Charge-velocity-dependent one-scale linear model

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We apply a recently developed formalism to study the evolution of a current-carrying string network under the simple but generic assumption of a linear equation of state. We demonstrate that the existence of a scaling solution with nontrivial current depends on the expansion rate of the Universe, the initial root-mean-square current on the string, and the available energy-loss mechanisms. We find that the fast expansion rate after radiation-matter equality will tend to rapidly dilute any preexisting current, and the network will evolve towards the standard Nambu-Goto scaling solution (provided there are no external current-generating mechanisms). During the radiation era, current growth is possible provided the initial conditions for the network generate a relatively large current and/or there is significant early string damping. The network can then achieve scaling with a stable nontrivial current, assuming large currents will be regulated by some leakage mechanism. The potential existence of current-carrying string networks in the radiation era, unlike the standard Nambu-Goto networks expected in the matter era, could have interesting phenomenological consequences.

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I. INTRODUCTION

Models of the early Universe suggest that the symmetry-breaking phase transitions can lead to the formation of one-dimensional topological defects, known as cosmic strings [1–3].

An evolving network of these objects will leave observationally discernible features, including anisotropies in the cosmic microwave background [4–9], gravitational lensing [10,11], and a stochastic background of gravitational waves [12–15], which can be probed by present and future observational programs. However, accurate observational predictions of such cosmic string features can only be obtained once one has a quantitatively accurate understanding of their evolution. Several analytic models, with different levels of detail, are available to describe the evolution of the string networks [16–22].

The physical properties of cosmic strings are determined by the specific details of the symmetry-breaking phase transition that produced them. The Nambu-Goto (NG) and Abelian-Higgs models provide the simplest descriptions, but one expects that physically realistic cosmic strings will have additional degrees of freedom. Of particular interest are superconducting cosmic strings, which initially were suggested in Ref. [23], since they are the expected outcome of various high-energy scenarios [24–28]; it was even argued that when the low-energy limit of the string-forming model contains the standard $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ model, currents must be produced, at least at the electroweak scale, but also at any intermediate scale structured in a similar way [29].

The properties of superconducting strings have been previously studied in Refs. [30–33]. These works suggest that effective models with an analogous NG limit can approximately mimic superconducting string behavior [34–37]. Nevertheless, a more systematic approach to the evolution of these networks is not yet fully developed. Indeed, current-carrying degrees of freedom will substantially complicate the numerical study of cosmic string networks, which are already challenging for Nambu-Goto strings. Moreover, it would require entirely different algorithms, which would need to be developed *ab initio*, rather than simply adapting an existing Nambu-Goto simulation, a problem which is compounded by increased memory requirements and the potential need to accurately resolve new phenomena on very different length scales. In other words, the motivations for developing a reliable analytic model for the evolution of current- and charge-carrying string

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networks are even more compelling than those for the Nambu-Goto case.

Our approach to modeling the evolution of superconducting string networks is based on the canonical velocity-dependent one-scale (VOS) model [20,38]. This approach already demonstrated flexibility and usefulness in modeling particular realizations of superconducting string networks [39–44]. Here, we draw on the recently developed general VOS formalism for superconducting string networks, which shall be referred to as charged or current-carrying VOS (CVOS), developed in Ref. [45], and explore some of its consequences. Specifically, the aim of this sequel is to study in detail the dynamics of superconducting cosmic string networks with a linear equation of state and to determine conditions when a scaling solution with nontrivial charge amplitude is possible.

We emphasize that while the assumption of a linear equation of state may seem too simple, it is not merely used for the sake of computational simplicity. Instead, it should be a good approximation when one has small charges and currents on the strings, which is expected to be generically the case for string-forming phase transitions in the early Universe [23]. On the other hand, the need to accurately describe the additional degrees of freedom evolving on the string worldsheet implies that the CVOS model will be more complex than its simpler VOS counterpart. The larger number of coupled evolution equations yields richer dynamical phenomena. This is unavoidable if one wants an accurate, self-consistent, and quantitative description, but we also emphasize that this averaged (macroscopic) approach is nevertheless far simpler than any analogous microscopic study, let alone direct numerical simulations. Admittedly, for the simpler string models, numerical simulations play an important role in calibrating (or at least providing bounds on) the analytic phenomenological parameters, which is a source of information which is not yet available for the CVOS model. However, this motivates our present approach of systematically studying the possible scaling behaviors of these networks (and in particular the behavior of the charges and currents) under various possible assumptions for the model's phenomenological parameters, so that we can at least develop a robust qualitative understanding.

Our analysis herein may be thought of as a stress test of the model's scaling solutions, in the sense that we will assume that asymptotic scaling solutions with nonzero charge exist and study their form under various assumptions for the available energy-loss mechanisms. Broadly speaking, we find that such solutions are physically problematic, except in limited regions of parameter space. In other words, these generalized scaling solutions are possible but not generic. Indeed, two other types of solution also exist, and are more common. The first of these recovers the standard NG scaling solution, while the second type includes solutions where the charge and current grow on the strings and would eventually lead to a "frozen" network. The generalized scaling solutions can therefore be thought of as equilibrium points between charge growth and its disappearance. Several factors impact the behavior of the network, with the most crucial one being the expansion rate of the Universe: the slower the expansion rate, the more likely it is that charges and currents survive on the strings. We study generic universes where the scale factor grows as a power of physical or conformal time and find, broadly speaking and for the physically expected values of the model parameters, that fast expansion rates (including the matter-dominated epoch) lead to the Nambu-Goto solution, while slow expansion rates (including the radiation-dominated epoch) can lead to charge growth or scaling. These three types of solution have previously been identified for chiral superconducting strings [41], and more recently also for wiggly strings [46]. Finally, through a stability analysis, we also identify another key factor—the initial conditions—that determines which of these dynamical outcomes for the network is physically realized. A scaling solution with nontrivial charge (or a growing charge solution) appears only to be possible for an initial state in which the current or charge density is significant (and/or the velocities are low).

II. MACROSCOPIC MODEL

In previous work [45], we proposed a formalism extending the VOS model to include current-carrying string networks (often referred to as superconducting) which exhibit microscopic dynamics on the string worldsheet described by a given equation of state (that is, the energy per unit length and tension specified as functions of a state parameter). This approach describes the network properties in terms of averaged quantities—in particular, the root-mean-square (rms) velocity v and the correlation length in comoving units ξ_c are sufficient to encode the Nambu-Goto network properties. Here, however, these must be supplemented with the averaged (timelike) charge Q and (spacelike) current J flowing along the strings, which are essentially the rms averages of the corresponding timelike and spacelike components of the microscopic current, respectively [45]. It proves convenient to use the averaged Lorentz-invariant two-current amplitude

$$K = Q^2 - J^2 \tag{1}$$

and to define the relative energy density due to the current and charge as

$$Y = \frac{1}{2}(Q^2 + J^2). \tag{2}$$

As the limit $K \to 0$ has been dubbed "chiral" in previous works [35,47], we shall in what follows refer to K as the distance to chirality—or, for the sake of conciseness, the *chirality*. We shall describe Y as the relative *charge*, encompassing the effective energy density trapped in both the string current and charges. Adding this internal energy to that of the bare string E_0, one can define the total energy E in a volume V and thereby extend the definition of the string characteristic length, $L_{\rm C}$, expressed in comoving units, as

$$E = \frac{\mu_0 V}{a^2 L_{\rm C}^2}, \tag{3}$$

where we continue to make a Brownian assumption for network correlations. Note that in the standard VOS model, the string characteristic length can be assumed to coincide with the string correlation length, but that assumption is clearly not applicable here—in other words, we will no longer have a one-scale model.

A. Linear model

Our extended CVOS model is based on several assumptions, already to be found in the structureless original VOS model; given that the model quantitatively describes many of the characteristic features of Nambu-Goto string networks, one can expect its current-carrying extension to similarly reproduce superconducting cosmic string network properties. The assumptions are that the microscopic variables are uncorrelated when averaged along the string, that they satisfy specific boundary conditions encoding a toruslike topology for the Universe (on scales much larger than the observed one), and that the string network, having been stochastically produced through a phase transition, is therefore Brownian. We will also consider additional charge and energy-loss mechanisms below—that is, beyond those due to direct loop production.

The most important ingredient that explicitly appears in the equations for our macroscopic variables is the underlying microscopic equation of state, which was also assumed to translate into a macroscopic one. Current-carrying strings are here assumed to be infinitely thin (after integration of the transverse degrees of freedom), with a local state parameter κ, made from the gradient of a scalar field (often a phase in the underlying microscopic theory) living on the two-dimensional worldsheet. The local energy per unit length and tension of the string are calculated by means of a two-dimensional surface Lagrangian $f(\kappa)$. Once integrated (averaged) over the full network, the state parameter κ yields the time-dependent chirality K, and the Lagrangian turns into a function of K, called $F(K)$, encoding the microphysics.

In the present work, we explore the consequences of the extended VOS model under the special assumption that the equation of state is of the linear kind, as is expected to be appropriate in the limit of small currents. In other words, we have

$$F(K) = 1 - \frac{\kappa_0}{2}K, \tag{4}$$

where κ_0 is a positive constant. We will comment on the applicability of our results to more generic equations of state in the concluding section. The equation of state [Eq. (4)] can be understood as a linear correction to the NG action [23,35]—i.e., the lowest-order expansion of a general function $F(K)$, valid for small values of the current $K \ll 1$. For this linear model [Eq. (4)], the averaged tension T and energy per unit length U are given by [34,44],

$$U = \mu_0\left(1 + \frac{\kappa_0}{2}|K|\right),$$
$$T = \mu_0\left(1 - \frac{\kappa_0}{2}|K|\right), \tag{5}$$

where μ_0 is a constant with units of mass squared.

One could argue against the use of such a model by considering the transverse and longitudinal averaged velocities of perturbations propagation, respectively given by

$$c_{\rm T}^2 \equiv \frac{T}{U} = \frac{1 - |\kappa_0 K|/2}{1 + |\kappa_0 K|/2} \quad \text{and} \quad c_{\rm L}^2 \equiv -\frac{\mathrm{d}T}{\mathrm{d}U} = 1, \tag{6}$$

implying that this is a subsonic ($c_{\rm L} > c_{\rm T}$) type of string, while the field theory of the original $U(1) \times U(1)$ Witten model, currently accepted as standard to describe superconducting cosmic strings, has been shown to be of the supersonic ($c_{\rm T} > c_{\rm L}$) type [30]. Other models of potential cosmological relevance—for instance, that obtained by integrating out the short-scale wiggles to consider only smooth strings with a nontrivial equation of state [39,48]—have been shown to be transonic [34].

The super- or subsonicity property, however, is known to be relevant for the stability [49] of would-be vortons [50]. It may have important consequences on the trajectories of individual strings, such as enhancing loop formation, energy losses, or even charge leakage [51,52]. As such effects are already taken care of at the phenomenological level, with parameters in principle to be determined by comparison with yet-to-be-done numerical simulations, our conclusions should hold for these models provided the equation of state itself remains, on average, in the linear regime.

B. Cosmological setup

The string network evolution will be studied on a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background with line element

$$\mathrm{d}s^2_{\rm FLRW} = a^2(\tau)(\mathrm{d}\tau^2 - \mathrm{d}\boldsymbol{x}^2), \tag{7}$$

where τ is the conformal time and a the scale factor. In Ref. [45], the extended VOS model with an arbitrary equation of state in such a FLRW background was derived; using the linear equation of state [Eq. (4)] reduces this system of equations to

$$\dot{L}_c = \frac{\dot{a}}{a}\frac{L_c}{1+\kappa_0 Y}(v^2+\kappa_0 Y) + \frac{gcv}{2\sqrt{1+\kappa_0 Y}}, \tag{8a}$$

$$\dot{v} = \frac{1-v^2}{1+\kappa_0 Y}\left[\frac{(1-\kappa_0 Y)k}{L_c\sqrt{1+Y}} - 2v\frac{\dot{a}}{a}\right], \tag{8b}$$

$$\kappa_0\dot{Y} = 2\kappa_0 Y\left(\frac{vk}{L_c\sqrt{1+\kappa_0 Y}} - \frac{\dot{a}}{a}\right) - \frac{v}{L_c}c(g-1)\sqrt{1+\kappa_0 Y}, \tag{8c}$$

$$\kappa_0\dot{K} = 2K\kappa_0\left(\frac{vk}{L_c\sqrt{1+\kappa_0 Y}} - \frac{\dot{a}}{a}\right) - 2\frac{v}{L_c}c(g-1)(1-2\rho)\sqrt{1+\kappa_0 Y}, \tag{8d}$$

where a dot denotes differentiation with respect to τ ($\dot{A} \equiv \mathrm{d}A/\mathrm{d}\tau$), the chirality is as defined above [Eq. (1)], along with the relative charge [Eq. (2)], while the relation between the characteristic lengths ξ_C and L_C for a linear equation of state [Eq. (4)] is given by

$$\xi_C = L_C\sqrt{1+\kappa_0 Y}. \tag{9}$$

One can immediately see that by rescaling $K \to K/\kappa_0$ and $Y \to Y/\kappa_0$, one can absorb the κ_0 dependence in the whole system; hence, and without loss of generality, we shall set the constant $\kappa_0 \to 1$ from now on, as this amounts to redefining the units in which the charge and chirality are measured.

There are six parameters we have introduced in the CVOS evolution equations, with four of these in Eq. (8). The first two essentially govern the dynamics of the NG network in the original VOS model [Eq. (8)], namely

(i) The momentum parameter k, which is, in principle, a function of the rms velocity $k = k(v)$, representing the averaged scalar product of the string velocity and normalized curvature vectors (discussed at length in Ref. [45]).

(ii) The loop-chopping efficiency c, describing the key network energy-loss mechanism.

To these, we add four new physically motivated quantities determining the loss mechanisms for current and charge during network evolution:

(iii) The current-chopping efficiency g, accounting for whether loops typically lose above or below the average current and charge from the network, together with the bias or relative proportion ρ of the current versus the charge that escapes with each loop.

(iv) A charge- and current-loss parameter A, describing direct leakage from long strings and the corresponding bias ρ_A between the relative current and charge that leave the string.

This last parameter should, in principle, be evaluated from the microscopic dynamics: when the curvature of the string leads to direct leakage of the current, interactions with background particles provide a model-dependent cross section for ejecting charged particles, or when string self-interactions such as reconnection disrupt the string currents and cause further charge losses [53]. While we can suggest the form of the dependence on macroscopic variables for the charge leakage (see Sec. II D), we will not fix the model-dependent parameter A, keeping our consideration generic. For accurate estimation of the charge and current-loss parameter A, we would require a detailed analytic evaluation or numerical simulations in the framework of a particular model. As no such treatment is currently available, A remains a purely phenomenological and undetermined parameter.

We shall define these additional parameters in greater detail below, but we note here that the generic network behaviors we find do not appear to be particularly sensitive to the precise values of g, ρ, ρ_A, and A.

C. CVOS model parameters

The extended, current-carrying, CVOS model includes phenomenological terms describing energy losses of the long string network into loops. Defining E_0 as the bare energy of the strings in a volume V with a correlation length (expressed in comoving units) ξ_C, the energy stored in the network for a vanishing current contribution is

$$E_0 = \frac{\mu_0 V}{a^2\xi_C^2}, \tag{10}$$

thereby identifying the average (conformal) distance ξ_C between strings. Recall that in the VOS model, loop production is modeled through the energy loss [20]

$$\left.\frac{\mathrm{d}E_0}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -cv\frac{E_0}{\xi_C}, \tag{11}$$

where the chopping efficiency c encodes the typical energy lost by the long string network into loops.

In addition to the bare string energy, however, the total energy E of a string network must include the charge contribution, which can be represented, for the linear equation of state, as [45]

$$E = E_0(1+Y). \tag{12}$$

This renormalization results in the modified current-carrying characteristic length L_C defined in Eq. (3). Following Eq. (11), the complete energy loss for the total energy E is modified but remains in a similar form,

$$\frac{\mathrm{d}E}{\mathrm{d}\tau}\bigg|_{\mathrm{loops}} = -gcv\frac{E}{\xi_{\mathrm{C}}}, \tag{13}$$

where we have introduced the current-chopping efficiency g to correct the usual bare loop-chopping efficiency c; given its definition, g is a parameter that represents how much of the charge is lost by loops in comparison with infinite strings. In other words, if $g < 1$ (respectively, if $g > 1$), there is typically less (more) charge on loops chopped off the network in comparison with the remaining "infinite" strings, the limiting case $g = 1$ producing loops containing the same amount of charge as the infinite strings. Phenomenological analytic modeling of such charge biases has also been considered [41]. Similarly to c, one naturally expects that g could depend on Y.

The charge Y is directly affected by the energy-loss term, given by Eq. (13), since it is a sum of timelike and spacelike components of the current. The chirality K, being defined as the difference between the squared timelike and spacelike components, can decrease or increase depending on which component of the current is dominantly lost in the form of loops. In order to allow for this possible bias between the losses in timelike and spacelike current components, a parameter $0 \leq \rho \leq 1$ is introduced: the unbiased case is represented by the midpoint $\rho_{\mathrm{bias}} = 1/2$. In other words, ρ represents a skewness between the timelike or spacelike current distribution on cosmic string loops: if $\rho = \rho_{\mathrm{bias}}$, each loop contains an equal proportion of timelike and spacelike contributions, while if $\rho < \rho_{\mathrm{bias}}$ (respectively, $\rho > \rho_{\mathrm{bias}}$), the timelike (spacelike) current loss is dominant.

Various intuitive arguments can be offered for why loop creation may favor or disfavor current losses, and even why there could be a bias for charge over current losses (or vice versa). While we have not yet come to compelling conclusions about these complex processes, we do offer general arguments at the beginning of Sec. III D which tend to disfavor charge or current loss by loops (i.e., $g > 1$ or $b < 0$). However, this dynamical mechanism remains to be tested directly by numerical simulations, which is ultimately required to calibrate the free parameters in the CVOS model. For this reason, a variety of possible forms of these parameter dependencies and their specific values will be explored throughout this work. A key goal is to investigate how sensitive the string network evolution is to their influence.

D. Linear charge leakage

One may anticipate that there should exist an additional energy-loss mechanism for a network of superconducting cosmic strings—namely, the so-called charge leakage, discussed, e.g., in Refs. [53–57]. It takes into account string curvature and the possibility that the current does not follow the string trajectory exactly and may leave the string [53]; by construction, there cannot be any such leakage from a straight string. Other effects should also be considered, such as a background of high-energy particles hitting the relativistic strings, thereby increasing the energy of the condensate particles until they can escape. Also, the reconnection of two strings will disrupt the currents flowing along uncorrelated string regions which need to adjust, again potentially allowing the affected charges or currents to move off of the strings.

In the VOS model, there is only one characteristic length scale—namely, the average comoving interstring distance or correlation length ξ_{c}, which can thus also be identified with the average conformal string curvature $R_{\mathrm{c}} \approx \xi_{\mathrm{c}}$. Using this, we assume that the charge loss can be embedded in the model by means of an additional term in the current amplitude dynamics of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau}\Big|_{\mathrm{leakage}} = -A\frac{Y}{\xi_{\mathrm{c}}} = -A\frac{Y}{L_{\mathrm{c}}\sqrt{1+Y}}, \tag{14}$$

where A is a positive definite constant, characterizing the amount of charge leakage. To show that this is indeed the appropriate form, let us consider the energy loss due to leakage. This has to be proportional to the current contribution in the total energy, which is $E - E_0$. It also ought to be inversely proportional to the average curvature R_{c}. Putting together the above considerations results in

$$\frac{\mathrm{d}E}{\mathrm{d}\tau}\bigg|_{\mathrm{leakage}} = -A\frac{E-E_0}{\xi_{\mathrm{c}}}, \tag{15}$$

which can be rewritten using the characteristic length L_{c} and Brownian assumption in Eq. (10), together with the relation in Eq. (12) between the bare and total energies, as

$$\frac{\mathrm{d}L_{\mathrm{c}}}{\mathrm{d}\tau}\bigg|_{\mathrm{leakage}} = \frac{A}{2}\frac{Y}{(1+Y)^{3/2}}. \tag{16}$$

Equations (14) and (16) are consistent with one another, provided that there is no bare string energy change due to charge leakage:

$$\frac{\mathrm{d}E_0}{\mathrm{d}\tau}\bigg|_{\mathrm{leakage}} = 0, \tag{17}$$

which should be the case, because the charge and current contribution is separate from the bare string energy.

The full system of dynamical equations for the extended VOS model [Eq. (8)] with the addition of charge leakage [Eq. (16)] then can be written as

$$\dot{L}_{\mathrm{c}} = \frac{L_{\mathrm{c}}(v^2+Y)}{1+Y}\frac{\dot{a}}{a} + \frac{gcv(1+Y)+AY}{2(1+Y)^{3/2}}, \tag{18a}$$

$$\dot{v} = \frac{(1-v^2)}{1+Y}\left[\frac{k(1-Y)}{L_{\mathrm{c}}\sqrt{1+Y}} - 2v\frac{\dot{a}}{a}\right], \tag{18b}$$

$$\dot{Y} = 2Y\left(\frac{vk}{L_c\sqrt{1+Y}} - \frac{\dot{a}}{a}\right) - \frac{AY}{L_c\sqrt{1+Y}} - \frac{v}{L_c}c(g-1)\sqrt{1+Y}, \tag{18c}$$

$$\dot{K} = 2K\left(\frac{vk}{L_c\sqrt{1+Y}} - \frac{\dot{a}}{a}\right) - \frac{2(1-2\rho_A)AY}{L_c\sqrt{1+Y}} - 2\frac{v}{L_c}c(g-1)(1-2\rho)\sqrt{1+Y}. \tag{18d}$$

These are the general equations describing the time evolution of a current-carrying string network with a linear equation of state with the phenomenology discussed above taken into account. This system is more complicated than the VOS model, as it seemingly doubles the number of degrees of freedom and adds extra undetermined parameters. It should, however, be emphasized that it represents a tremendous simplification of the true system (see Ref. [45]), which is a large set of strongly coupled nonlinear partial differential equations; an intractable problem can instead be solved on a laptop. The CVOS approach appears to be the only feasible route currently available for gaining insight into the time evolution of current-carrying cosmic strings.

We shall now consider special cases for the parameter choices to clarify the existence and stability of scaling solutions and their plausible cosmological consequences.

E. Time evolution

Consider the full system of equations for the superconducting network in Eq. (18). The first thing one notices is that, if the four phenomenological parameters k, c, g, and ρ do not depend on K, then Eq. (18d) decouples from the rest of the system (18)—that is, the chirality K is sourced by the other variables but without backreacting on them. Note that in general—i.e., for an arbitrary equation of state $F(K)$—couplings do exist that are proportional to $F'(K)$ and $F''(K)$, so the above statement is only strictly valid in the case of a linear equation of state [Eq. (4)] for which $F'(K) = -\frac{1}{2}$ (with $\kappa_0 \to 1$) and $F''(K) = 0$, both independent of K. Given this decoupling, we shall not consider the time evolution of K until later in Sec. V.

We also need to specify at this point the cosmological background evolution, which we assume to be first radiation- and then matter-dominated: we want to investigate power-law expansion rates, for which the scale factor evolves as $a \propto \tau^n$, so that we merely replace $\dot{a}/a \to n/\tau$, with n being a constant. The most relevant cosmological regimes are $n = 1$ for radiation domination and $n = 2$ for matter domination. Furthermore, we will also explore numerically the cosmological transition from radiation to matter domination, relying on the exact solution for the scale factor [58]

$$a(\tau) = a_{eq}\left[2\left(\frac{\tau}{\tau_{eq}}\right) + \left(\frac{\tau}{\tau_{eq}}\right)^2\right], \tag{19}$$

which has the limits $a_{rad} = 2(a_{eq}/\tau_{eq})\tau \propto \tau$ in the radiation era, and $a_{mat} = (a_{eq}/\tau_{eq}^2)\tau^2 \propto \tau^2$ in the matter era, as required. The power

$$n = \frac{1 + (\tau/\tau_{eq})}{1 + \frac{1}{2}(\tau/\tau_{eq})}$$

indeed connects $n = 1$ for $\tau \ll \tau_{eq}$ to $n = 2$ for $\tau \gg \tau_{eq}$.

A scaling solution is characterized by L_c being a constant fraction of τ, so one can conveniently set $L_c = \zeta\tau$ to find

$$\dot{\zeta}\tau = \frac{v^2 + Y}{1+Y}n\zeta + \frac{gcv(1+Y) + AY}{2(1+Y)^{3/2}} - \zeta, \tag{20a}$$

$$\dot{v}\tau = \frac{1-v^2}{1+Y}\left[\frac{k(1-Y)}{\zeta\sqrt{1+Y}} - 2nv\right], \tag{20b}$$

$$\dot{Y}\tau = 2Y\left(\frac{vk}{\zeta\sqrt{1+Y}} - n\right) - \frac{vc(g-1)}{\zeta}\sqrt{1+Y} - \frac{AY}{\zeta\sqrt{1+Y}}, \tag{20c}$$

which contains all the modifications included in Eq. (18) and should thus represent a reasonable approximation of the dynamical evolution of the current-carrying string network.

III. SCALING SOLUTIONS

We start our study of current-carrying string network evolution by making the simplest assumption—that all the parameters described in Sec. II C are constants; we then consider small linear deviations of these from the standard values of the VOS model. Our underlying assumption is that the linear equation of state [Eq. (4)] and the corresponding Eqs. (8) and (18) on which the present analysis rests will hold for small currents, and therefore for a nearly Nambu-Goto string network. Although, in principle, this corresponds to the small-K limit, we shall assume in what follows that Y is also expected to be small.

As discussed already above, one will also have to take into account the fact that these phenomenological parameters can have a charge dependence, which can play a role in the stability (or lack thereof) of scaling solutions. The system in Eq. (18) decouples K from the other variables L_c, v, and Y, provided these phenomenological parameters do not depend on K. Given our lack of a better understanding of the microphysics involved, we shall initially assume this below—i.e., that $g = g(Y)$ and $c = c(Y)$—returning to discuss the dynamics with possible chirality dependence in Sec. V.

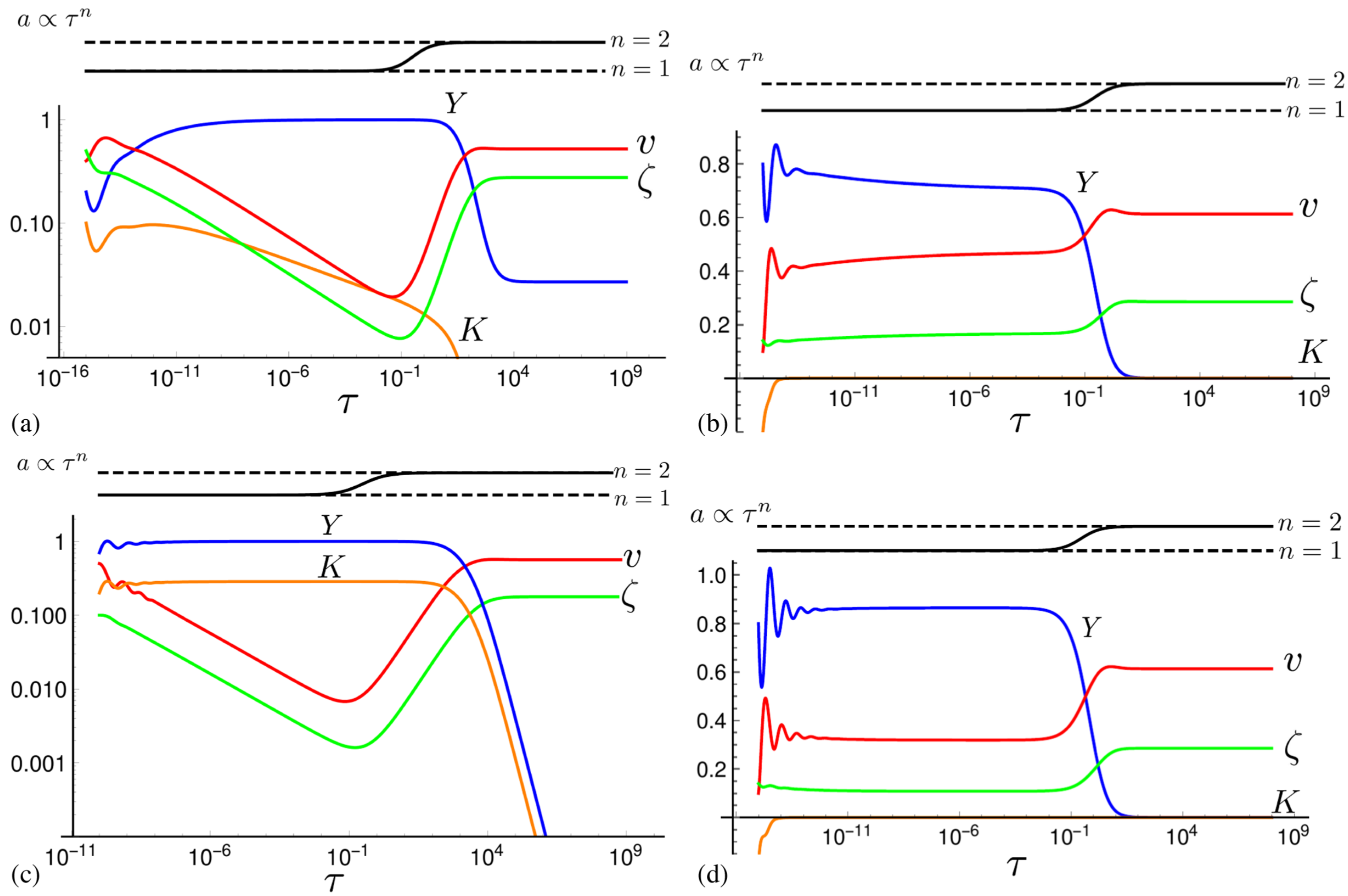

FIG. 1. Evolution of the velocity v, charge Y, chirality K, and characteristic length ζ in various cases through the radiation-to-matter transition (with $\tau_{\mathrm{eq}} = 1$, the time evolution of the expansion rate n being shown above the graphs). (a) Top-left panel: solution of Eq. (22) with constant parameter values set to $g_o = 0.9$, $c_o = 0.5$, $k_o = 0.6$ (with $\rho = 1/2$ throughout), showing a nonscaling growing charge solution with $Y \to 1$, $v \to 0$ and $\zeta \to 0$ during the radiation era, followed by a scaling charged configuration in the matter era. (b) Top-right panel: solution for Eq. (29) with parameters set to $c_o = 0.23$, $k_o = 0.7$, and $b = 0.6$, where a charged scaling solution in the radiation era evolves into a NG network in the matter era. (c) Bottom-left panel: solution with $c_o = 0.23$, $k_o = 0.4$ and $g_o = 1$, parameters for which the running solution of the radiation epoch, whose power laws are seen to satisfy Eq. (26), evolves into a NG solution in the matter era. (d) Bottom-right panel: solution for Eq. (36) with $c_o = 0.23$, $k_o = 0.7$, $b = 0$, and $A = 0.6$, in which case the growing charge solution is modulated into a charged scaling configuration in the radiation era, which then ends with an uncharged NG scaling solution in the matter era.

Figure 1 provides an overview of this section by displaying the possible cosmological evolution scenarios under various modeling assumptions, while the network passes from the radiation- to matter-dominated epochs. In what follows, we discuss each of these constant parameter scenarios in more detail, while noting that a richer phenomenology emerges in Sec. VI using full numerical solutions with variable parameters.

A. Constant parameters

We begin our analysis with a cosmological background dominated by a single fluid component for which n is constant, and we also neglect charge and current leakage at the outset, thereby setting $A \to 0$. A scaling solution is one satisfying

$$v \to v_{\mathrm{SC}} \quad \text{and} \quad L_{\mathrm{c}} \to \zeta_{\mathrm{SC}}\tau, \tag{21}$$

where v_{SC} and ζ_{SC} are constants.[1] Plugging the behaviors from Eq. (21) into the system of Eq. (18), one notices that in such a scaling regime, the variables Y and K should also asymptotically approach some constant values Y_{SC} and K_{SC}. We are thus led to find the equilibrium points for the system:

$$\dot{\zeta}\tau = \frac{v^2 + Y}{1 + Y} n\zeta + \frac{gcv}{2\sqrt{1+Y}} - \zeta, \tag{22a}$$

[1] A variable X at scaling will always be denoted by the same symbol with the scaling subscript, namely X_{SC}. Although these are constants, they are to be contrasted with constant parameters entering the dynamical equations, denoted with a subscript o as in Eq. (23).

$$\dot{v}\tau = \frac{1-v^2}{1+Y}\left[\frac{k(1-Y)}{\zeta\sqrt{1+Y}} - 2vn\right], \tag{22b}$$

$$\dot{Y}\tau = 2Y\left(\frac{vk}{\zeta\sqrt{1+Y}} - n\right) - \frac{vc(g-1)}{\zeta}\sqrt{1+Y}, \tag{22c}$$

$$\dot{K}\tau = 2K\left(\frac{vk}{\zeta\sqrt{1+Y}} - n\right) - \frac{2vc(g-1)}{\zeta}(1-2\rho)\sqrt{1+Y}. \tag{22d}$$

One way to obtain some constraints on the relevant parameter space is to first assume that, at scaling—i.e., at the point (if any) for which $\dot{\zeta} = \dot{v} = \dot{Y} = \dot{K} = 0$—the parameters of the system in Eq. (8) have reached constant values, depending on the scaling values $\zeta_{\rm SC}$, $v_{\rm SC}$, $Y_{\rm SC}$, and $K_{\rm SC}$, and so one sets

$$c \to c_o, \quad k \to k_o, \quad g \to g_o, \quad \text{and} \quad \rho \to \rho_o, \tag{23}$$

where c_o, k_o, g_o and ρ_o are constant. This is a natural requirement, as in any case they are expected to depend either on the usual variables ζ or v, or on the new Y and K, all of which should behave as constants at scaling.

Substituting the assumption from Eq. (23) into Eq. (22), and ignoring nonphysical equilibrium points (e.g., having $v_{\rm SC} = 1$ or $Y_{\rm SC}, \zeta_{\rm SC}, v_{\rm SC} < 0$), one obtains the following values for the equilibrium point:

$$\begin{aligned}
v_{\rm SC}^2 &= \frac{k_o}{n(c_o+k_o)}\frac{k_o(n-2)+c_o[2(g_o-1)+n]}{k_o(n-2)+c_o(g_o-1+n)},\\
\zeta_{\rm SC}^2 &= \frac{k_o(c_o+k_o)}{4n}\frac{k_o(n-2)+c_o[2(g_o-1)+n]}{k_o(n-2)+c_o n},\\
Y_{\rm SC} &= \frac{c_o(1-g_o)}{k_o(n-2)+c_o(g_o-1+n)},\\
K_{\rm SC} &= \frac{2c_o(1-g_o)(1-2\rho_o)}{k_o(n-2)+c_o(g_o-1+n)}.
\end{aligned} \tag{24}$$

This includes the standard non-current-carrying VOS solution, having $Y_{\rm SC} = 0$ and $K_{\rm SC} = 0$:

$$v_{\rm SC}^2 = v_{\rm NG}^2 \equiv \frac{k_o}{n(c_o+k_o)}, \tag{25a}$$

$$\zeta_{\rm SC}^2 = \zeta_{\rm NG}^2 \equiv \frac{k_o(c_o+k_o)}{4n}, \tag{25b}$$

provided one assumes $g_o \to 1$, indicating, as expected, that the model causes the cancellation of any averaged current if there is no current-chopping efficiency introduced.

It should be noted at this stage that Nambu-Goto and Abelian-Higgs field theory network simulations [59–61] have both shown convincingly not only that the relation in Eq. (25) applies when $v_{\rm SC}$ is seen as a constant obtained by solving the implicit Eq. (25a) with $k_o \to k(v_{\rm SC})$, but also that[2] for the cosmologically relevant expansion rates (e.g., the radiation and matter eras), one can safely assume that $c_o < k(v_{\rm SC})$, so that we can set $c_o < k_o$ when investigating the solutions. Moreover, the effect of charges or currents is expected to decrease string velocities, thus increasing the value of k. In this sense, the choice $k_o > c_o$ is physically plausible, and indeed, using the parameter values obtained from NG and Abelian-Higgs, it is a conservative assumption.

Let us begin with the radiation era, which, as we shall see below, is more susceptible to exhibiting a nontrivial charged solution. In this case, one finds a scaling charge

$$Y_{\rm SC}^{\rm rad} = \frac{c_o(1-g_o)}{c_o g_o - k_o},$$

together with the other parameters

$$v_{\rm SC}^{\rm rad} = v_{\rm NG}^{\rm rad}\sqrt{1-Y_{\rm SC}^{\rm rad}}$$

and

$$\zeta_{\rm SC}^{\rm rad} = \zeta_{\rm NG}^{\rm rad}\sqrt{1+\frac{2c_o(1-g_o)}{k_o-c_o}}.$$

Provided $0 \le Y_{\rm SC}^{\rm rad} < 1$, this implies a slower-moving network than the NG case, $v_{\rm SC}^{\rm rad} \le v_{\rm NG}^{\rm rad}$, as expected. If $g_o > 1$, the network is denser, with $\zeta_{\rm SC}^{\rm rad} \ge \zeta_{\rm NG}^{\rm rad}$, and one needs to ensure that $c_o g_o > k_o$, so there may exist a nontrivial scaling solution if the momentum parameter and the charge-chopping efficiency satisfy $c_o g_o > k_o > c_o$. On the other hand, $g_o < 1$ demands that $c_o g_o > k_o$ in order for $Y_{\rm SC}^{\rm rad}$ to be positive, but $g_o < 1$ also implies $c_o g_o < c_o < k_o$, in contradiction with our hypothesis. With these assumptions, the case $g_o < 1$ generically then leads to a nonscaling growing charge solution that we will discuss further below. In other words, if the usual NG relation $k_o > c_o$ holds, the solution can only yield a scaling solution with a charged network configuration, if $g_o > 1$ in the radiation era.

For the matter era—i.e., setting $n = 2$—Eq. (24) implies $Y_{\rm SC}^{\rm mat} = (1-g_o)/(1+g_o)$, and therefore a nontrivial scaling solution having $Y_{\rm SC} > 0$ requires $g_o < 1$, regardless of c_o and k_o. Hence, a charged solution in the matter era with $g_o < 1$ implies a nonscaling growing charge solution in the radiation era $Y_{\rm SC}^{\rm rad} \to 1$, as shown in Fig. 1(A). On the other hand, a charged scaling solution in the radiation era implies an uncharged NG scaling solution in the matter era—i.e., $Y_{\rm SC}^{\rm rad} \neq 0 \Rightarrow Y_{\rm SC}^{\rm mat} = 0$ [similar to Fig. 1(B)].

[2]For the case of global (axion) strings [62], the values of the two parameters are less clear, due to the numerical difficulty of disentangling the effects of loop production and radiation losses.

As a final point, it is worth clarifying the nature of the nonscaling growing charge solution: in Eq. (22b), one sees that if $Y(\tau)$ approaches unity, then the first term becomes negligible and the dynamics is driven by the expansion only. The rms velocity $v(\tau)$ in that case decays, and so does $\zeta(\tau)$. An example of this behavior is illustrated during the radiation era in Fig. 1 (on the left). Indeed, assuming power-law behaviors for all the variables and taking the limit $\tau \to \infty$, the leading terms in the CVOS model equations [Eq. (22)] lead to a growing charge solution of the form

$$v \sim \tau^{-\alpha}, \qquad \zeta \sim \tau^{-\alpha}, \quad \text{and} \quad Y \sim 1 - \tau^{-2\alpha}, \tag{26}$$

where the power α reads

$$\alpha = 1 - \frac{n}{2}\frac{k_o + c_o(2g_o - 1)}{k_o + c_o(g_o - 1)} \underset{g_o=1}{\to} 1 - \frac{c_o + k_o}{2k_o} n. \tag{27}$$

This nonscaling growing or running solution apparently yields a “frozen” network with maximum charge $Y \sim 1$; a comparison with Eq. (12) then shows that this means the string network energy is equally distributed between the bare string contribution and that due to the charge. As this breaks our assumption of small currents (the regime in which the linear equation of state should be valid), we do not consider such solutions to be necessarily physically realized, even if they exist mathematically. Although this effect might possibly be an artifact of the linear approximation, similar running solutions should also be present for better-motivated, nonlinear equations of state [45]. Most probably, these solutions will not come to a standstill, but rather lose charge and current through microphysical effects—i.e., achieving scaling through direct charge leakage, as we shall discuss in Sec. III C.

A comparison of the numerical evolution of Eq. (22) for $g = 1$ with the analytic expression in Eq. (26) for α given by Eq. (27) is shown in Fig. 2 and found to be in good agreement with numerical calculations. In practice, one solves Eq. (36) in the limit of vanishingly small leakage A (see Sec. III C below): fixing $A \to 0$ exactly leads to numerical issues, as the time required to reach this asymptotic behavior scales inversely to A, and we have set $A \to 10^{-7}$. To be consistent with Sec. VI, we also fix $k_o \approx k(0) = 2\sqrt{2}/\pi$ in Eq. (27) [see Eq. (59) for vanishing velocity]. It is interesting to note that a similar effect was found in Ref. [41] for a fixed momentum parameter k_o. Our growing charge solution with dynamical $k(v)$ is illustrated in Fig. 2. This explains the rather counterintuitive observation that some charge leakage is required to produce a nontrivial scaling solution for a current-carrying network, because charge growth needs to be modulated in some way, if dilution due to the expansion or loop charge losses are inadequate.

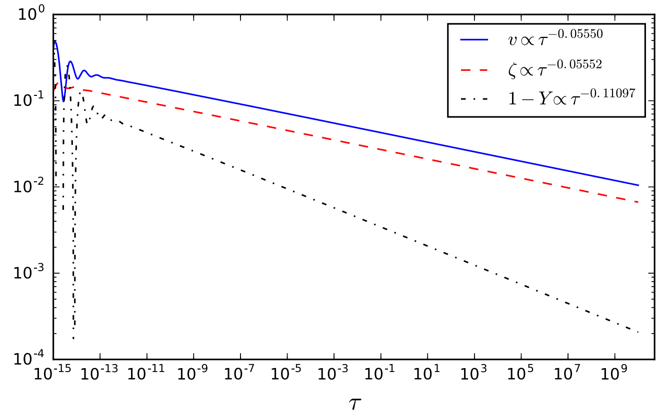

FIG. 2. Evolution of Eq. (36) with $c_o = 0.8$, $n = 1$, $A = 10^{-7}$, $g = 1$, and momentum parameter $k_o \to k(v)$ defined as in Eq. (59), and comparison with the power-law decrease [Eq. (26)]. The fitted values for the decay are obtained from the slope with $\tau > 1$. The analytic value for the decay exponent is given by Eq. (27), numerically given by the value 0.05571, where we use the fact that $k(0) = 2\sqrt{2}/\pi$.

B. Current-chopping bias

Before moving to charge leakage, we consider nonconstant values for the current-chopping efficiency parameter g, and in particular, we ascribe a linear behavior to g as a function of the current amplitude,

$$g \equiv 1 + 2bY, \tag{28}$$

where b is a constant. This linear dependence is motivated by the notion that the charges or currents can only have a significant influence on the nature of loop production if they are nonzero. In order to clarify each effect separately, we assume, in this section, that both the loop-chopping efficiency c and the momentum parameter k remain constant, and thus set $c \to c_o$ and $k \to k_o$.

Because of the decoupling of K discussed above, we begin with the subsystem

$$\dot{\zeta}\tau = \frac{v^2 + Y}{1 + Y} n\zeta + \frac{1 + 2bY}{2\sqrt{1 + Y}} c_o v - \zeta, \tag{29a}$$

$$\dot{v}\tau = \frac{(1 - v^2)}{1 + Y}\left[\frac{k_o(1 - Y)}{\zeta\sqrt{1 + Y}} - 2vn\right], \tag{29b}$$

$$\dot{Y}\tau = 2Y\left(\frac{vk_o}{\zeta\sqrt{1 + Y}} - n\right) - 2c_o b\frac{v}{\zeta} Y\sqrt{1 + Y}. \tag{29c}$$

It is obvious that $Y = 0$ is a solution of these and corresponds to the Nambu-Goto solution. Let us seek a different solution and assume that further scaling with nonzero charge exists, whose equilibrium point (ζ_{SC}, v_{SC}, Y_{SC}) is obtained as the solution of $\dot{\zeta} = \dot{v} = \dot{Y} = 0$. Writing the corresponding algebraic equations in the form of a linear system in c_o, n, and k_o, one finds

$$c_o = \frac{2\zeta_{SC}\sqrt{1+Y_{SC}}(2v_{SC}^2 + Y_{SC} - 1)}{v_{SC}\{2v_{SC}^2[1+b(1+Y_{SC})] + Y_{SC} - 1\}}, \tag{30a}$$

$$n = \frac{2b(1-Y_{SC})(1+Y_{SC})}{2v_{SC}^2[1+b(1+Y_{SC})] + Y_{SC} - 1}, \tag{30b}$$

$$k_o = \frac{4bv_{SC}(1+Y_{SC})^{3/2}\zeta_{SC}}{2v_{SC}^2[1+b(1+Y_{SC})] + Y_{SC} - 1}. \tag{30c}$$

This is, of course, not to be mistaken as some fine-tuning of the underlying parameters n, c_o, and k_o, which are fixed by either the cosmological evolution or the local string physics. Obtaining these relations and demanding that these parameters be positive definite is merely one way of determining relevant constraints on these parameters as well as on the scaling solution ζ_{SC}, v_{SC}, and Y_{SC}. It is also a way to show the limited applicability of such solutions.

One immediately notes that in the limiting case for which $b \to 0$, the solution in Eq. (30) yields $c_o = 2\zeta_{SC}\sqrt{1+Y_{SC}}/v_{SC}$ and $n = k_o = 0$, which would therefore only apply in a Minkowski universe, unless the determinant vanishes, which requires that $v_{SC}^2 = (1 - Y_{SC})/2$. This in turn implies $Y_{SC} < 1$, in accord with the assumption that the linear model ought to be valid for small-current amplitudes.[3] We shall keep this assumption in what follows.

Requiring n, c_o, and k_o to be positive, one obtains two possible situations, depending on both the sign of b and the sign of the denominator $2v_{SC}^2[1+b(1+Y_{SC})] + Y_{SC} - 1$ in Eq. (30). Note that under the assumption $0 < Y_{SC} < 1$ discussed above, this denominator is actually positive definite if $b > 0$, which must be the case, according to Eq. (30c), if we demand an expanding universe with $n > 0$. Then, according to Eq. (30a), we are left with the requirement that $2v_{SC}^2 + Y_{SC} - 1 > 0$, leading finally to

$$1 - 2v_{SC}^2 < Y_{SC} < 1. \tag{31}$$

From the above and Eq. (28), we see that in order to have a nontrivial current-carrying solution, we require $g > 1$, which, we recall, implies that chopped off loops carry more charge than the infinite strings. In other words, it would mean that the average charge is mostly carried by the loops. One therefore expects that in the absence of this bias, the charge on the long strings would grow. This can also be understood by our boundary conditions, discussed around Eq. (29) in Ref. [45], according to which the integral of the spatial derivative of any quantity over the full string network should vanish. Assuming the current to come from such a (phase) gradient, its overall value over the network should thus be initially vanishing. As a result, any leftover should come from loops being chopped off from the network.

Finally, if $b < 0$ and $2v_{SC}^2[1+b(1+Y_{SC})] + Y_{SC} - 1 < 0$, one can obtain a condition for the scaling value of the current amplitude as

$$Y_{SC} < 1 - 2v_{SC}^2 \tag{32}$$

to ensure $c_o > 0$; note that Eq. (32) is slightly more restrictive than the original assumption, as $b < 0$ also implies $2v_{SC}^2[1+b(1+Y_{SC})] + Y_{SC} - 1 < 2v_{SC}^2 + Y_{SC} - 1$. This describes the case for which chopped off loops carry less charge than the infinite strings. Note also that $b < 0$ implies, because of Eq. (28), that g could vanish or even become negative, which is impossible, as it would mean that the total energy [Eq. (13)] increases as the network forms loops. Clearly, while such solutions are mathematically allowed, they are physically unrealistic. In other words, with the scaling solution Y_{SC}, one must impose that b satisfies

$$b > -\frac{1}{2Y_{SC}} > -\frac{1}{2}, \tag{33}$$

where the last inequality was obtained for the limiting case $Y_{SC} \to 1$, which is the maximum possible value. Here, we note the caveat that at high charge levels $Y \to 1$, we expect all our linear approximations to require modification.

Let us now investigate how the system of Eq. (29) depends on the momentum k_o, chopping c_o, and b parameters. In this admittedly oversimplified case (k_o and c_o constant and no direct charge leakage $A = 0$), the scaling values (equilibrium points) for the system of differential equations in Eq. (29) can be found analytically (again excluding unphysical or nonscaling cases). There are two equilibrium points: the first is again the standard, non-current-carrying solution [Eq. (25)], while the second is a new solution with nontrivial current, namely

$$v_{SC}^2 = \frac{k_o[c_o(n+4b) - k_o(2-n)]}{2bc_o(c_o+k_o)n},$$
$$\zeta_{SC}^2 = \frac{k_o(c_o+k_o)[c_o(n+4b) - k_o(2-n)]}{4n[k_o(2-n) - c_o n]},$$
$$Y_{SC} = \frac{k_o(2-n) - c_o(n+2b)}{2bc_o}, \tag{34}$$

which reproduces Eq. (25) when setting $k_o(2-n) = c_o(n+2b)$. When $b \to 0$, the above scaling solution is

[3]The small-current limit for which the linear equation of state is supposed to be valid concerns the chirality K, so that there does not seem to be any actual constraint on the charge Y. However, for $Y \sim 1$, the charge contribution to the overall network energy [Eq. (12)] is comparable to the bare energy, and this entails that the linear regime is no longer appropriate.

obviously problematic, as it implies infinite velocity and charge. Since both $v \to 1$ and $Y \to 1$ are singular points, the dynamics prevents such solutions from being reached and naturally leads either to the growing charge solution, as shown by Eq. (26) and illustrated in the radiation era in Fig. 1(C), or to the uncharged NG solution. For small values of b, actually including $b = 0$, the latter solution is the only available, and there exists a threshold in b above which Eq. (34) becomes acceptable. This is somehow similar to a symmetry-breaking mechanism.

In the special case of matter domination ($n = 2$), one finds $Y_{\rm SC} = -(1+b)/b$, which requires $-1 < b < 0$. Adding the extra requirement that $v_{\rm SC}^2 > 0$ (or $\zeta_{\rm SC}^2 > 0$, both constraints being at this stage equivalent) further restricts the available domain, as it implies $-1 < b < -\frac{1}{2}$. So, in order to sustain a nontrivial current-carrying scaling solution during the matter era, one would need a significant loop-chopping bias against charge losses to counteract the dilution caused by the enhanced expansion rate. We do not expect such a scenario to be physically realistic.

For general values of $n < 2$, one finds first that for $b = 0$, the system is singular: there is no fixed point to the system in Eq. (29) unless one imposes $k_o = c_o$, a rather meaningless fine-tuning. Even then, the only solution would be trivial from the point of view of the current ($Y_{\rm SC} = 0$), while $\zeta_{\rm SC} = c_o/\sqrt{2n}$ and $v_{\rm SC} = 1/\sqrt{2n}$. Clearly, in this case, the physically realistic solution will be one where the charge grows.

For $b \neq 0$, one finds the following two possible cases:
(i) $b > 0 \Rightarrow c_o(n+4b) > k_o(2-n) > c_o(n+2b)$,
(ii) $b < 0 \Rightarrow c_o(n+4b) < k_o(2-n) < c_o(n+2b)$,
which can be summed up though the definition of the constant

$$B \equiv b\left[\frac{k_o}{c_o}(2-n) - n\right]^{-1}, \tag{35}$$

and the constraints now read as $\frac{1}{4} < |B| < \frac{1}{2}$. An example of time evolution for such a network is shown in Fig. 1(B).

Under these assumptions, we find that for a FLRW background with an expansion rate bigger than that for the matter-dominated epoch ($n > 2$), there cannot be any nontrivial scaling solution ($Y_{\rm SC} \neq 0$) with $b \geq 0$. In other words, for such fast expansion rates, the expansion alone is sufficient to force the decay of the charge and lead to the NG solution. As we shall see below, this is exactly what is found numerically, with current-carrying scaling solutions which are present in the radiation era quickly decaying towards a NG regime as matter begins to dominate and the expansion rate increases. As for the radiation epoch ($n = 1$), the constraint $b \geq 0$ implies that one must have $k_o \geq c_o$ to achieve scaling.

Figure 1(C) demonstrates an example of a growing charge solution, which appears when the standard NG equilibrium point [Eq. (25)] is a repeller and the equilibrium point associated with nontrivial charge [Eq. (34)] lies outside the physically meaningful region. In the case of the growing or running solution, we deviate from the usual string network scaling behavior and reach another power-law description, given by Eq. (26).

C. Linear charge leakage

Let us now focus on the effect of charge leakage—i.e., Eq. (20) with a nonvanishing current-loss parameter A, while again keeping $k = k_o$ and $c = c_o$ constant, but now neglecting the current-chopping efficiency with $g = 1$ ($b = 0$). The system now reads

$$\dot{\zeta}\tau = \frac{(v^2+Y)}{1+Y}n\zeta + \frac{c_o v(1+Y) + AY}{2(1+Y)^{3/2}} - \zeta, \tag{36a}$$

$$\dot{v}\tau = \frac{(1-v^2)}{1+Y}\left[\frac{k_o(1-Y)}{\zeta\sqrt{1+Y}} - 2vn\right], \tag{36b}$$

$$\dot{Y}\tau = 2Y\left(\frac{vk_o}{\zeta\sqrt{1+Y}} - n\right) - \frac{AY}{\zeta\sqrt{1+Y}}. \tag{36c}$$

One can find the equilibrium points for this system of differential equations analytically. One of these points is the standard uncharged scaling configuration defined by Eq. (25), while another one has the form

$$\begin{aligned} v_{\rm SC} &= \frac{A}{k_o(2-n) - c_o n}, \\ \zeta_{\rm SC} &= \frac{c_o + k_o}{2\sqrt{1+Y_{\rm SC}}} v_{\rm SC}. \\ Y_{\rm SC} &= 1 - \frac{A^2(c_o+k_o)n}{k_o[k_o(n-2) + c_o n]^2}, \end{aligned} \tag{37}$$

whose limit, when $A \to 0$, is the frozen network of Eq. (26), as expected. Similarly to the solution above [Eq. (34)], fixing A such that $Y_{\rm SC} \to 0$ and plugging back into $v_{\rm SC}$ and $\zeta_{\rm SC}$, one recovers Eq. (25). A particular example of such a string network evolution is shown in Fig. 1(D).

Again, one immediately notices that this nontrivial equilibrium can only be reached provided the expansion rate is $n \leq 2$, as a larger n necessarily implies $v_{\rm SC} < 0$. A similar behavior has been identified for wiggly strings [46]. More precisely, for $n \leq 2$, one has $0 \leq v_{\rm SC} < 1$—i.e.,

$$\frac{k_o}{c_o} \geq \frac{n}{2-n} \underset{n\to 1}{\longrightarrow} 1, \tag{38}$$

and

$$A \leq (2-n)k_o - c_o n, \tag{39}$$

showing that, as expected, too much charge leakage leads back to the non-current-carrying case. This agrees with the previous discussion, when there were net charge losses due to loops with $g > 1 (b > 0)$. Demanding that $Y_{\rm SC} > 0$ leads to

$$v_{\rm SC}^2 \leq \frac{k_o}{n(c_o + k_o)}, \tag{40}$$

a condition which is obviously satisfied as long as $n \geq 0$.

D. Loop-chopping parameter c

Having introduced biased loop-chopping and charge-leakage mechanisms, we briefly consider the possibility of a charge-dependent loop-chopping efficiency—i.e., a decrease (or increase) in the amount of loops produced depending on the presence of a charge along the string. Motivation for such a possibility stems from previous works on superconducting loop dynamics [40,41,51,63] showing that a current flowing along a string can prevent loops from collapsing. Indeed, there exists a special value for the loop radius—$r_{\rm V}$ say—for which the loop reaches an equilibrium state whereby the contraction due to the local string tension is balanced by the angular momentum made possible by the Lorentz symmetry breaking of the worldsheet due to the very existence of a current, with $r_{\rm V}$ actually depending on the current amplitude.

Clearly, for such a scenario, loop production would be strongly suppressed for loop sizes smaller than $r_{\rm V}$: the chopping efficiency should be greatly reduced for such loops. While the typical size of loops produced by superconducting networks is not well known, one may nevertheless expect that it depends on the charge, and in our modeling approach this would lead to a charge-dependent loop-chopping efficiency. Alternatively, one may reason by analogy with the case of wiggly strings, where one expects that there will be more loops at small scale in comparison with the standard description (see Refs. [42,43] for details). Again, the chopping efficiency could depend on the charge.

To lowest order, such an effect could be modeled by the following effective chopping parameter:

$$c = c_o(1 - \beta Y), \tag{41}$$

where c_o is the standard Nambu-Goto chopping efficiency and β is a constant: as one expects the chopping to be reduced as the charge increases, one could reasonably assume $\beta > 0$. Under the hypothesis (41), the VOS model equations take the form

$$\dot{\zeta}\tau = \frac{v^2 + Y}{1 + Y} n\zeta + \frac{c_o(1 - \beta Y)v}{2\sqrt{1 + Y}} - \zeta, \tag{42a}$$

$$\dot{v}\tau = \frac{(1 - v^2)}{1 + Y}\left[\frac{k_o(1 - Y)}{\zeta\sqrt{1 + Y}} - 2vn\right], \tag{42b}$$

$$\dot{Y}\tau = 2Y\left(\frac{vk_o}{\zeta\sqrt{1 + Y}} - n\right), \tag{42c}$$

where all the previously discussed charge-loss mechanisms are neglected—i.e., we set $g = 1$ and $A = 0$.

Clearly, the standard NG behavior with $Y = 0$ is a solution of this system. On the other hand, if one enforces $Y \neq 0$, one nominally finds the nontrivial equilibrium point of Eq. (42) as

$$\begin{aligned} v_{\rm SC}^2 &= \frac{k_o(2 - n) + (\beta - 1)nc_o}{2\beta nc_o}, \\ \zeta_{\rm SC}^2 &= \left(\frac{k_o^2}{2n^2}\right)\frac{k_o(2 - n) + (\beta - 1)nc_o}{k_o(n - 2) + (\beta + 1)nc_o}, \\ Y_{\rm SC} &= \frac{k_o(n - 2) + nc_o}{\beta nc_o}. \end{aligned} \tag{43}$$

This solution shares with that obtained in Eq. (34) the fact that it becomes unreachable for sufficiently small values of β, since both the charge and velocity would diverge as $\beta \to 0$, while in the same limit $\zeta_{\rm SC}^2$ would be negative. The same conclusion actually holds that for small values of β, only the NG attractor is dynamically attainable, and there exists a threshold in β above which the new charged solution becomes physically admissible as the symmetry-broken phase for low enough temperatures. It remains to be seen whether such a solution is realized in practice—i.e., if it is an attractor or a repeller.

The specific case of the matter-dominated regime with $n = 2$ yields the special solution $Y_{\rm SC} = 1/\beta$, which requires $\beta > 1$, in order to ensure that $v_{\rm SC} > 0$ and $\zeta_{\rm SC}^2 > 0$. Note at this point that Eq. (41) would then imply that the effective chopping parameter c vanishes at scaling, while in the limit $\beta \to \infty$, the rms velocity becomes $v_{\rm SC} \to 1/\sqrt{2}$. This is in agreement with the fact, known from both the VOS model and from NG numerical simulations, that a nonzero loop-chopping efficiency is not necessary for a string network's density and velocity to reach scaling: a fast enough expansion rate, including the matter era, is sufficient [38].

In summary, the outcome of this exercise is that for $n > 2$, the NG solution is the only possible one, while for $n < 2$, we would have an ill-defined solution if we insisted on a constant charge. Clearly in this regime, we should either expect a growing charge solution or will need some leakage mechanism to ensure charge scaling for small expansion rates. In other words, a charge-dependent loop-chopping efficiency does not lead to any qualitatively new behavior with respect to what has been discussed in the previous subsections. So, we note the three different solution classes, depending on the expansion rate: Nambu-Goto behavior for large expansion rates, growing charge for small expansion rates (unless modulated by charge leakage), and a transition between the two occurring at the matter-dominated era. Similar behavior has also been identified for wiggly cosmic strings [46]. We present further supporting evidence for these solutions in the sections that follow.

IV. SCALING STABILITY

In order to unveil the nature of the critical points described in the previous section, we expand the relevant quantities around each of these solutions.

A. General method

The existence of equilibrium points with nontrivial current, such as those given by Eqs. (34), (37), and (43), does not guarantee that these points can be dynamically reached, and even less that they represent attractors for the corresponding systems of differential equations. To understand the nature of these, we study the relevant Jacobian matrices. For convenience, we introduce the vector $\boldsymbol{V} = \{\zeta, v, Y\}$ and the vector function $\boldsymbol{Z}(\boldsymbol{V})$, which allows us to rewrite formally the systems in Eqs. (29), (36), and (42) as

$$\dot{\boldsymbol{V}}\tau = \boldsymbol{Z}(\boldsymbol{V}). \tag{44}$$

Denoting $\boldsymbol{V}_{\rm SC} \equiv \{\zeta_{\rm SC}, v_{\rm SC}, Y_{\rm SC}\}$ as the equilibrium (scaling) point—i.e., the solution of $\boldsymbol{Z}(\boldsymbol{V}_{\rm SC}) = 0$—the stability of $\boldsymbol{V}_{\rm SC}$ can be understood by finding the parameters for which small perturbations around the equilibrium points decay with time.

Let us consider a solution $\boldsymbol{V}$ representing a small deviation from the equilibrium point, namely with $|\delta\boldsymbol{V}| \ll |\boldsymbol{V}_{\rm SC}|$, where we define $\delta\boldsymbol{V} = \boldsymbol{V} - \boldsymbol{V}_{\rm SC}$. We expand the vector function $\boldsymbol{Z}(\boldsymbol{V})$ in Eq. (44) as

$$\dot{\boldsymbol{V}}\tau = \boldsymbol{Z}(\boldsymbol{V}_{\rm SC}) + \left.\frac{\partial \boldsymbol{Z}(\boldsymbol{V})}{\partial \boldsymbol{V}}\right|_{\boldsymbol{V}=\boldsymbol{V}_{\rm SC}} \delta\boldsymbol{V} + \mathcal{O}(\delta\boldsymbol{V}^2), \tag{45}$$

which can be reduced to

$$\delta\dot{\boldsymbol{V}}\tau = \mathcal{J}_{\rm SC}\delta\boldsymbol{V}, \tag{46}$$

where $\mathcal{J}_{\rm SC}$ is the Jacobian matrix $\partial\boldsymbol{Z}/\partial\boldsymbol{V}$ evaluated at the point $\boldsymbol{V} = \boldsymbol{V}_{\rm SC}$. Diagonalizing $\mathcal{J}_{\rm SC}$ through $\mathcal{J}_{\rm SC} = \mathcal{P}^{-1}\mathcal{D}\mathcal{P}$, thereby defining the matrix $\mathcal{P}$, and setting $\mathcal{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, one can integrate Eq. (46) to obtain the time dependence of the eigenvector components as

$$(\mathcal{P}\delta\boldsymbol{V})_i \propto \tau^{\lambda_i}. \tag{47}$$

The scaling solution $\boldsymbol{V}_{\rm SC}$ is an attractor if the perturbation $\delta\boldsymbol{V}$ decreases for large τ, which translates, through Eq. (47), into a requirement on the real part of the eigenvalues of the Jacobian $\mathcal{J}_{\rm SC}$, namely $\Re\mathrm{e}(\lambda_i) \leq 0, \;\; \forall\, i$. In what follows, we study the distribution of the maximum $\Re\mathrm{e}(\lambda_i)$ as a function of the relevant parameters to determine the attractor regions.

B. Current-chopping bias

The CVOS model, in the version described in Sec. III B by Eq. (29), contains four parameters, namely k, c, b and n, where k is in principle a function of the rms velocity v, and c and b could depend on the charge, as discussed in the previous sections.

While the expansion power index n is fixed by the background cosmological setup, the other quantities are in principle given by the microphysics of the strings themselves. Lacking knowledge of their actual numerical values, one must choose a set $\{c_o, k_o, b\}$ leading to physically meaningful solutions for the scaling rms velocity $v_{\rm SC}$, characteristic length $L_{\rm SC} = \zeta_{\rm SC}\tau$, and charge magnitude $Y_{\rm SC}$, given the restriction on B as defined in Eq. (35).

Figure 3 illustrates our procedure for $n = 1$ (radiation-dominated epoch) by setting $B \to \frac{1}{3}$ and plotting the maximum value of the real part of the eigenvalues λ_i in the (c_o, k_o) plane for both equilibrium points, using Eq. (25) (Nambu-Goto network) and Eq. (34).

Carrying out calculations for different values of the underlying parameters, one notices that only the expansion rate n and the ratio k_o/c_o are important to determine the nature (attractor, repeller, saddle point) of the equilibrium point. Moreover, the parameter space where the equilibrium point with the trivial magnitude of the current $Y_{\rm SC} = 0$, given by Eq. (25), is an attractor does not overlap with the space where the equilibrium point with nontrivial current

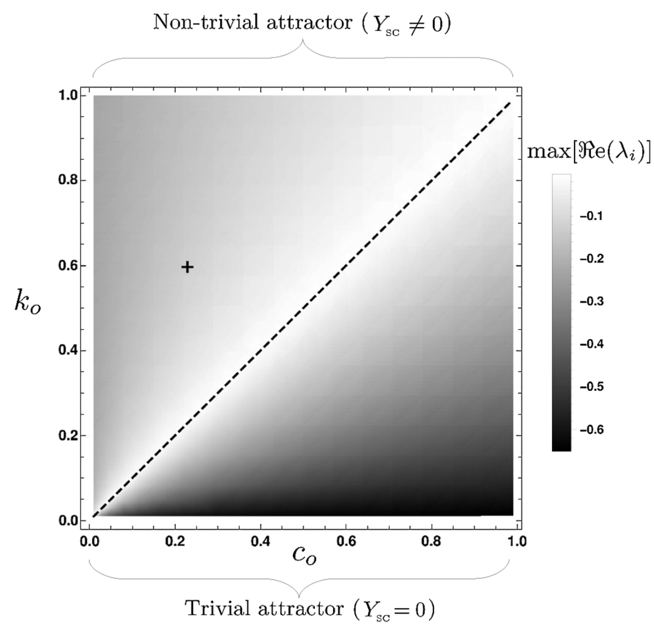

FIG. 3. Distribution of the maximal real part of the Jacobian eigenvalues [Eq. (46)] around the no-current [Eq. (25)] (lower-right triangle) and current-carrying [Eq. (34)] (upper-left triangle) equilibrium points for $B = \frac{1}{3}$. Note that changing the value of B in the range allowed by the constraint below Eq. (35) does not qualitatively change the plot. The dashed line represents the values of k_o and c_o for which $\max[\Re\mathrm{e}(\lambda_i)] = 0$, so that the upper-left triangle represents the region for which the equilibrium point of Eq. (34) is an attractor, while that of Eq. (25) is an attractor in the lower right triangle. Varying the expansion rate n changes the slope of the dashed line, which coincides with Eq. (48). An example of a trajectory for a nontrivial solution is shown in Fig. 4 for the parameter values at the point denoted by the + sign ($c_o = 0.23$ and $k_o = 0.6$).

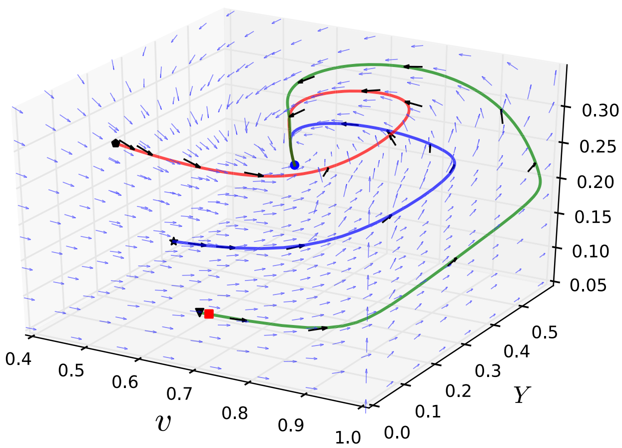

FIG. 4. Phase diagrams for the nontrivial equilibrium points shown in Fig. 3 with parameters $c_o = 0.23$, $k_o = 0.6$, $b = 0.4$, and $n = 1$ (radiation-dominated epoch). The nontrivial solution of Eq. (34) is an attractor in this case. It is shown by the blue round point, while the uncharged solution of Eq. (25), a repeller, is represented as a red square point. Different lines represent different evolutions (different initial conditions) of the system in Eq. (29). The gray dots represent initial conditions.

magnitude $Y_{SC} \neq 0$, given by Eq. (34), is an attractor. This behavior means that, depending on the phenomenological parameters, one expects only one solution to be realized, either the charged or uncharged. In Fig. 4, an example is shown of the phase-space trajectory of the system in Eq. (29) in the radiation era ($n = 1$) which approaches a nontrivial scaling charge magnitude; this uses constant parameter values c_o and k_o consistent with a NG scaling solution with some loop charge loss $b > 0$. Figure 4 also illustrates the independence of the initial conditions for the charged attractor solution under these assumptions.

The relevant regions of parameter space representing charged and uncharged attractors are separated by the line

$$k_o = \frac{n}{2 - n} c_o, \tag{48}$$

implying in particular that for the matter-dominated era ($n = 2$), the nontrivial charged scaling solution triangle apparently shrinks to the line $c_o = 0$. For $n > 2$, as both k_o and c_o are positive definite, there is no stable attractor solution with nontrivial current. Similar behaviors for the existence (or otherwise) of scaling solutions have also been observed for chiral superconducting strings [41] and wiggly strings [46]. Here, however, we reiterate the caveat that these assumptions should not exclude the possibility of a physically growing current solution discussed previously which can be modulated by other charge-loss mechanisms.

C. Linear charge leakage

We now return to the linear leakage model [Eq. (36)] of Sec. III C with fixed points given by Eqs. (25) and (37), to which we apply the eigenvalue method of Sec. IV A. We make the simplifying assumption again that there is no loop charge bias $g = 1$ (or $b = 0$). Since the parameter space for this model is three-dimensional, we choose to plot two

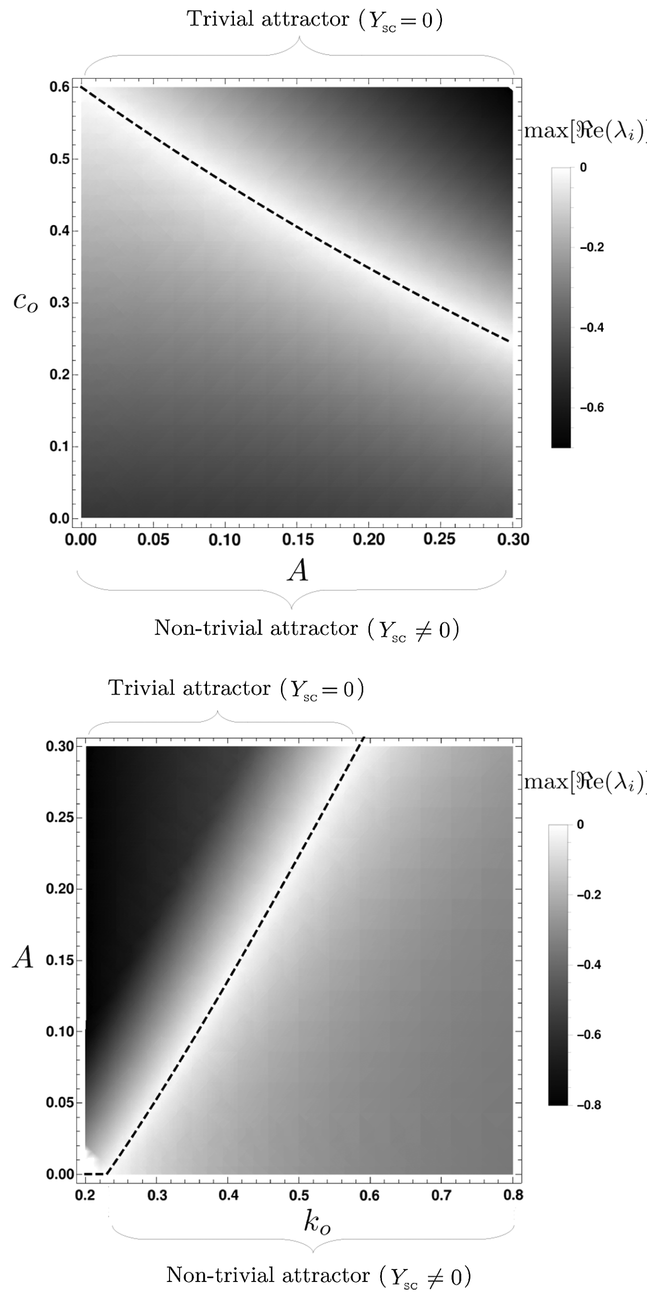

FIG. 5. Distribution of the maximal real part of the Jacobian eigenvalues [Eq. (46)] for the system of Eq. (36) with fixed points given by Eqs. (25) and (37), for $n = 1$ (radiation-dominated era). Only the regions with negative eigenvalues, representing attractors, are shown. Upper panel: (c, A) plane, $k_o = 0.6$. Lower panel: (k_o, A) plane, $c_o = 0.23$. The fixed point of Eq. (37) with $Y_{SC} \neq 0$ is a repeller in the region of parameter space where the trivial one, from Eq. (25), with $Y_{SC} = 0$, is an attractor, and vice versa. The dashed line that separates both regions is given by Eq. (49).

slices of the distribution of eigenvalues, namely in the (A, c_o) and (k_o, A) planes. Figure 5 shows the distribution of the maximal real part of the eigenvalues for the radiation-dominated epoch ($n = 1$).

As we see from this figure, there are two nonoverlapping regions of parameter space, one with the uncharged solution attractor with $Y_{SC} = 0$, and the other with $Y_{SC} \neq 0$. It is worth emphasizing at this point that the two solutions are actually exclusive of one another, so the system is completely deterministic and mostly independent of the initial conditions, the asymptotic solution features being determined by the values of the underlying parameters. Note also that there exist other mathematically acceptable solutions with nonphysical values of the variables, such as, e.g., having $Y_{SC} < 0$ or $\zeta_{SC} < 0$; we will discard these solutions, which can be found in the region for which the attractor is for $Y_{SC} = 0$.

Repeating the analysis of Fig. 5 for different values of n, one finds that the line that separates the two regions of parameter space is described by the equation

$$A^2 = \frac{k_o[k_o(2-n) - c_o n]^2}{(c_o + k_o)n}, \tag{49}$$

with the condition $k_o(2-n) - c_o n > 0$.

V. CHIRALITY

Up to this point, we have analyzed the stability of scaling solutions for different phenomenological scenarios of the system of Eq. (18) without the chirality K. As we argued already, provided the phenomenological parameters do not depend on K, and in the linear equation of state with which we are concerned here, this variable is decoupled from the rest of the system and does not contribute to the string network evolution. Nevertheless, one should check that it also has a stable scaling solution. If we start at the current-forming phase transition at time τ_0 with purely chiral initial conditions $K(\tau_0) = 0$, it is expected to remain zero throughout the subsequent evolution of the network, but if we start with arbitrary $K(\tau_0)$, we need to require that the scaling value K_{SC} be an attractor. Physically, as the initial current exists as a random fluctuation at a phase transition, one expects that its statistical average should vanish, and thus it is natural to assume $K(\tau_0) = 0$, or in any case $K(\tau_0) \ll 1$ if one understands K more as a variance than as a mean value. Provided it does not grow much during its subsequent evolution, the linear equation of state should thus remain a good approximation.

The parameter space for which the chirality K is stable/unstable can be easily understood if we fix L_c, v, and Y according to their scaling solution values and solve the equation for K. This reads

$$\dot{K}\tau = 2\bar{\alpha}K - 2K_o, \tag{50}$$

where

$$\bar{\alpha} = \frac{v_{SC}k(v_{SC})}{\zeta_{SC}\sqrt{1+Y_{SC}}} - n \tag{51}$$

and

$$K_o = \frac{(1-2\rho_A)AY_{SC}}{\zeta_{SC}\sqrt{1+Y_{SC}}} + \frac{v_{SC}}{\zeta_{SC}}c[g(Y_{SC}) - 1] \times (1-2\rho)\sqrt{1+Y_{SC}}. \tag{52}$$

The general solution for the chirality then has the form

$$K = \left[K(\tau_0) - \frac{K_o}{\bar{\alpha}}\right]\left(\frac{\tau}{\tau_0}\right)^{2\bar{\alpha}} + \frac{K_o}{\bar{\alpha}}, \tag{53}$$

in which the constant term should provide the scaling value $K_{SC} = K_o/\bar{\alpha}$, providing $\bar{\alpha} < 0$. In order for the $K(\tau)$ to vanish asymptotically—i.e., for $K_o = 0$—one must have both $\rho_A = \rho_{\text{bias}}$ (no bias towards either charge or current in the leakage mechanism) and either $g(Y_{SC}) = 1$ (loops and long strings having the same average charge or current) or $\rho = \rho_{\text{bias}}$ (charge and current losses of equal amounts), as expected. If either of these parameters takes a different value, then a fixed amount of chirality will be produced as the network evolves. This is, in fact, of no consequence as far as the other variables are concerned, and it should not affect the scaling solution, unless $K_o/\bar{\alpha}$ is large enough that it drives the model into its nonlinear regime.

The case $\bar{\alpha} \to 0$ must be treated separately. It yields a logarithmic divergence in time for $K(\tau)$ unless $K_o = 0$.

If $\bar{\alpha} > 0$ ($\bar{\alpha} < 0$), $K(\tau)$ grows (decays), and the distance to the equilibrium chirality increases (decreases). One can plug the analytic solutions given by Eqs. (25), (34), and (37) in to Eq. (51) to see that these lead to

$$\bar{\alpha} = \frac{2k_o}{c_o + k_o} - n = \frac{\alpha}{1-\alpha}n, \tag{54}$$

where we use Eq. (27). The expression (54) for the radiation epoch ($n = 1$) is always positive when $k_o > c_o$. Bearing in mind that scaling solutions with nontrivial current amplitude are attractors for ζ_{SC}, v_{SC}, and Y_{SC} only when $k_o > c_o$ (as shown in Sec. IV A), we can conclude that K does not have a stable scaling solution when the current amplitude is nontrivial, $Y_{SC} \neq 0$, for a linear charge leakage or a linear perturbation of g.

By studying the behavior of $\bar{\alpha}$, we conclude from Eq. (51) that $K(\tau)$ is growing in the parameter space where the nontrivial charge is an attractor. Hence, there is no stable scaling solution with nontrivial charge Y_{SC} due to the instability of the chirality K. This could be due to the behavior of the parameters ρ and ρ_A in Eq. (18), which we have not explored so far.

As was described in Ref. [45], the parameter ρ was introduced as a possible skew factor of charge/current loss due to the production of loops. In other words, each loop contains some amount of charge Q (timelike contribution) or current J (spacelike contribution), and this amount is controlled by the parameter ρ. If it is constant, each loop contains the same relative amounts of charge and current, so that $\rho = \frac{1}{2}$ implies that each loop contains the same amount of charge and current. If we initially have a small deviation form chirality—i.e., $K(0) \neq 0$—and there is a nonvanishing Y, the string network will go away from chirality and $K(\infty)$ will tend to diverge from zero.

A more generic behavior should allow for the increase or decrease of charge loss in comparison to current if either one of them dominates over the other initially. If $Q^2 > J^2$, the string network should more likely lose charge rather than current, and this can be modeled through

$$\rho = \frac{1}{2} - sK = \frac{1}{2} - s(Q^2 - J^2), \tag{55}$$

where $s > 0$ is a constant. Plugging Eq. (55) in to the time evolution equations for Q and J, given by Eq. (42) of Ref. [45], one sees that if $Q^2 > J^2$, ρ yields a positive contribution to J^2 and a negative one to Q^2, meaning that the averaged charge per string length decreases and the current per string length increases. Substituting ρ given by Eq. (55) into Eq. (8d) and again extracting the parameter $\bar{\alpha}$, still using the analytic form Eq. (34), one obtains

$$\bar{\alpha} = \frac{k_o(2-n) - nc_o}{2B^2(c_o + k_o)c_o}(B - 2s + 4Bs), \tag{56}$$

where we have assumed that $k_o(2-n) - nc_o > 0$, since only in that parameter region does the scaling solution with nontrivial current amplitude exist. We also use Eq. (35) for b, assuming $\frac{1}{4} < B < \frac{1}{2}$. To ensure that α is negative, one then needs to set $s > B/[2(1-2B)]$, a relation which is always possible, demanding $s > \frac{1}{4}$, provided $B \neq 0$.

An example of a time-dependent solution of Eq. (8) with g and ρ given by Eqs. (28) and (55), and $k(v) = k_o$, is shown in Fig. 1(B) for the radiation- and matter-dominated eras. We find that the solution is indeed dynamically driven to a nonvanishing constant charge in the radiation era, charge which subsequently vanishes during matter domination; the chirality also vanishes ($K \to 0$). One should keep in mind, however, that our model concerns average values, so the actual superconducting cosmic string network may contain both timelike and spacelike and/or chiral current-carrying strings.

An analogous treatment can be done for ρ_A—i.e., one can set

$$\rho_A = \frac{1}{2} - s_A K = \frac{1}{2} - s_A(Q^2 - J^2), \tag{57}$$

with a similar interpretation as for ρ: if the string network contains a timelike contribution larger than the spacelike one, it is more likely that charge, rather than current, might escape the network. By substituting Eq. (57) into Eq. (18d), setting $g = 1$, and assuming scaling for L_c, v, and Y, one finds that

$$\bar{\alpha} = \frac{k_o(2-n) - nc_o}{(c_o + k_o)c_o}(1 - 4s_A Y_{SC}), \tag{58}$$

where Y_{SC} is given by Eq. (37). One sees that there are choices of s_A for which one can ensure stability—i.e., with $\bar{\alpha} < 0$ for K. An example of such an evolution for the system in Eq. (18) is shown in Fig. 1(D), again showing a charged scaling during radiation domination followed by a transition to the NG scaling when matter kicks in.

Let us conclude this section by stating that we have shown the CVOS system [Eq. (18)] to have only NG uncharged, $Y_{SC} = 0$, fixed points for $n \geq 2$: provided the phenomenological parameters are in the relevant domain, the charge Y is dynamically driven to a nonvanishing constant $Y_{SC} \neq 0$ while the radiation sources the Universe's expansion, and it subsequently decays after the radiation-to-matter transition. This is illustrated in Figs. 1(B) and 1(D). It is interesting to note that analogous classes of solutions were also found in the chiral superconducting and wiggly string cases [41,46], so one could be led to conjecture that it might be a generic behavior for a more universal current-carrying string equation of state [45].

VI. THE STANDARD MOMENTUM PARAMETER

So far, we have made the simplifying assumption that the momentum parameter is constant, $k \to k_o$, although it is known, at least in the NG case for which there are numerical simulations allowing the evaluation of k, that it is dependent on the rms velocity $k = k(v)$. Lacking similar simulations for the current-carrying case, one cannot decide whether it should or could depend on either the charge Y and/or the chirality K, and we will therefore stick with the simplifying assumption that k depends on neither.

At scaling, with all the relevant functions of time reaching constant values, one can safely assume $k \to k_o$, although one might still ask if the solution for the scaling variables is a good approximation to the exact case. Further, there is *a priori*, for an arbitrary functional dependence $k(v)$, no particular reason why its derivative should remain small around the scaling solution, which could potentially undermine our previous approximate stability analysis. In this section, we accordingly discuss the validity of the results obtained in the previous sections in the case for which the momentum parameter does, as in the usual NG case, depend on the rms velocity. As discussed above, we will assume that it retains the standard form given by [21]

$$k_{\rm NG}(v) = \frac{2\sqrt{2}}{\pi}\frac{1-8v^6}{1+8v^6}(1-v^2)(1+2\sqrt{2}v^3). \qquad (59)$$

It should be emphasized that this form has been obtained by comparison with NG simulations, but we could also modify it further to a slightly different velocity dependence, such as that which was found more recently by comparison with high-resolution field theory simulations [61]. However, the specific form of the velocity-dependent function should not impact our results too significantly, as long as the function $k(v)$ is monotonic (which is true for both of these forms), not least because we are seeking scaling solutions where the velocity is a constant—i.e., in which k itself becomes a constant. We also want to emphasize that the shape in Eq. (59) for $k(v)$, plotted, e.g., as Fig. 3 of Ref. [45], is varying only very slowly over a wide range of velocities, and therefore one might expect the stability analysis assuming $k(v) \sim k_o$ to be mostly valid over this regime of velocities.

For $k(v)$ given by Eq. (59), the CVOS model needs to be studied numerically. However, we expect the most important difference with the cases discussed in Secs IV A and V to stem from the fact that the constant ratio k_o/c_o, that mattered so much for the stability analysis, is now promoted to a dynamical variable. As a result, for a given set of underlying parameters, we will find that is now possible for two different attractors to simultaneously exist. In particular, the solution $Y_{\rm SC} = 0$, which is always present, could then be always stable, thereby restricting the possible set of initial conditions leading to a charged network: if the mechanism by which the current originally forms is such that at that time, the system is close to the $Y = 0$ solution, it would then return to that state, and even though in principle a charged scaling solution exists, it may never be actually reached. What we shall find is that, for a given set of phenomenological parameters allowing for a charged scaling configuration, the space of initial conditions for v, ζ, and Y contains two clearly separated regions, shown, e.g., in Fig. 8, in which the subsequent evolution is fully deterministic, leading inevitably to a charged or an uncharged network.

We repeat the analysis of Sec. IV A by numerically searching for all possible fixed points with physically meaningful values—e.g., satisfying $0 < v_{\rm SC} < 1$, $\zeta_{\rm SC} > 0$, and $Y_{\rm SC} \geq 0$. As a first result in the varying momentum parameter case, we recover the conclusion that a charge-loss mechanism, such as the leakage in Eq. (14), is again needed to modulate growing charge solutions to ensure there is a stable nontrivial equilibrium point with $Y_{\rm SC} \neq 0$.

Having found that modifying the chopping parameter c by means of the charge dependence [Eq. (41)] does not lead (on its own) to new scaling solutions for the constant momentum parameter k_o case, we assume that this is still the case when it takes the form of Eq. (59): were a different scaling solution to exist, upon reaching it, the parameters would be mostly constant, and our previous analysis, showing no $Y_{\rm SC} \neq 0$ solution to exist, should apply. For this reason, we assume initially that $\beta = 0$ in what follows and explore the two-dimensional parameter space (A, b).

In comparison with our previous analysis with $k(v) = k_o$, the important difference with the dynamical momentum parameter $k(v)$ is that for the same fixed parameters it is possible to have two scaling attractors: Nambu-Goto and charged. The configuration that is actually dynamically reached is determined by the network's initial conditions.

Figure 6 shows an example of the region in the (A, b) plane where a $Y_{\rm SC} \neq 0$ attractor solution exists with the chopping efficiency set to $c_o = 0.23$. As expected from the previous discussion, the regions with one solution $Y_{\rm SC} = 0$ only and those with two solutions, including both $Y_{\rm SC} = 0$ and $Y_{\rm SC} \neq 0$ (depending on the initial conditions), are neatly separated. Repeating the analysis for various values of the expansion rate n as shown in Fig. 6, we affirm the conclusion that the matter-dominated era cannot sustain a charged network configuration. As n is lowered, the region in parameter space which can sustain a charged solution increases: numerical investigations reveal that the boundary curves between the charged and uncharged networks are well approximated by the straight lines

$$b = c_1(n) + c_2(n), \qquad (60)$$

where

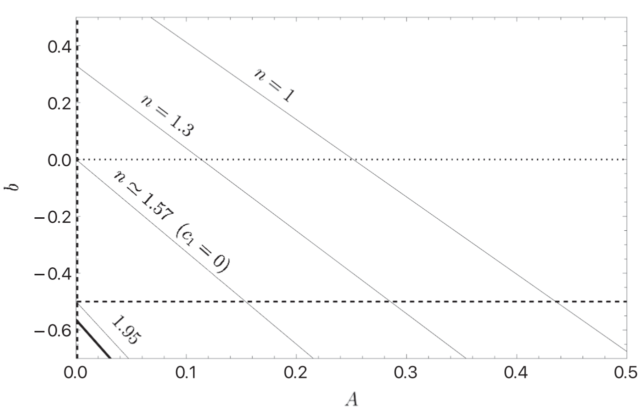

FIG. 6. Boundary lines, in the (A, b) plane, between regions of potentially charged and certainly uncharged network solutions, for various expansion rates n, assuming a velocity-dependent momentum parameter $k(v)$ given by Eq. (59). On the top-right side of each line, only solutions having $Y_{\rm SC} = 0$ are attractors, whereas on the bottom-left side, attractor solutions with $Y_{\rm SC} = 0$ as well as $Y_{\rm SC} \neq 0$ exist that may be reached depending on the initial conditions (see Fig. 8); the dashed vertical axis $A = 0$ is excluded from the region for which there is a nontrivial solution. The best fit [Eqs. (60) and (61)], in the displayed case for which $c_o = 0.23$, has $\alpha_1 = 1.616$, $\beta_1 = -0.932$, $\gamma_1 = 1.226$, $\alpha_2 = -2.624$, $\beta_2 = -0.095$, and $\gamma_2 = 4.198$. The thick line in the bottom-left corner represents the matter-dominated era $n = 2$, and the thin line labeled "1.95" is that for which the $Y_{\rm SC} \neq 0$ attractor demands $b < -\frac{1}{2}$ (dashed line), in contradiction with the constraint of Eq. (33).

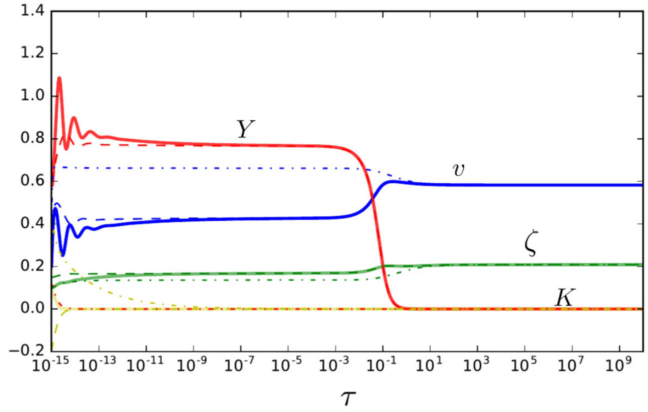

FIG. 7. Time evolution of the velocity v, charge Y, and ζ for parameter values $c_o = 0.23$, $b = 0$, $A = 0.25$, $\beta = 0$, $s = 1$, and $s_{\mathrm{A}} = 1$, during the radiation- and matter-dominated epochs, for different initial conditions: the solid and the dashed trajectories correspond, in Fig. 8, to initial conditions inside the dark (blue) region leading to a nonvanishing scaling charge, and the dash-dotted line corresponds to the NG network configuration. Even in the cases for which the $Y_{\mathrm{SC}} \neq 0$ attractor is reached while $n = 1$ before the radiation-to-matter transition, the subsequent dynamics then drives the solution to an uncharged one ($Y_{\mathrm{SC}} = 0$ attractor) in the matter era.

$$c_i(n) = \alpha_i(c_o) + \beta_i(c_o) n^{\gamma_i(c_o)}, \tag{61}$$

so that $c_1(n_0) = 0$ yields the value of $n = n_0$ such that only for $b < 0$ do nontrivial solutions exist (for $n > n_0$,

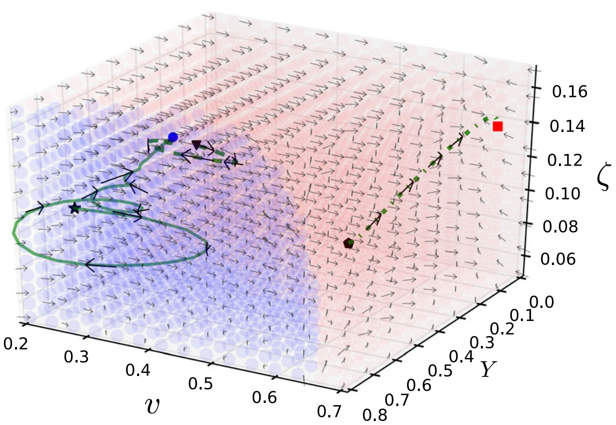

FIG. 8. Phase diagrams showing the nature of the fixed points —i.e., leading to a charged $Y_{\mathrm{SC}} \neq 0$ or NG $Y_{\mathrm{SC}} = 0$ scaling configuration in the radiation era ($n = 1$). Setting initial conditions in the blue (dark) region leads to a trajectory ending in a charged scaling solution, whereas starting in the red region leads inevitably to a NG network. The three curves shown represent three different trajectories, which are also shown in Fig. 7 for the same parameter set. The scaling value with nontrivial charge leads to the momentum parameter $k(v_{\mathrm{SC}}) \approx 0.81$, which lies inside the area with $Y_{\mathrm{SC}} \neq 0$ of Fig. 5, while that with vanishing charge yields $k(v_{\mathrm{SC}}) \approx 0.18$, lying inside the area with $Y_{\mathrm{SC}} = 0$ of Fig. 5. The black dots show the initial conditions, chosen close to the boundary surface and on both sides.

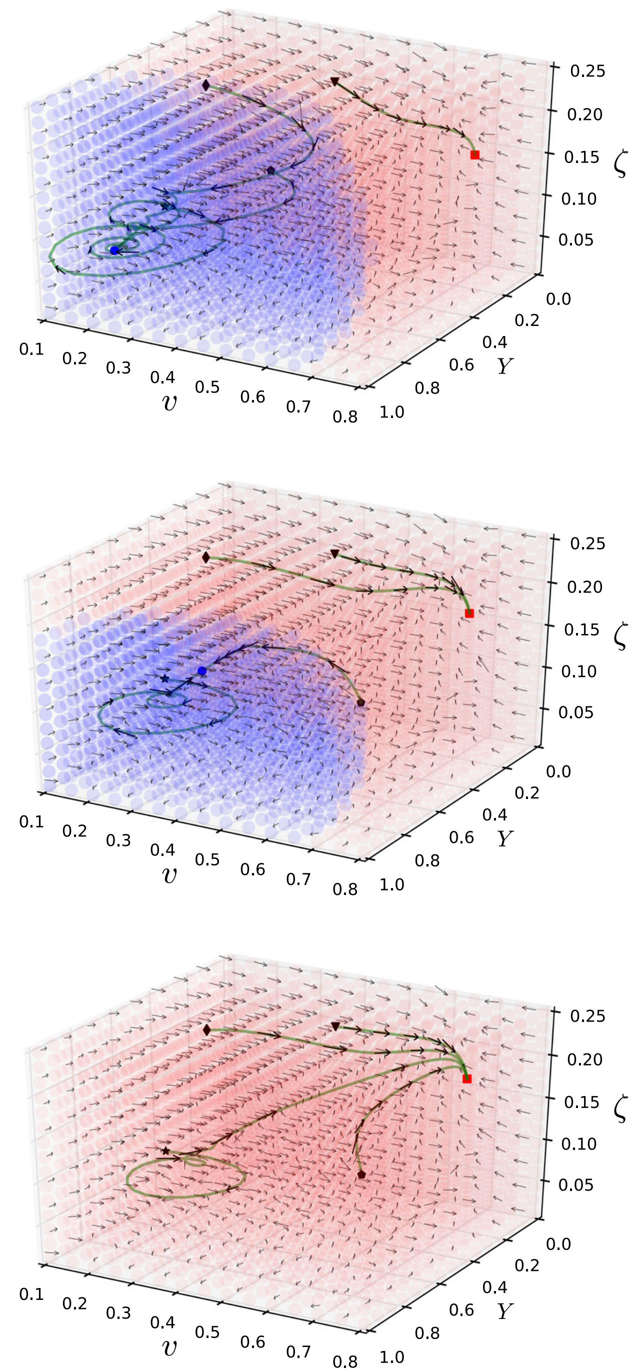

FIG. 9. Same as Fig. 8 for $n = 1$ (top panel), $n = 1.2$ (mid panel), and $n = 1.3$ (bottom panel), with identical parameters except $A \to 0.15$. As expected from the discussion above and shown in Fig. 6, the available region (in blue/dark) for which initial conditions for v, ζ, and Y may lead to a nontrivial charge $Y_{\mathrm{SC}} \neq 0$, the scaling solution is getting smaller as n increases. The charged configuration attractor points are shown as dark blue circles, and the standard uncharged configuration is shown as a red square. The initial conditions shown as *triangle*, *pentagon*, *diamond*, and *star* are the same for all three graphs. In this example, for $n \geq 1.3$, there are no charged solutions, and the would-be attractor becomes a repeller.

see Sec. III B). The constraint in Eq. (33) also implies that $\exists\, n_{\mathrm{crit}}$ such that $c_1(n_{\mathrm{crit}}) = -\frac{1}{2}$, and $\forall\, n \geq n_{\mathrm{crit}}$, $Y_{\mathrm{SC}} = 0$ whatever the initial conditions.

These dynamics, which change in different cosmological eras and depend on initial conditions, can give rise to an interesting phenomenology, especially as a charged network crosses the radiation-matter transition. For a set of values of A and b located between the boundary lines $n = 1$ and $n = n_{\mathrm{crit}}$ (as illustrated in Fig. 6), one may find initial conditions for v, ζ, and Y such that the dynamics leads to the $Y_{\mathrm{SC}} \neq 0$ attractor during the radiation-dominated era, followed, after the radiation-to-matter transition, by a new time evolution towards the $Y_{\mathrm{SC}} = 0$ attractor. An example of such an evolution during the radiation-to-matter transition, with initial conditions appropriately chosen, is illustrated in Fig. 7. The corresponding configuration space trajectories for the radiation epoch are shown in Fig. 8, emphasizing how initial conditions define the choice of scaling behavior. The region of initial conditions that leads to the scaling solution with nontrivial current amplitude depends on the parameters A and b (and possibly on β if included, although the phenomenology will be quite similar).

Figure 9 illustrates the behavior of the regions in configuration space having both charged and uncharged, or only uncharged attractors. As discussed above, the smaller the value of n, the larger the region in which $Y_{\mathrm{SC}} \neq 0$ can be reached. Above a threshold value ($n = 1.3$ in this case), there is no longer any charged configuration available. This explains why the typical time evolution of a superconducting string network will be to first form a charged but chiral state, rapidly evolving, after the radiation-matter transition, into a non-current-carrying standard network.

VII. CONCLUSIONS

The time evolution of a superconducting cosmic string network in the framework of the extended VOS model involves, on top of the usual rms string network velocity v and characteristic length $L_{\mathrm{C}} = \zeta\tau$, not only the chirality parameter $K = Q^2 - J^2$ (the difference between the integrated squared charge Q and current J), but also the overall charge amplitude $Y = \frac{1}{2}(Q^2 + J^2)$. By examining the time evolution, we found that the linear equation of state approximation, expected to be valid for small currents, arguably leads to a natural evolution towards the chiral case; this is for arbitrary initial conditions and in the absence of some bias between the charge- and current-loss mechanisms. In other words, starting with a nontrivial value for K describing the distance to chirality (with $K > 0$ for a spacelike current and $K < 0$ for timelike), we found that in all regimes studied—i.e., in both the radiation- and the matter-dominated epochs, this chirality parameter tends to rapidly vanish, leading to an effective chiral model. In any case, the essential decoupling of the chirality K from the other parameters v, L_{C}, and Y meant that their dynamics could be considered separately.

The CVOS dynamical system is found here to possess two distinct scaling solutions, if the expansion rate is sufficiently slow—that is, typically including the cosmologically relevant radiation-dominated epoch. The specific scaling solution that emerges depends on the initial conditions of the network: the standard NG scaling solution with no currents or charges arises from small initial currents, while there is a possible nontrivial attractor with non-vanishing current amplitude Y, which emerges from relatively large initial currents (and/or low string velocities). We leave for further investigation the question of whether such high-current initial conditions are feasible in realistic scenarios. On the other hand, for fast expansion rates, including the matter-dominated epoch, we find that there is no nontrivial scaling solution, and all initial conditions are unavoidably driven towards the NG uncharged scaling solution.

The nontrivial attractor with nonvanishing charge Y is characterized, as compared to the NG solution, by a lower rms velocity and a smaller characteristic length, leading to an overall larger energy of the network (as compared to uncharged strings), but with part of the energy being in the charge/current contribution. In order for this scaling solution to be stabilized, however, one needs to introduce a charge-leakage mechanism so that some charge can move away from the network—e.g., due to local losses in high-curvature string regions. Under these circumstances, the predicted scaling charge Y will be a strongly model-dependent quantity, depending on microphysical dynamics. On the other hand, in the absence of such charge leakage, the attractor solution would be one with a continuously growing charge/current in which the network eventually becomes "frozen" with $v = 0$. We note that such nonscaling growing solutions have also been found for chiral superconducting strings [41] and, in a slightly different context, for wiggly strings [46]. However, this growing charge solution is unlikely to be physically realized because, as discussed, we expect charge-loss mechanisms to intervene to yield a scaling solution.

The fact that we obtain either charged or Nambu-Goto attractor solutions raises an issue that deserves further consideration, namely the question of initial conditions for the charge Y and chirality K. As we have shown, the space of the initial condition shows two separated regions, at least for slowly expanding universes, in which the dynamics leads to very different outcomes: either a charged scaling solution or one without any charge. Unlike the Nambu-Goto case, current-carrying strings do exhibit some sensitivity to their initial conditions. Nevertheless, for fast expansion, we recover the usual (charge-free) scaling solution, so initial conditions again become irrelevant.

From the point of view of early-Universe cosmology, our main result is that charges and currents could play a significant role in the evolution of string networks in the radiation era (quite probably leaving behind astrophysical fingerprints), but after the radiation-to-matter transition, the time evolution drives the charge amplitude Y towards zero—that is, the uncharged NG scaling configuration. In particular, this could lead to a radically different spectrum of gravitational waves in both eras—e.g., potentially suppressing short-wavelength signals. For this reason, one could anticipate that realistic string networks, possessing internal degrees of freedom, might generate enhanced observational signatures that are linked to the radiation-matter transition.

Finally, we note that we have obtained these results in the framework of the small-current limit for which the linear equation of state should be a valid approximation. While we expect this to be representative of string-forming phase transitions in the early Universe, a question that may legitimately be asked is whether our result of no charge or current in the matter era is a consequence of the smallness of the current in the radiation era, and of the specific use of the linear equation of state, or a fully generic result that will also apply for other, possibly more realistic (Witten-like) worldsheet actions. The fact that growing charge solutions clearly exist for slow expansion rates, and previous knowledge of the evolution of chiral superconducting and wiggly string networks [41,46] leads us to suspect that this behavior is qualitatively generic: regardless of the equation of state, a fast enough expansion rate will always suffice to make charges and currents disappear. Quantitatively, the question is then whether the frontier between slow and fast expansion rates depends on the equation of state (or only on the available energy-loss mechanisms), and whether the matter-dominated epoch is always on the fast (no-current) side of this frontier. A more detailed study of this issue is left for future work.

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