Effect of $K^0-\bar{K}^0$ mixing on *CP* and *CPT* violations in $B_c^{\pm}\to B^{\pm}K^0_{S,L}$ decays

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The large data sample of the B_c meson collected at the LHC experiment and the HL-LHC experiment provides us the opportunity to study the B_c decays and the related physics. In this paper, we investigate the effect of $K^0-\bar{K}^0$ mixing on the branching ratios, *CP* violations and *CPT* violations in the $B_c^{\pm}\to B^{\pm}K^0_{S,L}$ decays. We find that some of the $B_c^{\pm}\to B^{\pm}K^0_{S,L}\to f_{B^{\pm}}f_{K^0_{S,L}}$ decay chains have large branching ratios, whose maximum value can exceed the order of 10^{-6}, the minimum number of $B_c^{\pm}$ events times efficiency for observing the decays at three standard deviations (3σ) level is about 10^6. We study the *CP* asymmetries in the $B_c^{\pm}\to B^{\pm}K^0_{S,L}$ decays and find that the *CP* asymmetries can exceed the order of 10^{-3}, which are dominated by $K^0-\bar{K}^0$ mixing. We give the most promising processes to observe the *CP* violations and the ranges of the numbers of $B_c^{\pm}$ events-times-efficiency needed to observe the *CP* asymmetries at a significance of 3σ in these decays. We investigate the possibility to constraint the *CPT* violation parameter z in the $B_c^{\pm}\to B^{\pm}K^0_{S,L}\to f_{B^{\pm}}f_{K^0_{S,L}}$ decays and give the most promising processes to constraint the parameter z.

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I. INTRODUCTION

CP violation provides deep insights into the nature and plays an important role in explaining for our matter-dominated universe [1,2]. In the Standard Model (SM), effects of *CP* violations in the charm sector, unlike the kaon and B meson systems, are expected to be rather small [3–5]. During the past decade, many theoretical and experimental efforts have so far been made to study *CP* violations in the D system and charm baryon [6–20]. In 2019, the LHCb collaboration reported a first confirmed observation of the *CP* violations in charm decays via measuring the difference of time-integrated *CP* asymmetries of $D^0\to K^+K^-$ and $D^0\to\pi^+\pi^-$ decays with the result of $(1.54\pm0.29)\times10^{-3}$ [21], the significance of the measured deviation from zero is 5.3σ. However, studies of *CP*-violating processes in the B_c decays that proceed via the c quark decay with the b quark as a spectator are very scarce.

CPT invariance is one of the most fundamental symmetries in physics and is based on three assumptions: unitarity, locality, and Lorentz invariance [22]. Therefore, a strong motivation for experimental and theoretical studies on *CPT* theorem is to test the *CPT* symmetry [23–27]. The $K^0-\bar{K}^0$ mixing system is one of the most intriguing processes to study *CP* and *CPT* violations.

The decays with final states including K^0_S or K^0_L can be used to study *CP* violation [28] and *CPT* violation. In these decays, $K^0-\bar{K}^0$ mixing have a non-negligible effect on *CP* violation and *CPT* violation, even play a dominant role. The *CP* asymmetries in the decays $D^+\to K^0_S\pi^+$ and $\tau^+\to\pi^+K^0_S\bar{\nu}_\tau$ have been measured by Belle [29,30], *BABAR* [31,32], CLEO [33,34], and FOCUS [35] collaborations. There exist 2.8σ discrepancy observed between the *BABAR* measurement and the SM prediction of the *CP* asymmetry in the $\tau^+\to\pi^+K_S\bar{\nu}_\tau$ decay [36–38]. Because the direct *CP* violation in this decay is absented at the tree level in the SM, the discrepancy could be a hint of the physics beyond the SM, several possible new-physics proposals are put forwarded [39]. However, the result for the *CP* asymmetry in $\tau^+\to\pi^+K_S\bar{\nu}_\tau$ decay suffers large uncertainty and no unambiguous conclusion can be drawn,

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so more precise data and more reactions are needed in both experiment and theory.

The B_c meson is the lowest bound state of the doubly heavy-flavored ($\bar{b}c$) system, here both the b and c quark can decay while the other serves as a spectator, so the B_c physics must be very rich [40–43]. At the LHC experiment, around 5×10^{10} B_c mesons per year could be produced [44,45], which provides us the opportunity to get more information of this particle and study the related physics. In this paper, we investigate the effect of $K^0 - \bar{K}^0$ mixing on CP and CPT violations in the $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ decays. We predict the branching ratio of these decays and calculate the CP and CPT asymmetries which are dominated by the effect of $K^0 - \bar{K}^0$ mixing. In addition, we also explore the sensitivity of these measurements in experiment.

II. BRANCHING FRACTIONS OF THE $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ DECAYS

Within the SM, the decays $B_c^+ \to B^+ \bar{K}^0$ and its charge conjugate can occur through the Cabibbo-favored channels, which are shown in Fig. 1. The decays $B_c^+ \to B^+ K^0$ and its charge conjugate can proceed via the doubly Cabibbo-suppressed channels, as shown in Fig 2. The resulting effective Hamiltonian for these decays can be written as

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} [V_{cs}^* V_{ud} \bar{s}\gamma^\mu(1-\gamma_5) c \bar{u}\gamma_\mu(1-\gamma_5) d + V_{cd}^* V_{us} \bar{d}\gamma^\mu(1-\gamma_5) c \bar{u}\gamma_\mu(1-\gamma_5) s] + \text{H.c.}, \quad (1)$$

where G_F is the Fermi coupling constant; V_{ij} denotes the CKM matrix element. To calculate the hadronic matrix elements of the effective Hamiltonian, we work with the factorization assumption [5,46,47], then, we need to reconstruct the effective Hamiltonian in a form suitable for the use of this assumption by Fierz transformation [48]. By making use of the Fierz transformation, we obtain the factorized Hamiltonian

$$\mathcal{H}_{\rm eff}^{\rm fac} = -\frac{G_F}{\sqrt{2}N_c} [V_{cs}^* V_{ud} \bar{s}\gamma^\mu(1-\gamma_5) d \bar{u}\gamma_\mu(1-\gamma_5) c + V_{cd}^* V_{us} \bar{d}\gamma^\mu(1-\gamma_5) s \bar{u}\gamma_\mu(1-\gamma_5) c] + \text{H.c.}, \quad (2)$$

with $N_c = 3$ is the number of colors.

The decay constant for pseudoscalar meson K^0 are defined by [49]

$$\langle K^0(q_t)|\bar{d}\gamma^\mu\gamma_5 s|0\rangle = i q_t^\mu f_K, \quad (3)$$

$$\langle |\bar{K}^0(q_t)|\bar{s}\gamma^\mu\gamma_5 d|0\rangle = i q_t^\mu f_K, \quad (4)$$

here, the CP transformation property $\mathcal{C}P|K^0(q_t^0, \vec{q}_t)\rangle = -|K^0(q_t^0, -\vec{q}_t)\rangle$ is used. The matrix elements $\langle B^+(p_2)|\bar{u}\gamma^\mu c|B_c^+(p_1)\rangle$ and $\langle B^-(p_2)|\bar{c}\gamma^\mu u|B_c^-(p_1)\rangle$ are parametrized in terms of various form factors as [50]

$$\langle B^+(p_2)|\bar{u}\gamma^\mu c|B_c^+(p_1)\rangle = f_0(q_t^2)\left(\frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu\right) + f_+(q_t^2)\left(p_1^\mu + p_2^\mu - \frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu\right), \quad (5)$$

$$\langle B^-(p_2)|\bar{c}\gamma^\mu u|B_c^-(p_1)\rangle = -f_0(q_t^2)\left(\frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu\right) - f_+(q_t^2)\left(p_1^\mu + p_2^\mu - \frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu\right), \quad (6)$$

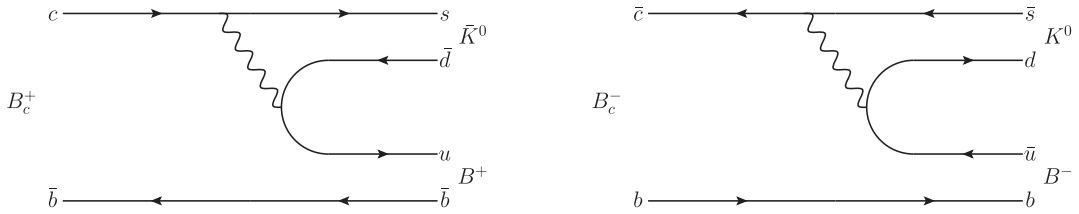

FIG. 1. Quark diagrams for the Cabibbo-allowed $B_c^+ \to B^+ \bar{K}^0$ and $B_c^- \to B^- K^0$ decays.

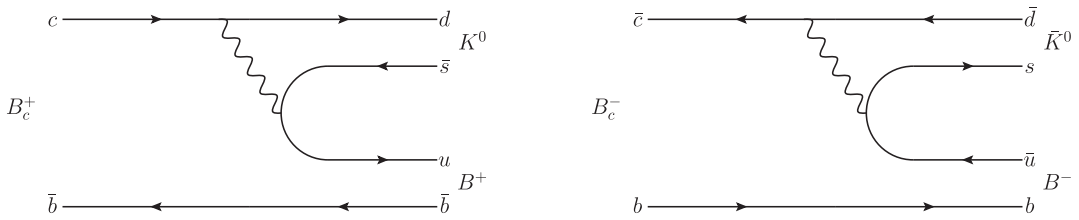

FIG. 2. Quark diagrams for the doubly Cabibbo-suppressed $B_c^+ \to B^+ K^0$ and $B_c^- \to B^- \bar{K}^0$ decays.

where $q_t = p_1 - p_2$ is the 4-momentum transfer. The form factors $f_0(q_t^2)$ and $f_+(q_t^2)$ read

$$f_0(q_t^2) = \frac{1}{1-\frac{q_t^2}{m_{D_0^*}^2}}(b_0^{(0)} + b_0^{(1)} Z_p + b_0^{(2)} Z_p^2 + b_0^{(3)} Z_p^3), \quad (7)$$

$$f_+(q_t^2) = \frac{1}{1-\frac{q_t^2}{m_{D^*}^2}}(b_+^{(0)} + b_+^{(1)} Z_p + b_+^{(2)} Z_p^2 + b_+^{(3)} Z_p^3), \quad (8)$$

here, $m_{D_0^*}$ and m_{D^*} is respectively the mass of the neutral scalar meson $D_0^*(2300)^0$ and the neutral vector meson $D^*(2007)^0$. The variable Z_p is defined as

$$Z_p = \left(\frac{\sqrt{t_+ - q_t^2} - \sqrt{t_+}}{\sqrt{t_+ - q_t^2} + \sqrt{t_+}}\right) \cdot \left| \frac{\sqrt{t_+ - m_{D^*}^2} + \sqrt{t_+}}{\sqrt{t_+ - m_{D^*}^2} - \sqrt{t_+}} \right|, \quad (9)$$

with $t_+ = (m_{B_c^+} + m_{B^+})^2$. The polynomial coefficients $b_{0,+}^{(i)}$ $(i = 0, 1, 2, 3)$ appearing in Eq. (7) and Eq. (8) have been calculated in Ref. [50]

$$b_0^{(0)} = 0.548 \pm 0.023, \qquad b_0^{(1)} = -0.19 \pm 0.22,$$
$$b_0^{(2)} = 0.05 \pm 0.74, \qquad b_0^{(3)} = 0, \quad (10)$$

$$b_+^{(0)} = 0.548 \pm 0.023, \qquad b_+^{(1)} = -0.48 \pm 0.21,$$
$$b_+^{(2)} = 0.12 \pm 0.77, \qquad b_+^{(3)} = 0. \quad (11)$$

Now, we can proceed to calculate the hadronic matrix elements $\langle K^0(q_t) B^\pm(p_2)|\mathcal{H}_{\rm eff}^{\rm fac}|B_c^\pm(p_1)\rangle$ and $\langle \bar{K}^0(q_t) B^\pm(p_2)|\mathcal{H}_{\rm eff}^{\rm fac}|B_c^\pm(p_1)\rangle$. Because of the parity conservation in strong interaction, the matrix elements $\langle K^0(q_t)|\bar{d}\gamma^\mu s|0\rangle$, $\langle B^+(p_2)|\bar{u}\gamma^\mu\gamma^5 c|B_c^+(p_1)\rangle$ and their charge conjugate have no contribution to the corresponding the hadronic matrix elements. By combining Eq. (2), Eq. (3), Eq. (4), Eq. (5), Eq. (6) and $q_t = p_1 - p_2$, we can obtain

$$\langle K^0(q_t) B^+(p_2)|\mathcal{H}_{\rm eff}^{\rm fac}|B_c^+(p_1)\rangle = \frac{G_F f_K}{3\sqrt{2}} V_{cd}^* V_{us} e^{i\phi} f_0(q_t^2)(p_1^2 - p_2^2), \quad (12)$$

$$\langle \bar{K}^0(q_t) B^+(p_2)|\mathcal{H}_{\rm eff}^{\rm fac}|B_c^+(p_1)\rangle = \frac{G_F f_K}{3\sqrt{2}} V_{cs}^* V_{ud} e^{i\phi} f_0(q_t^2)(p_1^2 - p_2^2), \quad (13)$$

$$\langle K^0(q_t) B^-(p_2)|\mathcal{H}_{\rm eff}^{\rm fac}|B_c^-(p_1)\rangle = \frac{G_F f_K}{3\sqrt{2}} V_{cs} V_{ud}^* e^{i\phi} f_0(q_t^2)(p_1^2 - p_2^2), \quad (14)$$

$$\langle \bar{K}^0(q_t) B^-(p_2)|\mathcal{H}_{\rm eff}^{\rm fac}|B_c^-(p_1)\rangle = \frac{G_F f_K}{3\sqrt{2}} V_{cd} V_{us}^* e^{i\phi} f_0(q_t^2)(p_1^2 - p_2^2), \quad (15)$$

where ϕ denotes the strong phase of the involved matrix elements. In the $B_c^\pm \to B^\pm K^0$ and $B_c^\pm \to B^\pm \bar{K}^0$ decays, there are only color-suppressed tree-emission diagrams contributing to both the Cabibbo-favored and doubly Cabibbo-suppressed decay amplitudes, without W-annihilation or W-exchange diagrams. In this case, the strong phases in the Cabibbo-favored decay and Cabibbo-suppressed decay are identical [51], i.e., the strong phases are identical in the $B_c^\pm \to B^\pm K^0$ and $B_c^\pm \to B^\pm \bar{K}^0$ decays.

In the $K^0 - \bar{K}^0$ system, the two mass eigenstates, K_S^0 of mass m_S and width Γ_S and K_L^0 of mass m_L and width Γ_L, are linear combinations of the flavor eigenstates K^0 and $\bar{K}^0$ [52]

$$|K_L^0\rangle = p\sqrt{1+z}|K^0\rangle - q\sqrt{1-z}|\bar{K}^0\rangle, \quad (16)$$

$$|K_S^0\rangle = p\sqrt{1-z}|K^0\rangle + q\sqrt{1+z}|\bar{K}^0\rangle, \quad (17)$$

where p, q and z are complex mixing parameters. CPT invariant requires $z = 0$, while CP conservation requires both $p = q = \sqrt{2}/2$ and $z = 0$. The mass and width eigenstates, $K_{S,L}^0$, may also be described with the popular notations

$$|K_L^0\rangle = \frac{1+\epsilon-\delta}{\sqrt{2(1+|\epsilon-\delta|^2)}}|K^0\rangle - \frac{1-\epsilon+\delta}{\sqrt{2(1+|\epsilon-\delta|^2)}}|\bar{K}^0\rangle, \quad (18)$$

$$|K_S^0\rangle = \frac{1+\epsilon+\delta}{\sqrt{2(1+|\epsilon+\delta|^2)}}|K^0\rangle + \frac{1-\epsilon-\delta}{\sqrt{2(1+|\epsilon+\delta|^2)}}|\bar{K}^0\rangle, \quad (19)$$

where the complex parameter ϵ signifies deviation of the mass eigenstates from the CP eigenstates, δ is the CPT violating complex parameter. The parameters p, q and z can be expressed in terms of ϵ and δ [neglecting terms of $\epsilon\delta$ and $\mathcal{O}(\delta)$]

$$p = \frac{1+\epsilon}{\sqrt{2(1+|\epsilon|^2)}}, \qquad q = \frac{1-\epsilon}{\sqrt{2(1+|\epsilon|^2)}}, \qquad z = -2\delta. \quad (20)$$

In this paper, we adopt the notations p, q and z in order to ensuring the rephasing invariance of all analytical results [19].

The time-evolved states of the $K^0 - \bar{K}^0$ system can be expressed by the mass eigenstates

$$|K_{\rm phys}^0(t)\rangle = \frac{\sqrt{1+z}}{2p} e^{-im_L t - \frac{1}{2}\Gamma_L t}|K_L^0\rangle + \frac{\sqrt{1-z}}{2p} e^{-im_S t - \frac{1}{2}\Gamma_S t}|K_S^0\rangle, \quad (21)$$

$$|\bar{K}^0_{\rm phys}(t)\rangle = -\frac{\sqrt{1-z}}{2q}e^{-im_L t-\frac{1}{2}\Gamma_L t}|K^0_L\rangle + \frac{\sqrt{1+z}}{2q}e^{-im_S t-\frac{1}{2}\Gamma_S t}|K^0_S\rangle. \tag{22}$$

With Eq. (21) and Eq. (22), the time-dependent amplitudes of the cascade decays $B_c^\pm \to B^\pm K^0 \to B^\pm f_{K^0}$ can be written as

$$\begin{aligned} A(B_c^\pm \to B^\pm K^0(t) \to B^\pm f_{K^0}(t)) = {} & \langle K^0(q_t)B^\pm(p_2)|\mathcal{H}^{\rm fac}_{\rm eff}|B_c^\pm(p_1)\rangle \cdot A(K^0_{\rm phys}(t) \to f_{K^0}) \\ & + \langle \bar{K}^0(q_t)B^\pm(p_2)|\mathcal{H}^{\rm fac}_{\rm eff}|B_c^\pm(p_1)\rangle \cdot A(\bar{K}^0_{\rm phys}(t) \to f_{K^0}), \end{aligned} \tag{23}$$

where f_{K^0} denotes the final state from the decay of the K^0 or $\bar{K}^0$ meson. $A(K^0_{\rm phys}(t) \to f_{K^0})$ and $A(\bar{K}^0_{\rm phys}(t) \to f_{K^0})$ denotes the amplitude of the $K^0_{\rm phys}(t) \to f_{K^0}$ and $\bar{K}^0_{\rm phys}(t) \to f_{K^0}$ decays, respectively. They have the following forms

$$A(K^0_{\rm phys}(t) \to f_{K^0}) = \frac{\sqrt{1+z}}{2p}e^{-im_L t-\frac{1}{2}\Gamma_L t}A(K^0_L \to f_{K^0}) + \frac{\sqrt{1-z}}{2p}e^{-im_S t-\frac{1}{2}\Gamma_S t}A(K^0_S \to f_{K^0}), \tag{24}$$

$$A(\bar{K}^0_{\rm phys}(t) \to f_{K^0}) = -\frac{\sqrt{1-z}}{2q}e^{-im_L t-\frac{1}{2}\Gamma_L t}A(K^0_L \to f_{K^0}) + \frac{\sqrt{1+z}}{2q}e^{-im_S t-\frac{1}{2}\Gamma_S t}A(K^0_S \to f_{K^0}). \tag{25}$$

Making use of Eqs. (12), (13), (23), (24), and (25) and performing integration over phase space, we can obtain

$$\begin{aligned} \Gamma(B_c^+ \to B^+ K(t) \to B^+ f_{K^0}(t)) = {} & \frac{G_F^2 f_K^2 |f_0(m^2_{K^0})|^2}{1152\pi|p|^2|q|^2 m^3_{B_c^+}} \cdot f(m_{B_c^+}, m_{B^+}, m_{K^0}) \cdot (m^2_{B_c^+} - m^2_{B^+})^2 \\ & \cdot [e^{-\Gamma_L t} \cdot \Gamma(K^0_L \to f_{K^0}) \cdot g_{K^0_L} + e^{-\Gamma_S t} \cdot \Gamma(K^0_S \to f_{K^0}) \cdot g_{K^0_S} \\ & + e^{-i\Delta m t-\Gamma t} \cdot A^*(K^0_S \to f_{K^0}) \cdot A(K^0_L \to f_{K^0}) \cdot g_{K^0_S - K^0_L} \\ & + e^{i\Delta m t-\Gamma t} \cdot A(K^0_S \to f_{K^0}) \cdot A^*(K^0_L \to f_{K^0}) \cdot g^*_{K^0_S - K^0_L}], \end{aligned} \tag{26}$$

where m_r with $r = B_c^+, B^+, K^0$ denotes the mass of the resonance, Δm denotes the difference in masses of K^0_L and K^0_S, Γ denotes the average in widths of K^0_L and K^0_S

$$\Delta m = m_L - m_S, \qquad \Gamma = \frac{\Gamma_L + \Gamma_S}{2}. \tag{27}$$

The terms $g_{K^0_L}$, $g_{K^0_S}$ and $g_{K^0_S - K^0_L}$ are related to the effect of the K^0_L decay, the K^0_S decay and their interference, respectively, and have the following forms

$$\begin{aligned} g_{K^0_L} = {} & \Big[|q|^2|\sqrt{1+z}|^2|V_{cd}|^2|V_{us}|^2 + |p|^2|\sqrt{1-z}|^2|V_{cs}|^2|V_{ud}|^2 \\ & - pq^*\sqrt{1-z}(\sqrt{1+z})^* V^*_{cs}V_{ud}V_{cd}V^*_{us} - p^*q\sqrt{1+z}(\sqrt{1-z})^* V^*_{cd}V_{us}V_{cs}V^*_{ud}\Big], \end{aligned} \tag{28}$$

$$\begin{aligned} g_{K^0_S} = {} & \Big[|q|^2|\sqrt{1-z}|^2|V_{cd}|^2|V_{us}|^2 + |p|^2|\sqrt{1+z}|^2|V_{cs}|^2|V_{ud}|^2 \\ & + pq^*\sqrt{1+z}(\sqrt{1-z})^* V^*_{cs}V_{ud}V_{cd}V^*_{us} + p^*q\sqrt{1-z}(\sqrt{1+z})^* V^*_{cd}V_{us}V_{cs}V^*_{ud}\Big], \end{aligned} \tag{29}$$

$$\begin{aligned} g_{K^0_S - K^0_L} = {} & \Big[|q|^2\sqrt{1+z}(\sqrt{1-z})^*|V_{cd}|^2|V_{us}|^2 - |p|^2\sqrt{1-z}(\sqrt{1+z})^*|V_{cs}|^2|V_{ud}|^2 \\ & - pq^*|\sqrt{1-z}|^2 V^*_{cs}V_{ud}V_{cd}V^*_{us} + p^*q|\sqrt{1+z}|^2 V^*_{cd}V_{us}V_{cs}V^*_{ud}\Big], \end{aligned} \tag{30}$$

The function $f(x, y, z)$ in Eq. (26) is defined as

$$f(x, y, z) = \sqrt{x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2}. \tag{31}$$

In experiment, the K_S^0 state is defined via a final state $\pi^+\pi^-$ with $m_{\pi\pi} \approx m_S$ and a time difference between the B_c decay and the K_S^0 decay [36]. By taking into account these experimental features, the partial decay width for the $B_c^+ \to B^+ K_S^0$ decay can be defined as

$$\Gamma(B_c^+ \to B^+ K_S^0) = \frac{\int_{t_0}^{t_1} \Gamma(B_c^+ \to B^+ K(t) \to B^+\pi^+\pi^-(t))dt}{(e^{-\Gamma_S t_0} - e^{-\Gamma_S t_1}) \cdot \mathcal{B}(K_S^0 \to \pi^+\pi^-)}, \tag{32}$$

where $t_0 = 0.1\tau_S$ and $t_1 = 2\tau_S \sim 20\tau_S$ with τ_S is the K_S^0 lifetime, we adopt $t_1 = 10\tau_S$ in our calculation. Combining Eq. (26) and Eq. (32), we can obtain

$$\Gamma(B_c^+ \to B^+ K_S^0) = \frac{G_F^2 f_K^2 |f_0(m_S^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_S) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2 \cdot \left[g_{K_S^0} + \frac{e^{-\Gamma_L t_0} - e^{-\Gamma_L t_1}}{e^{-\Gamma_S t_0} - e^{-\Gamma_S t_1}} \cdot \frac{\mathcal{B}(K_L^0 \to \pi^+\pi^-)}{\mathcal{B}(K_S^0 \to \pi^+\pi^-)} \cdot g_{K_L^0} \right.$$
$$\left. + 2\mathrm{Re}\left(\frac{e^{-i\Delta m t_0 - \Gamma t_0} - e^{-i\Delta m t_1 - \Gamma t_1}}{e^{-\Gamma_S t_0} - e^{-\Gamma_S t_1}} \cdot \frac{\Gamma_S}{\Gamma + i\Delta m} \cdot \frac{A(K_L^0 \to \pi^+\pi^-)}{A(K_S^0 \to \pi^+\pi^-)} \cdot g_{K_S^0 - K_L^0} \right) \right]. \tag{33}$$

From Particle Data Group [52], $\Gamma_L/\Gamma_S = (1.75 \pm 0.01) \times 10^{-3}$. If we adopt $t_1 = 10\tau_S = 10/\Gamma_S$, then we can obtain

$$\frac{e^{-\Gamma_L t_0} - e^{-\Gamma_L t_1}}{e^{-\Gamma_S t_0} - e^{-\Gamma_S t_1}} \approx 0.019, \tag{34}$$

the value of $\mathcal{B}(K_L^0 \to \pi^+\pi^-)/\mathcal{B}(K_S^0 \to \pi^+\pi^-)$ can also obtained from Particle Data Group [52]

$$\frac{\mathcal{B}(K_L^0 \to \pi^+\pi^-)}{\mathcal{B}(K_S^0 \to \pi^+\pi^-)} = (2.84 \pm 0.01) \times 10^{-3}, \tag{35}$$

so the second term in the bracket in Eq. (33), which corresponds to the effect of the K_L^0 decay, can be neglected.

Using Eq. (16), Eq. (17) and neglecting the tiny direct CP asymmetry in the $K^0 \to \pi^+\pi^-$ decay, we can derive

$$\frac{A(K_L^0 \to \pi^+\pi^-)}{A(K_S^0 \to \pi^+\pi^-)} = \frac{p\sqrt{1+z} - q\sqrt{1-z}}{p\sqrt{1-z} + q\sqrt{1+z}}. \tag{36}$$

For convenience, we introduce the following substitution

$$t_{K_S^0 - K_L^0} = \frac{e^{-i\Delta m t_0 - \Gamma t_0} - e^{-i\Delta m t_1 - \Gamma t_1}}{e^{-\Gamma_S t_0} - e^{-\Gamma_S t_1}} \cdot \frac{\Gamma_S}{\Gamma + i\Delta m}. \tag{37}$$

Combining Eqs. (29), (30), (33), (34), (35), and (36), we can derive

$$\Gamma(B_c^+ \to B^+ K_S^0) = \frac{G_F^2 f_K^2 |f_0(m_S^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_S) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2$$
$$\cdot \left\{ [|q|^2 |\sqrt{1-z}|^2 |V_{cd}|^2 |V_{us}|^2 + |p|^2 |\sqrt{1+z}|^2 |V_{cs}|^2 |V_{ud}|^2 \right.$$
$$+ pq^* \sqrt{1+z}(\sqrt{1-z})^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + p^* q \sqrt{1-z}(\sqrt{1+z})^* V_{cd}^* V_{us} V_{cs} V_{ud}^*]$$
$$+ 2\mathrm{Re}\left(t_{K_S^0 - K_L^0} \cdot \frac{p\sqrt{1+z} - q\sqrt{1-z}}{p\sqrt{1-z} + q\sqrt{1+z}} \cdot [|q|^2 \sqrt{1+z}(\sqrt{1-z})^* |V_{cd}|^2 |V_{us}|^2 \right.$$
$$- |p|^2 \sqrt{1-z}(\sqrt{1+z})^* |V_{cs}|^2 |V_{ud}|^2 - pq^* |\sqrt{1-z}|^2 V_{cs}^* V_{ud} V_{cd} V_{us}^*$$
$$\left. \left. + p^* q |\sqrt{1+z}|^2 V_{cd}^* V_{us} V_{cs} V_{ud}^*] \right) \right\}. \tag{38}$$

Similarly, we can obtain the partial decay width for the $B_c^- \to B^- K_S^0$ decay

$$\Gamma(B_c^- \to B^- K_S^0) = \frac{G_F^2 f_K^2 |f_0(m_S^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_S) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2$$
$$\cdot \Big\{ [|q|^2 |\sqrt{1-z}|^2 |V_{cs}|^2 |V_{ud}|^2 + |p|^2 |\sqrt{1+z}|^2 |V_{cd}|^2 |V_{us}|^2$$
$$+ pq^* \sqrt{1+z}(\sqrt{1-z})^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + p^* q \sqrt{1-z}(\sqrt{1+z})^* V_{cs} V_{ud}^* V_{cd}^* V_{us}]$$
$$+ 2\mathrm{Re}\Big(t_{K_S^0 - K_L^0} \cdot \frac{p\sqrt{1+z} - q\sqrt{1-z}}{p\sqrt{1-z} + q\sqrt{1+z}} \cdot [|q|^2 \sqrt{1+z}(\sqrt{1-z})^* |V_{cs}|^2 |V_{ud}|^2$$
$$- |p|^2 \sqrt{1-z}(\sqrt{1+z})^* |V_{cd}|^2 |V_{us}|^2 - pq^* |\sqrt{1-z}|^2 V_{cs}^* V_{ud} V_{cd} V_{us}^*$$
$$+ p^* q |\sqrt{1+z}|^2 V_{cd}^* V_{us} V_{cs} V_{ud}^*] \Big) \Big\}. \tag{39}$$

In experiment, the K_L^0 state is defined via a large time difference between the B_c decay and the K_L^0 decay and mostly decay outside the detector [53]. Basing on these experimental features, the partial decay width for the $B_c^+ \to B^+ K_L^0$ decay can be defined as

$$\Gamma(B_c^+ \to B^+ K_L^0) = \frac{\int_{t_2}^{+\infty} \Gamma(B_c^+ \to B^+ K(t) \to B^+ f_{K_L^0}(t)) dt}{e^{-\Gamma_L t_2} \cdot \mathcal{B}(K_L^0 \to f_{K_L^0})}, \tag{40}$$

where $f_{K_L^0}$ denotes the final state of the K_L^0 decay, $t_2 \geq 100\tau_S$. Combining Eq. (26) and Eq. (40), we can obtain

$$\Gamma(B_c^+ \to B^+ K_L^0) = \frac{G_F^2 f_K^2 |f_0(m_L^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_L) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2 \cdot \Big[g_{K_L^0} + e^{-(\Gamma_S - \Gamma_L)t_2} \cdot \frac{\mathcal{B}(K_S^0 \to f_{K_L^0})}{\mathcal{B}(K_L^0 \to f_{K_L^0})} \cdot g_{K_S^0}$$
$$+ 2\mathrm{Re}\Big(e^{i\Delta m t_2 - \frac{\Gamma_S - \Gamma_L}{2} t_2} \cdot \frac{\Gamma_L}{\Gamma - i\Delta m} \cdot \frac{A(K_S^0 \to f_{K_L^0})}{A(K_L^0 \to f_{K_L^0})} \cdot g^*_{K_S^0 - K_L^0} \Big) \Big]. \tag{41}$$

Using the result from Particle Data Group [52]: $\Gamma_L/\Gamma_S = (1.75 \pm 0.01) \times 10^{-3}$, we can obtain

$$e^{-(\Gamma_S - \Gamma_L)t_2} \leq 4.4 \times 10^{-44}, \qquad e^{-\frac{\Gamma_S - \Gamma_L}{2} t_2} \leq 2.1 \times 10^{-22}, \tag{42}$$

so the second and the third terms in the bracket in Eq. (41), which corresponds to the effects of the K_S^0 decay and the interference between the K_S^0 decay and the K_L^0 decay, can be neglected. Substituting Eq. (28) into Eq. (41) and neglecting the second and the third terms in the bracket in Eq. (41), we obtain

$$\Gamma(B_c^+ \to B^+ K_L^0) = \frac{G_F^2 f_K^2 |f_0(m_L^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_L) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2$$
$$\cdot [|q|^2 |\sqrt{1+z}|^2 |V_{cd}|^2 |V_{us}|^2 + |p|^2 |\sqrt{1-z}|^2 |V_{cs}|^2 |V_{ud}|^2$$
$$- pq^* \sqrt{1-z}(\sqrt{1+z})^* V_{cs}^* V_{ud} V_{cd} V_{us}^* - p^* q \sqrt{1+z}(\sqrt{1-z})^* V_{cd}^* V_{us} V_{cs} V_{ud}^*]. \tag{43}$$

Similarly, we can obtain the partial decay width for the $B_c^- \to B^- K_L^0$ decay

$$\Gamma(B_c^- \to B^- K_L^0) = \frac{G_F^2 f_K^2 |f_0(m_L^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_L) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2$$
$$\cdot [|q|^2 |\sqrt{1+z}|^2 |V_{cs}|^2 |V_{ud}|^2 + |p|^2 |\sqrt{1-z}|^2 |V_{cd}|^2 |V_{us}|^2$$
$$- pq^* \sqrt{1-z}(\sqrt{1+z})^* V_{cs}^* V_{ud} V_{cd} V_{us}^* - p^* q \sqrt{1+z}(\sqrt{1-z})^* V_{cs} V_{ud}^* V_{cd}^* V_{us}]. \tag{44}$$

From Eqs. (38), (39), (43) and (44), we can obtain the branching ratios of $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays, which are the products of the partial decay widths for these decays and the mean life of B_c meson.

III. CP VIOLATIONS, $K_S^0 - K_L^0$ ASYMMETRIES, AND CPT VIOLATIONS

In the $B_c^{\pm} \to B^{\pm} K_{S,L}^0 \to f_{B^{\pm}} f_{K_{S,L}^0}$ decays, the time-independent CP asymmetries are defined as

$$\mathcal{A}_{CP}^{K_{S,L}^0} = \frac{\Gamma(B_c^- \to B^- K_{S,L}^0) \cdot \Gamma(B^- \to f_{B^-}) - \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B_c^- \to B^- K_{S,L}^0) \cdot \Gamma(B^- \to f_{B^-}) + \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot \Gamma(B^+ \to \bar{f}_{B^-})}, \tag{45}$$

here, f_r with $r = B^{\pm}, K_{S,L}^0$ denote the final states from the decays of the $B^{\pm}$ and $K_{S,L}^0$ mesons, $\bar{f}_{B^{\pm}}$ are the CP-conjugate states of $f_{B^{\pm}}$, $\Gamma(B^- \to f_{B^-})$ and $\Gamma(B^+ \to \bar{f}_{B^-})$ is the partial decay width of the $B^- \to f_{B^-}$ and $B^+ \to \bar{f}_{B^-}$ decays respectively. Using the relations in Eqs. (A1) and (A2) listed in Appendix A, we can obtain

$$\mathcal{A}_{CP}^{K_{S,L}^0} = \frac{\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})} + \frac{\Gamma(B_c^- \to B^- K_{S,L}^0) - \Gamma(B_c^+ \to B^+ K_{S,L}^0)}{\Gamma(B_c^- \to B^- K_{S,L}^0) + \Gamma(B_c^+ \to B^+ K_{S,L}^0)}, \tag{46}$$

where $(\Gamma(B_c^- \to B^- K_{S,L}^0) - \Gamma(B_c^+ \to B^+ K_{S,L}^0))/(\Gamma(B_c^- \to B^- K_{S,L}^0) + \Gamma(B_c^+ \to B^+ K_{S,L}^0))$ are the CP asymmetries in the $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays and hereinafter denoted as $\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_{S,L}^0)$. Under the assumption of CPT invariance, $\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_{S,L}^0)$ can be calculated from Eqs. (38), (39), (43), and (44)

$$\begin{aligned}\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_S^0) = & \frac{(|q|^2 - |p|^2) \cdot (|V_{cs}|^2 |V_{ud}|^2 - |V_{cd}|^2 |V_{us}|^2)}{(|p|^2 + |q|^2) \cdot (|V_{cs}|^2 |V_{ud}|^2 + |V_{cd}|^2 |V_{us}|^2) + 2pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + 2p^* q V_{cs} V_{ud}^* V_{cd}^* V_{us}} \\ & + \frac{2(|V_{cs}|^2 |V_{ud}|^2 - |V_{cd}|^2 |V_{us}|^2) \cdot (|p|^2 + |q|^2) \cdot \mathrm{Re}(t_{K_S^0 - K_L^0} \cdot \frac{p-q}{p+q})}{(|p|^2 + |q|^2) \cdot (|V_{cs}|^2 |V_{ud}|^2 + |V_{cd}|^2 |V_{us}|^2) + 2pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + 2p^* q V_{cs} V_{ud}^* V_{cd}^* V_{us}},\end{aligned} \tag{47}$$

$$\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_L^0) = \frac{(|q|^2 - |p|^2) \cdot (|V_{cs}|^2 |V_{ud}|^2 - |V_{cd}|^2 |V_{us}|^2)}{(|p|^2 + |q|^2) \cdot (|V_{cs}|^2 |V_{ud}|^2 + |V_{cd}|^2 |V_{us}|^2) - 2pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* - 2p^* q V_{cs} V_{ud}^* V_{cd}^* V_{us}}. \tag{48}$$

From Eq. (47) and Eq. (48), we can see that the dominant contributions to the CP asymmetries in the $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays arise indirectly from $K^0 - \bar{K}^0$ mixing, the time evolution effect of neutral kaons can significantly affect the CP asymmetry in the $B_c^{\pm} \to B^{\pm} K_S^0$ decays, but has little effect on the CP asymmetry in the $B_c^{\pm} \to B^{\pm} K_L^0$ decays. This is very different from the D meson decays. The reason is that the decay amplitudes of $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ only receive the contributions of the color-suppressed tree-emission diagrams, which can be calculated in the naive factorization approach and have no strong phase difference. In D meson decays, the naive factorization does not work, due to the large nonperturbative contributions at the charm scale. There are different strong phases between the Cabibbo-favored and doubly Cabibbo-suppressed amplitudes, the time-evolution effect of neutral kaons can contribute to a new and large CP violation in the D meson decays [15,51].

In Eq. (46), $(\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-}))/(\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-}))$ is the CP asymmetry in $B^- \to f_{B^-}$ and $B^+ \to \bar{f}_{B^-}$ decays and hereinafter denoted as $\mathcal{A}_{CP}(B^- \to f_{B^-})$. For the semileptonic decays of the $B^{\pm}$ to charmed meson, such as $B^{\pm} \to \bar{D}^0 l^{\pm} \nu_l$ and $B^{\pm} \to \bar{D}^*(2007)^0 l^{\pm} \nu_l$, the direct CP violations do not occur in the SM [54], so $\mathcal{A}_{CP}^{K_S^0}$ and $\mathcal{A}_{CP}^{K_L^0}$ are only determined by $\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_{S,L}^0)$. Here, we note that $\mathcal{A}_{CP}(B^- \to f_{B^-})$ should also receive the contribution from the CP violations in charm meson decays, for example, the CP violations in $B^{\pm} \to \bar{D}^0 l^{\pm} \nu_l \to \bar{f}_D l^{\pm} \nu_l$ decay chains can be derived as

$$\mathcal{A}_{CP}(B^- \to D^0 l^- \bar{\nu}_l) = \frac{\Gamma(B^- \to D^0 l^- \bar{\nu}_l) - \Gamma(B^+ \to \bar{D}^0 l^+ \nu_l)}{\Gamma(B^- \to D^0 l^- \bar{\nu}_l) + \Gamma(B^+ \to \bar{D}^0 l^+ \nu_l)} + \frac{\Gamma(D^0 \to f_D) - \Gamma(\bar{D}^0 \to \bar{f}_D)}{\Gamma(D^0 \to f_D) + \Gamma(\bar{D}^0 \to \bar{f}_D)}, \tag{49}$$

here, f_D and $\bar{f}_D$ denote the final states of D^0 and $\bar{D}^0$ decays, respectively. Fortunately, the second term on the right side of Eq. (49) is zero in the semileptonic and leptonic decays of the D mesons in the SM. However, because the final states of $B^{\pm} \to D$ semileptonic decays followed by the semileptonic and leptonic charm decays contain two neutrinos, the experiment environment is imperfect. The CP asymmetries in the nonleptonic decays of the $B^{\pm}$ mesons have nonzero

values, some of these values can reach the order of 10^{-1}, so the CP asymmetries in the nonleptonic decays of the $B^{\pm}$ mesons couldn't be neglected, we will discuss the possibility to observe the CP violations in the $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays followed by the nonleptonic decays of the $B^{\pm}$ mesons in section IV.

Next, we turn to calculate the $K_S^0 - K_L^0$ asymmetries in the $B_c^+ \to B^+ K_{S,L}^0$ and $B_c^- \to B^- K_{S,L}^0$ decays, which are defined as [15,55–57]

$$R(B_c^+ \to B^+ K_{S,L}^0) = \frac{\Gamma(B_c^+ \to B^+ K_S^0) - \Gamma(B_c^+ \to B^+ K_L^0)}{\Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^+ \to B^+ K_L^0)}, \tag{50}$$

$$R(B_c^- \to B^- K_{S,L}^0) = \frac{\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^- \to B^- K_L^0)}{\Gamma(B_c^- \to B^- K_S^0) + \Gamma(B_c^- \to B^- K_L^0)}, \tag{51}$$

By substituting Eqs. (38), (39), (43), and (44) into Eqs. (50)–(51) and neglecting the $K_S^0 - K_L^0$ mass difference and the CPT violation parameter z, we can obtain

$$R(B_c^+ \to B^+ K_{S,L}^0) = \frac{pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + p^* q V_{cs} V_{ud}^* V_{cd}^* V_{us}}{|q|^2|V_{cd}|^2|V_{us}|^2 + |p|^2|V_{cs}|^2|V_{ud}|^2} + \frac{\mathrm{Re}(t_{K_S^0 - K_L^0} \cdot \frac{p-q}{p+q} \cdot [|q|^2|V_{cd}|^2|V_{us}|^2 - |p|^2|V_{cs}|^2|V_{ud}|^2 - pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + p^* q V_{cd}^* V_{us} V_{cs} V_{ud}^*])}{|q|^2|V_{cd}|^2|V_{us}|^2 + |p|^2|V_{cs}|^2|V_{ud}|^2}, \tag{52}$$

$$R(B_c^- \to B^- K_{S,L}^0) = \frac{pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + p^* q V_{cs} V_{ud}^* V_{cd}^* V_{us}}{|q|^2|V_{cs}|^2|V_{ud}|^2 + |p|^2|V_{cd}|^2|V_{us}|^2} + \frac{\mathrm{Re}(t_{K_S^0 - K_L^0} \cdot \frac{p-q}{p+q} \cdot [|q|^2|V_{cs}|^2|V_{ud}|^2 - |p|^2|V_{cd}|^2|V_{us}|^2 - pq^* V_{cs}^* V_{ud} V_{cd} V_{us}^* + p^* q V_{cd}^* V_{us} V_{cs} V_{ud}^*])}{|q|^2|V_{cs}|^2|V_{ud}|^2 + |p|^2|V_{cd}|^2|V_{us}|^2}, \tag{53}$$

here, we note that the $K_S^0 - K_L^0$ mass difference is involved in the following expression in $R(B_c^{\pm} \to B^{\pm} K_{S,L}^0)$

$$\frac{|f_0(m_S^2)|^2 f(m_{B_c^+}, m_{B^+}, m_S) - |f_0(m_L^2)|^2 f(m_{B_c^+}, m_{B^+}, m_L)}{|f_0(m_S^2)|^2 f(m_{B_c^+}, m_{B^+}, m_S)}, \tag{54}$$

the experimental value of the $K_S^0 - K_L^0$ mass difference is $(3.48 \pm 0.01) \times 10^{-15}$ GeV [52], so the above expression is of the order of 10^{-16} and can be safely neglected.

Now, we proceed to study the CPT violation parameter z in the $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays. From Eqs. (38), (39), (43), and (44), we can obtain

$$\begin{aligned}
&\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^+ \to B^+ K_S^0) \\
&= \frac{G_F^2 f_K^2 |f_0(m_S^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_S) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2 \\
&\quad \cdot (|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2) \cdot \Big\{ (|q|^2|\sqrt{1-z}|^2 - |p|^2|\sqrt{1+z}|^2) \\
&\quad + 2\mathrm{Re}\left(t_{K_S^0 - K_L^0} \cdot \frac{p\sqrt{1+z} - q\sqrt{1-z}}{p\sqrt{1-z} + q\sqrt{1+z}} \cdot [|q|^2 (\sqrt{1-z})^* \sqrt{1+z} + |p|^2 \sqrt{1-z} (\sqrt{1+z})^*] \right) \Big\},
\end{aligned} \tag{55}$$

$$\begin{aligned}
\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0) = &\frac{G_F^2 f_K^2 |f_0(m_L^2)|^2}{1152\pi |p|^2 |q|^2 m_{B_c^+}^3} \cdot f(m_{B_c^+}, m_{B^+}, m_L) \cdot (m_{B_c^+}^2 - m_{B^+}^2)^2 \\
&\cdot (|q|^2|\sqrt{1+z}|^2 - |p|^2|\sqrt{1-z}|^2) \cdot (|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2),
\end{aligned} \tag{56}$$

we define the follow asymmetry

$$\mathcal{A}_{CPT}^{K^0_{S,L}}(B_c^{\pm} \to B^{\pm} K^0_{S,L}) = \frac{[\Gamma(B_c^- \to B^- K^0_S) - \Gamma(B_c^+ \to B^+ K^0_S)] - [\Gamma(B_c^- \to B^- K^0_L) - \Gamma(B_c^+ \to B^+ K^0_L)]}{\Gamma(B_c^- \to B^- K^0_S) + \Gamma(B_c^+ \to B^+ K^0_S) + \Gamma(B_c^- \to B^- K^0_L) + \Gamma(B_c^+ \to B^+ K^0_L)}. \tag{57}$$

Substituting Eqs. (38), (39), (43), (44), (55), and (56) into Eq. (57), we can obtain

$$\begin{aligned}
&\mathcal{A}_{CPT}^{K^0_{S,L}}(B_c^{\pm} \to B^{\pm} K^0_{S,L}) \\
&= -\frac{\mathrm{Re}(z)}{(1+\frac{|z|^2}{4})} \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2} + \mathrm{Re}\left(t_{K^0_S - K^0_L} \cdot \left(\frac{p-q}{p+q} + \frac{z}{2}\right)\right) \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2} \\
&\quad + \frac{|q|^2 - |p|^2}{|q|^2 + |p|^2} \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2} \cdot \frac{|f_0(m_S^2)|^2 f(m_{B_c^+}, m_{B^+}, m_S) - |f_0(m_L^2)|^2 f(m_{B_c^+}, m_{B^+}, m_L)}{|f_0(m_S^2)|^2 f(m_{B_c^+}, m_{B^+}, m_S) + |f_0(m_L^2)|^2 f(m_{B_c^+}, m_{B^+}, m_L)},
\end{aligned} \tag{58}$$

here, $(|q|^2 - |p|^2)/(|q|^2 + |p|^2)$ is the mixing-induced CP violation, which is of the order of 10^{-3} [19,52]. $(|f_0(m_S^2)|^2 f(m_{B_c^+}, m_{B^+}, m_S) - |f_0(m_L^2)|^2 f(m_{B_c^+}, m_{B^+}, m_L))/(|f_0(m_S^2)|^2 f(m_{B_c^+}, m_{B^+}, m_S) + |f_0(m_L^2)|^2 f(m_{B_c^+}, m_{B^+}, m_L))$ denotes the result of the $K_S^0 - K_L^0$ mass difference and is at the order of 10^{-16}, which can be calculated by Eq. (54) and its result, so the third term on the right side of Eq. (58) is immeasurably small (on the order of 10^{-19}), we can safely neglect this term. In this approach, we obtain

$$\mathcal{A}_{CPT}^{K^0_{S,L}}(B_c^{\pm} \to B^{\pm} K^0_{S,L}) = -\frac{\mathrm{Re}(z)}{(1+\frac{|z|^2}{4})} \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2} + \mathrm{Re}\left(t_{K^0_S - K^0_L} \cdot \left(\frac{p-q}{p+q} + \frac{z}{2}\right)\right) \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2}. \tag{59}$$

Obviously, $\mathcal{A}_{CPT}^{K^0_{S,L}}(B_c^{\pm} \to B^{\pm} K^0_{S,L})$ contains the term $\mathrm{Re}(t_{K^0_S - K^0_L} \cdot \frac{p-q}{p+q})$, which is independent from the CPT violation parameter z, so the precise calculation of $\mathrm{Re}(t_{K^0_S - K^0_L} \cdot \frac{p-q}{p+q})$, which is the function of the parameters $m_L - m_S, \Gamma_L, \Gamma_S, p, q, t_0$ and t_1, is crucial to constraint the CPT violation parameter z in the $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ decays. Here, we note that the values of t_0 and t_1 must be consistent with the event selection criteria in experiment. In a word, $\mathcal{A}_{CPT}^{K^0_{S,L}}(B_c^{\pm} \to B^{\pm} K^0_{S,L})$ can in principle constraint the parameter of CPT violation only if the measurements of the parameters $m_L - m_S, \Gamma_L, \Gamma_S, p$ and q have high precision and the values of t_0 and t_1 are consistent with the event selection criteria in experiment.

In experiment, the intermediate $B^{\pm}$ states are reconstructed in the final states $\bar{f}_{B^-}$ or f_{B^-}, so we defined the following observable

$$\mathcal{A}_{CPT} = \frac{\mathcal{A}_{CPT}^{K^0_S,-} - \mathcal{A}_{CPT}^{K^0_L,-}}{\mathcal{A}_{CPT}^{K^0_S,+} + \mathcal{A}_{CPT}^{K^0_L,+}}, \tag{60}$$

where $\mathcal{A}_{CPT}^{K^0_S,\pm}$ and $\mathcal{A}_{CPT}^{K^0_L,\pm}$ are defined as

$$\mathcal{A}_{CPT}^{K^0_S,\pm} = \Gamma(B_c^- \to B^- K^0_S)\Gamma(B^- \to f_{B^-}) \pm \Gamma(B_c^+ \to B^+ K^0_S)\Gamma(B^+ \to \bar{f}_{B^-}), \tag{61}$$

$$\mathcal{A}_{CPT}^{K^0_L,\pm} = \Gamma(B_c^- \to B^- K^0_L)\Gamma(B^- \to f_{B^-}) \pm \Gamma(B_c^+ \to B^+ K^0_L)\Gamma(B^+ \to \bar{f}_{B^-}). \tag{62}$$

By using Eqs. (B1)–(B5) listed in Appendix B, we can obtain

$$\begin{aligned}
\mathcal{A}_{CPT} &= R(B_c^+ \to B^+ K^0_{S,L}) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-}) + \mathcal{A}_{CPT}^{K^0_{S,L}}(B_c^{\pm} \to B^{\pm} K^0_{S,L})) \\
&= \frac{\Gamma(B_c^+ \to B^+ K^0_S) - \Gamma(B_c^+ \to B^+ K^0_L)}{\Gamma(B_c^+ \to B^+ K^0_S) + \Gamma(B_c^+ \to B^+ K^0_L)} \cdot \frac{\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})} \\
&\quad + \frac{[\Gamma(B_c^- \to B^- K^0_S) - \Gamma(B_c^+ \to B^+ K^0_S)] - [\Gamma(B_c^- \to B^- K^0_L) - \Gamma(B_c^+ \to B^+ K^0_L)]}{\Gamma(B_c^- \to B^- K^0_S) + \Gamma(B_c^+ \to B^+ K^0_S) + \Gamma(B_c^- \to B^- K^0_L) + \Gamma(B_c^+ \to B^+ K^0_L)}.
\end{aligned} \tag{63}$$

For the semileptonic decays of $B^{\pm}$ to charm meson, $\mathcal{A}_{CP}(B^- \to f_{B^-})$ is zero in the SM, so $\mathcal{A}_{CPT}$ can in principle constraint the parameter of CPT violation in the semileptonic decays of $B^{\pm}$ to charm meson, provided the new physics effects are negligible small. By combining Eq. (59) and Eq. (63), we can obtain

$$\begin{aligned}&\mathcal{A}_{CPT}(B_c^+ \to B^+ K_{S,L}^0 \to \bar{D}^0(\bar{D}^*(2007)^0) l^+ \nu_l K_{S,L}^0)\\&= -\frac{\mathrm{Re}(z)}{(1+\frac{|z|^2}{4})} \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2} + \mathrm{Re}\left(t_{K_S^0 - K_L^0} \cdot \left(\frac{p-q}{p+q} + \frac{z}{2}\right)\right) \cdot \frac{|V_{cs}|^2|V_{ud}|^2 - |V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2}. \end{aligned} \tag{64}$$

For the nonleptonic decays of $B^{\pm}$, $\mathcal{A}_{CP}(B^- \to f_{B^-})$ is nonzero, and what is more, the values of $R(B_c^+ \to B^+ K_{S,L}^0)$ have not been precisely measured up to now, so it is difficult to constraint the CPT violating parameter z by measuring the value of $\mathcal{A}_{CPT}$ via the nonleptonic decays of the $B^{\pm}$ mesons.

IV. NUMERICAL CALCULATION

Now, we are going to calculate the values of the observables. The input parameters used in this paper are collected as below [52,58,59]

$$\begin{aligned} m_{B_c^+} &= 6.275\ \mathrm{GeV}, \qquad m_{B^+} = 5.279\ \mathrm{GeV},\\ m_S &= 0.498\ \mathrm{GeV}, \qquad m_L = 0.498\ \mathrm{GeV},\\ \Delta m &= (3.48 \pm 0.01) \times 10^{-15}\ \mathrm{GeV}, \qquad G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2},\\ \Gamma_S &= 7.35 \times 10^{-15}\ \mathrm{GeV}, \qquad \Gamma_L = (1.29 \pm 0.01) \times 10^{-17}\ \mathrm{GeV},\\ m_{D^*} &= 2.007\ \mathrm{GeV}, \qquad m_{D_0^*} = 2.300\ \mathrm{GeV},\\ \tau_{B_c} &= (0.510 \pm 0.009) \times 10^{-12}\ \mathrm{s}, \qquad f_K = (0.154 \pm 0.002)\ \mathrm{GeV},\\ \mathrm{Re}(\epsilon) &= (1.66 \pm 0.02) \times 10^{-3}, \qquad \mathrm{Im}(\epsilon) = (1.57 \pm 0.02) \times 10^{-3}. \end{aligned} \tag{65}$$

We use the Wolfenstein parametrization to write the CKM matrix elements [52,60]

$$\begin{aligned} V_{ud} &= \sqrt{1-\lambda^2}\sqrt{1 - A^2\lambda^6(\rho^2+\eta^2)}, \qquad V_{cd} = -\lambda\sqrt{1 - A^2\lambda^4} - A^2\lambda^5\sqrt{1-\lambda^2}(\rho + i\eta),\\ V_{us} &= \lambda\sqrt{1 - A^2\lambda^6(\rho^2+\eta^2)}, \qquad V_{cs} = \sqrt{1-\lambda^2}\sqrt{1 - A^2\lambda^4} - A^2\lambda^6(\rho + i\eta), \end{aligned} \tag{66}$$

where λ, A, ρ and η are the real parameters. The latest results fitted by the UTfit collaboration [61] are presented as following

$$\lambda = 0.225 \pm 0.001, \qquad A = 0.826 \pm 0.012, \qquad \rho = 0.152 \pm 0.014, \qquad \eta = 0.357 \pm 0.010. \tag{67}$$

With all these parameters and Eqs. (38), (39), (43), and (44), we obtain the branching fractions of the $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays

$$\mathcal{B}(B_c^{\pm} \to B^{\pm} K_S^0) = (2.05^{-0.20}_{+0.21}) \times 10^{-3}, \tag{68}$$

$$\mathcal{B}(B_c^{\pm} \to B^{\pm} K_L^0) = (2.54^{-0.25}_{+0.26}) \times 10^{-3}, \tag{69}$$

here, the above results are the averaged branching ratios of the decay and its charge conjugate. By using the branching ratios of the consequent decays of the $B^{\pm}$ and $K_{S,L}^0$ mesons given by the Particle Data Group, we can calculate the results of $\mathcal{B}(B_c^{\pm} \to B^{\pm} K_{S,L}^0 \to f_{B^{\pm}} f_{K_{S,L}^0})$, which are listed in Table I. Here, we only consider the decay channels with the branching ratios are larger than 5.0×10^{-7}, which are hopefully to be marginally observed by the current experiments. We also list the $B_c^+ \to B^+ K_{S,L}^0 \to J/\psi K^+ K_{S,L}^0 \to e^+e^- K^+\pi^+\pi^-(\pi^{\pm}e^{\mp}\nu_e)$ decays in Table I, because these decays have a small number of final states and can be easily detected.

By substituting the values of the parameters in Eqs. (65), (66), and (67) into Eqs. (47) and (48), we can proceed to calculate the numerical results of the CP asymmetries in $B_c^{\pm} \to B^{\pm} K_{S,L}^0$ decays

TABLE I. The branching fractions of the $B_c^{\pm} \to B^{\pm} K_{S,L}^0 \to f_{B^{\pm}} f_{K_{S,L}^0}$ decays.

The decay channel	The branching ratio
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l \pi^+ \pi^-)$	$(1.18^{-0.12}_{+0.13}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- l^+ \nu_l \pi^+ \pi^-)$	$(1.32 \pm 0.14) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-)$	$(4.80^{-0.53}_{+0.55}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^+ \pi^-)$	$(1.84 \pm 0.20) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-)$	$(2.05^{-0.22}_{+0.23}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^+ \pi^-)$	$(7.48^{-0.83}_{+0.87}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 \pi^+ K_S^0 \to K^+ \pi^- \pi^0 \pi^+ \pi^+ \pi^-)$	$(9.56^{-1.03}_{+1.07}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 \rho^+ K_S^0 \to K^+ e^- \bar{\nu}_e \pi^+ \pi^0 \pi^+ \pi^-)$	$(6.73^{-1.12}_{+1.14}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 \rho^+ K_S^0 \to K^+ \pi^- \pi^+ \pi^0 \pi^+ \pi^-)$	$(7.51^{-1.25}_{+1.27}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 \rho^+ K_S^0 \to K^+ \pi^- \pi^0 \pi^+ \pi^0 \pi^+ \pi^-)$	$(2.74^{-0.46}_{+0.47}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 \pi^+ K_S^0 \to \bar{D}^0 \pi^0 \pi^+ K_S^0 \to K^+ \pi^- \pi^0 \pi^0 \pi^+ \pi^+ \pi^-)$	$(6.48^{-0.71}_{+0.74}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 \pi^+ \pi^+ \pi^- K_S^0 \to \bar{D}^0 \pi^0 \pi^+ \pi^+ \pi^- K_S^0 \to K^+ \pi^- \pi^0 \pi^0 \pi^+ \pi^+ \pi^- \pi^+ \pi^-)$	$(1.36^{-0.21}_{+0.22}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_S^0 \to J/\psi K^+ K_S^0 \to e^+ e^- K^+ \pi^+ \pi^-)$	$(8.52^{-0.86}_{+0.90}) \times 10^{-8}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e)$	$(8.57^{-0.91}_{+0.94}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e)$	$(9.56^{-1.01}_{+1.05}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e)$	$(3.49^{-0.39}_{+0.40}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^- \pi^0)$	$(1.08 \pm 0.12) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e)$	$(1.34^{-0.14}_{+0.15}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e)$	$(1.49 \pm 0.16) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e)$	$(5.43^{-0.61}_{+0.63}) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^+ \pi^- \pi^0)$	$(1.68 \pm 0.19) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 \pi^+ K_L^0 \to K^+ \pi^- \pi^0 \pi^+ \pi^{\pm} e^{\mp} \nu_e)$	$(6.94^{-0.75}_{+0.77}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 \rho^+ K_L^0 \to K^+ \pi^- \pi^+ \pi^0 \pi^{\pm} e^{\mp} \nu_e)$	$(5.45^{-0.91}_{+0.92}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 \rho^+ K_L^0 \to K^+ \pi^- \pi^0 \pi^+ \pi^0 \pi^{\pm} e^{\mp} \nu_e)$	$(1.99 \pm 0.34) \times 10^{-6}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 \rho^+ K_L^0 \to K^+ \pi^- \pi^0 \pi^+ \pi^0 \pi^+ \pi^- \pi^0)$	$(6.15^{-1.05}_{+1.06}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 \pi^+ \pi^+ \pi^- K_L^0 \to \bar{D}^0 \pi^0 \pi^+ \pi^+ \pi^- K_L^0 \to K^+ \pi^- \pi^0 \pi^0 \pi^+ \pi^+ \pi^- \pi^{\pm} e^{\mp} \nu_e)$	$(9.88^{-1.55}_{+1.57}) \times 10^{-7}$
$\mathcal{B}(B_c^+ \to B^+ K_L^0 \to J/\psi K^+ K_L^0 \to e^+ e^- K^+ \pi^{\pm} e^{\mp} \nu_e)$	$(6.19^{-0.63}_{+0.66}) \times 10^{-8}$

$$\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_S^0) = \frac{\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^+ \to B^+ K_S^0)}{\Gamma(B_c^- \to B^- K_S^0) + \Gamma(B_c^+ \to B^+ K_S^0)} = (4.05 \pm 0.05) \times 10^{-3}, \tag{70}$$

$$\mathcal{A}_{CP}(B_c^{\pm} \to B^{\pm} K_L^0) = \frac{\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0)}{\Gamma(B_c^- \to B^- K_L^0) + \Gamma(B_c^+ \to B^+ K_L^0)} = (-2.98 \pm 0.04) \times 10^{-3}. \tag{71}$$

Combining Eqs. (46), (49), (70), and (71), we can obtain the time-independent CP asymmetries $\mathcal{A}_{CP}^{K_S^0}$ and $\mathcal{A}_{CP}^{K_L^0}$ in the $B_c^{\pm} \to B^{\pm} K_{S,L}^0 \to f_{B^-}(\bar{f}_{B^-}) f_{K_{S,L}^0}$ decays. As for the $B^{\pm} \to \bar{D}^0(D^0) l^{\pm} \nu_l$ and $B^{\pm} \to \bar{D}^*(2007)^0(D^*(2007)^0) l^{\pm} \nu_l$ decays, the CP asymmetries in these decays are zero in the SM, so we can obtain

$$\mathcal{A}_{CP}^{K_S^0}(B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0) = (4.05 \pm 0.05) \times 10^{-3} + \frac{\Gamma(D^0 \to f_D) - \Gamma(\bar{D}^0 \to \bar{f}_D)}{\Gamma(D^0 \to f_D) + \Gamma(\bar{D}^0 \to \bar{f}_D)}, \tag{72}$$

$$\mathcal{A}_{CP}^{K_L^0}(B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0) = (-2.98 \pm 0.04) \times 10^{-3} + \frac{\Gamma(D^0 \to f_D) - \Gamma(\bar{D}^0 \to \bar{f}_D)}{\Gamma(D^0 \to f_D) + \Gamma(\bar{D}^0 \to \bar{f}_D)}, \tag{73}$$

$$\mathcal{A}_{CP}^{K_S^0}(B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0) = (4.05 \pm 0.05) \times 10^{-3} + \frac{\Gamma(D^0 \to f_D) - \Gamma(\bar{D}^0 \to \bar{f}_D)}{\Gamma(D^0 \to f_D) + \Gamma(\bar{D}^0 \to \bar{f}_D)}, \tag{74}$$

$$\mathcal{A}_{CP}^{K_L^0}(B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0) = (-2.98 \pm 0.04) \times 10^{-3} + \frac{\Gamma(D^0 \to f_D) - \Gamma(\bar{D}^0 \to \bar{f}_D)}{\Gamma(D^0 \to f_D) + \Gamma(\bar{D}^0 \to \bar{f}_D)}, \tag{75}$$

of course, $(\Gamma(D^0 \to f_D) - \Gamma(\bar{D}^0 \to \bar{f}_D))/(\Gamma(D^0 \to f_D) + \Gamma(\bar{D}^0 \to \bar{f}_D))$ is zero in the semileptonic decay of the D^0 meson in the SM. Meanwhile, the CP violations in the nonleptonic decays of charm mesons, such as $D^0 \to K^-\pi^+$ and $D^0 \to K^-\pi^+\pi^0$, are of order of 10^{-4}, or even smaller [52,55,62–65], so the CP violations in the nonleptonic decays of charm mesons are smaller than that in $B_c^\pm \to B^\pm K_{S,L}^0$ decays and can be safely neglected. From Table I, we can see that the branching ratios of the following decay channels are of the order of 10^{-6}: $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^\pm e^\mp \nu_e$, $B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^- \pi^0$, $B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^\pm e^\mp \nu_e$, $B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^\pm e^\mp \nu_e$, $B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^\pm e^\mp \nu_e$ and $B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^+ \pi^- \pi^0$, so the magnitude of the branching ratio times $\mathcal{A}_{CP}^{K_{S,L}^0}$ can reach 10^{-9} or larger for these channels, which are possible to be observed. Here, one can also consider the possibility to sum over the available final states of D^0 decays to obtain a statistically significant signal of CP violation.

For the case of the nonleptonic decays of $B^\pm$ mesons, $\mathcal{A}_{CP}^{K_{S,L}^0}$ can receive contributions from both the CP asymmetries in $B^\pm \to \bar{f}_{B^-}(f_{B^-})$ decays and the CP asymmetries in $B_c^\pm \to B^\pm K_{S,L}^0$ decays, so it is possible to extract the CP asymmetries in $B_c^\pm \to B^\pm K_{S,L}^0$ decays in the channels which have the accurate measurements of the CP asymmetries in $B^\pm \to \bar{f}_{B^-}(f_{B^-})$ decays and the large branching ratios of the $B_c^\pm \to B^\pm K_{S,L}^0 \to \bar{f}_{B^-}(f_{B^-}) f_{K_{S,L}^0}$ decay chains. However, one can see from Particle Data Group [52] and Table I that some decay channels, such as $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 \pi^+ \pi^+ \pi^- \to K^+ \pi^- \pi^0 \pi^+ \pi^+ \pi^-$ and $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 \rho^+ \pi^+ \pi^- \to K^+ \pi^- \pi^0 \pi^+ \pi^0 \pi^+ \pi^-$, have the large branching ratio (at the level of 10^{-6}) but an inaccurate value of $\mathcal{A}_{CP}(B^- \to f_{B^-})$. Other decays, such as $B_c^+ \to B^+ K_S^0 \to K^+ K^- \pi^+ \pi^+ \pi^-$ and $B_c^+ \to B^+ K_S^0 \to \pi^+ \pi^- \pi^+ \pi^+ \pi^-$, have the accurate measurement of $\mathcal{A}_{CP}(B^- \to f_{B^-})$ but a small branching ratio, further more, $\mathcal{A}_{CP}(B^- \to f_{B^-})$ in these channels is at the level of 10^{-2} or larger, which dominant the $\mathcal{A}_{CP}^{K_{S,L}^0}$, so it is difficult to extract the CP asymmetries in $B_c^\pm \to B^\pm K_{S,L}^0$ decays by measuring the CP violations in the $B_c^\pm \to B^\pm K_{S,L}^0 \to f_{B^\pm} f_{K_{S,L}^0}$ processes with the nonleptonic decays of $B^\pm$.

The explicit expressions for the $K_S^0 - K_L^0$ asymmetries $R(B_c^+ \to B^+ K_{S,L}^0)$ and $R(B_c^- \to B^- K_{S,L}^0)$ have been given in Eqs. (52) and (53). Using the values of the parameters in Eqs. (65), (66) and (67), the numerical results of $R(B_c^+ \to B^+ K_{S,L}^0)$ and $R(B_c^- \to B^- K_{S,L}^0)$ can be obtained

$$R(B_c^+ \to B^+ K_{S,L}^0) = -0.109 \pm 0.001, \tag{76}$$

$$R(B_c^- \to B^- K_{S,L}^0) = -0.103 \pm 0.001. \tag{77}$$

Obviously, the large values of the $K_S^0 - K_L^0$ asymmetries indicate that there exist a large difference between the branching ratios of $B_c^\pm \to B^\pm K_S^0$ and the branching ratios of $B_c^\pm \to B^\pm K_L^0$. Combining Eqs. (76) and (77) with the branching ratios of the $B_c^\pm \to B^\pm K_{S,L}^0 \to f_{B^\pm} K_{S,L}^0$ decays from Particle Data Group [52] and Table I, we can see that the maximum order of the branching ratios for the decays $B_c^\pm \to B^\pm K_{S,L}^0 \to f_{B^\pm} K_{S,L}^0$ times $R(B_c^\pm \to B^\pm K_{S,L}^0)$ can reach 10^{-7}, so $R(B_c^\pm \to B^\pm K_{S,L}^0)$ are hopefully to be detected in the current experiments.

Now, we move on to the determination or constraint of the parameter of CPT violation in $B_c^\pm \to B^\pm K_{S,L}^0 \to f_{B^\pm} f_{K_{S,L}^0}$ decays. As discussed in Section III, the most promising process to constraint the parameter of CPT violation are the $B_c^+ \to B^+ K_{S,L}^0 \to \bar{D}^0 l^+ \nu_l K_{S,L}^0 \to K^+ l^- \bar{\nu}_l l^+ \nu_l K_{S,L}^0$ and $B_c^+ \to B^+ K_{S,L}^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_{S,L}^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_{S,L}^0 \to K^+ l^- \bar{\nu}_l \pi^0 l^+ \nu_l K_{S,L}^0$ decays. By combining Eq. (64), Eq. (66) and Eq. (67), we can obtain

$$\frac{|V_{cs}|^2 |V_{ud}|^2 - |V_{cd}|^2 |V_{us}|^2}{|V_{cs}|^2 |V_{ud}|^2 + |V_{cd}|^2 |V_{us}|^2} = 0.9943 \pm 0.0001. \tag{78}$$

From Table I, we can see that the branching ratios of the $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ l^- \bar{\nu}_l l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ l^- \bar{\nu}_l l^+ \nu_l \pi^\pm e^\mp \nu_e$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ l^- \bar{\nu}_l \pi^0 l^+ \nu_l \pi^+ \pi^-$ and $B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ l^- \bar{\nu}_l \pi^0 l^+ \nu_l \pi^\pm e^\mp \nu_e$ decays can arrive at 1.18×10^{-6},

8.57×10^{-7}, 1.84×10^{-6} and 1.34×10^{-6}, respectively, so the sensitivity for the measurement of $\mathcal{A}_{CPT}$ is expected to be at the level of 10^{-3} if we assume the selection efficiency is 10^{-3} and the total number of $B_c^{\pm}$ events is 10^{12} after the LHC run 5 [66]. With the values of the parameters in Eqs. (65), (66) and (67), we can obtain

$$\mathrm{Re}\left(t_{K_S^0-K_L^0}\cdot\frac{p-q}{p+q}\right)\cdot\frac{|V_{cs}|^2|V_{ud}|^2-|V_{cd}|^2|V_{us}|^2}{|V_{cs}|^2|V_{ud}|^2+|V_{cd}|^2|V_{us}|^2} = (3.46\pm0.03)\times10^{-3}, \tag{79}$$

which accuracy can reach $10^{-4}\sim10^{-5}$. According to Eq. (64), we can obtain that the sensitivity for the measurement of the CPT violation parameter z is expected to be at the level of 10^{-3}.

V. PROSPECTS FOR THE MEASUREMENT AT LHC

At the LHC experiment, around 5×10^{10} B_c mesons per years could be produced [44,45], therefore, one could expect 5×10^{11} B_c events with 10 years' run and 50 fb^{-1} data at LHCb [67]. Moreover, LHCb will accumulate a data sample corresponding to a minimum of 300 fb^{-1} in the HL-LHC era [66], so the total number of B_c events may exceed 3×10^{12} by the end of the HL-LHC era. Basing on these samples, the B_c decay chains with a branching ratio at the level of 10^{-8}–10^{-6} can be measured with a good precision.

Following Refs. [44,68,69], we give an estimation of how many B_c events-times-efficiency are needed to establish the branching ratio, the CP asymmetry and the $K_S^0 - K_L^0$ asymmetry to three standard deviations (3σ). When the decays are observed at three standard deviations (3σ) level, the numbers of B_c events-times-efficiency needed read

$$(\epsilon_f N)_{\mathcal{B}} = \frac{9}{\mathcal{B}(B_c^{\pm}\to B^{\pm}K_{S,L}^0\to f_{B^{\pm}}f_{K_{S,L}^0})}, \tag{80}$$

where ϵ_f is the detecting efficiency of the final states. With the numerical results of the branching ratios in Table I, we can obtain the numerical results of $(\epsilon_f N)_{\mathcal{B}}$ for the $B_c^{\pm}\to B^{\pm}K_{S,L}^0\to f_{B^{\pm}}f_{K_{S,L}^0}$ channels, as shown in Table II.

TABLE II. The numerical results of $(\epsilon_f N)_{\mathcal{B}}$ for the $B_c^{\pm}\to B^{\pm}K_{S,L}^0\to f_{B^{\pm}}f_{K_{S,L}^0}$ decays.

The decay channel	$(\epsilon_f N)_{\mathcal{B}}$
$B_c^+\to B^+K_S^0\to\bar{D}^0l^+\nu_lK_S^0\to K^+e^-\bar{\nu}_el^+\nu_l\pi^+\pi^-$	$(6.87\sim8.52)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^0l^+\nu_lK_S^0\to K^+\pi^-l^+\nu_l\pi^+\pi^-$	$(6.16\sim7.64)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^0l^+\nu_lK_S^0\to K^+\pi^-\pi^0l^+\nu_l\pi^+\pi^-$	$(1.68\sim2.11)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^*(2007)^0l^+\nu_lK_S^0\to\bar{D}^0\pi^0l^+\nu_lK_S^0\to K^+e^-\bar{\nu}_e\pi^0l^+\nu_l\pi^+\pi^-$	$(4.40\sim5.47)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^*(2007)^0l^+\nu_lK_S^0\to\bar{D}^0\pi^0l^+\nu_lK_S^0\to K^+\pi^-\pi^0l^+\nu_l\pi^+\pi^-$	$(3.95\sim4.91)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^*(2007)^0l^+\nu_lK_S^0\to\bar{D}^0\pi^0l^+\nu_lK_S^0\to K^+\pi^-\pi^0\pi^0l^+\nu_l\pi^+\pi^-$	$(1.08\sim1.35)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^0\pi^+K_S^0\to K^+\pi^-\pi^0\pi^+\pi^+\pi^-$	$(0.85\sim1.05)\times10^7$
$B_c^+\to B^+K_S^0\to\bar{D}^0\rho^+K_S^0\to K^+e^-\bar{\nu}_e\pi^+\pi^0\pi^+\pi^-$	$(1.14\sim1.60)\times10^7$
$B_c^+\to B^+K_S^0\to\bar{D}^0\rho^+K_S^0\to K^+\pi^-\pi^+\pi^0\pi^+\pi^-$	$(1.03\sim1.44)\times10^7$
$B_c^+\to B^+K_S^0\to\bar{D}^0\rho^+K_S^0\to K^+\pi^-\pi^0\pi^+\pi^0\pi^+\pi^-$	$(2.80\sim3.96)\times10^6$
$B_c^+\to B^+K_S^0\to\bar{D}^*(2007)^0\pi^+K_S^0\to\bar{D}^0\pi^0\pi^+K_S^0\to K^+\pi^-\pi^0\pi^0\pi^+\pi^+\pi^-$	$(1.25\sim1.56)\times10^7$
$B_c^+\to B^+K_S^0\to\bar{D}^*(2007)^0\pi^+\pi^+\pi^-K_S^0\to\bar{D}^0\pi^0\pi^+\pi^+\pi^-K_S^0\to K^+\pi^-\pi^0\pi^0\pi^+\pi^+\pi^-\pi^+\pi^-$	$(5.70\sim7.84)\times10^6$
$B_c^+\to B^+K_S^0\to J/\psi K^+K_S^0\to e^+e^-K^+\pi^+\pi^-$	$(0.95\sim1.18)\times10^8$
$B_c^+\to B^+K_L^0\to\bar{D}^0l^+\nu_lK_L^0\to K^+e^-\bar{\nu}_el^+\nu_l\pi^{\pm}e^{\mp}\nu_e$	$(0.95\sim1.17)\times10^7$
$B_c^+\to B^+K_L^0\to\bar{D}^0l^+\nu_lK_L^0\to K^+\pi^-l^+\nu_l\pi^{\pm}e^{\mp}\nu_e$	$(0.85\sim1.05)\times10^7$
$B_c^+\to B^+K_L^0\to\bar{D}^0l^+\nu_lK_L^0\to K^+\pi^-\pi^0l^+\nu_l\pi^{\pm}e^{\mp}\nu_e$	$(2.32\sim2.91)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^0l^+\nu_lK_L^0\to K^+\pi^-\pi^0l^+\nu_l\pi^+\pi^-\pi^0$	$(7.49\sim9.40)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^*(2007)^0l^+\nu_lK_L^0\to\bar{D}^0\pi^0l^+\nu_lK_L^0\to K^+e^-\bar{\nu}_e\pi^0l^+\nu_l\pi^{\pm}e^{\mp}\nu_e$	$(6.06\sim7.55)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^*(2007)^0l^+\nu_lK_L^0\to\bar{D}^0\pi^0l^+\nu_lK_L^0\to K^+\pi^-\pi^0l^+\nu_l\pi^{\pm}e^{\mp}\nu_e$	$(5.44\sim6.77)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^*(2007)^0l^+\nu_lK_L^0\to\bar{D}^0\pi^0l^+\nu_lK_L^0\to K^+\pi^-\pi^0\pi^0l^+\nu_l\pi^{\pm}e^{\mp}\nu_e$	$(1.49\sim1.87)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^*(2007)^0l^+\nu_lK_L^0\to\bar{D}^0\pi^0l^+\nu_lK_L^0\to K^+\pi^-\pi^0\pi^0l^+\nu_l\pi^+\pi^-\pi^0$	$(4.80\sim6.04)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^0\pi^+K_L^0\to K^+\pi^-\pi^0\pi^+\pi^{\pm}e^{\mp}\nu_e$	$(1.17\sim1.45)\times10^7$
$B_c^+\to B^+K_L^0\to\bar{D}^0\rho^+K_L^0\to K^+\pi^-\pi^+\pi^0\pi^{\pm}e^{\mp}\nu_e$	$(1.41\sim1.98)\times10^7$
$B_c^+\to B^+K_L^0\to\bar{D}^0\rho^+K_L^0\to K^+\pi^-\pi^0\pi^+\pi^0\pi^{\pm}e^{\mp}\nu_e$	$(3.86\sim5.46)\times10^6$
$B_c^+\to B^+K_L^0\to\bar{D}^0\rho^+K_L^0\to K^+\pi^-\pi^0\pi^+\pi^0\pi^+\pi^-\pi^0$	$(1.25\sim1.76)\times10^7$
$B_c^+\to B^+K_L^0\to\bar{D}^*(2007)^0\pi^+\pi^+\pi^-K_L^0\to\bar{D}^0\pi^0\pi^+\pi^+\pi^-K_L^0\to K^+\pi^-\pi^0\pi^0\pi^+\pi^+\pi^-\pi^{\pm}e^{\mp}\nu_e$	$(0.79\sim1.08)\times10^7$
$B_c^+\to B^+K_L^0\to J/\psi K^+K_L^0\to e^+e^-K^+\pi^{\pm}e^{\mp}\nu_e$	$(1.32\sim1.62)\times10^8$

In the same way, the numbers of $B_c^{\pm}$ events-times-efficiency needed for observing the $K_S^0 - K_L^0$ asymmetries at three standard deviations (3σ) level are

$$(\epsilon_f N)^{+}_{K^0_{S,L}} = \frac{9}{[\mathcal{B}(B_c^+ \to B^+ K_S^0 \to f_{B^+} K_S^0) + \mathcal{B}(B_c^+ \to B^+ K_L^0 \to f_{B^+} K_L^0)] \cdot |R(B_c^+ \to B^+ K^0_{S,L})|}, \tag{81}$$

and

$$(\epsilon_f N)^{-}_{K^0_{S,L}} = \frac{9}{[\mathcal{B}(B_c^- \to B^- K_S^0 \to f_{B^-} K_S^0) + \mathcal{B}(B_c^- \to B^- K_L^0 \to f_{B^-} K_L^0)] \cdot |R(B_c^- \to B^- K^0_{S,L})|}. \tag{82}$$

From Particle Data Group [52], Table I, Eq. (76) and Eq. (77), we can calculate the numerical results of $(\epsilon_f N)^{+}_{K^0_{S,L}}$ and $(\epsilon_f N)^{-}_{K^0_{S,L}}$, which are listed in Table III.

Again, the numbers of $B_c^{\pm}$ events-times-efficiency needed for testing CP violation with 3σ significance can be derived as follows

TABLE III. The numerical results of $(\epsilon_f N)^{\pm}_{K^0_{S,L}}$ for the $B_c^{\pm} \to B^{\pm} K^0_{S,L} \to f_{B^{\pm}} K^0_{S,L}$ decays.

The decay channel	$(\epsilon_f N)^{+}_{K^0_{S,L}}$ $((\epsilon_f N)^{-}_{K^0_{S,L}})$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 l^+ \nu_l K^0_{S,L} \to K^+ e^- \bar{\nu}_e l^+ \nu_l K^0_{S,L}$	$(1.99 \sim 2.35) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^0 l^- \bar{\nu}_l K^0_{S,L} \to K^- e^+ \nu_e l^- \bar{\nu}_l K^0_{S,L}$	$(2.11 \sim 2.49) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 l^+ \nu_l K^0_{S,L} \to K^+ \pi^- l^+ \nu_l K^0_{S,L}$	$(1.79 \sim 2.11) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^0 l^- \bar{\nu}_l K^0_{S,L} \to K^- \pi^+ l^- \bar{\nu}_l K^0_{S,L}$	$(1.89 \sim 2.23) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 l^+ \nu_l K^0_{S,L} \to K^+ \pi^- \pi^0 l^+ \nu_l K^0_{S,L}$	$(4.88 \sim 5.82) \times 10^6$
$B_c^- \to B^- K^0_{S,L} \to D^0 l^- \bar{\nu}_l K^0_{S,L} \to K^- \pi^+ \pi^0 l^- \bar{\nu}_l K^0_{S,L}$	$(5.16 \sim 6.16) \times 10^6$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^*(2007)^0 l^+ \nu_l K^0_{S,L} \to \bar{D}^0 \pi^0 l^+ \nu_l K^0_{S,L} \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l K^0_{S,L}$	$(1.28 \sim 1.51) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^*(2007)^0 l^- \bar{\nu}_l K^0_{S,L} \to D^0 \pi^0 l^- \bar{\nu}_l K^0_{S,L} \to K^- e^+ \nu_e \pi^0 l^- \bar{\nu}_l K^0_{S,L}$	$(1.35 \sim 1.60) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^*(2007)^0 l^+ \nu_l K^0_{S,L} \to \bar{D}^0 \pi^0 l^+ \nu_l K^0_{S,L} \to K^+ \pi^- \pi^0 l^+ \nu_l K^0_{S,L}$	$(1.15 \sim 1.35) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^*(2007)^0 l^- \bar{\nu}_l K^0_{S,L} \to D^0 \pi^0 l^- \bar{\nu}_l K^0_{S,L} \to K^- \pi^+ \pi^0 l^- \bar{\nu}_l K^0_{S,L}$	$(1.21 \sim 1.43) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^*(2007)^0 l^+ \nu_l K^0_{S,L} \to \bar{D}^0 \pi^0 l^+ \nu_l K^0_{S,L} \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l K^0_{S,L}$	$(3.13 \sim 3.74) \times 10^6$
$B_c^- \to B^- K^0_{S,L} \to D^*(2007)^0 l^- \bar{\nu}_l K^0_{S,L} \to D^0 \pi^0 l^- \bar{\nu}_l K^0_{S,L} \to K^- \pi^+ \pi^0 \pi^0 l^- \bar{\nu}_l K^0_{S,L}$	$(3.31 \sim 3.96) \times 10^6$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 \pi^+ K^0_{S,L} \to K^+ \pi^- \pi^0 \pi^+ K^0_{S,L}$	$(2.46 \sim 2.91) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^0 \pi^- K^0_{S,L} \to K^- \pi^+ \pi^0 \pi^- K^0_{S,L}$	$(2.60 \sim 3.08) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 \rho^+ K^0_{S,L} \to K^+ e^- \bar{\nu}_e \pi^+ \pi^0 K^0_{S,L}$	$(3.29 \sim 4.47) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^0 \rho^- K^0_{S,L} \to K^- e^+ \nu_e \pi^- \pi^0 K^0_{S,L}$	$(3.48 \sim 4.73) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 \rho^+ K^0_{S,L} \to K^+ \pi^- \pi^+ \pi^0 K^0_{S,L}$	$(2.95 \sim 4.01) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^0 \rho^- K^0_{S,L} \to K^- \pi^+ \pi^- \pi^0 K^0_{S,L}$	$(3.12 \sim 4.24) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^0 \rho^+ K^0_{S,L} \to K^+ \pi^- \pi^0 \pi^+ \pi^0 K^0_{S,L}$	$(0.81 \sim 1.10) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^0 \rho^- K^0_{S,L} \to K^- \pi^+ \pi^0 \pi^- \pi^0 K^0_{S,L}$	$(0.85 \sim 1.17) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^*(2007)^0 \pi^+ K^0_{S,L} \to \bar{D}^0 \pi^0 \pi^+ K^0_{S,L} \to K^+ \pi^- \pi^0 \pi^0 \pi^+ K^0_{S,L}$	$(3.62 \sim 4.32) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^*(2007)^0 \pi^- K^0_{S,L} \to D^0 \pi^0 \pi^- K^0_{S,L} \to K^- \pi^+ \pi^0 \pi^0 \pi^- K^0_{S,L}$	$(3.83 \sim 4.57) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to \bar{D}^*(2007)^0 \pi^+ \pi^+ \pi^- K^0_{S,L} \to \bar{D}^0 \pi^0 \pi^+ \pi^+ \pi^- K^0_{S,L} \to K^+ \pi^- \pi^0 \pi^0 \pi^+ \pi^+ \pi^- K^0_{S,L}$	$(1.64 \sim 2.18) \times 10^7$
$B_c^- \to B^- K^0_{S,L} \to D^*(2007)^0 \pi^- \pi^- \pi^+ K^0_{S,L} \to D^0 \pi^0 \pi^- \pi^- \pi^+ K^0_{S,L} \to K^- \pi^+ \pi^0 \pi^0 \pi^- \pi^- \pi^+ K^0_{S,L}$	$(1.74 \sim 2.31) \times 10^7$
$B_c^+ \to B^+ K^0_{S,L} \to J/\psi K^+ K^0_{S,L} \to e^+ e^- K^+ K^0_{S,L}$	$(2.78 \sim 3.24) \times 10^8$
$B_c^- \to B^- K^0_{S,L} \to J/\psi K^- K^0_{S,L} \to e^+ e^- K^- K^0_{S,L}$	$(2.94 \sim 3.43) \times 10^8$

TABLE IV. The numerical results of $(\epsilon_f N)_{CP}$ for the $B_c^{\pm} \to B^{\pm} K^0_{S,L} \to f_{B^{\pm}} f_{K^0_{S,L}}$ decays.

The decay channel	$(\epsilon_f N)_{CP}$
$B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l \pi^+ \pi^-$	$(0.85 \sim 1.05) \times 10^9$
$B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- l^+ \nu_l \pi^+ \pi^-$	$(7.60 \sim 9.44) \times 10^8$
$B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$	$(2.08 \sim 2.60) \times 10^8$
$B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^+ \pi^-$	$(5.43 \sim 6.76) \times 10^8$
$B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$	$(4.87 \sim 6.06) \times 10^8$
$B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^+ \pi^-$	$(1.33 \sim 1.67) \times 10^8$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e$	$(1.59 \sim 1.97) \times 10^9$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e$	$(1.42 \sim 1.77) \times 10^9$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e$	$(3.88 \sim 4.88) \times 10^8$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^- \pi^0$	$(1.26 \sim 1.58) \times 10^9$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e$	$(1.02 \sim 1.27) \times 10^9$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e$	$(0.91 \sim 1.14) \times 10^9$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^{\pm} e^{\mp} \nu_e$	$(2.49 \sim 3.13) \times 10^8$
$B_c^+ \to B^+ K_L^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_L^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_L^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^+ \pi^- \pi^0$	$(0.81 \sim 1.01) \times 10^9$

$$(\epsilon_f N)_{CP} = \frac{9}{[\mathcal{B}(B_c^+ \to B^+ K^0_{S,L} \to f_{B^+} f_{K^0_{S,L}}) + \mathcal{B}(B_c^- \to B^- K^0_{S,L} \to f_{B^-} f_{K^0_{S,L}})] \cdot \left| \mathcal{A}_{CP}^{K^0_{S,L}} \right|} \approx \frac{9}{2 \cdot \mathcal{B}(B_c^+ \to B^+ K^0_{S,L} \to f_{B^+} f_{K^0_{S,L}}) \cdot \left| \mathcal{A}_{CP}^{K^0_{S,L}} \right|}. \tag{83}$$

As discussed in Sec. III and Sec. IV, the CP asymmetries can be observed only in the $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ decays and the semileptonic decays of the $B^{\pm}$ to charmed meson, so we only calculate the numbers of $B_c^{\pm}$ events-times-efficiency needed in these decays,which are shown in Table IV. Meanwhile, we note that the decays $B_c^{\pm} \to B^{\pm} K_S^0 \to f_{B^{\pm}} \pi^+ \pi^-$ are more favorable than the decays $B_c^{\pm} \to B^{\pm} K_L^0 \to f_{B^{\pm}} \pi^{\pm} e^{\mp} \nu_e (\pi^+ \pi^- \pi^0)$ to observe in experiment, because the number of the final state particles in $K_S^0 \to \pi^+ \pi^-$ decay is less than that in $K_L^0 \to \pi^{\pm} e^{\mp} \nu_e (\pi^+ \pi^- \pi^0)$ decays.

From Table II, Table III, and Table IV, we can see that the numbers of the $B_c^{\pm}$ events-times-efficiency, which are needed to observe the branching ratios, $K_S^0 - K_L^0$ asymmetries and CP asymmetries, are of the order of $10^6 \sim 10^9$ in the listed processes, so they are possible to be observed at the LHC experiment and the HL-LHC experiment.

VI. CONCLUSIONS

In conclusion, the large $B_c^{\pm}$ sample, which can be produced at the LHC experiment and the HL-LHC experiment, will make it an ideal place to study the $B_c^{\pm}$ decays and investigate the related physics. We study the effect of $K^0 - \bar{K}^0$ mixing on the branching ratios, CP asymmetries, $K_S^0 - K_L^0$ asymmetries and CPT violations in the $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ decays. We find that some of the $B_c^{\pm} \to B^{\pm} K^0_{S,L} \to f_{B^{\pm}} f_{K^0_{S,L}}$ decay chains have large branching ratios, whose maximum value can reach 7.48×10^{-6}. The CP asymmetries in the $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ decays are dominated by $K^0 - \bar{K}^0$ mixing and can exceed the order of 10^{-3}. The most promising processes to observe the CP violation are $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l \pi^+ \pi^-$, $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 l^+ \nu_l \pi^+ \pi^-$ and $B_c^+ \to B^+ K_S^0 \to \bar{D}^*(2007)^0 l^+ \times \nu_l K_S^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_S^0 \to K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l \pi^+ \pi^-$, for which about 1.05×10^9, 9.44×10^8, 2.60×10^8, 6.76×10^8, 6.06×10^8 and 1.67×10^8 $B_c^{\pm}$ events-times-efficiency are needed respectively, if CP violation is observed at 3σ level.

The $K_S^0 - K_L^0$ asymmetries in the $B_c^{\pm} \to B^{\pm} K^0_{S,L}$ decays are also studied. The numerical results of the $K_S^0 - K_L^0$ asymmetries can reach as large as 0.11. Together with the branching ratios for the $B_c^{\pm} \to B^{\pm} K^0_{S,L} \to f_{B^{\pm}} K^0_{S,L}$ decay chains and the $K_S^0 - K_L^0$ asymmetries, we calculate the numbers of the $B_c^{\pm}$ events-times-efficiency needed to establish the $K_S^0 - K_L^0$ asymmetries to 3 standard deviations in the decay processes with a large branching ratios.

The range of the numbers of the $B_c^\pm$ events-times-efficiency needed to observe $K_S^0 - K_L^0$ asymmetries at a significance of 3σ in these decays is from 3.74×10^6 to 3.43×10^8, which is measurable at the LHC experiment. We investigate the possibility of extracting the CPT violation parameter z in the $B_c^\pm \to B^\pm K_{S,L}^0 \to f_{B^\pm} f_{K_{S,L}^0}$ decays. We find that the decays $B_c^+ \to B^+ K_{S,L}^0 \to \bar{D}^0 l^+ \nu_l K_{S,L}^0 \to K^+ e^- \bar{\nu}_e l^+ \nu_l K_{S,L}^0$ and $B_c^+ \to B^+ K_{S,L}^0 \to \bar{D}^*(2007)^0 l^+ \nu_l K_{S,L}^0 \to \bar{D}^0 \pi^0 l^+ \nu_l K_{S,L}^0 \to K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l K_{S,L}^0$ can in principle constraint the parameter z if the measurements of the parameters $m_L - m_S$, Γ_L, Γ_S, p and q have high precision and the values of t_0 and t_1 are consistent with the event selection criteria in experiment.

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APPENDIX A: THE RELATIONS USED IN $\mathcal{A}_{CP}^{K_{S,L}^0}$ CALCULATIONS

Below we present the relations used in evaluating the expression of $\mathcal{A}_{CP}^{K_{S,L}^0}$ in Sec. III

$$\begin{aligned}
&\Gamma(B_c^- \to B^- K_{S,L}^0) \cdot \Gamma(B^- \to f_{B^-}) - \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot \Gamma(B^+ \to \bar{f}_{B^-}) \\
&= \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot (\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})) + (\Gamma(B_c^- \to B^- K_{S,L}^0) - \Gamma(B_c^+ \to B^+ K_{S,L}^0)) \cdot \Gamma(B^- \to f_{B^-}) \\
&= \Gamma(B_c^- \to B^- K_{S,L}^0) \cdot (\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})) + (\Gamma(B_c^- \to B^- K_{S,L}^0) - \Gamma(B_c^+ \to B^+ K_{S,L}^0)) \cdot \Gamma(B^+ \to \bar{f}_{B^-}),
\end{aligned} \tag{A1}$$

and

$$\begin{aligned}
&\Gamma(B_c^- \to B^- K_{S,L}^0) \cdot \Gamma(B^- \to f_{B^-}) + \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot \Gamma(B^+ \to \bar{f}_{B^-}) \\
&= \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot (\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})) + (\Gamma(B_c^- \to B^- K_{S,L}^0) - \Gamma(B_c^+ \to B^+ K_{S,L}^0)) \cdot \Gamma(B^- \to f_{B^-}) \\
&= (\Gamma(B_c^- \to B^- K_{S,L}^0) + \Gamma(B_c^+ \to B^+ K_{S,L}^0)) \cdot \Gamma(B^- \to f_{B^-}) - \Gamma(B_c^+ \to B^+ K_{S,L}^0) \cdot (\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})).
\end{aligned} \tag{A2}$$

APPENDIX B: THE $\mathcal{A}_{CPT}$ CALCULATIONS

To calculate the expression of $\mathcal{A}_{CPT}$, the following relations are employed

$$\begin{aligned}
&[\Gamma(B_c^- \to B^- K_S^0)\Gamma(B^- \to f_{B^-}) - \Gamma(B_c^+ \to B^+ K_S^0)\Gamma(B^+ \to \bar{f}_{B^-})] \\
&\quad - [\Gamma(B_c^- \to B^- K_L^0)\Gamma(B^- \to f_{B^-}) - \Gamma(B_c^+ \to B^+ K_L^0)\Gamma(B^+ \to \bar{f}_{B^-})] \\
&= [\Gamma(B_c^+ \to B^+ K_S^0) - \Gamma(B_c^+ \to B^+ K_L^0)] \cdot [\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})] \\
&\quad + [(\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^+ \to B^+ K_S^0)) - (\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0))] \cdot \Gamma(B^- \to f_{B^-}),
\end{aligned} \tag{B1}$$

and

$$\begin{aligned}
&\Gamma(B_c^- \to B^- K_S^0)\Gamma(B^- \to f_{B^-}) + \Gamma(B_c^+ \to B^+ K_S^0)\Gamma(B^+ \to \bar{f}_{B^-}) \\
&\quad + \Gamma(B_c^- \to B^- K_L^0)\Gamma(B^- \to f_{B^-}) + \Gamma(B_c^+ \to B^+ K_L^0)\Gamma(B^+ \to \bar{f}_{B^-}) \\
&= [\Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^+ \to B^+ K_L^0)] \cdot [\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})] \\
&\quad + 2[\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0)] \cdot \Gamma(B^- \to f_{B^-}) + [(\Gamma(B_c^- \to B^- K_S^0) \\
&\quad - \Gamma(B_c^+ \to B^+ K_S^0)) - (\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0))] \cdot \Gamma(B^- \to f_{B^-}) \\
&= [\Gamma(B_c^- \to B^- K_S^0) + \Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^- \to B^- K_L^0) + \Gamma(B_c^+ \to B^+ K_L^0)] \\
&\quad \cdot \Gamma(B^- \to f_{B^-}) - [\Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^+ \to B^+ K_L^0)] \cdot [\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})].
\end{aligned} \tag{B2}$$

By combining Eqs. (60), (61), (62), (B1), and (B2), we can obtain

$$\begin{aligned}\mathcal{A}_{CPT} \approx & \frac{\Gamma(B_c^+ \to B^+ K_S^0) - \Gamma(B_c^+ \to B^+ K_L^0)}{\Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \cdot \frac{\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})} \\ & - \frac{\Gamma(B_c^+ \to B^+ K_S^0) - \Gamma(B_c^+ \to B^+ K_L^0)}{\Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \cdot \frac{\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})} \\ & \cdot \frac{\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0)}{\Gamma(B_c^- \to B^- K_L^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \\ & - \frac{\Gamma(B_c^+ \to B^+ K_S^0) - \Gamma(B_c^+ \to B^+ K_L^0)}{\Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \cdot \frac{\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})} \\ & \cdot \frac{[\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^+ \to B^+ K_S^0)] - [\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0)]}{\Gamma(B_c^- \to B^- K_S^0) + \Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^- \to B^- K_L^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \\ & + \frac{[\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^+ \to B^+ K_S^0)] - [\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0)]}{\Gamma(B_c^- \to B^- K_S^0) + \Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^- \to B^- K_L^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \\ & + \frac{[\Gamma(B_c^- \to B^- K_S^0) - \Gamma(B_c^+ \to B^+ K_S^0)] - [\Gamma(B_c^- \to B^- K_L^0) - \Gamma(B_c^+ \to B^+ K_L^0)]}{\Gamma(B_c^- \to B^- K_S^0) + \Gamma(B_c^+ \to B^+ K_S^0) + \Gamma(B_c^- \to B^- K_L^0) + \Gamma(B_c^+ \to B^+ K_L^0)} \\ & \cdot \frac{\Gamma(B^- \to f_{B^-}) - \Gamma(B^+ \to \bar{f}_{B^-})}{\Gamma(B^- \to f_{B^-}) + \Gamma(B^+ \to \bar{f}_{B^-})}, \end{aligned} \tag{B3}$$

then we obtain

$$\begin{aligned}\mathcal{A}_{CPT} = & R(B_c^+ \to B^+ K_{S,L}^0) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-}) - R(B_c^+ \to B^+ K_{S,L}^0) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-}) \cdot \mathcal{A}_{CP}(B_c^\pm \to B^\pm K_L^0) \\ & - R(B_c^+ \to B^+ K_{S,L}^0) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-}) \cdot \mathcal{A}_{CPT}^{K_{S,L}^0}(B_c^\pm \to B^\pm K_{S,L}^0) \\ & + \mathcal{A}_{CPT}^{K_{S,L}^0}(B_c^\pm \to B^\pm K_{S,L}^0) + \mathcal{A}_{CPT}^{K_{S,L}^0}(B_c^\pm \to B^\pm K_{S,L}^0) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-}). \end{aligned} \tag{B4}$$

Neglecting the high order terms of $R(B_c^+ \to B^+ K_{S,L}^0) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-})$ and $\mathcal{A}_{CPT}^{K_{S,L}^0}(B_c^\pm \to B^\pm K_{S,L}^0)$, we can obtain

$$\mathcal{A}_{CPT} = R(B_c^+ \to B^+ K_{S,L}^0) \cdot \mathcal{A}_{CP}(B^- \to f_{B^-}) + \mathcal{A}_{CPT}^{K_{S,L}^0}(B_c^\pm \to B^\pm K_{S,L}^0)). \tag{B5}$$

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