

# Canonical analysis of covariant unimodular gravity and an extension of the Kodama state

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We carry out the canonical analysis of a covariant version of unimodular gravity in terms of the connection representation. We then proceed to quantize this theory by implementing the Dirac procedure. We confirm whether and how the Kodama state, which is a solution of quantum general relativity, can be extended into covariant unimodular gravity. Finally, we discuss the difference of quantum states between covariant unimodular gravity, the original unimodular gravity, and general relativity.

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## I. INTRODUCTION

Unimodular gravity is a simple modification of general relativity (GR) in which the determinant of the spacetime metric is restricted to be constant. Due to this restriction, unimodular gravity does not preserve the full diffeomorphism invariance. Nevertheless, the classical field equations in unimodular gravity are almost the same as in GR. A subtle but crucial difference from GR is that the cosmological constant is treated as an arbitrary integration constant [1]. This arbitrariness brings a different perspective on the cosmological constant problem [2,3].

The Hamiltonian analysis of unimodular gravity in terms of the Arnowitt-Deser-Misner (ADM) variables has been performed in Refs. [4,5]. In contrast to ordinary GR, the lapse function is not regarded as an independent variable due to the unimodular condition, and the Hamiltonian constraint is a second-class constraint. Additionally, the total Hamiltonian does not vanish on the constraint surface. In canonical quantum theory, these differences from GR cause the differences in the physical states. Specifically, the physical state of unimodular gravity is constructed from the eigenstates of the cosmological constant. Furthermore, unimodular gravity can have an appropriate time variable, and the physical state obeys the Schrödinger-like equation rather than the Wheeler-DeWitt one. In this sense, one can avoid the problem of time in quantum gravity [2,6].

In this paper, we perform the canonical analysis of unimodular gravity and its quantization; however, the theory we will discuss has two different points from the original unimodular gravity explained above. The first point is that we employ a covariant version of unimodular

gravity that was suggested in Ref. [7]. In this framework, the square root of the determinant of the spacetime metric is equal to the divergence of a densitized vector field, and the full diffeomorphism invariance is retained. Moreover, one can introduce time as spacetime volume [8]. This theory gives the same physics as the original unimodular gravity at least at the classical level, while we can expect that these two unimodular theories provide different quantum theories because of the difference of the constraints.

The second point is that we describe the theory in terms of the connection representation instead of the ADM one. Within the framework, one of the configuration variables is the Ashtekar-Barbero connection with the Barbero-Immirzi parameter  $\beta$ , and its conjugate momentum is the densitized triad [9–11]. The advantage of this representation is that the constraints are somewhat simpler than those in the ADM representation. Thanks to the simplicity, several solutions that satisfy quantum first-class constraints of GR have been found. In particular, the Kodama state, which is also called the Chern-Simons state, is a well-known solution of quantum GR with a nonvanishing cosmological constant in the case of  $\beta = i$  (the imaginary unit) [12,13]. A generalization of the Kodama state for real values of  $\beta$  was also suggested in Ref. [14]. On the other hand, the Kodama state is not regarded as a physical state in the original unimodular gravity [15].

The aim of this paper is to confirm the difference between covariant unimodular gravity, the original unimodular gravity, and GR, especially at the quantum level. The manuscript is organized as follows. In Sec. II, we perform the Hamiltonian analysis of covariant unimodular gravity in terms of the connection representation. While some work along this line has been done for  $\beta = i$  [16,17], we further develop the theory for real values of  $\beta$ . Although this choice makes the Hamiltonian constraint more

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complicated than in the case of  $\beta = i$ , it facilitates the construction of the inner product in quantum theory. In Sec. III, we quantize the theory by applying the Dirac procedure [18,19]. Then, we consider whether and how the Kodama state can be extended into covariant unimodular gravity, mainly following Ref. [14]. In Sec. IV, we summarize our results including the comparison between covariant unimodular gravity, the original unimodular gravity, and GR.

We use the following notation. Greek letters  $\mu, \nu, \dots \in \{\tau, 1, 2, 3\}$  indicate four-dimensional spacetime indices where  $\tau$  is the time flow component. Capital letters  $I, J, \dots \in \{0, 1, 2, 3\}$  are Lorentz indices. Letters  $a, b, \dots \in \{1, 2, 3\}$  are three-dimensional spatial indices, and  $i, j, \dots \in \{1, 2, 3\}$  are internal  $\text{su}(2)$  Lie algebra indices. We employ a four-metric signature  $(-, +, +, +)$ , and use units in which the speed of light is unity.

## II. CANONICAL ANALYSIS

The simplest action of the original unimodular gravity without matter is obtained by modifying the Einstein-Hilbert action

$$S_{\text{UG}}(g_{\mu\nu}, \Lambda) = \frac{1}{2k} \int d^4x \left[ \sqrt{-\det g} R^{(4)} - \Lambda(\sqrt{-\det g} - \alpha) \right], \quad (1)$$

where  $k$  is Newton's constant times  $8\pi$ ,  $R^{(4)}$  is a scalar curvature of four-dimensional spacetime,  $\Lambda$  is a scalar field that plays the role of a Lagrange multiplier, and  $\alpha$  is a fixed scalar density. The variation with respect to  $\Lambda$  leads to the unimodular condition  $\sqrt{-\det g} - \alpha = 0$ . In this framework, the spacetime diffeomorphism is restricted so that the value of the determinant of the four-metric remains unchanged.

The reformulation of unimodular gravity that ensures full diffeomorphism invariance was introduced by Henneaux and Teitelboim [7]. The action has the form

$$S_{\text{HT}}(g_{\mu\nu}, \Lambda, \phi^\mu) = \frac{1}{2k} \int d^4x \left[ \sqrt{-\det g} R^{(4)} - \Lambda(\sqrt{-\det g} - \partial_\mu \phi^\mu) \right], \quad (2)$$

where  $\phi^\mu$  is a densitized vector field of weight one. The unimodular condition is rewritten as  $\sqrt{-\det g} - \partial_\mu \phi^\mu = 0$ . Let us write the action corresponding to (2) in terms of the connection representation. This can be done by modifying the Holst action [20]

$$\begin{aligned} S(\omega_\mu^{IJ}, e_\mu^I, \Lambda, \phi^\mu) &= \int d^4x L \\ &= -\frac{1}{2k\beta} \int e^I \wedge e^J \wedge \left( R_{IJ}^{(4)} - \frac{\beta}{2} \epsilon_{IJKL} R^{(4)KL} \right) \\ &\quad - \frac{1}{48k} \int \Lambda \epsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \\ &\quad + \frac{1}{2k} \int d^4x \Lambda \partial_\mu \phi^\mu, \end{aligned} \quad (3)$$

where  $e^I$  is a cotetrad,  $R^{(4)IJ} = d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ}$  is a curvature of the spin connection  $\omega_\mu^{IJ}$ , and  $\beta$  is the Barbero-Immirzi parameter that takes nonvanishing real values. The 3 + 1 form of the above action under the time gauge  $e_a^0 = 0$  is written as

$$\begin{aligned} S &= \frac{1}{k\beta} \int d^4x (E_i^a \dot{A}_a^i - A_i^a G_i - N^a V_a - NC) \\ &\quad + \frac{1}{2k} \int d^4x (\Lambda \dot{\phi}^\tau - \Lambda N \det e - \phi^a \partial_a \Lambda), \end{aligned} \quad (4)$$

where  $\det e$  is a determinant of a cotriad  $e_a^i$ ,  $E_i^a = (\det e) e_i^a$  is a densitized triad,  $A_\mu^i = -\frac{1}{2} \epsilon^i_{jk} \omega_\mu^{jk} - \beta \omega_\mu^{0i}$ ,  $N^a$  is a shift vector, and  $N$  is a lapse function. The spatial component of  $A_\mu^i$  is usually expressed as  $A_a^i = \Gamma_a^i + \beta K_a^i$ , where  $\Gamma_a^i = -\frac{1}{2} \epsilon^i_{jk} \omega_a^{jk}$  is a three-dimensional spin connection compatible with  $e_a^i$ , and  $K_a^i = -\omega_a^{0i} = K_{ab} e_j^b \delta^{ij}$  is related with the extrinsic curvature  $K_{ab}$  and  $e_a^i$ . In addition,

$$G_i = -(D_a E^a)_i = -(\partial_a E_a^i + \epsilon_{ij}^k A_a^j E_k^i), \quad (5)$$

$$V_a = -E_a^b F_{ba}^i, \quad (6)$$

$$C = \frac{1}{2\beta \sqrt{\det E}} \epsilon^{ijk} E_i^a E_j^b [(1 + \beta^2) R_{abk}(E) - F_{abk}], \quad (7)$$

where  $F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon^i_{jk} A_a^j A_b^k$  is a curvature of  $A_a^i$ ,  $R_{ab}^i(E) = \partial_a \Gamma_b^i - \partial_b \Gamma_a^i + \epsilon^i_{jk} \Gamma_a^j \Gamma_b^k$  is a curvature of  $\Gamma_a^i$  that is constructed from  $E_a^i$ , and  $\det E = (\det e)^2$  is a determinant of  $E_a^i$ . Note that while  $C$  (7) is often expressed as

$$C = \frac{\beta}{2\sqrt{\det E}} \epsilon^{ijk} E_i^a E_j^b [F_{abk} - (1 + \beta^2) \epsilon_{klm} K_a^l K_b^m], \quad (8)$$

we use the former expression (7) for latter convenience.

The configuration variables of this theory are  $(A_\tau^i, A_a^i, N, N^a, \Lambda, \phi^\tau, \phi^a)$ . The canonical conjugate momenta (multiplied by  $k\beta$  or  $2k$ ) are given by

$$\pi_i = k\beta \frac{\partial L}{\partial A_\tau^i} = 0, \quad E_i^a = k\beta \frac{\partial L}{\partial A_a^i}, \quad (9)$$

$$\pi_N = k\beta \frac{\partial L}{\partial \dot{N}} = 0, \quad \pi_a = k\beta \frac{\partial L}{\partial \dot{N}^a} = 0, \quad (10)$$

$$\pi_\Lambda = 2k \frac{\partial L}{\partial \dot{\Lambda}} = 0, \quad (11)$$

$$p_\tau = 2k \frac{\partial L}{\partial \dot{\phi}^\tau} = \Lambda, \quad p_a = 2k \frac{\partial L}{\partial \dot{\phi}^a} = 0. \quad (12)$$

These momenta yield primary constraints

$$\pi_i \approx \pi_N \approx \pi_a \approx \pi_\Lambda \approx \Pi \approx p_a \approx 0, \quad (13)$$

where  $\Pi = p_\tau - \Lambda$ , and the symbol  $\approx$  is weak equality, which indicates that the equality holds on the constraint surface. The fundamental Poisson bracket relations are

$$\begin{aligned} \{A_\tau^i(x), \pi_j(y)\} &= k\beta \delta_j^i \delta^3(x-y), \\ \{A_a^i(x), E_j^b(y)\} &= k\beta \delta_a^i \delta_j^b \delta^3(x-y), \\ \{N(x), \pi_N(y)\} &= k\beta \delta^3(x-y), \\ \{N^a(x), \pi_b(y)\} &= k\beta \delta_b^a \delta^3(x-y), \\ \{\Lambda(x), \pi_\Lambda(y)\} &= 2k \delta^3(x-y), \\ \{\phi^\tau(x), p_\tau(y)\} &= 2k \delta^3(x-y), \\ \{\phi^a(x), p_b(y)\} &= 2k \delta_b^a \delta^3(x-y). \end{aligned} \quad (14)$$

The total Hamiltonian  $H_T$  is a combination of the ordinary Hamiltonian and the primary constraints with Lagrange multipliers  $v^i, v^a, v_N, v_\Lambda, \tilde{w}, \tilde{w}^a$ :

$$\begin{aligned} H_T(A_\tau^i, \pi_i, A_a^i, E_i^a, N, \pi_N, N^a, \pi_a, \Lambda, \pi_\Lambda, \phi^\tau, p_\tau, \phi^a, p_a) \\ = \int d^3x \left[ \frac{1}{k\beta} (A_\tau^i G_i + N^a V_a + NC) \right. \\ \left. + \frac{1}{2k} (\Lambda N \det e + \phi^a \partial_a \Lambda) + \frac{1}{k\beta} (v^i \pi_i + v^a \pi_a + v_N \pi_N) \right. \\ \left. + \frac{1}{2k} (v_\Lambda \pi_\Lambda + \tilde{w} \Pi + \tilde{w}^a p_a) \right], \end{aligned} \quad (15)$$

where  $\tilde{w}$  and  $\tilde{w}^a$  are densities of weight one. In general, every constraint must satisfy the stability condition, that is, each constraint must hold under time evolution on the constraint surface. Applying this condition to the primary constraints (13), we have

$$\{\pi_i, H_T\} = -G_i \approx 0, \quad (16)$$

$$\{\pi_a, H_T\} = -V_a \approx 0, \quad (17)$$

$$\{\pi_N, H_T\} = -\Phi \approx 0, \quad (18)$$

$$\{\pi_\Lambda, H_T\} = -(N \det e - \partial_a \phi^a - \tilde{w}) \approx 0, \quad (19)$$

$$\{\Pi, H_T\} = -v_\Lambda \approx 0, \quad (20)$$

$$\{p_a, H_T\} = -\Sigma_a \approx 0, \quad (21)$$

where

$$\begin{aligned} \Phi &= \frac{1}{2\beta \sqrt{\det E}} \epsilon^{ijk} E_i^a E_j^b \\ &\times \left[ (1 + \beta^2) R_{abk} - F_{abk} + \frac{\beta^2 \Lambda}{6} \epsilon_{abc} E_k^c \right], \end{aligned} \quad (22)$$

$$\Sigma_a = \partial_a \Lambda. \quad (23)$$

While  $v^i, v^a, v_N$ , and  $\tilde{w}^a$  remain unspecified,  $v_\Lambda$  and  $\tilde{w}$  are determined by Eqs. (19) and (20) as

$$v_\Lambda = 0, \quad \tilde{w} = N \det e - \partial_a \phi^a. \quad (24)$$

The secondary constraints  $G_i \approx 0$  (16),  $V_a \approx 0$  (17), and  $\Phi \approx 0$  (18) are the Gauss, vector, and Hamiltonian constraints, respectively, which are the same as those in GR. The secondary constraint  $\Sigma_a \approx 0$  (21) implies that  $\Lambda$  is a spatial constant. We define the smeared versions of the secondary constraints:

$$G[X^i] = \frac{1}{k\beta} \int d^3x X^i G_i(x), \quad (25)$$

$$V[X^a] = \frac{1}{k\beta} \int d^3x X^a V_a(x), \quad (26)$$

$$\Phi[X] = \frac{1}{k\beta} \int d^3x X \Phi(x), \quad (27)$$

$$\Sigma[\tilde{X}^a] = \frac{1}{2k} \int d^3x \tilde{X}^a \Sigma_a(x), \quad (28)$$

where  $X^i, X^a, X$  are test functions, and  $\tilde{X}^a$  is a densitized test function. One can check that every secondary constraint has a weakly vanishing Poisson bracket with the total Hamiltonian (15). Then, every secondary constraint automatically satisfies the stability condition, and no more constraints arise.

Now, we can classify the primary constraints  $(\pi_i, \pi_a, \pi_N, \pi_\Lambda, \Pi, p_a)$  and the secondary constraints  $(G[X^i], V[X^a], \Phi[X], \Sigma[\tilde{X}^a])$  into the first- and second-class constraints. The weakly nonvanishing Poisson brackets among these constraints are

$$\{\pi_\Lambda(x), \Pi(y)\} = 2k \delta^3(x-y), \quad (29)$$

$$\{\pi_\Lambda, \Phi[X]\} = -X \det e, \quad (30)$$

$$\{\pi_\Lambda, \Sigma[\tilde{X}^a]\} = \partial_a \tilde{X}^a. \quad (31)$$

Hence constraints  $(\pi_\Lambda, \Pi, \Phi[X], \Sigma[\tilde{X}^a])$  are second class, and the remaining constraints are first class.

The number of the second-class constraints can be reduced by replacing  $\Phi[X]$  and  $\Sigma[\tilde{X}^a]$  with the modified constraints  $\Phi'[X]$  and  $\Sigma'[\tilde{X}^a]$ , respectively:

$$\begin{aligned}\Phi'[X] &= \Phi[X] + \frac{1}{2k} \int d^3x X \Pi \det e \\ &= \frac{1}{2k\beta^2} \int d^3x \frac{X}{\sqrt{\det E}} e^{ijk} E_i^a E_j^b \\ &\quad \times \left[ (1 + \beta^2) R_{abk} - F_{abk} + \frac{\beta^2}{6} \epsilon_{abc} p_\tau E_k^c \right],\end{aligned}\quad (32)$$

$$\begin{aligned}\Sigma'[\tilde{X}^a] &= \Sigma[\tilde{X}^a] + \frac{1}{2k} \int d^3x \tilde{X}^a \partial_a \Pi \\ &= \frac{1}{2k} \int d^3x \tilde{X}^a \partial_a p_\tau.\end{aligned}\quad (33)$$

In fact, the weakly nonvanishing Poisson bracket among the primary constraints  $(\pi_i, \pi_a, \pi_N, \pi_\Lambda, \Pi, p_a)$  and the secondary constraints  $(G[X^i], V[X^a], \Phi'[X], \Sigma'[\tilde{X}^a])$  is only one:

$$\{\pi_\Lambda(x), \Pi(y)\} = 2k\delta^3(x-y). \quad (34)$$

Then, constraints  $(\pi_\Lambda, \Pi)$  are second class, and the remaining constraints are first class. Let us count the degrees of freedom of this theory. Variables  $(A_\tau^i, A_a^i, N, N^a, \Lambda, \phi^\tau, \phi^a)$  have  $3 + 9 + 1 + 3 + 1 + 1 + 3 = 21$  components. The first-class constraints  $(\pi_i, \pi_a, \pi_N, p_a, G[X^i], V[X^a], \Phi'[X], \Sigma'[\tilde{X}^a])$  constrain  $3 + 3 + 1 + 3 + 3 + 3 + 1 + 1 = 18$  components. Note that  $\Sigma'[\tilde{X}^a]$  constrains only one component, because this constraint is parametrized by  $\partial_a \tilde{X}^a$  rather than  $\tilde{X}^a$ . The second-class constraints  $(\pi_\Lambda, \Pi)$  constrain  $(1 + 1)/2 = 1$  component. Therefore, the local degrees of freedom in configuration space are  $21 - 18 - 1 = 2$ . This result is consistent with GR and the original unimodular gravity [15].

Using the second-class constraints  $\Pi \approx 0$  and  $\pi_\Lambda \approx 0$ , we can eliminate variables  $\Lambda$  and  $\pi_\Lambda$  by substituting

$$\Lambda = p_\tau, \quad \pi_\Lambda = 0. \quad (35)$$

After the elimination, the total Hamiltonian (15) is rewritten as

$$\begin{aligned}H_T(A_\tau^i, \pi_i, A_a^i, E_i^a, N, \pi_N, N^a, \pi_a, \phi^\tau, p_\tau, \phi^a, p_a) \\ = G[A_\tau^i] + V[N^a] + \Phi'[N] + \Sigma'[\phi^a] \\ + \int d^3x \left[ \frac{1}{k\beta} (v^i \pi_i + v^a \pi_a + v_N \pi_N) + \frac{1}{2k} \tilde{w}^a p_a \right].\end{aligned}\quad (36)$$

The constraint  $\Sigma'[\tilde{X}^a] \approx 0$  (33) and the evolution equation  $\{p_\tau, H_T\} \approx 0$  imply that  $p_\tau$  is a spacetime constant. In addition, the evolution equation  $\{\phi^\tau, H_T\} = N \det e - \partial_a \phi^a$  leads to the covariant version of the unimodular condition  $N \det e - \partial_\mu \phi^\mu = 0$ . Obviously,  $p_\tau$  and  $\Lambda$  correspond to the cosmological constant (times two) in GR.

We can introduce the spatial diffeomorphism constraint  $\mathcal{D}[X^a]$  by extending the vector constraint  $V[X^a]$ :

$$\mathcal{D}[X^a] = V[X^a] + G[X^a A_a^i] - \Sigma'[X^a \phi^\tau]. \quad (37)$$

This constraint generates the spatial diffeomorphism for the dynamical variables

$$\{A_a^i, \mathcal{D}[X^b]\} = \mathcal{L}_{\vec{X}} A_a^i, \quad \{E_i^a, \mathcal{D}[X^b]\} = \mathcal{L}_{\vec{X}} E_i^a, \quad (38)$$

$$\{\phi^\tau, \mathcal{D}[X^a]\} = \mathcal{L}_{\vec{X}} \phi^\tau, \quad \{p_\tau, \mathcal{D}[X^a]\} = \mathcal{L}_{\vec{X}} p_\tau, \quad (39)$$

where  $\mathcal{L}_{\vec{X}}$  is a Lie derivative along  $\vec{X}$ . The constraint  $\mathcal{D}[X^a]$  is first class, and holds the stability condition

$$\begin{aligned}\{\mathcal{D}[X^a], H_T\} &= G[\mathcal{L}_{\vec{X}} A_\tau^i] + V[\mathcal{L}_{\vec{X}} N^a] \\ &\quad + \Phi'[\mathcal{L}_{\vec{X}} N] + \Sigma'[\mathcal{L}_{\vec{X}} \phi^a] \\ &\approx 0.\end{aligned}\quad (40)$$

We employ  $\mathcal{D}[X^a]$  instead of  $V[X^a]$  as an element of the first-class constraints.

### III. QUANTIZATION

After the reduction (35), all the remaining constraints  $(\pi_i, \pi_a, \pi_N, p_a, G[X^i], \mathcal{D}[X^a], \Phi'[X], \Sigma'[\tilde{X}^a])$  belong to first class. Hence, we can proceed to quantize this theory by replacing Poisson brackets  $\{\bullet, \bullet\}$  with quantum commutators  $(i\hbar)^{-1}[\hat{\bullet}, \hat{\bullet}]$ . The quantum operators corresponding to the canonical variables are given by

$$\hat{A}_\tau^i = A_\tau^i, \quad \hat{\pi}_i = -i\hbar k\beta \frac{\delta}{\delta A_\tau^i}, \quad (41)$$

$$\hat{A}_a^i = A_a^i, \quad \hat{E}_i^a = -i\hbar k\beta \frac{\delta}{\delta A_a^i}, \quad (42)$$

$$\hat{N} = N, \quad \hat{\pi}_N = -i\hbar k\beta \frac{\delta}{\delta N}, \quad (43)$$

$$\hat{N}^a = N^a, \quad \hat{\pi}_a = -i\hbar k\beta \frac{\delta}{\delta N^a}, \quad (44)$$

$$\hat{\phi}^\tau = \phi^\tau, \quad \hat{p}_\tau = -2i\hbar k \frac{\delta}{\delta \phi^\tau}, \quad (45)$$

$$\hat{\phi}^a = \phi^a, \quad \hat{p}_a = -2i\hbar k \frac{\delta}{\delta \phi^a}. \quad (46)$$

A physical state  $\Psi$  must satisfy the quantized first-class constraints

$$\hat{\pi}_i \Psi = \hat{\pi}_N \Psi = \hat{\pi}_a \Psi = \hat{p}_a \Psi = 0, \quad (47)$$

$$\hat{G}[X^i] \Psi = \hat{D}[X^a] \Psi = \hat{\Phi}'[X] \Psi = \hat{\Sigma}'[\tilde{X}^a] \Psi = 0. \quad (48)$$

Constraints (47) indicate that  $\Psi$  should be independent from  $A_\tau^i, N, N^a$ , and  $\phi^a$ , that is,

$$\Psi = \Psi[\phi^\tau, A_a^i]. \quad (49)$$

We assume that the wave functional is variable-separable, namely,  $\Psi[\phi^\tau, A_a^i] = \Psi[\phi^\tau] \Psi[A_a^i]$ . Constraint  $\hat{\Sigma}'[\tilde{X}^a] \Psi = (2k)^{-1} \int d^3x \tilde{X}^a \partial_a \hat{p}_\tau \Psi \approx 0$  implies that  $\Psi$  has the form

$$\Psi[\phi^\tau, A_a^i] = \exp \left[ \frac{\lambda}{-2i\hbar k} \int d^3x \phi^\tau \right] \Psi[A_a^i], \quad (50)$$

that satisfies

$$\hat{p}_\tau \Psi[\phi^\tau, A_a^i] = \lambda \Psi[\phi^\tau, A_a^i], \quad (51)$$

where  $\lambda$  is an unspecified constant. Note that  $\hat{p}_\tau$  weakly commutes with every quantum first-class constraint; therefore,  $\hat{p}_\tau$  is the physical observable in the sense of Dirac [19].

A possible solution of the constraints (47) and (48) is expressed as

$$\begin{aligned} \Psi_{\lambda,R}[\phi^\tau, A_a^i] &= \Psi_\lambda[\phi^\tau] \Psi_{\lambda,R}[A_a^i] \\ &= \exp \left[ \frac{\lambda}{-2i\hbar k} \int d^3x \phi^\tau \right] \\ &\quad \times \exp \left[ \frac{6}{i\hbar k \beta^3 \lambda} \left( Y_{\text{CS}}[A_a^i] - 2(1 + \beta^2) \int \text{Tr}(A \wedge R) \right) \right], \end{aligned} \quad (52)$$

where

$$Y_{\text{CS}}[A_a^i] = \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (53)$$

is the Chern-Simons functional,  $\text{Tr}$  indicates the trace of  $\text{SU}(2)$  generators  $T_i$ , which is normalized as  $\text{Tr}(T_i T_j) = -\frac{1}{2} \delta_{ij}$ , and  $\lambda$  and  $R_{ab}^i$  are parameters associated with the cosmological constant and the spatial curvature, respectively. We would like to emphasize that the form of the second factor in (52),  $\Psi_{\lambda,R}[A_a^i]$ , was originally proposed in Ref. [14] as a generalization of the Kodama state for real values of  $\beta$ .

Since this state is a pure phase, we can define a naive inner product

$$\begin{aligned} \langle \Psi_{\lambda',R'} | \Psi_{\lambda,R} \rangle &= \int \mathcal{D}\phi^\tau \mathcal{D}A \Psi_{\lambda',R'}^*[\phi^\tau, A_a^i] \Psi_{\lambda,R}[\phi^\tau, A_a^i] \\ &\sim \delta(\lambda - \lambda') \delta(R - R'), \end{aligned} \quad (54)$$

where

$$\delta(R - R') = \prod_x \prod_{a,b,i} \delta(R_{ab}^i(x) - R'_{ab}^i(x)). \quad (55)$$

The inner product (54) has an undesirable property. Due to the factor  $\delta(R - R')$ , when  $R_{ab}^i$  and  $R'_{ab}^i$  have different values, this inner product vanishes even if  $R_{ab}^i$  and  $R'_{ab}^i$  are connected by the gauge and spatial diffeomorphism transformations. We can improve this inner product by using the group averaging technique [14]

$$\begin{aligned} (\Psi_{\lambda',R'} | \Psi_{\lambda,R}) &= \int \mathcal{D}g \langle \Psi_{\lambda',\varphi_g R'} | \Psi_{\lambda,R} \rangle \\ &\sim \delta(\lambda - \lambda') \int \mathcal{D}g \delta(R - \varphi_g R'), \end{aligned} \quad (56)$$

where  $\varphi_g$  is the gauge and spatial diffeomorphism transformations parametrized by  $g$ , and  $\int \mathcal{D}g$  is an integral over both transformations. The inner product (56) does not vanish when  $\lambda = \lambda'$  and  $R'_{ab}^i$  can reach  $R_{ab}^i$  by these transformations. Note that these transformations do not affect  $\lambda$ . We find that the state

$$(\Psi_{\lambda,R} | = \int \mathcal{D}g \langle \Psi_{\lambda,\varphi_g R} | \quad (57)$$

is invariant under the gauge and spatial diffeomorphism transformations, because

$$(\Psi_{\lambda,R} | \hat{U}(\varphi_g) = \int \mathcal{D}g \langle \Psi_{\lambda,\varphi_g \circ \varphi_g R} | = (\Psi_{\lambda,R} |, \quad (58)$$

where  $\hat{U}(\varphi_g)$  is the operator corresponding to these transformations. This is an analog of the strategy to obtain the gauge and spatial diffeomorphism invariant state in loop quantum gravity [21,22].

One can write the curvature operator  $\hat{R}_{ab}^i$  by using the inner product (56) as

$$\begin{aligned} &\int d^3x \tilde{X}_i^{ab} \hat{R}_{ab}^i \\ &= \int d^3x \tilde{X}_i^{ab} \int \mathcal{D}g \mathcal{D}R' \mathcal{D}\lambda' \varphi_g R_{ab}^i | \Psi_{\lambda',\varphi_g R'} \rangle \langle \Psi_{\lambda',\varphi_g R'} |, \end{aligned} \quad (59)$$

where  $\tilde{X}_i^{ab}$  is a densitized test function, and  $\int \mathcal{D}R'$  is an integral over the curvature parameter  $R'$  modulo the

gauge and spatial diffeomorphism transformations. From Eq. (59), we have

$$\int d^3x \tilde{X}_i^{ab} \hat{R}_{ab}^i |\Psi_{\lambda,R}\rangle = \int d^3x \tilde{X}_i^{ab} R_{ab}^i |\Psi_{\lambda,R}\rangle. \quad (60)$$

Using (51), (60), and

$$\hat{E}_i^a \Psi_{\lambda,R}[\phi^\tau, A_a^i] = \frac{3}{\beta^2 \lambda} e^{abc} [F_{bci} - (1 + \beta^2) R_{bci}] \Psi_{\lambda,R}[\phi^\tau, A_a^i], \quad (61)$$

we find that the state  $\Psi_{\lambda,R}[\phi^\tau, A_a^i]$  satisfies the Hamiltonian constraint

$$\begin{aligned} \hat{\Phi}'[X] \Psi_{\lambda,R}[\phi^\tau, A_a^i] &= \frac{1}{2k\beta^2} \int d^3x \frac{X}{\sqrt{\det \hat{E}}} \epsilon^{ijk} \hat{E}_i^a \hat{E}_j^b \\ &\times \left[ (1 + \beta^2) \hat{R}_{abk} - \hat{F}_{abk} + \frac{\beta^2}{6} \epsilon_{abc} \hat{p}_\tau \hat{E}_k^c \right] \Psi_{\lambda,R}[\phi^\tau, A_a^i] \\ &= 0. \end{aligned} \quad (62)$$

Thus, under the appropriate inner product (56), the state  $|\Psi_{\lambda,R}\rangle$  is a solution of quantum covariant unimodular gravity.

#### IV. CONCLUSIONS

In this work, we have analyzed the full theory of covariant unimodular gravity in terms of the connection representation with real values of the Barbero-Immirzi parameter  $\beta$ . Unlike the original unimodular gravity, the Hamiltonian constraint of covariant unimodular gravity (32) is first class. Therefore, the constraint structure of covariant unimodular gravity is closer to that of GR than that of the original unimodular gravity. The subtle difference from GR is that the cosmological constant in GR is replaced with the canonical momentum  $p_\tau$  which is regarded as a constant of motion and a Dirac observable.

In the original unimodular gravity, the Kodama state is not a solution of the quantum constraints [15]. On the other hand, in covariant unimodular gravity, the solution of the constraints can be obtained by extending the Kodama state. The state (52) is regarded as a natural extension of the state proposed in Ref. [14]. Since this state is a pure phase, it is delta-function normalizable. In addition, since all the canonical variables are real, we can avoid the reality condition problem. The main difference from the quantum states of GR in Ref. [14] is that each state is labeled not only by the spatial curvature  $R_{ab}^i$  (modulo the gauge and spatial diffeomorphism transformations) but also by the cosmological constant. This implies that a general solution of the physical state can be written as a superposition of the eigenstates of  $\hat{p}_\tau$  and  $\hat{R}_{ab}^i$ . Thus, at least in this framework,

covariant unimodular gravity is different from GR at the quantum level. Note that such a superposition of the different values of the cosmological constant has already appeared in previous studies both within the Hamiltonian formalism [6,16] and the path integral formalism [3,5,23].

It is still unclear whether the original unimodular gravity and covariant unimodular gravity differ at the quantum level. In the original unimodular gravity, only the wave functional with zero cosmological constant has been found [15]. On the other hand, in covariant unimodular gravity,  $\Psi_{\lambda,R}[\phi^\tau, A_a^i]$  (52) is a state with a non-vanishing cosmological constant. To compare the two theories, one needs to find a solution with a non-vanishing cosmological constant in the original unimodular gravity. Note that even if such a state is found, it is not so obvious whether the difference in the physical states provides different physical predictions.

Another approach to confirm the difference between the two unimodular theories is to compare the observables. In general, a physical observable must commute with every first-class constraint [19], while the two unimodular theories have different first-class constraints. If we can define appropriate observables, it will be easier to compare the two theories. Let us consider  $V_\tau = \int d^3x s N \det e$  as an example of a candidate for the physical observable. Here,  $s = \text{sign}(\det e)$ , and the integral is over the spacelike surface parametrized by  $\tau$ . In covariant unimodular gravity, the Poisson bracket between  $V_\tau$  and the Hamiltonian constraint does not weakly vanish:  $\{V_\tau, \Phi[X]\} = \int d^3x s X N K_a^i E_i^a \neq 0$ . This implies that, in quantum theory, not all the constraints commute with  $\hat{V}_\tau$ , and a physical state is not an eigenstate of  $\hat{V}_\tau$ . Hence,  $V_\tau$  and the four-volume  $\int d\tau V_\tau$  are not observables in covariant unimodular gravity. On the other hand, in the original unimodular gravity, the unimodular constraint  $N \det e - \alpha = 0$  is a second-class constraint. This means that  $N$  is not an independent variable, and can be eliminated as  $N = \alpha(\det e)^{-1}$ . Therefore,  $V_\tau = \int d^3x s \alpha$ . In this case, the Poisson bracket between  $V_\tau$  and every first-class constraint weakly vanishes. However,  $V_\tau$  and the four-volume in this theory should be regarded as constants associated with the Hamiltonian and the Lagrangian rather than observables. Thus,  $V_\tau$  is not an appropriate quantity to compare the two theories. To confirm the (in)equivalence of the two theories, one needs to find some other quantities that can be physical observables.

It is worthwhile to investigate how covariant unimodular gravity is extended into loop quantum gravity. Both covariant unimodular gravity discussed above and loop quantum gravity employ real values of the Barbero-Immirzi parameter  $\beta$ . Therefore, one can expect that this theory can be naturally extended into a full theory of loop quantum gravity. Note that the symmetry-reduced models of covariant unimodular gravity in the context of loop quantum cosmology has been studied in Refs. [24–26].

Another direction for future research is to investigate the relation between the extended Kodama state (52) and other physical states of unimodular gravity. In Refs. [27,28], the Hartle-Hawking state, which is the solution of quantum GR in the ADM representation, is interpreted as the Fourier dual of the Kodama state. On the other hand, there is also an argument that it is difficult to translate the Hartle-Hawking state straightforwardly into the connection

representation [29]. The question of how these arguments are modified in covariant unimodular gravity is left for future study.

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