol{\eta}|^{2m_1} \zeta_{j_1 k_1} \dots \zeta_{j_{m_2} k_{m_2}} \rangle_c \left(-\frac{d}{d\nu} \right)^{m_0} \\ & \times \left[e^{-\nu^2/2} \langle D_Z^{j_1 k_1} \dots D_Z^{j_{m_2} k_{m_2}} \det(\nu I - Z) \rangle_G \right], \quad (50) \end{aligned}$$

where both the indices of ζ_{jk} on the rhs are summed over $j_i, k_i = 1, \dots, d$ (without constraints $j_i \leq k_i$).

The partial derivatives $\partial/\partial Z_{ij}$ always appear with cumulants $C_{\mu_1 \dots \mu_m}^{(m)}$ in Eq. (15). These cumulants are tensors that consist of the Kronecker delta with spatial indices. Applying the property of Eq. (48), we only need a form $\langle \text{Tr}(D_Z^{m_1}) \dots \text{Tr}(D_Z^{m_k}) \det A \rangle_G$ to evaluate Eq. (15), where m_1, \dots, m_k , and k are non-negative integers. There is a remarkable identity,

$$\begin{aligned} & \langle \text{Tr}(D_Z^{m_1}) \dots \text{Tr}(D_Z^{m_k}) \det(\nu I - Z) \rangle_G \\ & = \frac{(-1)^k}{2^{m-k}} \frac{d!}{(d-m)!} H_{d-m}(\nu), \quad (51) \end{aligned}$$

where $m = m_1 + \dots + m_k$. Equation (45) is a special case of $m = k = 0$ of this identity. The proof of Eq. (51) is given in Appendix B. This formula plays a central role in this paper.

C. Evaluations of each order

1. The Gaussian term

The first term on the rhs of Eq. (15) corresponds to the Gaussian contribution and is immediately calculated by applying Eq. (45). Putting $m_0 = m_1 = m_2 = 0$ in Eq. (50), this term is given by

$$\langle F \rangle_G = \frac{1}{(2\pi)^{(d+1)/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^d e^{-\nu^2/2} H_d(\nu). \quad (52)$$

2. The first-order term

The second term on the rhs of Eq. (15) corresponds to the first-order contribution. We should evaluate

$$\begin{aligned} W^{(1)} & \equiv C_{\mu_1 \mu_2 \mu_3}^{(3)} \left\langle \frac{\partial^3 F}{\partial X_{\mu_1} \partial X_{\mu_2} \partial X_{\mu_3}} \right\rangle_G \sigma_0 \\ & = \langle \alpha^3 \rangle_c \left\langle \frac{\partial^3 F}{\partial \alpha^3} \right\rangle_G + 3 \sum_{i \leq j} \langle \alpha^2 \zeta_{ij} \rangle_c \left\langle \frac{\partial^3 F}{\partial \alpha^2 \partial \zeta_{ij}} \right\rangle_G \\ & \quad + 3 \sum_{i,j} \langle \alpha \eta_i \eta_j \rangle_c \left\langle \frac{\partial^3 F}{\partial \alpha \partial \eta_i \partial \eta_j} \right\rangle_G \\ & \quad + 3 \sum_{i \leq j} \sum_{k \leq l} \langle \alpha \zeta_{ij} \zeta_{kl} \rangle_c \left\langle \frac{\partial^3 F}{\partial \alpha \partial \zeta_{ij} \partial \zeta_{kl}} \right\rangle_G \\ & \quad + 3 \sum_{i,j} \sum_{k \leq l} \langle \eta_i \eta_j \zeta_{kl} \rangle_c \left\langle \frac{\partial^3 F}{\partial \eta_i \partial \eta_j \partial \zeta_{kl}} \right\rangle_G \\ & \quad + \sum_{i \leq j} \sum_{k \leq l} \sum_{p \leq q} \langle \zeta_{ij} \zeta_{kl} \zeta_{pq} \rangle_c \left\langle \frac{\partial^3 F}{\partial \zeta_{ij} \partial \zeta_{kl} \partial \zeta_{pq}} \right\rangle_G. \quad (53) \end{aligned}$$

The cumulants with odd numbers of spatial indices are zero for rotational symmetry. The cumulants are independent of the spatial position, thus we have

$$\langle \alpha^2 \zeta_{ij} \rangle_c = \frac{\sigma_0}{\sigma_2} \langle \alpha^2 \alpha_{,ij} \rangle_c = -2 \frac{\sigma_0}{\sigma_2} \langle \alpha \alpha_{,i,j} \rangle_c = -2\gamma \langle \alpha \eta_i \eta_j \rangle_c \quad (54)$$

in the second term of the rhs. Due to Eq. (50), the summation in the fourth term reduces to

$$\begin{aligned} & \sum_{i \leq j} \sum_{k \leq l} \langle \alpha \zeta_{ij} \zeta_{kl} \rangle_c \left\langle \frac{\partial^3 F}{\partial \alpha \partial \zeta_{ij} \partial \zeta_{kl}} \right\rangle_G \\ & = \frac{1}{(2\pi)^{(d+1)/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^d \left(\frac{\gamma}{d} \right)^{-2} \langle \alpha \zeta_{ij} \zeta_{kl} \rangle_c \\ & \quad \times \left(-\frac{d}{d\nu} \right) [e^{-\nu^2/2} \langle D_Z^{ij} D_Z^{kl} \det(\nu I - Z) \rangle_G]. \quad (55) \end{aligned}$$

The cumulant in this equation reduces to

$$\langle \alpha \zeta_{ij} \zeta_{kl} \rangle_c = -\gamma \langle \eta_i \eta_j \zeta_{kl} \rangle_c - \frac{\sigma_0 \sigma_1}{\sigma_2^2} \langle \alpha \eta_i \alpha_{,jkl} \rangle_c. \quad (56)$$

The last term on the rhs of this equation does not contribute in Eq. (55). This property is seen as follows: due to the relation of Eq. (B3), $\langle D_Z^{ij} D_Z^{kl} \det A \rangle$ can be replaced by $\epsilon_{i k i_3 \dots i_d} \epsilon_{j l j_3 \dots j_d} \langle A_{i_3 j_3} \dots A_{i_d j_d} \rangle_G / (d-2)!$ in Eq. (55). Since the last factor is antisymmetric with respect to (j, l) , while $\alpha_{,jkl}$ is symmetric, the last term of Eq. (56) does vanish in Eq. (55), and only the first term of Eq. (56) survives. In the same manner, it is observed that the last term of Eq. (53) does not contribute. In fact, the cumulant of the corresponding term is given by

$$\langle \zeta_{ij} \zeta_{kl} \zeta_{pq} \rangle_c = -\frac{\sigma_0^3}{\sigma_2^3} (\langle \alpha_{,i,jkl} \alpha_{,pq} \rangle_c + \langle \alpha_{,i} \alpha_{,kl} \alpha_{,jpq} \rangle_c). \quad (57)$$

Because the terms on the rhs are symmetric with respect to (j, l) or (j, q) , all the terms of Eq. (57) vanish when they are substituted in the last term of Eq. (53).

Therefore, the only cumulants we need to evaluate are $\langle \alpha^3 \rangle_c$, $\langle \alpha \eta_i \eta_j \rangle_c$, and $\langle \eta_i \eta_j \zeta_{kl} \rangle_c$. For rotational symmetry, these cumulants are parametrized as

$$\begin{aligned} \langle \alpha^3 \rangle_c &= \tilde{S}^{(0)} \sigma_0, & \langle \alpha \eta_i \eta_j \rangle_c &= \frac{1}{d} \tilde{S}^{(1)} \delta_{ij} \sigma_0, \\ \langle \eta_i \eta_j \zeta_{kl} \rangle_c &= \frac{\gamma}{d^2} [\tilde{S}_1^{(2)} \delta_{ij} \delta_{kl} + \tilde{S}_2^{(2)} \Delta_{ij;kl}] \sigma_0, \end{aligned} \quad (58)$$

where

$$\Delta_{ij;kl} \equiv \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (59)$$

From the identity

$$\langle \eta_i \eta_j \zeta_{kl} \rangle_c + \langle \eta_i \eta_l \zeta_{jk} \rangle_c + \langle \eta_j \eta_l \zeta_{ik} \rangle_c = 0, \quad (60)$$

we have a relation

$$\tilde{S}_1^{(2)} + \tilde{S}_2^{(2)} = 0. \quad (61)$$

Substituting Eqs. (54) and (56)–(58) into Eq. (53), omitting the vanishing terms as indicated above, and using Eqs. (50) and (51), we finally derive the expression

$$\begin{aligned} W^{(1)} &= \frac{1}{(2\pi)^{(d+1)/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^d e^{-\nu^2/2} \\ &\times [S^{(0)} H_{d+3}(\nu) + 2dS^{(1)} H_{d+1}(\nu) \\ &+ d(d-1)S_1^{(2)} H_{d-1}(\nu)] \sigma_0, \end{aligned} \quad (62)$$

where we introduce a set of parameters

$$S^{(0)} \equiv \tilde{S}^{(0)}, \quad S^{(1)} \equiv \frac{3}{2} \tilde{S}^{(1)}, \quad S^{(2)} \equiv -\frac{3}{2} \tilde{S}_1^{(2)} \quad (63)$$

for an aesthetic reason.

3. The second-order terms

The second-order terms of Eq. (15) are evaluated similar to the first-order term. We define

$$W_1^{(2)} \equiv C_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4)} \left\langle \frac{\partial^4 F}{\partial X_{\mu_1} \partial X_{\mu_2} \partial X_{\mu_3} \partial X_{\mu_4}} \right\rangle_G \sigma_0^2, \quad (64)$$

$$W_2^{(2)} \equiv C_{\mu_1 \mu_2 \mu_3}^{(3)} C_{\mu_4 \mu_5 \mu_6}^{(3)} \left\langle \frac{\partial^6 F}{\partial X_{\mu_1} \partial X_{\mu_2} \partial X_{\mu_3} \partial X_{\mu_4} \partial X_{\mu_5} \partial X_{\mu_6}} \right\rangle_G \sigma_0^2. \quad (65)$$

The evaluation of these terms is similar to the first-order case and is straightforward by applying the same

techniques. The four-point cumulants that are required to evaluate $W_1^{(2)}$ are parametrized as

$$\begin{aligned} \langle \alpha^4 \rangle_c &= \tilde{K}^{(0)} \sigma_0^2, & \langle \alpha^2 \eta_i \eta_j \rangle_c &= \frac{1}{d} \tilde{K}^{(1)} \delta_{ij} \sigma_0^2, \\ \langle \alpha \eta_i \eta_j \zeta_{kl} \rangle_c &= \frac{\gamma}{d^2} [\tilde{K}_1^{(2)} \delta_{ij} \delta_{kl} + \tilde{K}_2^{(2)} \Delta_{ij;kl}] \sigma_0^2, \\ \langle \eta_i \eta_j \zeta_{kl} \zeta_{pq} \rangle_c &= \frac{\gamma^2}{d^3} \left[\tilde{K}_1^{(3)} \delta_{ij} \delta_{kl} \delta_{pq} \right. \\ &+ \frac{1}{2} \tilde{K}_2^{(3)} (\Delta_{ij;kl} \delta_{pq} + \Delta_{ij;pq} \delta_{kl}) \\ &+ \tilde{K}_3^{(3)} \Delta_{kl;pq} \delta_{ij} + \tilde{K}_4^{(3)} \Delta_{ij;kl;pq} \left. \right] \sigma_0^2, \end{aligned} \quad (66)$$

where

$$\begin{aligned} \Delta_{ij;kl;pq} &\equiv \frac{1}{8} (\delta_{jk} \delta_{lp} \delta_{qi} + \delta_{jk} \delta_{lq} \delta_{pi} + \delta_{jl} \delta_{kp} \delta_{qi} + \delta_{jl} \delta_{kq} \delta_{pi} \\ &+ \delta_{ik} \delta_{lp} \delta_{qj} + \delta_{ik} \delta_{lq} \delta_{pj} + \delta_{il} \delta_{kp} \delta_{qj} + \delta_{il} \delta_{kq} \delta_{pj}). \end{aligned} \quad (67)$$

Other cumulants are given by the above ones as

$$\begin{aligned} \langle \alpha^3 \zeta_{ij} \rangle_c &= -3\gamma \langle \alpha^2 \eta_i \eta_j \rangle_c, \\ \langle \alpha^2 \zeta_{ij} \zeta_{kl} \rangle_c &= -2\gamma \langle \alpha \eta_i \eta_j \zeta_{kl} \rangle_c + \dots, \\ \langle \alpha \zeta_{ij} \zeta_{kl} \zeta_{pq} \rangle_c &= -\gamma \langle \eta_i \eta_j \zeta_{kl} \zeta_{pq} \rangle_c + \dots, \\ \langle \eta_i \eta_j \eta_k \eta_l \rangle_c &= -3\gamma^{-1} \langle \alpha \eta_i \eta_j \zeta_{kl} \rangle_c, \end{aligned} \quad (68)$$

where \dots represent terms that do not contribute in Eq. (50). The cumulant $\langle \zeta_{ij} \zeta_{kl} \zeta_{pq} \zeta_{rs} \rangle_c$ does not contribute in Eq. (50), because of the same reason we described around Eq. (57). From the identity,

$$\begin{aligned} \langle \eta_i \eta_j \zeta_{kl} \zeta_{pq} \rangle_c + \langle \eta_i \eta_k \zeta_{jl} \zeta_{pq} \rangle_c + \langle \eta_j \eta_k \zeta_{il} \zeta_{pq} \rangle_c \\ = \langle \eta_i \eta_j \zeta_{kq} \zeta_{lp} \rangle_c + \langle \eta_i \eta_k \zeta_{jq} \zeta_{lp} \rangle_c + \langle \eta_j \eta_k \zeta_{iq} \zeta_{lp} \rangle_c, \end{aligned} \quad (69)$$

we have a relation,

$$4\tilde{K}_1^{(3)} + 2\tilde{K}_2^{(3)} - 2\tilde{K}_3^{(3)} - \tilde{K}_4^{(3)} = 0 \quad (70)$$

for $d \geq 2$.

The calculations of Eqs. (64) and (65) are similar to the first-order case and are straightforward using Eqs. (50) and (51) but tedious. The results are

$$\begin{aligned} W_1^{(2)} &= \frac{1}{(2\pi)^{(d+1)/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^d e^{-\nu^2/2} \left\{ K^{(0)} H_{d+4}(\nu) \right. \\ &+ 3dK^{(1)} H_{d+1}(\nu) + \frac{3d}{2} [(d-2)K_1^{(2)} + dK_2^{(2)}] H_{d-1}(\nu) \\ &+ d(d-1)(d-2)K^{(3)} H_{d-2}(\nu) \left. \right\} \sigma_0^2, \end{aligned} \quad (71)$$

and

$$W_2^{(2)} = \frac{1}{(2\pi)^{(d+1)/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^d e^{-\nu^2/2} \{ (S^{(0)})^2 H_{d+6}(\nu) + 4dS^{(0)}S^{(1)}H_{d+4}(\nu) + 2d(d-1)S^{(0)}S^{(2)}H_{d+2}(\nu) \\ + 4d(d-2)(S^{(1)})^2 H_{d+2}(\nu) + 4d(d-1)(d-4)S^{(1)}S^{(2)}H_d(\nu) + d(d-1)(d-2)(d-7)(S^{(2)})^2 H_{d-2}(\nu) \} \sigma_0^2, \quad (72)$$

where we introduce a set of new parameters,

$$K^{(0)} \equiv \tilde{K}^{(0)}, \quad K^{(1)} \equiv 2\tilde{K}^{(1)}, \quad K_1^{(2)} \equiv -2\tilde{K}_1^{(2)}, \\ K_2^{(2)} \equiv 2\tilde{K}_2^{(2)}, \quad K^{(3)} \equiv 2\tilde{K}_1^{(3)} - \tilde{K}_3^{(3)}, \quad (73)$$

for an aesthetic reason.

Equation (15) is given by

$$\langle F \rangle = \langle F \rangle_G + \frac{1}{6} W^{(1)} + \frac{1}{24} W_1^{(2)} + \frac{1}{72} W_2^{(2)}. \quad (74)$$

Substituting Eqs. (62), (71), and (72), into Eq. (74), and using the integral

$$\int_{\nu}^{\infty} d\nu e^{-\nu^2/2} H_n(\nu) = e^{-\nu^2/2} H_{n+1}(\nu), \quad (75)$$

one obtains the result for the expression of $n_{\chi}(\nu)$ of Eq. (33).

D. Skewness and kurtosis parameters

The skewness and kurtosis parameters can be explicitly given in the form of rotationally invariant averages of field variables. By taking all the possible contractions of spatial indices in Eqs. (58) and (66) and solving the resulting linear equations, one obtains

$$S^{(0)} = \frac{\langle f^3 \rangle_c}{\sigma_0^4}, \quad S^{(1)} = \frac{3 \langle f |\nabla f|^2 \rangle_c}{2 \sigma_0^2 \sigma_1^2}, \\ S^{(2)} = \frac{-3d}{2(d-1)} \frac{\langle |\nabla f|^2 \Delta f \rangle_c}{\sigma_1^4}, \quad (76)$$

and

$$K^{(0)} = \frac{\langle f^4 \rangle_c}{\sigma_0^6}, \quad K^{(1)} = 2 \frac{\langle f^2 |\nabla f|^2 \rangle_c}{\sigma_0^4 \sigma_1^2}, \\ K_1^{(2)} = \frac{-2d}{(d+2)(d-1)} \frac{(d+2) \langle f |\nabla f|^2 \Delta f \rangle_c + \langle |\nabla f|^4 \rangle_c}{\sigma_0^2 \sigma_1^4}, \\ K_2^{(2)} = \frac{-2d}{(d+2)(d-1)} \frac{(d+2) \langle f |\nabla f|^2 \Delta f \rangle_c + d \langle |\nabla f|^4 \rangle_c}{\sigma_0^2 \sigma_1^4}, \\ K^{(3)} = \frac{2d^2}{(d-1)(d-2)} \frac{\langle |\nabla f|^2 (\Delta f)^2 \rangle_c - \langle |\nabla f|^2 f_{ij} f_{ij} \rangle_c}{\sigma_1^6}. \quad (77)$$

The parameters $K_1^{(2)}$ and $K_2^{(2)}$ are undetermined in the case of $d = 1$, since only the combination $K_1^{(2)} - K_2^{(2)}$ can

be determined. In this case, by noticing that $\langle f_1^2 \rangle = -3 \langle f f_1^2 f_{11} \rangle$, we have

$$K_1^{(2)} - K_2^{(2)} = \frac{2 \langle f_1^4 \rangle_c}{3 \sigma_0^2 \sigma_1^4}, \quad (d = 1), \quad (78)$$

which is substituted into Eq. (71) in the case of $d = 1$. The parameter $K^{(3)}$ is undetermined in the cases of $d = 1, 2$, but does not appear in Eq. (71). We can ignore the term of $K^{(3)}$ in these cases.

In Eq. (76), the third-order cumulants are the same as the third-order mean values, $\langle f^3 \rangle_c = \langle f^3 \rangle$, $\langle f |\nabla f|^2 \rangle_c = \langle f |\nabla f|^2 \rangle$, $\langle |\nabla f|^2 \Delta f \rangle_c = \langle |\nabla f|^2 \Delta f \rangle$ because the mean values are zero, $\langle f \rangle = \langle f_i \rangle = \langle f_{ij} \rangle = 0$. In Eq. (77), the fourth-order cumulants are related to the fourth-order mean values by

$$\langle f^4 \rangle_c = \langle f^4 \rangle - 3\sigma_0^4, \quad (79)$$

$$\langle f^2 |\nabla f|^2 \rangle_c = \langle f^2 |\nabla f|^2 \rangle - \sigma_0^2 \sigma_1^2, \quad (80)$$

$$\langle f |\nabla f|^2 \Delta f \rangle_c = \langle f |\nabla f|^2 \Delta f \rangle + \sigma_1^4, \quad (81)$$

$$\langle |\nabla f|^4 \rangle_c = \langle |\nabla f|^4 \rangle - \frac{d+2}{d} \sigma_1^4, \quad (82)$$

$$\langle |\nabla f|^2 (\Delta f)^2 \rangle_c = \langle |\nabla f|^2 (\Delta f)^2 \rangle - \sigma_1^2 \sigma_2^2, \quad (83)$$

$$\langle |\nabla f|^2 f_{ij} f_{ij} \rangle_c = \langle |\nabla f|^2 f_{ij} f_{ij} \rangle - \sigma_1^2 \sigma_2^2, \quad (84)$$

which are followed by the definition of cumulants and the fact that mean values with an odd number of spatial derivatives vanish because of rotational symmetry.

E. Minkowski functionals

From Crofton's formula, Eq. (23), the k th Minkowski functional in d dimensions, expectation values of the Minkowski functionals are given by Eq. (25), and $\langle V_k^{(k)} \rangle$ is the expectation value of the density of the Euler characteristic. Because the formula for the Euler characteristic we have derived thus far is for general d dimensions, the last quantity $\langle V_k^{(k)} \rangle$ can simply replace $n_{\chi}(\nu)$ with $d = k$, assuming that all the parameters are calculated in k -dimensional subspace. As such, the parameters in the derived formula $\sigma_0, \sigma_1, S^{(0)}, \dots, K^{(0)}, \dots$ should be replaced by the

corresponding parameters in the k -dimensional subspace in d -dimensional space.

We denote the corresponding parameters as ${}^k\sigma_0, {}^k\sigma_1, {}^kS^{(0)}, \dots, {}^kK^{(0)}, \dots$. These parameters are represented by corresponding ones in d -dimensional space as

$${}^k\sigma_0^2 = \langle f^2 \rangle = \sigma_0^2, \quad (85)$$

$${}^k\sigma_1^2 = -\langle f \Delta_k f \rangle = -k \langle f f_{11} \rangle = -\frac{k}{d} \langle f \Delta f \rangle = \frac{k}{d} \sigma_1^2, \quad (86)$$

where Δ_k is the Laplacian operator in k -dimensional subspace. Similarly, we have

$${}^kS^{(0)} = \frac{\langle f^3 \rangle_c}{k \sigma_0^4} = \frac{\langle f^3 \rangle_c}{\sigma_0^4} = \frac{\langle \alpha^3 \rangle_c}{\sigma_0} = S^{(0)}, \quad (87)$$

$${}^kS^{(1)} = \frac{3 \langle f |\nabla_k f|^2 \rangle_c}{2 k \sigma_0^2 k \sigma_1^2} = \frac{3}{2} \frac{k \langle f f_{11}^2 \rangle_c}{\sigma_0^2 (k/d) \sigma_1^2} = \frac{3}{2} \frac{d \langle \alpha \eta_1^2 \rangle_c}{\sigma_0} = S^{(1)}, \quad (88)$$

$$\begin{aligned} {}^kS^{(2)} &= \frac{-3k}{2(k-1)} \frac{\langle |\nabla_k f|^2 \Delta_k f \rangle_c}{k \sigma_1^4} = \frac{-3k}{2(k-1)} \frac{k \langle f_1^2 \Delta_k f \rangle_c}{(k/d)^2 \sigma_1^4} \\ &= \frac{-3d^2}{2(k-1)} \frac{\sigma_1^2 \sigma_2}{\sigma_1^4} [\langle \eta_1^2 \zeta_{11} \rangle_c + (k-1) \langle \eta_1^2 \zeta_{22} \rangle_c] = S^{(2)}, \end{aligned} \quad (89)$$

where ∇_k is the gradient in the subspace, and Eq. (58) is applied to derive the last expressions. For the kurtosis parameters, similar calculations show that

$$\begin{aligned} {}^kK^{(0)} &= K^{(0)}, \quad {}^kK^{(1)} = K^{(1)}, \quad {}^kK_1^{(2)} = K_1^{(2)}, \\ {}^kK_2^{(2)} &= K_2^{(2)}, \quad {}^kK^{(3)} = K^{(3)}. \end{aligned} \quad (90)$$

Combining Eqs. (33), (62), (71)–(75), and (85)–(90), a weakly non-Gaussian formula for the density of Minkowski functionals is finally derived as

$$\begin{aligned} \bar{V}_k^{(d)}(\nu) &= \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_d}{\omega_{d-k} \omega_k} \left(\frac{\sigma_1}{\sqrt{d} \sigma_0} \right)^k e^{-\nu^2/2} \left[H_{k-1}(\nu) + \left[\frac{1}{6} S^{(0)} H_{k+2}(\nu) + \frac{k}{3} S^{(1)} H_k(\nu) + \frac{k(k-1)}{6} S^{(2)} H_{k-2}(\nu) \right] \sigma_0 \right. \\ &\quad + \left\{ \frac{1}{72} (S^{(0)})^2 H_{k+5}(\nu) + \left(\frac{1}{24} K^{(0)} + \frac{k}{18} S^{(0)} S^{(1)} \right) H_{k+3}(\nu) + k \left[\frac{1}{8} K^{(1)} + \frac{k-1}{36} S^{(0)} S^{(2)} + \frac{k-2}{18} (S^{(1)})^2 \right] H_{k+1}(\nu) \right. \\ &\quad + k \left[\frac{k-2}{16} K_1^{(2)} + \frac{k}{16} K_2^{(2)} + \frac{(k-1)(k-4)}{18} S^{(1)} S^{(2)} \right] H_{k-1}(\nu) \\ &\quad \left. \left. + k(k-1)(k-2) \left[\frac{1}{24} K^{(3)} + \frac{k-7}{72} (S^{(2)})^2 \right] H_{k-3}(\nu) \right\} \sigma_0^2 + \mathcal{O}(\sigma_0^3) \right]. \end{aligned} \quad (91)$$

This is the main result of this paper. We confirm that specific cases of this general result agree with all the known results in the literature. The lowest-order term, i.e., the Gaussian part, agrees with Tomita's formula [106]. The first-order term $\mathcal{O}(\sigma_0)$ is in exact agreement with the result of Ref. [108], which is a conjectured equation suggested by lower-dimensional calculations. Therefore, the newly obtained result is a proof of this conjecture for general dimensions. The second-order term $\mathcal{O}(\sigma_0^2)$ in $d = 2$ dimensions is in exact agreement with the result of Ref. [111]. The second-order term of $V_d^{(d)}(\nu)$, which is equivalent to the genus statistic up to the overall amplitude, exactly agrees with the results of Ref. [110] in $d = 2, 3$ dimensions, after the conversion of the cumulants in this reference to the skewness and kurtosis parameters in this paper.

F. Spectral representation of parameters

In cosmological applications of Minkowski functionals, it is convenient to represent the parameters in the derived formula in terms of the power spectrum $P(k)$, bispectrum

$B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, and trispectrum $T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ of the field f , because these polyspectra can be directly predicted from theories such as the higher-order perturbation theory of nonlinear gravitational evolution, etc. Although the relations are relatively straightforward, we explicitly provide the relations in the following for convenience.

Denoting the Fourier transform of the field as

$$\tilde{f}(\mathbf{k}) = \int d^d x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}), \quad (92)$$

the polyspectra up to fourth order are defined as

$$\langle \tilde{f}(\mathbf{k}) \tilde{f}(\mathbf{k}') \rangle_c = (2\pi)^d \delta^d(\mathbf{k} + \mathbf{k}') P(k), \quad (93)$$

$$\langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c = (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \quad (94)$$

$$\begin{aligned} \langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \tilde{f}(\mathbf{k}_4) \rangle_c \\ = (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \end{aligned} \quad (95)$$

The parameters defined by Eqs. (11), (76), and (77) are represented by these spectra as

$$\sigma_j^2 = \int \frac{d^d k}{(2\pi)^d} k^{2j} P(k), \quad (96)$$

$$S^{(a)} = \frac{1}{\sigma_0^{4-2a} \sigma_1^{2a}} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \\ \times (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) s^{(a)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \quad (97)$$

$$K^{(a)} = \frac{1}{\sigma_0^{6-2a} \sigma_1^{2a}} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \\ \times (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \\ \times \kappa^{(a)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4), \quad (98)$$

where

$$s^{(0)} = 1, \quad s^{(1)} = -\frac{3}{2} \mathbf{k}_1 \cdot \mathbf{k}_2, \\ s^{(2)} = -\frac{3d}{2(d-1)} (\mathbf{k}_1 \cdot \mathbf{k}_2) k_3^2, \\ \kappa^{(0)} = 1, \quad \kappa^{(1)} = -2\mathbf{k}_1 \cdot \mathbf{k}_2, \\ \kappa_1^{(2)} = \frac{-2d}{(d+2)(d-1)} (\mathbf{k}_1 \cdot \mathbf{k}_2) [(d+2)k_3^2 + \mathbf{k}_3 \cdot \mathbf{k}_4], \\ \kappa_2^{(2)} = \frac{-2d}{(d+2)(d-1)} (\mathbf{k}_1 \cdot \mathbf{k}_2) [(d+2)k_3^2 + d\mathbf{k}_3 \cdot \mathbf{k}_4], \\ \kappa^{(3)} = \frac{-2d^2}{(d-1)(d-2)} (\mathbf{k}_1 \cdot \mathbf{k}_2) [k_3^2 k_4^2 - (\mathbf{k}_3 \cdot \mathbf{k}_4)^2]. \quad (99)$$

Once the functional forms of the power spectrum, bispectrum, and trispectrum in a model are given, the parameters of the model can be calculated by the above equations. The dimensionality of the integrals of Eqs. (97) and (98) are too large to evaluate straightforwardly in higher-dimensional spaces. For the nonlinear perturbation theory of gravitational evolution [11] in three-dimensional space, one can in principle apply a technique developed in Refs. [124–127] to reduce the dimensionality of multidimensional integrations of perturbation kernels.

V. CONCLUSIONS

In this paper, we present a method to analytically calculate the non-Gaussian corrections of the Minkowski functionals for the excursion set of smoothed fields in general dimensions. We explicitly derive analytic formulas for first- and second-order corrections of non-Gaussianity for the Minkowski functionals, Eq. (91), which is the main result of the paper. In the derivation, the formula of Eq. (51)

plays a central role. It is straightforward to generalize our calculations to higher-order corrections.

The findings of this paper are quite general. Non-Gaussian corrections to the expected Minkowski functionals of an excursion set is generally given in arbitrary dimensions d , based on the assumptions of statistical homogeneity and isotropy of space only. In cosmology, the cases $d = 1, 2, 3$ are of particular interest for the analyses of cosmic fields. The formulas for the first-order corrections with $d = 1, 2, 3$, which were derived in a previous work [107,108], are reproduced from our general formula as special cases. The formulas for second-order corrections with $d = 2$, which have been reported in a previous work [111], are also reproduced as a special case. Moreover, the formulas for the second-order corrections for the Euler characteristic with $d = 2, 3$, which were derived in a previous work [110], are also reproduced as special cases of the general formula. Thus, our formula contains all the previously known formulas as special cases and unifies them into a single formula, generalizing them to arbitrary dimensions.

The non-Gaussian corrections to the Minkowski functionals are parametrized by the skewness and kurtosis parameters defined by Eqs. (76) and (77). In cosmic fields, these parameters can be theoretically predicted in principle, provided that the bispectrum and trispectrum are known. The relations are given by Eqs. (96)–(98). They involve multidimensional integrations that are not easy to evaluate numerically in a straightforward manner, especially for the kurtosis parameters in higher-dimensional space. In practice, one can apply a technique developed in Refs. [124–127] to reduce the dimensionality of integration, and the multidimensional integration is reduced to be evaluated by one-dimensional fast Fourier transforms with FFTLOG developed by Hamilton [128]. Future work should focus on an approach along this line to theoretically predict the skewness and kurtosis parameters for the case of cosmic fields, such as the three-dimensional density field, two-dimensional weak lensing fields, and so on, with bispectra and trispectra predicted from various theoretical models.

The derived formula relies on the weak field expansion with a small parameter σ_0 . It is mathematically difficult to see if the expansion converges to the true value, as is always the case with the weak field expansions. However, the expansion is asymptotic if the probability distribution function of derivative fields f_μ exists and is bounded and the derivative fields have finite moments. The asymptotic nature of the expansion is mathematically described in a separate paper [114].

Comparisons of the predicted Minkowski functionals with those calculated from numerical realizations of weakly non-Gaussian random fields in two dimensions have already been performed in Ref. [111] with first- and second-order corrections of the non-Gaussianity. The results are in complete agreement with each other within the limit of numerical

errors of the realizations. Detailed comparisons of the first- and second-order corrections with numerical realizations for three dimensions and comparisons with data of cosmological N -body simulations will be presented in a subsequent paper [129]. The analytic formula of this paper well reproduce numerical evaluations of the Minkowski functionals on large scales in quasilinear regime, $\gtrsim 30h^{-1}$ Mpc. In the mildly nonlinear regime, the non-Gaussianity becomes too large to describe the Minkowski functionals by the analytic formula of this paper.

The utility of the analytic formula in cosmology mainly resides in theoretical understanding of how the cosmological information is contained in the Minkowski functionals in general. In weakly nonlinear regime, the deviations from the Gaussian predictions can be naturally described by superpositions of Hermite polynomials as functions of the threshold ν , and coefficients are given by lower-order cumulants of fields, for which values depend on cosmological models that we need to distinguish among. While quantitative analysis of the Minkowski functionals in realistic data still requires some numerical methods, the analytic formula of the present work provides qualitative guides for how one can extract underlying information in the measured Minkowski functionals.

Considering the derived formula beyond three dimensions, it would be extraordinarily interesting if the cases of $d \geq 4$ could be applied to some sort of abstract data analysis or higher-dimensional theories in fundamental physics, etc.

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APPENDIX A: PROOF OF EQ. (45)

In this Appendix, Eq. (45),

$$\langle \det(\nu I - Z) \rangle_G = H_d(\nu). \quad (\text{A1})$$

is proven. This formula is already known (see, e.g., Refs. [116,121]). Here, we provide an alternative proof. We define

$$A = \nu I - Z \quad (\text{A2})$$

below.

Considering the derivative of $\det A$ with respect to ν , we have

$$\frac{\partial}{\partial \nu} \det A = \frac{\partial A_{ij}}{\partial \nu} \frac{\partial \det A}{\partial A_{ij}} = \sum_{i=1}^d \hat{A}_i, \quad (\text{A3})$$

where \hat{A}_i is the (i, i) minor of the matrix A . Given the statistical isotropy, we have

$$\langle \hat{A}_i \rangle_G = \langle \det A^{(d-1)} \rangle_G, \quad (\text{A4})$$

where $A^{(d-1)} = \nu I^{(d-1)} - Z^{(d-1)}$ is the matrix A in $(d-1)$ -dimensional subspace. Thus, we have

$$\frac{d}{d\nu} \langle \det A \rangle_G = d \langle \det A^{(d-1)} \rangle_G. \quad (\text{A5})$$

Using the same approach, we can show a recursion relation

$$\frac{d}{d\nu} \langle \det A^{(m)} \rangle_G = m \langle \det A^{(m-1)} \rangle_G. \quad (\text{A6})$$

for $m = 1, 2, \dots$, where $\det A^{(0)} \equiv 1$. Thereby,

$$\tilde{H}_m(\nu) \equiv \langle \det A^{(m)} \rangle_G \quad (\text{A7})$$

satisfies the same relation as the Appell sequence of Hermite polynomials,

$$H'_m(\nu) = m H_{m-1}(\nu). \quad (\text{A8})$$

Therefore, if the integration constant $\tilde{H}_m(0)$ is the same as $H_m(0)$, $\tilde{H}_m(\nu)$ is identified with $H_m(\nu)$ by induction. The relation $\tilde{H}_m(0) = H_m(0)$ can be shown as follows. We have

$$\tilde{H}_m(0) = (-1)^m \langle \det Z^{(m)} \rangle_G. \quad (\text{A9})$$

The determinant of the matrix $Z^{(m)}$ in m -dimensional subspace is given by

$$\det Z^{(m)} = \frac{1}{m!} \epsilon_{i_1 \dots i_m} \epsilon_{j_1 \dots j_m} Z_{i_1 j_1}^{(m)} \dots Z_{i_m j_m}^{(m)}. \quad (\text{A10})$$

The Gaussian average of this equation with odd m is zero. For odd m , we have $H_m(0) = 0$ and Eq. (A9) is trivially identified with $H_m(0)$. For even m , we have

$$\begin{aligned} \langle \det Z^{(m)} \rangle_G &= \frac{(m-1)!!}{m!} \epsilon_{i_1 \dots i_m} \epsilon_{j_1 \dots j_m} \langle Z_{i_1 j_1}^{(m)} Z_{i_2 j_2}^{(m)} \rangle \dots \langle Z_{i_{m-1} j_{m-1}}^{(m)} Z_{i_m j_m}^{(m)} \rangle, \\ & \quad (\text{A11}) \end{aligned}$$

where Wick's probability theorem for the multivariate normal distribution is applied. Equation (37) also holds for $Z^{(m)}$ in the m -dimensional subspace with $1 \leq i, j, k, l \leq m$,

$$\langle Z_{ij}^{(m)} Z_{kl}^{(m)} \rangle = -\delta_{ij}\delta_{kl} + \frac{d}{(d+2)\gamma^2} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (\text{A12})$$

In Eq. (A11), symmetric components with respect to the permutations of (i_1, i_2) , (j_1, j_2) , etc., should vanish, and thus we can substitute

$$\begin{aligned} \langle Z_{i_1 j_1}^{(m)} Z_{i_2 j_2}^{(m)} \rangle &\rightarrow \frac{1}{4} [2\langle Z_{i_1 j_1}^{(m)} Z_{i_2 j_2}^{(m)} \rangle - \langle Z_{i_2 j_1}^{(m)} Z_{i_1 j_2}^{(m)} \rangle - \langle Z_{i_1 j_2}^{(m)} Z_{i_2 j_1}^{(m)} \rangle] \\ &= -\frac{1}{2} (\delta_{i_1 j_1} \delta_{i_2 j_2} - \delta_{i_1 j_2} \delta_{i_2 j_1}), \end{aligned} \quad (\text{A13})$$

etc. in Eq. (A11). Consequently, we have

$$\langle \det Z^{(m)} \rangle_G = (m-1)! (-1)^{m/2} = H_m(0), \quad (\text{A14})$$

where $\epsilon_{i_1 \dots i_m} \epsilon_{i_1 \dots i_m} = m!$ is used. Therefore, $\tilde{H}_m(0)$ in Eq. (A9) is identified with $H_m(0)$ for even m . Thus,

we have $\langle \det A^{(m)} \rangle_G = H_m(\nu)$. Setting $m = d$ in this relation completes the proof of Eq. (A1).

APPENDIX B: PROOF OF EQ. (51)

In this Appendix, we prove Eq. (51),

$$\begin{aligned} &\langle \text{Tr}(D_Z^{m_1}) \dots \text{Tr}(D_Z^{m_k}) \det(\nu I - Z) \rangle_G \\ &= \frac{(-1)^k d!}{2^{m-k} (d-m)!} H_{d-m}(\nu), \end{aligned} \quad (\text{B1})$$

where $m = m_1 + \dots + m_k$ and m_1, \dots, m_k are non-negative integers, and D_Z is given by Eq. (49).

The determinant of $A = \nu I - Z$ is given by

$$\det A = \frac{1}{n!} \epsilon_{i_1 \dots i_d} \epsilon_{j_1 \dots j_d} A_{i_1 j_1} \dots A_{i_d j_d}. \quad (\text{B2})$$

From this expression, we calculate

$$\begin{aligned} D_Z^{i_1 j_1} \dots D_Z^{i_m j_m} \det A &= \frac{(-1)^m}{2^m} \left(\frac{\partial}{\partial A_{i_1 j_1}} + \frac{\partial}{\partial A_{j_1 i_1}} \right) \dots \left(\frac{\partial}{\partial A_{i_m j_m}} + \frac{\partial}{\partial A_{j_m i_m}} \right) \det A \\ &= \frac{(-1)^m}{2^m (d-m)!} \epsilon_{i_1 \dots i_d} \epsilon_{j_1 \dots j_d} A_{i_{m+1} j_{m+1}} \dots A_{i_d j_d} + \text{sym.} \left(\begin{array}{c} i_1 \leftrightarrow j_1 \\ \vdots \\ i_m \leftrightarrow j_m \end{array} \right). \end{aligned} \quad (\text{B3})$$

Thus, we have

$$\begin{aligned} \text{Tr}(D_Z^m) \det A &= D_Z^{k_1 k_2} D_Z^{k_2 k_3} \dots D_Z^{k_m k_1} \det A \\ &= \frac{2(-1)^m}{2^m (d-m)!} \epsilon_{k_1 k_2 \dots k_m i_{m+1} \dots i_d} \epsilon_{k_2 k_3 \dots k_m k_1 j_{m+1} \dots j_d} A_{i_{m+1} j_{m+1}} \dots A_{i_d j_d} \\ &= \frac{-1}{2^{m-1} (d-m)!} \epsilon_{k_1 k_2 \dots k_m i_{m+1} \dots i_d} \epsilon_{k_1 k_2 \dots k_m j_{m+1} \dots j_d} A_{i_{m+1} j_{m+1}} \dots A_{i_d j_d}, \end{aligned} \quad (\text{B4})$$

where the second equality is derived since there is no contribution when the same indices appear in the antisymmetric tensor.

We also calculate

$$\begin{aligned} \frac{\partial^m}{\partial \nu^m} \det A &= \frac{\partial}{\partial A_{k_1 k_1}} \dots \frac{\partial}{\partial A_{k_m k_m}} \det A \\ &= \frac{1}{(d-m)!} \epsilon_{k_1 k_2 \dots k_m i_{m+1} \dots i_d} \epsilon_{k_1 k_2 \dots k_m j_{m+1} \dots j_d} A_{i_{m+1} j_{m+1}} \dots A_{i_d j_d}. \end{aligned} \quad (\text{B5})$$

Comparing Eqs. (B4) and (B5), we have

$$\text{Tr}(D_Z^m) \det A = \frac{-1}{2^{m-1}} \frac{\partial^m}{\partial \nu^m} \det A, \quad (\text{B6})$$

and consequently, we have

$$\text{Tr}(D_Z^{m_1}) \dots \text{Tr}(D_Z^{m_k}) \det A = \frac{(-1)^k}{2^{m-k}} \frac{\partial^m}{\partial \nu^m} \det A, \quad (\text{B7})$$

where $m = m_1 + \dots + m_k$. Taking the Gaussian average of the last equation and applying Eq. (A1), we have

$$\langle \text{Tr}(D_Z^{m_1}) \dots \text{Tr}(D_Z^{m_k}) \det A \rangle_G = \frac{(-1)^k d^m}{2^{m-k} d \nu^m} H_d(\nu). \quad (\text{B8})$$

Equation (B1) immediately follows from the repeated use of the Appell sequence of Eq. (A8).

An alternative proof of Eq. (B1) is also provided in a separate paper [114].

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